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Research article

Investigating a generalized Hilfer-type fractional differential equation with two-point and integral boundary conditions

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Abstract: In this paper, we investigate a nonlinear generalized fractional differential equation with two-point and integral boundary conditions in the frame of κ-Hilfer fractional derivative. The existence and uniqueness results are obtained using Krasnoselskii and Banach's fixed point theorems. We analyze different types of stability results of the proposed problem by using some mathematical methodologies. At the end of the paper, we present a numerical example to demonstrate and validate our findings.

Keywords: κ-Hilfer fractional derivative; boundary conditions; fixed point theorem
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1. Introduction

Fractional differential equations (FDEs) provide more advantages than integer-order differential equations. When it comes to explaining the changing laws of nature, these equations are both flexible and exact. As a result, FDEs are commonly used in real-world situations [1–3]. However, the development of the theory of FDEs is still in its early stages because various physical interpretations of

FDEs are still unknown due to the intricacies of their initial values. Nonetheless, these equations have become a valid topic of debate among a number of researchers due to their wide range of practical applications and theoretical significance. Atangana and Baleanu [4] introduced a new definition of fractional derivatives with the nonlocal and non-singular kernel. In contrast, the properties of this operator , such as convolution of the power law, exponential decay law, and generalized Mittag-Leffler law with fractal derivative, are introduced by Atangana in [5]. For more details about the application of the Atangana-Baleanu operator see [6]. Zhu [7] studied the stabilization problem of stochastic nonlinear delay systems. Also, Zhu et al. [8] discussed the moment exponential stability problem for a class of stochastic delay nonlinear systems. Hu et al. [9] studied the Razumikhin stability theorem for a class of impulsive stochastic delay differential systems. In the last three decades, some researchers introduced definitions of fractional calculus, including definitions of Riemann-Liouville (RL) and Caputo, and less well-known definitions such as Erdelyi-Kober and Hadamard. In [10], Hilfer was given generalization of FDs of RL and Caputo, which so-called the Hilfer FD of order η and a type $q \in [0, 1]$. When we give $q = 0$ and $q = 1$ respectively in the formula of Hilfer FD can get RL and Caputo FDs. Such a derivative interpolate between the RL and Caputo FDs. For more details on this FD above-mentioned can be found in [11, 12]. In 2018, Sousa and de Oliveira [13] introduced a new FD with respect to another function ψ called "ψ-Hilfer FD" generalizes most of the previous FDs. Some of the existence and stability results of fractional boundary value problems (BVPs) are addressed in the recent literature; for instance, Benchohra et al. [14] studied the existence of solutions of a class of BVPs for a nonlinear FDE

$$\begin{cases} {}^{C}\mathcal{D}^{\alpha}\upsilon(\sigma) = f(\sigma, \upsilon(\sigma)), \quad \sigma \in [0, T], \\ a\upsilon(0) + b\upsilon(T) = c, \end{cases} \tag{1.1}$$

where $0 < \alpha < 1$, $a, b, c \in \mathbb{R}$, $a + b \neq 0$, and ${}^{C}\mathcal{D}^{\alpha}$ is the Caputo FD of order α. Salim et al. [15] discussed some existence and Ulam stability results for the implicit problem (1.1) with Caputo-Fabrizio FD. Ashyralyev et al. [16] studied the existence and uniqueness of a fractional BVP of the form

$$\begin{cases} {}^{C}\mathcal{D}^{\alpha}\upsilon(\sigma) = f(\sigma, \upsilon(\sigma)), \quad \sigma \in [0, b], \\ A\upsilon(0) + B\upsilon(b) = \int_0^b g(s, \upsilon(s))ds, \end{cases} \tag{1.2}$$

where $0 < \alpha \leq 1$, ${}^{C}\mathcal{D}^{\alpha}$ is the Caputo FD of order α, $A, B \in \mathbb{R}^{n\times n}$, $f, g : [0, b] \times \mathbb{R}^n \to \mathbb{R}^n$ are continuous. Sharifov et al. [17] investigated the existence and uniqueness of the solutions to the following fractional BVP

$$\begin{cases} {}^{C}\mathcal{D}^{\alpha}\upsilon(\sigma) = f(\sigma, \upsilon(\sigma)), \quad \sigma \in [0, b], \\ A\upsilon(0) + \int_0^b n(s)\upsilon(s)ds + B\upsilon(b) = C, \end{cases} \tag{1.3}$$

where $0 < \alpha < 1$, ${}^{C}\mathcal{D}^{\alpha}$ is the Caputo FD of order α, $A, B, C \in \mathbb{R}^{n\times n}$, $f : [0, b] \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous, and $n(s) : [0, b] \to \mathbb{R}^{n\times n}$ is given matrices.

On the other hand, Sousa and de Oliveira [18] proved the existence and uniqueness of solutions of the following Cauchy-type problem

$$\begin{cases} \mathcal{D}_{a^+}^{\alpha,\beta,\psi}\upsilon(\sigma) = f(\sigma, \upsilon(\sigma)), \quad \sigma \in (a, b], \\ \mathcal{I}^{1-\gamma,\psi}\upsilon(\sigma) = \upsilon_a, \ \gamma = \alpha + \beta(1 - \alpha), \end{cases} \tag{1.4}$$

where $0 < \alpha < 1$, $0 \le \beta \le 1$, $\upsilon_a \in \mathbb{R}$, $\mathcal{D}_{a^+}^{\alpha,\beta,\psi}$ is the ψ-Hilfer FD, and $f : [0, b] \times \mathbb{R} \to \mathbb{R}$ is continuous. The authors in [19] discussed the Ulam-Hyers (UH) stability of (1.4) for an integro-differential-type. Furthermore, they proved the Ulam-Hyers-Rassias stability results of (1.4) for an implicit type in [20]. Abdo et al., in [21, 22] investigated the existence and various types of stability theorems of fractional Cauchy problem involving ψ-Hilfer FD, whereas, some qualitative analyses of ψ-Hilfer type nonlocal Cauchy problem have been investigated in [23]. Nonlocal fractional BVPs with ψ-Hilfer FDs have been considered in [24, 25].

In this paper, we study the existence, uniqueness and UH stability results of the class of BVPs for the following nonlinear FDE

$$\begin{cases} {}_H\mathcal{D}^{\eta,q;\kappa}\upsilon(\sigma) = F(\sigma, \upsilon(\sigma)), \quad \sigma \in J := [a, b], \\ d_1\upsilon(a) + d_2\int_a^b \kappa'(s)\upsilon(s)ds + d_3\upsilon(b) = d_4, \end{cases} \tag{1.5}$$

where $0 < \eta < 1$, $0 \le q \le 1$, ${}_H\mathcal{D}^{\eta,q;\kappa}$ is the κ-Hilfer FD of order η, and type q, $F : J \times \mathbb{R} \to \mathbb{R}$ is continuous, $d_i \in \mathbb{R}$ $(i = 1, 2, 3, 4)$, $\kappa'(\sigma) \in C^1(J, \mathbb{R})$ be an increasing function with $\kappa'(\sigma) \neq 0$ for all $\sigma \in J$.

