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Research article

Stability analysis for (ω, c)-periodic non-instantaneous impulsive differential equations

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Abstract: In this paper, the stability of (ω, c)-periodic solutions of non-instantaneous impulses differential equations is studied. The exponential stability of homogeneous linear non-instantaneous impulsive problems is studied by using Cauchy matrix, and some sufficient conditions for exponential stability are obtained. Further, by using Gronwall inequality, sufficient conditions for exponential stability of (ω, c)-periodic solutions of nonlinear noninstantaneous impulsive problems are established. Finally, some examples are given to illustrate the correctness of the conclusion.

Keywords: non-instantaneous impulsive differential equations; (ω, c)-periodic solutions; stability
Mathematics Subject Classification: 34A08, 34A37, 34C25

1. Introduction

Since 2013, Hernández et al. [1] introduced non-instantaneous impulse differential equations, and their basic theories and applications have become an important research field. The qualitative analysis for the non-instantaneous impulse differential system has attracted more and more researchers. Abundant results have been obtained in relevant studies on non-instantaneous impulse systems for reference [2–8]. It is well known that the impulsive periodic motion is a very important and special phenomenon. We can see that periodic phenomenon and non-instantaneous impulsive phenomenon often occurs together in a system. The concept of (ω, c)-periodic functions was proposed by Alvarez et al. [12], who studied the properties of $x(t)$ of the Mathieu equation $x''(t) + [a - 2qcos(2t)]x = 0$. When $c = 1$, the (ω, c)-periodic function becomes the standard ω-periodic function. When $c = -1$, the (ω, c)-periodic function becomes antiperiodic. Are there any other $|c| \neq 1$ unbounded function and Bloch functions. (ω, c)-periodic functions are more general and attract a large number of scholars to study them. Abundant results have been obtained for periodic solutions, almost periodic solutions and (ω, c)-periodic solutions of noninstantaneous impulses, see [9–18] and teferences therein. In addition, in many practical problems, because fractional differential model can describe some phenomena more effectively than ordinary differential model, it attracts a large number of scholars to study the dynamics of fractional system. Wang et al. [19] studied the controllability for a fractional noninstantaneous impulsive semilinear differential inclusion with delay. By Banach fixed point theorem, Kaliraj et al. [20] study the controllability of a class of fractional impulsive integro-differential equations with finite delay with initial conditions and non-local conditions. Wang et al. [21] study integral boundary value problems for integer order and fractional order of nonlinear non-instantaneous impulsive ordinary differential equations. Ravichandran et al. [22] studied the existence of solutions of impulsive neutral fractional integro-differential equations by atangana-Baleanu fractional derivatives. Kumar et al. [23] studied the existence of solutions for nonautonomous fractional differential equations by using the fixed point theory of noncompactness measure. Machado et al. [24] established the controllability of a class of abstract impulsive mixed-type functional integro-differential equations with finite delay in a Banach space.

With the development of control theory, the stability of differential equations have always been the focus of researchers. Guan et al. [25] proved the existence and uniqueness of periodic solutions for inhomogeneous systems by using matrix theory, and proved Hyers-Ulam stability results for classical problems of atmospheric ekman layer stroke in stationary eddy viscous atmosphere under mild conditions. Liu et al. [26] studied the Hyers-Ulam stability of linear Caputo-Fabrizio fractional differential equations with Mittag-Leffler kernel by using the Laplace transform method. Wang [27] established the sufficient conditions to guarantee the asymptotic stability of linear and semilinear problems for noninstantaneous impulsive evolution operator. Yang et al. [28] established the stability conditions for the periodic solutions of the noninstantaneous impulsive evolution equations by using the Grownwall-coppel inequality. Wang et al. [29] discussed Lyapunov regularity and stability of linear non-instantaneous impulsive differential systems, and gave some criteria for the existence of nonuniform exponential stability. Wang et al. [30] studied Ulam-Hyers-Rassias stability for nonlinear non-instantaneous impulsive equations under the restriction of exponential growth or stability conditions for non-instantaneous impulsive Cauchy matrix, respectively.