We concentrate on non-local problems because, in many cases, the non-local condition better captures physical phenomena than classical initial (border) conditions. So, utilising Banach's and Krasnoseliskii's fixed point theorems under the minimum assumptions, we analyze the result of existence and uniqueness as well as UH stability results of the BVP (1.5). The work presented in this article is current and adds to the literature, particularly in the area of nonlinear problems of the κ-Hilfer type.

In general, our results remain valid for various values of the function κ and cover many corresponding problems, for instance (Hilfer-Hadamard type problem for $\kappa(\sigma) = \log\sigma$), (Hilfer-Katugampola type problem for $\kappa(\sigma) = \sigma^\rho$, $\rho > 0$), (Caputo-type problem for $\kappa(\sigma) = \sigma$, and $q = 1$), and (RL-type problem for $\kappa(\sigma) = \sigma$, and $q = 0$).

The content of this paper is organized as follows: Section 2 presents some required results and preliminaries about κ-Hilfer FD. Our main results for the κ-Hilfer type BVP (1.5) are addressed in Section 3. Some examples to explain the acquired results are constructed in Section 4. Ultimately, we summarize our work in the conclusion section and suggest future directions.

2. Preliminaries

We set notations and certain fundamental facts in this part, which will be used in the proofs of the following results.

Let $C(J, \mathbb{R})$ and $L(J, \mathbb{R})$ are the Banach space of continuous functions and Lebesgue integrable functions from J into $\mathbb{R}$ with the norms

$$\|\upsilon\|_\infty = \sup\{|\upsilon| : \sigma \in J\}, \textit{ and}$$

$$\|\upsilon\|_L = \int_a^b |\upsilon(\sigma)|\, d\sigma,$$

respectively.

Definition 2.1. *[1] Let $\eta > 0$ and $g \in L^1(J,\mathbb{R})$. The following expression*

$$\mathcal{I}^{\eta;\kappa}g(\sigma) = \frac{1}{\Gamma(\eta)}\int_a^\sigma \kappa'(t)(\kappa(\sigma)-\kappa(t))^{\eta-1}g(t)dt,$$

is called left sided κ-RL fractional integral of order η.

Definition 2.2. *[13] The κ-Hilfer FD of order η and parameter q is defined by*

$${}_H\mathcal{D}^{\eta,q;\kappa}g(\sigma) = \mathcal{I}^{q(n-\eta);\kappa}(\frac{1}{\kappa'(\sigma)}\frac{d}{d\sigma})^n\ \mathcal{I}^{(1-q)(n-\eta);\kappa}g(\sigma),$$

where $n-1<\eta<n$, $0\le q\le 1$, $\sigma > a$.

Lemma 2.3. *[1, 13] Let η,χ, and $\delta > 0$. Then*
(1) $I^{\eta;\kappa}I^{\eta;\chi}g(\sigma) = I^{\eta+\chi;\kappa}g(\sigma)$.
(2) $I^{\eta;\kappa}(\kappa(\sigma)-\kappa(a))^{\delta-1} = \frac{\Gamma(\delta)}{\Gamma(\eta+\delta)}(\kappa(\sigma)-\kappa(a))^{\eta+\delta-1}$.

We note also that ${}_HD^{\eta,q;\kappa}(\kappa(\sigma)-\kappa(a))^{\gamma-1} = 0$.

Lemma 2.4. *[13] Let $g \in L(a,b)$, $\eta \in (n-1,n]$ $(n\in\mathbb{N})$, $q\in[0,1]$, then*

$$(\mathcal{I}^{\eta;\kappa}\ {}_H\mathcal{D}^{\eta,q;\kappa}g)(\sigma) = g(\sigma) - \sum_{k=0}^{n}\frac{(\kappa(\sigma)-\kappa(a))^{\gamma-k}}{\Gamma(\gamma-k+1)}g_\kappa^{[n-k]}\mathcal{I}^{(1-q)(n-\eta);\kappa}g(a),$$

where $g_\kappa^{[n-k]} = (\frac{1}{\kappa'(\sigma)}\frac{d}{d\sigma})^{[n-k]}g(\sigma)$.

Lemma 2.5. *Let $\upsilon \in C(J,\mathbb{R})$. Then, the unique solution of the κ-Hilfer type BVP (1.5) is given by*

$$\begin{aligned}\upsilon(\sigma) \ &= \ \frac{1}{\Lambda\Gamma(\gamma)}(\kappa(\sigma)-\kappa(a))^{\gamma-1}\\ &\left[d_4 - d_2\int_a^b \kappa'(s)\mathcal{I}^{\eta;\kappa}F(s,\upsilon(s))ds - d_3\mathcal{I}^{\eta;\kappa}F(b,\upsilon(b))\right]\\ &+\mathcal{I}^{\eta;\kappa}F(\sigma,\upsilon(\sigma)),\end{aligned} \tag{2.1}$$

where

$$\Lambda = \left[d_2 + \frac{\gamma d_3}{(\kappa(b)-\kappa(a))}\right]\frac{(\kappa(b)-\kappa(a))^\gamma}{\Gamma(\gamma+1)} \neq 0, \tag{2.2}$$

Proof. Let υ be a solution of the first equation of κ-Hilfer type BVP (1.5). Applying $\mathcal{I}^{\eta;\kappa}$ on the first equation of κ-Hilfer type BVP (1.5) with Lemma 2.4, and setting $I^{1-\eta;\kappa}\upsilon(a) = c_0$, we obtain

$$\upsilon(\sigma) = \frac{c_0}{\Gamma(\gamma)}(\kappa(\sigma)-\kappa(a))^{\gamma-1} + \mathcal{I}^{\eta;\kappa}F(\sigma,\upsilon(\sigma)), \tag{2.3}$$

where $c_{0,}$ is an arbitrary constant. From condition $d_1\upsilon(a) + d_2\int_a^b \kappa'(s)\upsilon(s)ds + d_3\upsilon(b) = d_4$, we have

$$d_4 \ = \ d_1\upsilon(a) + d_2\int_a^b \kappa'(s)\upsilon(s)ds + d_3\upsilon(b)$$

$$
\begin{aligned}
&= d_2 \int_a^b \kappa'(s)\left[\frac{c_0}{\Gamma(\gamma)}(\kappa(s)-\kappa(a))^{\gamma-1} + \mathcal{I}^{\eta;\kappa}F(s,\upsilon(s))\right]ds \\
&\quad + d_3\left[\frac{c_0}{\Gamma(\gamma)}(\kappa(b)-\kappa(a))^{\gamma-1} + \mathcal{I}^{\eta;\kappa}F(b,\upsilon(b))\right] \\
&= \frac{c_0}{\Gamma(\gamma)}\left[d_2\int_a^b \kappa'(s)(\kappa(s)-\kappa(a))^{\gamma-1}ds + d_3(\kappa(b)-\kappa(a))^{\gamma-1}\right] \\
&\quad + d_2\int_a^b \kappa'(s)\mathcal{I}^{\eta;\kappa}F(s,\upsilon(s))ds + d_3\mathcal{I}^{\eta;\kappa}F(b,\upsilon(b)) \\
&= c_0\left[d_2 + \frac{\gamma d_3}{(\kappa(b)-\kappa(a))}\right]\frac{(\kappa(b)-\kappa(a))^{\gamma}}{\Gamma(\gamma+1)} \\
&\quad + d_2\int_a^b \kappa'(s)\mathcal{I}^{\eta;\kappa}F(s,\upsilon(s))ds + d_3\mathcal{I}^{\eta;\kappa}F(b,\upsilon(b)).
\end{aligned}
$$

Hence

$$
c_{0,} = \frac{1}{\Lambda}\left[d_4 - d_2\int_a^b \kappa'(s)\mathcal{I}^{\eta;\kappa}F(s,\upsilon(s))ds - d_3\mathcal{I}^{\eta;\kappa}F(b,\upsilon(b))\right].
$$

Therefore,

$$
\begin{aligned}
\upsilon(\sigma) &= \frac{1}{\Lambda\Gamma(\gamma)}(\kappa(\sigma)-\kappa(a))^{\gamma-1} \\
&\quad \left[d_4 - d_2\int_a^b \kappa'(s)\mathcal{I}^{\eta;\kappa}F(s,\upsilon(s))ds - d_3\mathcal{I}^{\eta;\kappa}F(b,\upsilon(b))\right] \\
&\quad + \mathcal{I}^{\eta;\kappa}F(\sigma,\upsilon(\sigma)).
\end{aligned}
$$

This completes the proof. □

Here we can suffice to refer to Banach's fixed point theorem [26] and Krasnoselskii's fixed point theorem [27].

3. Existence and uniqueness results

In this part, we demonstrate the results of the existence and uniqueness of the κ-Hilfer type BVP (1.5) by employing Banach's and Krasnoselskii's fixed point theorems.

To obtain our main results, the following conditions must be satisfied:

$(H1)$ The function F is continuous and there exists $\lambda > 0$ such that

$$
|F(\sigma,\upsilon_1) - F(\sigma,\upsilon_2)| \le \lambda\,|\upsilon_1 - \upsilon_2|\,,
$$

for any $\upsilon_1, \upsilon_2 \in \mathbb{R}$ and $\sigma \in J$.

For convenience purpose, we are setting two constant:

$$
\Upsilon_1 = \lambda\left[d_2\frac{(\kappa(b)-\kappa(a))^{\eta+\gamma}}{\Lambda\Gamma(\gamma)\Gamma(\eta+2)} + \frac{d_3}{\Lambda\Gamma(\gamma)}\frac{(\kappa(b)-\kappa(a))^{\eta}}{\Gamma(\eta+1)} + \frac{(\kappa(b)-\kappa(a))^{\eta}}{\Gamma(\eta+1)}\right]. \tag{3.1}
$$

Theorem 3.1. *Assume that (H_1) holds. Then the κ-Hilfer type BVP (1.5) has at least one solution on J, provided that*

$$\Upsilon_1 < 1, \tag{3.2}$$

where Υ_1 was defined by (3.1).

Proof. In view of Lemma 2.5, we define operator $\Pi : C(J, \mathbb{R}) \to C(J, \mathbb{R})$ by

$$\begin{aligned}(\Pi\upsilon)(\sigma) &= \frac{(\kappa(\sigma)-\kappa(a))^{\gamma-1}}{\Lambda\Gamma(\gamma)}\left[d_4 - d_2\int_a^b \kappa'(s)\mathcal{I}^{\eta;\kappa}F(s,\upsilon(s))ds - d_3\mathcal{I}^{\eta;\kappa}F(\sigma,\upsilon(\sigma))(b)\right]\\ &\quad + \mathcal{I}^{\eta;\kappa}F(\sigma,\upsilon(\sigma)).\end{aligned}$$

Consider the closed ball $B_\delta = \{\upsilon \in C(J, \mathbb{R}) : \|\upsilon\| \le \delta\}$, with

$$\delta \ge \frac{\Upsilon_2}{1-\Upsilon_1},$$

where

$$\begin{aligned}\Upsilon_2 &= N\left[d_2\frac{(\kappa(b)-\kappa(a))^{\eta+\gamma}}{\Lambda\Gamma(\gamma)\Gamma(\eta+2)} + \frac{d_3}{\Lambda\Gamma(\gamma)}\frac{(\kappa(b)-\kappa(a))^{\eta}}{\Gamma(\eta+1)} + \frac{(\kappa(b)-\kappa(a))^{\eta}}{\Gamma(\eta+1)}\right]\\ &\quad + d_4\frac{(\kappa(b)-\kappa(a))^{\gamma-1}}{\Lambda\Gamma(\gamma)},\end{aligned}$$

and $N = \max_{\sigma\in J}|F(\sigma,0)|$. Now, we define the operators Π_1, Π_2 such that $\Pi = \Pi_1 + \Pi_2$ on B_δ as follows

$$(\Pi_1\upsilon)(\sigma) = \mathcal{I}^{\eta;\kappa}F(\sigma,\upsilon(\sigma)),\ \sigma \in J$$

and

$$(\Pi_2\upsilon)(\sigma) = \frac{(\kappa(\sigma)-\kappa(a))^{\gamma-1}}{\Lambda\Gamma(\gamma)}\left[d_4 - d_2\int_a^b \kappa'(s)\mathcal{I}^{\eta;\kappa}F(s,\upsilon(s))ds - d_3\mathcal{I}^{\eta;\kappa}F(\sigma,\upsilon(\sigma))(b)\right].$$

By using (H_1), we obtain

$$\begin{aligned}|F(\sigma,\upsilon(\sigma))| &\le |F(\sigma,\upsilon(\sigma)) - F(\sigma,0) + F(\sigma,0)|\\ &\le \lambda|\upsilon(\sigma)| + |F(\sigma,0)|\\ &\le \lambda|\upsilon(\sigma)| + N.\end{aligned}$$