Although a large number of literatures have been reported on the stability of non-instantaneous impulsive systems, there is no study on the stability of (ω, c)-periodic solutions of non-instantaneous impulsive systems. Based on the wide application of non-instantaneous pulses and the generality of (ω, c)-periodic functions, we are interested in the stability of (ω, c)-periodic solutions for non-instantaneous impulsive systems.

In this paper, we study the stability of the homogeneous linear non-instantaneous impulsive equations

$$\begin{cases} x'(t) = Ax(t),\ t \in (s_i, t_{i+1}],\ i = 0, 1, 2, \cdots, \\ x(t_i^+) = Bx(t_i^-),\ i = 1, 2, \cdots, \\ x(t) = Bx(t_i^-),\ t \in (t_i, s_i],\ i = 1, 2, \cdots, \\ x(s_i^+) = x(s_i^-),\ i = 1, 2, \cdots, \end{cases} \tag{1.1}$$

and the nonlinear non-instantaneous impulsive equations

$$\begin{cases} x'(t) = Ax(t) + g(t, x(t)),\ t \in (s_i, t_{i+1}],\ i = 0, 1, 2, \cdots, \\ x(t_i^+) = Bx(t_i^-),\ i = 1, 2, \cdots, \\ x(t) = Bx(t_i^-),\ t \in (t_i, s_i],\ i = 1, 2, \cdots, \\ x(s_i^+) = x(s_i^-),\ i = 1, 2, \cdots, \end{cases} \tag{1.2}$$

where $A, B \in \mathbb{R}^{n\times n}$, $0 = s_0 < t_1 < s_1 < t_2 < \cdots < t_i < s_i < t_{i+1} \cdots, i \in \mathbb{N} := \{1, 2, \cdots\}$ and $\{t_i\}_{i\in\mathbb{N}}$ and $\{s_i\}_{i\in\mathbb{N}\bigcup\{0\}}$ are ω-periodic sequences, which will be specified later. Let $\mathbb{I} = \bigcup_{i=1}^{\infty}(s_{i-1}, t_i]$ and $\mathbb{J} = \bigcup_{i=1}^{\infty}(t_i, s_i]$, $g \in C(\mathbb{I}, \mathbb{R}^n)$, $g(\cdot, x) \in C(\mathbb{I}, \mathbb{I}\times\mathbb{R}^n)$.

From [30, Theorem 2.1], any solution $x(\cdot; 0, x_0) \in PC(\mathbb{D}, \mathbb{R}^n)$, $\mathbb{D} = [0, +\infty)$ of (1.1) with $x(0) = x_0 \neq \mathbf{0}$ has the following form

$$x(t; 0, x_0) = W(t, 0)x_0,\ t \geq 0,$$

where non-instantaneous impulsive Cauchy matrix $W(\cdot,\cdot) : \{(t, s) \in \mathbb{D}\times\mathbb{D}\} \to \mathbb{R}^{n\times n}$ of (1.1) is defined as

$$W(t, s) := B^{i(t,0)-i(s,0)} \exp\Big(A[(t - s_{i(t,0)})^+ - (s - s_{i(s,0)})^+ + \sum_{k=i(s,0)}^{i(t,0)-1}(t_{k+1} - s_k)]\Big), \tag{1.3}$$

where $i(t, s)$ denotes the number of impulsive points $t_i \in (s, t)$, $z^+ := \max\{0, z\}, \cdot \in \mathbb{R}$. If $i(s, 0) = i(t, 0)$ then we set $\sum_{k=i(s,0)}^{i(t,0)-1} = 0$.

We impose the following assumptions:

$[A_1]$ A, B are permutable matrices.

$[A_2]$ $b_{i+m} = b_i$, $t_{i+m} = t_i + \omega$, $s_{i+m} = s_i + \omega$ for some fixed $m, i \in \mathbb{N}$ and $m = i(\omega, 0)$.