For any $\upsilon, \upsilon^* \in B_\delta$, we have

$$\begin{aligned}&\|(\Pi_1\upsilon) + (\Pi_2\upsilon)\|\\ \le\ & \sup_{\sigma\in J}\left\{\frac{(\kappa(\sigma)-\kappa(a))^{\gamma-1}}{\Lambda\Gamma(\gamma)}\left[d_4 + d_2\int_a^b \kappa'(s)\mathcal{I}^{\eta;\kappa}|F(s,\upsilon(s))|ds\right.\right.\\ &\left.\left.+ d_3\mathcal{I}^{\eta;\kappa}|F(\sigma,\upsilon(\sigma))(b)|\right] + \mathcal{I}^{\eta;\kappa}|F(\sigma,\upsilon(\sigma))|\right\}\\ \le\ & (\lambda\|\upsilon\| + N)\left[d_2\frac{(\kappa(b)-\kappa(a))^{\eta+\gamma}}{\Lambda\Gamma(\gamma)\Gamma(\eta+2)} + \frac{d_3}{\Lambda\Gamma(\gamma)}\frac{(\kappa(b)-\kappa(a))^{\eta}}{\Gamma(\eta+1)}\right]\end{aligned}$$

$$
\begin{aligned}
&+d_4\frac{(\kappa(b)-\kappa(a))^{\gamma-1}}{\Lambda\Gamma(\gamma)}+\frac{(\kappa(b)-\kappa(a))^{\eta}}{\Gamma(\eta+1)}(\lambda\|\upsilon\|+N)\\
\leq\ &\lambda\|\upsilon\|\left[d_2\frac{(\kappa(b)-\kappa(a))^{\eta+\gamma}}{\Lambda\Gamma(\gamma)\Gamma(\eta+2)}+\frac{d_3}{\Lambda\Gamma(\gamma)}\frac{(\kappa(b)-\kappa(a))^{\eta}}{\Gamma(\eta+1)}+\frac{(\kappa(b)-\kappa(a))^{\eta}}{\Gamma(\eta+1)}\right]\\
&+N\left[d_2\frac{(\kappa(b)-\kappa(a))^{\eta+\gamma}}{\Lambda\Gamma(\gamma)\Gamma(\eta+2)}+\frac{d_3}{\Lambda\Gamma(\gamma)}\frac{(\kappa(b)-\kappa(a))^{\eta}}{\Gamma(\eta+1)}+\frac{(\kappa(b)-\kappa(a))^{\eta}}{\Gamma(\eta+1)}\right]\\
&+d_4\frac{(\kappa(b)-\kappa(a))^{\gamma-1}}{\Lambda\Gamma(\gamma)}\\
\leq\ &\lambda\delta\left[d_2\frac{(\kappa(b)-\kappa(a))^{\eta+\gamma}}{\Lambda\Gamma(\gamma)\Gamma(\eta+2)}+\frac{d_3}{\Lambda\Gamma(\gamma)}\frac{(\kappa(b)-\kappa(a))^{\eta}}{\Gamma(\eta+1)}+\frac{(\kappa(b)-\kappa(a))^{\eta}}{\Gamma(\eta+1)}\right]\\
&+N\left[d_2\frac{(\kappa(b)-\kappa(a))^{\eta+\gamma}}{\Lambda\Gamma(\gamma)\Gamma(\eta+2)}+\frac{d_3}{\Lambda\Gamma(\gamma)}\frac{(\kappa(b)-\kappa(a))^{\eta}}{\Gamma(\eta+1)}+\frac{(\kappa(b)-\kappa(a))^{\eta}}{\Gamma(\eta+1)}\right]\\
&+d_4\frac{(\kappa(b)-\kappa(a))^{\gamma-1}}{\Lambda\Gamma(\gamma)}\\
\leq\ &\delta\Upsilon_1+\Upsilon_2\leq\delta.
\end{aligned}
$$

This shows that $\Pi_1\upsilon+\Pi_2\upsilon^*\in B_\delta$.

Next, due to continuity of F, we conclude that Π_1 is continuous too. Also, Π_1 is uniformly bounded on B_δ as

$$\|\Pi_2\upsilon\|\leq\frac{(\kappa(b)-\kappa(a))^{\eta}}{\Gamma(\eta+1)}\lambda\delta.$$

In addition, we prove the compactness of Π_1 as follows. Let $\sigma_1,\sigma_2\in J$ such that $\sigma_1<\sigma_2$. Then

$$
\begin{aligned}
&|(\Pi_1\upsilon)(\sigma_2)-(\Pi_1\upsilon)(\sigma_1)|\\
\leq\ &\frac{1}{\Gamma(\eta)}\left|\int_a^{\sigma_1}\kappa'(s)[(\kappa(\sigma_2)-\kappa(s))^{\eta-1}-((\kappa(\sigma_1)-\kappa(s))^{\eta-1}]\,|F(s,\upsilon(s))|)ds\right.\\
&\left.+\int_{\sigma_1}^{\sigma_2}\kappa'(s)((\kappa(\sigma_2)-\kappa(s))^{\eta-1}\,|F(s,\upsilon(s))|\,ds\right|\\
\leq\ &\frac{(\lambda\delta+N)}{\Gamma(\eta+1)}\left[|(\kappa(\sigma_2)-\kappa(a))^{\eta}-(\kappa(\sigma_1)-\kappa(a))^{\eta}|\right].
\end{aligned}
$$

The last inequality with $\sigma_2-\sigma_1\to 0$, gives

$$|(\Pi_1\upsilon)(\sigma_2)-(\Pi_1\upsilon)(\sigma_1)|\to 0,\ \text{as}\ \sigma_2\to\sigma_1,\ \upsilon\in\mathcal{B}_\delta.$$

Then, Π_1 is relatively compact on B_δ. An application of the Arzel-Ascoli theorem, Π_1 is compact on B_δ.

Now, we will show that Π_2 is a contraction. Let $\upsilon,\upsilon^*\in B_\delta$. Then, by ($\mathrm{H}_1$) for $\sigma\in J$, we have

$$
\begin{aligned}
&|(\Pi_2\upsilon)(\sigma)-(\Pi_2\upsilon^*)(\sigma)|\\
\leq\ &\sup_{\sigma\in J}\left\{\frac{(\kappa(\sigma)-\kappa(a))^{\gamma-1}}{\Lambda\Gamma(\gamma)}\left[d_4+d_2\int_a^b\kappa'(s)\mathcal{I}^{\eta;\kappa}\,|F(s,\upsilon(s))-F(s,\upsilon^*(s))|\,ds\right.\right.\\
&\left.\left.+d_3\mathcal{I}^{\eta;\kappa}\,|F(\sigma,\upsilon(\sigma))(b)-F(\sigma,\upsilon^*(\sigma))(b)|\right]\right\}
\end{aligned}
$$

$$
\begin{aligned}
&\leq \lambda \|\upsilon-\upsilon^*\| \left[d_2 \frac{(\kappa(b)-\kappa(a))^{\eta+\gamma}}{\Lambda\Gamma(\gamma)\Gamma(\eta+2)} + \frac{d_3}{\Lambda\Gamma(\gamma)} \frac{(\kappa(b)-\kappa(a))^{\eta}}{\Gamma(\eta+1)} \right] \\
&\leq \Upsilon_1 \|\upsilon-\upsilon^*\|.
\end{aligned}
$$

Hence, Π_2 is a contraction in B_δ. Thus, all conditions in Krasnoselskii's fixed point theorem are satisfied. So, the κ-Hilfer type BVP (1.5) has at least one solution on J. □

Theorem 3.2. *Assume that* (H_1) *holds. Then the κ-Hilfer type BVP (1.5) has unique solution on J, provided that*

$$
\Upsilon_1 < 1, \tag{3.3}
$$

where Υ_1 was defined by (3.1).

Proof. We shall show that Π has a unique fixed point by using Banach theorem [26]. By Theorem 3.1, we have

$$
\begin{aligned}
\|\Pi\upsilon\| &\leq \|\Pi_1\upsilon\| + \|\Pi_2\upsilon\| \\
&\leq \delta.
\end{aligned}
$$

Thus, $\Pi(B_\delta) \subset B_\delta$. Now, we show that Π is a contraction. For $\upsilon, \upsilon^* \in B_\delta$ and $\sigma \in J$, we have

$$
\begin{aligned}
& \|(\Pi\upsilon) - (\Pi\upsilon^*)\| \\
\leq\ & \|(\Pi_1\upsilon) - (\Pi_1\upsilon^*)\| + \|(\Pi_2\upsilon) - (\Pi_2\upsilon^*)\| \\
\leq\ & \sup_{\sigma\in J} \left\{ \frac{(\kappa(\sigma)-\kappa(a))^{\gamma-1}}{\Lambda\Gamma(\gamma)} \left[d_4 + d_2 \int_a^b \kappa'(s) \mathcal{I}^{\eta;\kappa} |F(s,\upsilon(s)) - F(s,\upsilon^*(s))|\, ds \right.\right. \\
& \left.\left. + d_3 \mathcal{I}^{\eta;\kappa} |F(\sigma,\upsilon(\sigma)) - F(\sigma,\upsilon^*(\sigma))| \right] + \mathcal{I}^{\eta;\kappa} |F(\sigma,\upsilon(\sigma)) - F(\sigma,\upsilon^*(\sigma))| \right\} \\
\leq\ & \lambda \|\upsilon-\upsilon^*\| \left[d_2 \frac{(\kappa(b)-\kappa(a))^{\eta+\gamma}}{\Lambda\Gamma(\gamma)\Gamma(\eta+2)} + \frac{d_3}{\Lambda\Gamma(\gamma)} \frac{(\kappa(b)-\kappa(a))^{\eta}}{\Gamma(\eta+1)} + \frac{(\kappa(b)-\kappa(a))^{\eta}}{\Gamma(\eta+1)} \right] \\
\leq\ & \Upsilon_1 \|\upsilon-\upsilon^*\|,
\end{aligned}
$$

which implies that $\|\Pi\upsilon - \Pi\upsilon^*\| \leq \Upsilon_1 \|\upsilon-\upsilon^*\|$. By (3.2), we realize that Π is a contraction. Then, by Krasnoselskii Theorem [26], the κ-Hilfer type BVP (1.5) has a unique solution on J. □

Special cases

According to our previous results, in this subsection we present several special cases:

Case (1): If $\kappa(\sigma) = \sigma$, then the κ-Hilfer type BVP (1.5) is reduced to the following Hilfer type problem

$$
\begin{cases}
{}_H\mathcal{D}^{\eta,q}\upsilon(\sigma) = F(\sigma,\upsilon(\sigma)), \quad \sigma \in J := [a,b], \\
d_1\upsilon(a) + d_2 \int_a^b \upsilon(s)ds + d_3\upsilon(b) = d_4,
\end{cases} \tag{3.4}
$$

where ${}_H\mathcal{D}^{\eta,q,\sigma}$ is the Hilfer FD of order η, $F : J \times \mathbb{R} \to \mathbb{R}$ is continuous function, $d_i \in \mathbb{R}$ $(i = 1,2,3,4)$. The solution of the Hilfer type BVP (3.4) is given by

$$\begin{aligned}\upsilon(\sigma) &= \frac{(\sigma-a)^{\gamma-1}}{\Lambda\Gamma(\gamma)}\left[d_4 - d_2\int_a^b \mathcal{I}^{\eta}F(s,\upsilon(s))ds - d_3\mathcal{I}^{\eta;\kappa}F(b,\upsilon(b))\right] \\ &\quad +\mathcal{I}^{\eta}F(\sigma,\upsilon(\sigma)),\end{aligned}$$

where

$$\Lambda = \left[d_2 + \frac{\gamma d_3}{(b-a)}\right]\frac{(b-a)^{\gamma}}{\Gamma(\gamma+1)} \neq 0.$$

Then the following corollary is extracted from the Theorem 3.2.

Corollary 3.3. *Assume that* (H_1) *is satisfied. Then the BVP (3.4) has a unique solution on J provided that* $\Upsilon_1^* < 1$*, where*

$$\Upsilon_1^* = \lambda\left[d_2\frac{(b-a)^{\eta+\gamma}}{\Lambda\Gamma(\gamma)\Gamma(\eta+2)} + \frac{d_3}{\Lambda\Gamma(\gamma)}\frac{(b-a)^{\eta}}{\Gamma(\eta+1)} + \frac{(b-a)^{\eta}}{\Gamma(\eta+1)}\right].$$

Case (2): If $\kappa(\sigma) = \log\sigma$, then the κ-Hilfer type BVP (1.5) is reduced to the following Hilfer-Hadamard type problem

$$\begin{cases} {}_H\mathcal{D}^{\eta,q,\log\sigma}\upsilon(\sigma) = F(\sigma,\upsilon(\sigma)), \quad \sigma\in J := [a,b], \\ \quad d_1\upsilon(a) + d_2\int_a^b \upsilon(s)ds + d_3\upsilon(b) = d_4, \end{cases} \tag{3.5}$$

where ${}_H\mathcal{D}^{\eta,q,\log\sigma}$ is the Hilfer-Hadamard FD of order η, $F : J\times\mathbb{R}\to\mathbb{R}$ is continuous function, $d_i\in\mathbb{R}$ $(i = 1,2,3,4)$. The solution of the Hilfer-Hadamard type BVP (3.5) is given by

$$\begin{aligned}\upsilon(\sigma) &= \frac{(\log\frac{\sigma}{a})^{\gamma-1}}{\Lambda\Gamma(\gamma)}\left[d_4 - d_2\int_a^b \mathcal{I}^{\eta;\log\sigma}F(s,\upsilon(s))ds - d_3\mathcal{I}^{\eta;\kappa}F(b,\upsilon(b))\right] \\ &\quad +\mathcal{I}^{\eta;\log\sigma}F(\sigma,\upsilon(\sigma)),\end{aligned}$$

where

$$\Lambda = \left[d_2 + \frac{\gamma d_3}{(\log\frac{b}{a})}\right]\frac{(\log\frac{b}{a})^{\gamma}}{\Gamma(\gamma+1)} \neq 0.$$

Then the following corollary is deduced from the Theorem 3.2.