$[A_3]$ $c \notin \sigma(W(\omega, 0))$.

$[A_4]$ There exist constants $\lambda \in \mathbb{R}$ and $M \geq 1$ such that $\|\exp(At)\| \leq M \exp(\lambda t)$ for any $t \geq 0$.

$[A_5]$ For all $t \in \mathbb{I}$ and $x \in \mathbb{R}^n$, $g(t + \omega, cx) = cg(t, x)$ where $c > 0$.

$[A_6]$ There exists $L > 0$ such that $\|g(t, x_1) - g(t, x_2)\| \leq L\|x_1 - x_2\|$ for all $t \in \mathbb{I}$ and $x_1, x_2 \in \mathbb{R}^n$.

$[A_7]$ Let $\sigma(A) = \{\lambda_1, \lambda_2, \cdots, \lambda_N\}$ be the eigenvalues of A and $Re\lambda_1 \leq Re\lambda_2 \leq \cdots \leq Re\lambda_N \leq -k < 0$, $k > 0$, i.e., there exist $\tilde{K}, k > 0$ such that $\|\exp At\| \leq \tilde{K} \exp(-kt)$ for $t \geq 0$.

$[A_8]$ For any $t \geq 0$ and all $x \in \mathbb{R}^n$, there exists $L_g > 0$ such that $\|g(t, x)\| < L_g\|x\|$.

$[A_9]$ For any $t \geq 0$ and all $x \in \mathbb{R}^n$, there exist $\varrho \in [0, 1)$ and $N > 0$ such that $\|g(t, x)\| \leq N\|x\|^{\varrho}$.

The rest of this paper is organized as follows. In Section 2, we collect some necessary definitions. In Section 3, we establish norm estimation and exponential stability results for (1.1). In Section 4, we obtain some sufficient conditions for the (ω, c)-periodic solutions of (1.2) to be exponentially and asymptotically stable.

2. Preliminaries

Throughout this paper, set $PC(\mathbb{D}, \mathbb{R}^n) = \{x : \mathbb{D} \to \mathbb{R}^n : x \in C((t_i, t_{i+1}], \mathbb{R}^n),\ x(t_i^+),\ x(t_i^-)$ exists and $x(t_i^-) = x(t_i)$ for every $i \in \mathbb{N}\}$ endowed with the norm $\|x\| = \sup_{t\in\mathbb{R}} \|x(t)\|$. Let I be the identity matrix. Let $\|x\| = \sum_{i=1}^{n} |x_i|$ and $\|B\| = \max_{1\leq j\leq n} \sum_{i=1}^{n} |b_{ij}|$ denote the vector norm and matrix norm of the n-dimensional Euclidean space $\mathbb{R}^n$, where x_i and b_{ij} are the elements of the vector x and the matrix B , respectively.

Set $\Psi_{\omega,c} := \{x : x \in PC(\mathbb{D}, \mathbb{R}^n)$ and $cx(\cdot) = x(\cdot + \omega)\}$, i.e., $\Psi_{\omega,c}$ denotes the set of all piecewise continuous and (ω, c)-periodic functions.

Definition 2.1. (1.1) *is exponentially stable if there exists constants $K > 0$ and $\gamma > 0$ such that $\|W(t, s)\| \leq K \exp(-\gamma(t - s))$, $0 \leq s < t$.*

Clearly, $W(\cdot,\cdot)$ is exponentially stable if and only if (1.1) is exponentially stable.

Definition 2.2. $x(\cdot;0,x_0) \in \Psi_{\omega,c}$ *is called exponentially stable, if there exist positive constants* k_1, k_2, *such that*

$$\|x(t;0,x_0)\| \le k_1 e^{-k_2 t}, \quad t \ge 0.$$

Definition 2.3. $x(\cdot;0,x_0) \in \Psi_{\omega,c}$ *is called asymptotically stable, if there exists* $\delta > 0$ *such that for any* $y_0 \in \mathbb{R}^n$ *with* $\|x_0 - y_0\| \le \delta$, *the following holds:*

$$\lim_{t\to+\infty} \|x(t;0,x_0) - x(t;0,y_0)\| = 0. \tag{2.1}$$

If $\delta > 0$ *can be arbitrary then* (ω, c)*-periodic functions* $x(\cdot;0,x_0)$ *is globally asymptotically stable.*

3. Exponentially stability of (1.1)

In this section, we give a set of sufficient conditions to guarantee (1.1) is exponential stable.