Corollary 3.4. *Assume that* (H_1) *is satisfied. Then the BVP (3.5) has a unique solution on J provided that* $\Upsilon_1^{**} < 1$*, where*

$$\Upsilon_1^{**} = \lambda\left[d_2\frac{(\log\frac{b}{a})^{\eta+\gamma}}{\Lambda\Gamma(\gamma)\Gamma(\eta+2)} + \frac{d_3}{\Lambda\Gamma(\gamma)}\frac{(\log\frac{b}{a})^{\eta}}{\Gamma(\eta+1)} + \frac{(\log\frac{b}{a})^{\eta}}{\Gamma(\eta+1)}\right].$$

Case (3): If $\kappa(\sigma) = \sigma^{\rho}, \rho > 0$, then the κ-Hilfer type BVP (1.5) is reduced to the following Hilfer-Katugumpola type problem

$$\begin{cases} {}_H\mathcal{D}^{\eta,q,\sigma^\rho}\upsilon(\sigma) = F(\sigma,\upsilon(\sigma)), \quad \sigma \in J := [a,b], \\ d_1\upsilon(a) + d_2\int_a^b \upsilon(s)ds + d_3\upsilon(b) = d_4, \end{cases} \tag{3.6}$$

where ${}_H\mathcal{D}^{\eta,q,\sigma^\rho}$ is the Hilfer-Katugumpola FD of order $\eta,\rho > 0, F : J\times\mathbb{R} \to \mathbb{R}$ is continuous function, $d_i \in \mathbb{R}$ $(i = 1,2,3,4)$. The solution of the Hilfer-Katugumpola type BVP (3.6) is given by

$$\begin{aligned} \upsilon(\sigma) \;=\; & \frac{(\sigma^\rho - a^\rho)^{\gamma-1}}{\Lambda\Gamma(\gamma)}\left[d_4 - d_2\int_a^b \mathcal{I}^{\eta,\sigma^\rho}F(s,\upsilon(s))ds - d_3\mathcal{I}^{\eta;\kappa}F(b,\upsilon(b))\right] \\ & +\mathcal{I}^{\eta,\sigma^\rho}F(\sigma,\upsilon(\sigma)), \end{aligned}$$

where

$$\Lambda = \left[d_2 + \frac{\gamma d_3}{(b^\rho - a^\rho)}\right]\frac{(b^\rho - a^\rho)^\gamma}{\Gamma(\gamma+1)} \neq 0.$$

Then the next corollary is a special case of the Theorem 3.2.

Corollary 3.5. *Assume that* (H_1) *is satisfied. Then the BVP (3.6) has a unique solution on J provided that* $\Upsilon_1^{***} < 1$, *where*

$$\Upsilon_1^{***} = \lambda\left[d_2\frac{(b^\rho - a^\rho)^{\eta+\gamma}}{\Lambda\Gamma(\gamma)\Gamma(\eta+2)} + \frac{d_3}{\Lambda\Gamma(\gamma)}\frac{(b^\rho - a^\rho)^\eta}{\Gamma(\eta+1)} + \frac{(b^\rho - a^\rho)^\eta}{\Gamma(\eta+1)}\right].$$

4. Stability analysis

In this part, we discuss various types of stability like UH, GUH, UHR and GUHR. First of all, we introduce the following definitions. Let $\epsilon > 0$ such that

$$|{}_H\mathcal{D}^{\eta,q;\kappa}\overline{\upsilon}(\sigma) - F(\sigma,\overline{\upsilon}(\sigma))| \leq \epsilon, \tag{4.1}$$

$$|{}_H\mathcal{D}^{\eta,q;\kappa}\overline{\upsilon}(\sigma) - F(\sigma,\overline{\upsilon}(\sigma))| \leq \epsilon\delta_\kappa(\sigma), \delta_\kappa \in C(J,\mathbb{R}). \tag{4.2}$$

Definition 4.1. *Let* $\overline{\upsilon} \in C(J,\mathbb{R})$ *be a function satisfies (4.1) corresponding to a solution* $\upsilon \in C(J,\mathbb{R})$ *of* κ*-Hilfer type BVP (1.5). If there exists* $0 < \mathcal{T} \in \mathbb{R}$ *such that*

$$|\overline{\upsilon}(\sigma) - \upsilon(\sigma)| \leq \mathcal{T}\epsilon, \qquad \sigma \in J,\ \epsilon > 0,$$

then,

$${}_H\mathcal{D}^{\eta,q;\kappa}\overline{\upsilon}(\sigma) = F(\sigma,\overline{\upsilon}(\sigma)), \tag{4.3}$$

is UH stable.

Remark 4.2. *A function* $\overline{\upsilon} \in C(J,\mathbb{R})$ *satisfies (4.1) if and only if there exist a functions* $z \in C(J,\mathbb{R})$ *such that*

(i) $|z(\sigma)| \leq \epsilon,\ \sigma \in J$,

(ii) ${}_HD^{\eta,q;\kappa}\overline{\upsilon}(\sigma) = F(\sigma,\overline{\upsilon}(\sigma)) + z(\sigma),\ \varkappa \in J.$

Lemma 4.3. *If $\upsilon \in C(J,\mathbb{R})$ is a solution to inequality (4.1), then υ satisfies the following inequality*

$$|\upsilon(\sigma) - \Theta_{\upsilon}| \leq \epsilon\Pi,$$

where

$$\begin{aligned}\Theta_{\overline{\upsilon}} &= \frac{1}{\Lambda\Gamma(\gamma)}(\kappa(\sigma)-\kappa(a))^{\gamma-1}\\ &\left[d_4 - d_2\int_a^b \kappa'(s)\mathcal{I}^{\eta;\kappa}F(s,\upsilon(s))ds - d_3\mathcal{I}^{\eta;\kappa}F(b,\upsilon(b))\right]\\ &+\mathcal{I}^{\eta;\kappa}F(\sigma,\upsilon(\sigma)),\end{aligned}$$

and

$$\Pi = \left(\frac{1}{\Lambda\Gamma(\gamma)}\left[\frac{d_4}{\epsilon} + d_2\frac{(\kappa(b)-\kappa(a))^{\eta+\gamma}}{\Gamma(\eta+2)} + d_3\frac{(\kappa(b)-\kappa(a))^{\eta}}{\Gamma(\eta+1)}\right] + \frac{(\kappa(b)-\kappa(a))^{\eta}}{\Gamma(\eta+1)}\right)$$

Proof. In view of Remark 4.2, we have

$$\begin{aligned}&{}_H\mathcal{D}^{\eta,q;\kappa}\upsilon(\sigma) = F(\sigma,\upsilon(\sigma)) + z(\varkappa)\\ &d_1\upsilon(a) + d_2\int_a^b \kappa'(s)\upsilon(s)ds + d_3\upsilon(b) = d_4.\end{aligned}$$

Then, by Lemma 2.5, we get

$$\begin{aligned}\upsilon(\sigma) &= \Theta_{\upsilon} + \frac{(\kappa(\sigma)-\kappa(a))^{\gamma-1}}{\Lambda\Gamma(\gamma)}\\ &\left[d_4 - d_2\int_a^b \kappa'(s)\mathcal{I}^{\eta;\kappa}z(s)ds - d_3\mathcal{I}^{\eta;\kappa}z(b)\right]\\ &+\mathcal{I}^{\eta;\kappa}z(\sigma),\end{aligned}$$

which implies

$$\begin{aligned}|\upsilon(\sigma)-\Theta_{\upsilon}| &\leq \frac{\epsilon}{\Lambda\Gamma(\gamma)}(\kappa(b)-\kappa(a))^{\gamma-1}\\ &\left[\frac{d_4}{\epsilon} + d_2\frac{(\kappa(b)-\kappa(a))^{\eta}}{\Gamma(\eta+1)} + d_3\frac{(\kappa(b)-\kappa(a))^{\eta}}{\Gamma(\eta+1)}\right]\\ &+\epsilon\frac{(\kappa(b)-\kappa(a))^{\eta}}{\Gamma(\eta+1)}\\ &\leq \epsilon\left(\frac{1}{\Lambda\Gamma(\gamma)}\left[\frac{d_4}{\epsilon} + d_2\frac{(\kappa(b)-\kappa(a))^{\eta+\gamma}}{\Gamma(\eta+2)} + d_3\frac{(\kappa(b)-\kappa(a))^{\eta}}{\Gamma(\eta+1)}\right] + \frac{(\kappa(b)-\kappa(a))^{\eta}}{\Gamma(\eta+1)}\right)\\ &\leq \epsilon\Pi.\end{aligned}$$

□

Theorem 4.4. *Assume that (H1) holds. Under the Lemma 4.3, the following equation*

$$ {}_H\mathcal{D}^{\eta,q;\kappa}\upsilon(\sigma) = F(\sigma,\upsilon(\sigma)), \sigma \in [a,b], \tag{4.4}$$

is UH stable as well as GUH provided that $\Upsilon_1 < 1$.

Proof. $\overline{\upsilon} \in C(J,\mathbb{R})$ be a function satisfies (4.1), let $\upsilon \in C(J,\mathbb{R})$ be the unique solution of the following problem

$$\begin{cases} {}_H\mathcal{D}^{\eta,q;\kappa}\upsilon(\sigma) = F(\sigma,\upsilon(\sigma)), \quad \sigma \in J := [a,b], \\ d_1\upsilon(a) + d_2\int_a^b \kappa'(s)\upsilon(s)ds + d_3\upsilon(b) = d_4, \\ \upsilon(\sigma) = \overline{\upsilon}(\sigma), \sigma \in J := [a,b]. \end{cases}$$

Then, by Lemma 2.5, we get

$$\upsilon(\sigma) = \Theta_\upsilon.$$

It follows from Theorem 3.2, that

$$\begin{aligned} \|\upsilon - \overline{\upsilon}\| &= \sup_{\sigma\in J} |\upsilon(\sigma) - \Theta_{\overline{\upsilon}}| \le \sup_{\tau\in\mho} |\upsilon(\sigma) - \Theta_\upsilon| + \sup_{\tau\in\mho} |\Theta_\upsilon - \Theta_{\overline{\upsilon}}| \\ &\le \epsilon\Pi + \Upsilon_1 \|\upsilon - \overline{\upsilon}\|. \end{aligned}$$

Thus

$$\|\upsilon - \overline{\upsilon}\| \le \mathcal{T}\epsilon,$$

where

$$\mathcal{T} = \frac{\Pi}{1-\Upsilon_1} > 0.$$

Now, by choosing $\varphi_\Upsilon(\epsilon) = C_\Upsilon\epsilon$ such that $\varphi_\Upsilon(0) = 0$, then the problem (4.4) is GUH stability. □

To prove the Ulam-Heyrs-Rassias stability, we need the following hypotheses:

(H_2) There exists an increasing function $\delta_\kappa \in C(J,\mathbb{R})$ and there exists $\mathcal{W} > 0$ such that for any $\sigma \in J$

$$\mathcal{I}^{\eta;\kappa}\delta_\kappa(\sigma) \le \mathcal{W}\delta_\kappa(\sigma). \tag{4.5}$$

Definition 4.5. *Let $\overline{\upsilon} \in C(J,\mathbb{R})$ be a function satisfies inequality (4.2) corresponding to a solution $\upsilon \in C(J,\mathbb{R})$ of κ-Hilfer type BVP (1.5). If there exists $0 < \mathcal{N} \in \mathbb{R}$ and non-decreasing function $\delta_\kappa(\sigma)$ such that*

$$\|\upsilon - \overline{\upsilon}\| \le \mathcal{N}\epsilon\delta_\kappa(\sigma), \quad \tau \in \mho, \ \epsilon > 0,$$

then, the problem (4.4) is UHR stable with respect to $\delta_\kappa(\sigma)$.

Remark 4.6. *A function $\overline{\upsilon} \in C(J,\mathbb{R})$ satisfies (4.2) if and only if there exist a functions $z \in C(J,\mathbb{R})$ such that*

(i) $|z(\tau)| \le \epsilon\delta_\kappa(\sigma), \ \sigma \in J,$

(ii) ${}_H\mathcal{D}^{\eta,q;\kappa}\overline{\upsilon}(\sigma) = F(\sigma,\overline{\upsilon}(\sigma)) + z(\tau), \tau \in \mho.$

Lemma 4.7. *If $\upsilon \in C(J,\mathbb{R})$ is a solution to inequality (4.2), then υ satisfies the following inequality*

$$\|\upsilon(\sigma) - \Theta_\upsilon\| \le \epsilon\mathcal{W}\delta_\kappa(\sigma),$$

Proof. Indeed, by Remark 4.6 and Lemma 2.5, one can easily prove that

$$\|\upsilon(\sigma) - \Theta_\upsilon\| \le \epsilon\mathcal{W}\delta_\kappa(\sigma),$$

where

$$\mathcal{W} = \left(\frac{1}{\Lambda\Gamma(\gamma)}\left[\frac{d_4}{\epsilon} + d_2\kappa(b) + d_3\right] + 1\right).$$

□

Theorem 4.8. *Suppose that (H1) and (H2) are satisfied. If $\Upsilon_1 < 1$, then Eq (4.4) is UHR and generalized UHR stable.*

Proof. Let $\overline{\upsilon}$ be satisfies (4.2), let $\overline{\upsilon} \in C(J, \mathbb{R})$ be unique solution of the following problem

$$\begin{cases} {}_H\mathcal{D}^{\eta,q;\kappa}\upsilon(\sigma) = F(\sigma, \upsilon(\sigma)), \quad \sigma \in J := [a, b], \\ d_1\upsilon(a) + d_2 \int_a^b \kappa'(s)\upsilon(s)ds + d_3\upsilon(b) = d_4, \\ \upsilon(\sigma) = \overline{\upsilon}(\sigma), \sigma \in J := [a, b]. \end{cases}$$

Then, by Lemma 2.5, we get

$$\upsilon(\sigma) = \Theta_\upsilon,$$

It follows from Theorem 3.2, that

$$\begin{aligned} \|\upsilon - \overline{\upsilon}\| &= \sup_{\sigma\in J} |\upsilon(\sigma) - \Theta_{\overline{\upsilon}}| \leq \sup_{\sigma\in J} |\upsilon(\sigma) - \Theta_\upsilon| + \sup_{\tau\in\mho} |\Theta_\upsilon - \Theta_{\overline{\upsilon}}| \\ &\leq \epsilon\mathcal{W}\delta_\kappa(\sigma) + \Upsilon_1 \|\upsilon - \overline{\upsilon}\|. \end{aligned}$$

Thus

$$\|\upsilon - \overline{\upsilon}\| \leq \mathcal{N}\epsilon\delta_\kappa(\sigma),$$

where

$$\mathcal{N} = \frac{\mathcal{W}}{1 - \Upsilon_1} > 0.$$

Hence, the problem (4.4) is UHR stable as well as generalized UHR stable. □

5. An example

In the end, we support our main results by suggesting an example to show the applicability of the outcomes numerically. In fact, the reported results acquired in Theorems 3.1, 3.2, 4.4 and 4.8 are guaranteed by an example.