We give two important exponentially estimation for $W(\cdot,\cdot)$.

Lemma 3.1. *Suppose* $[A_1]$ *and* $[A_4]$ *hold. For any* $0 \le s < t$,

$$\|W(t,s)\| \le M \exp\Big\{\lambda[(t - s_{i(t,0)})^+ - (s - s_{i(s,0)})^+ + \sum_{k=i(s,0)}^{i(t,0)-1}(t_{k+1} - s_k)] + i(t,s)\ln\|B\|\Big\}. \tag{3.1}$$

Proof. The proof is similar to [30, Lemma 2.7], however, for the completeness, we give the details of the proof. Clearly, $[A_1]$ implies (1.3) is well defined. By $[A_4]$,

$$\begin{aligned}
\|W(t,s)\| &= \Big\|B^{i(t,0)-i(s,0)} \exp\Big(A[(t - s_{i(t,0)})^+ - (s - s_{i(s,0)})^+ + \sum_{k=i(s,0)}^{i(t,0)-1}(t_{k+1} - s_k)]\Big)\Big\| \\
&\le \|B\|^{i(t,s)} M \exp\Big(\lambda[(t - s_{i(t,0)})^+ - (s - s_{i(s,0)})^+ + \sum_{k=i(s,0)}^{i(t,0)-1}(t_{k+1} - s_k)]\Big) \\
&\le M \exp\Big\{\lambda[(t - s_{i(t,0)})^+ - (s - s_{i(s,0)})^+ + \sum_{k=i(s,0)}^{i(t,0)-1}(t_{k+1} - s_k)] + i(t,s)\ln\|B\|\Big\}.
\end{aligned}$$

The proof is finish. □

Lemma 3.2. *Assumption* $[A_1]$, $[A_2]$ *and* $[A_4]$ *hold. Then for any* $t > s \ge 0$,

$$\|W(t,s)\| \le M \exp[i(t,s)(\lambda u + \ln\|B\|) + |\lambda| u]. \tag{3.2}$$

Proof. From $[A_2]$, one has $\omega > u_1 = \min_{m-1\ge k\ge 0}(t_{k+1} - s_k) > 0$, $\omega > u_2 = \max_{m-1\ge k\ge 0}(t_{k+1} - s_k) > 0$. Set

$$u = \begin{cases} u_1, & \lambda < 0, \\ u_2, & \lambda \ge 0. \end{cases}$$

Using (3.1), we have two possible cases:

If $\lambda < 0$ then we have

$$\begin{aligned}
\|W(t,s)\| &\leq M\exp\left\{\lambda[(t-s_{i(t,0)})^{+}-(s-s_{i(s,0)})^{+}+\sum_{k=i(s,0)}^{i(t,0)-1}(t_{k+1}-s_k)]+i(t,s)\ln\|B\|\right\}\\
&\leq M\exp(\lambda(i(t,s)-1)u_1+i(t,s)\ln\|B\|)\\
&\leq M\exp[i(t,s)(\lambda u_1+\ln\|B\|)-\lambda u_1]. \qquad (3.3)
\end{aligned}$$

If $\lambda \geq 0$ then we have

$$\begin{aligned}
\|W(t,s)\| &\leq M\exp\left\{\lambda[(t-s_{i(t,0)})^{+}-(s-s_{i(s,0)})^{+}+\sum_{k=i(s,0)}^{i(t,0)-1}(t_{k+1}-s_k)]+i(t,s)\ln\|B\|\right\}\\
&\leq M\exp(\lambda(i(t,s)+1)u_2+i(t,s)\ln\|B\|)\\
&\leq M\exp[i(t,s)(\lambda u_2+\ln\|B\|)+\lambda u_2]. \qquad (3.4)
\end{aligned}$$