Example 5.1. *Consider the following problem*

$$\begin{cases} {}_H\mathcal{D}^{\frac{1}{3},\frac{5}{6};\sqrt{\sigma+1}}\upsilon(\sigma) = \left(\frac{\sigma}{9}\right)\frac{|\upsilon(\sigma)|}{1+|\upsilon(\sigma)|} + \frac{1}{\sqrt{2}}, \quad \sigma \in [1, \frac{3}{2}], \\ \frac{1}{10}\upsilon(1) + \frac{2}{5}\int_1^{\frac{3}{2}} \frac{1}{2\sqrt{\sigma+1}}\upsilon(\sigma)d\sigma + \frac{5}{7}\upsilon(\frac{3}{2}) = 1, \end{cases} \tag{5.1}$$

where

$$\eta = \frac{1}{3}, q = \frac{5}{6}, \ \kappa(\sigma) = \sqrt{\sigma + 1}, \ J = [1, \frac{3}{2}],$$

$$d_1 = \frac{1}{10}, d_2 = \frac{2}{5}, d_3 = \frac{5}{7}, d_4 = 1.$$

From these settings, we compute constants as $\gamma = \frac{8}{9}$, $\Lambda = 0.89 \neq 0$. For $\upsilon_1, \upsilon_2 \in \mathbb{R}^+$, we have

$$|F(\sigma, \upsilon_1) - F(\sigma, \upsilon_2)| = \left(\frac{\sigma}{9}\right)\left|\frac{\upsilon_1}{1 + \upsilon_1} - \frac{\upsilon_2}{1 + \upsilon_2}\right|$$

$$\begin{aligned}
&\le \left(\frac{\sigma}{9}\right)|\upsilon_1-\upsilon_2| \\
&\le \frac{1}{6}\,|\upsilon_1-\upsilon_2|\,.
\end{aligned}$$

Hence, (H_1)*holds with* $\lambda = \frac{1}{6} > 0$ *. Also, the condition (3.2) is fulfilled, i.e.* $\Upsilon_1 = 0.18 < 1$*. Therefore, by the applying Banach's fixed point theorem, we conclude that the problem (5.1) has a unique solution* υ *on* $[1, \frac{3}{2}]$*. Finally, we see that the inequality*

$$|{}_H\mathcal{D}^{\eta,q;\kappa}\overline{\upsilon}(\sigma) - F(\sigma, \overline{\upsilon}(\sigma))| \le \epsilon$$

is satisfied. Then the Eq (5.1) is Ulam-Hyers stable with

$$\|\upsilon - \overline{\upsilon}\| \le \mathcal{T}\epsilon, \qquad \sigma \in J,\ \epsilon > 0,$$

where

$$\mathcal{T} = \frac{\Pi}{1-\Upsilon_1} > 0,$$

$$\Pi = \left(\frac{1}{\Lambda\Gamma(\gamma)}\left[\frac{d_4}{\epsilon} + d_2\frac{(\kappa(b)-\kappa(a))^{\eta+\gamma}}{\Gamma(\eta+2)} + d_3\frac{(\kappa(b)-\kappa(a))^{\eta}}{\Gamma(\eta+1)}\right] + \frac{(\kappa(b)-\kappa(a))^{\eta}}{\Gamma(\eta+1)}\right) > 0,$$

and $\Upsilon_1 = 0.18 < 1$*. Finally, we consider* $\delta_\kappa(\sigma) = \kappa(\sigma) - \kappa(a)$*, for* $\sigma \in [1, \frac{3}{2}]$*. Then,* $\delta_\kappa : [1, \frac{3}{2}] \to \mathbb{R}$ *is continuous nondecreasing function. Hence by Lemma 2.3, we get*

$$\begin{aligned}
I_{a^+}^{\eta;\kappa}\delta_\kappa(\sigma) &= I_{a^+}^{\eta;\kappa}(\kappa(\sigma)-\kappa(a)) \\
&= \frac{\Gamma(2)}{\Gamma(\eta+2)}(\kappa(\sigma)-\kappa(a))^{\eta+1} \\
&\le \frac{\left[\kappa(\frac{3}{2})-\kappa(1)\right]^{\eta}}{\Gamma(\eta+2)}\delta_\kappa(\sigma) \\
&= \mathcal{W}\delta_\kappa(\sigma), \text{ for all } \varkappa \in \mathcal{J},
\end{aligned}$$

where $\mathcal{W} = \frac{\left[\kappa(\frac{3}{2})-\kappa(1)\right]^{\eta}}{\Gamma(\eta+2)} > 0$*. Therefore, Theorem 4.8 applicable. Moreover, for* $\epsilon > 0$ *and a continuous function* $\delta_\kappa : [1, \frac{3}{2}] \to \mathbb{R}^+$ *we find that*

$$|{}_H\mathcal{D}^{\eta,q;\kappa}\overline{\upsilon}(\sigma) - F(\sigma, \overline{\upsilon}(\sigma))| \le \epsilon\delta_\kappa(\sigma)$$

is satisfied. Then Eq (5.1) is UHR stable with

$$\|\upsilon - \overline{\upsilon}\| \le \mathcal{N}\epsilon\delta_\kappa(\sigma)$$

where

$$\mathcal{N} = \frac{\mathcal{W}}{1-\Upsilon_1} > 0.$$

6. Conclusions

In this research work, a newly nonlinear fractional differential equation with two-point and integral boundary conditions in the κ-Hilfer fractional derivative frame has been investigated. The existence and uniqueness results of the κ-Hilfer type BVP (1.5) have been obtained using Krasnoselskii and Banach's fixed point theorems. Different types of stability of the κ-Hilfer type BVP (1.5) have been discussed by using some mathematical methodologies. An example is presented to confirm the viability of our obtained results.

The acquired results in this paper are more general and cover many of the parallel problems that contain special cases of function κ, because our proposed system contains a global fractional derivative that integrates many classic fractional derivatives, for instance, the κ-Hilfer type BVP (1.5) for various values of a function κ and parameter q includes the study of a problem of FDEs involving the Hilfer, Hadamard, Katugampola, and many other fractional derivative operators which are described in the introduction. In future work, we will extend this work by replacing the constants d_i ($i = 1, 2, 3, 4$) with matrices.

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Conflict of interest

The authors declare that they have no conflict of interest.

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