Linking (3.3) and (3.4), (3.2) holds. □

Theorem 3.3. *Suppose* $[A_1]$ *and* $[A_2]$ *hold. If there exist constants* $K_0,\ \lambda_0,\ \lambda_1$ *and* $0<\lambda_1<\lambda_0$ *such that* $\|\exp(At)\|\leq K_0\exp(-\lambda_0 t),\ t>0$ *and*

$$\prod_{k=1}^{m}\exp\{\lambda_0(s_k-t_k)+\ln\|B\|\}<1,$$

then $\{W(t,s), t>s\geq 0\}$ *is exponentially stable.*

Proof. By Lemma 3.1, we have

$$\begin{aligned}
&\|W(t,s)\|\\
\leq\ & K_0\exp\left\{-\lambda_0[(t-s_{i(t,0)})^{+}-(s-s_{i(s,0)})^{+}+\sum_{k=i(s,0)}^{i(t,0)-1}(t_{k+1}-s_k)]+i(t,s)\ln\|B\|\right\}\\
=\ & K_0\exp(-(\lambda_0-\lambda_1)(t-s))\exp(-\lambda_1(t-s))\exp\left\{-\lambda_0[(t-s_{i(t,0)})^{+}-(s-s_{i(s,0)})^{+}\right.\\
&\left.+\sum_{k=i(s,0)}^{i(t,0)-1}(t_{k+1}-s_k)]+i(t,s)\ln\|B\|+\lambda_0(t-s)\right\}\\
\leq\ & K_0\exp(-(\lambda_0-\lambda_1)(t-s))\exp(-\lambda_1(t-s))\exp\left\{\lambda_0\sum_{k=i(s,0)}^{i(t,0)}(s_k-t_k)+i(t,s)\ln\|B\|\right\}\\
\leq\ & K_0\exp(-(\lambda_0-\lambda_1)(t-s))\prod_{0\leq s\leq t_k<s_k\leq t}\exp\{\lambda_0(s_k-t_k)+\ln\|B\|\}\exp(-\lambda_1(t-s)).
\end{aligned}$$

For any $n\omega<s<t<(n+1)\omega$,

$$\begin{aligned}
&\prod_{0\leq s\leq t_k<s_k\leq t}\exp\{\lambda_0(s_k-t_k)+\ln\|B\|\}\exp(-\lambda_1(t-s))\\
\leq\ &\prod_{0\leq t_k<s_k\leq n\omega}\exp\{\lambda_0(s_k-t_k)+\ln\|B\|\}\exp(-\lambda_1 n\omega)
\end{aligned}$$

$$\times\Big(\prod_{n\omega\le t_k<s_k\le t}\exp\{\lambda_0(s_k-t_k)+\ln\|B\|\}\Big)\exp(-\lambda_1(t-n\omega))\exp(\lambda_1 s)$$
$$\le \Big[\prod_{k=1}^{m}\exp\{\lambda_0(s_k-t_k)+\ln\|B\|\}\Big]^n b\exp(\lambda_1 s)\exp(-\lambda_1 n\omega)$$
$$\le b\exp(\lambda_1\omega),$$

where

$$b=\max_{0\le s<t\le\omega}\Big\{\prod_{s\le t_i<s_i\le t}\exp(\lambda_0(s_k-t_k)+\ln\|B\|)\Big\}.$$

From above, we have

$$\|W(t,s)\|\le K_0 b\exp(\lambda_1 s)\exp\left(-(\lambda_0-\lambda_1)(t-s)\right):=K\exp(-\gamma(t-s)),$$

where $K=K_0 b\exp(\lambda_1\omega)>0$, and $\gamma=\lambda_0-\lambda_1>0$. The proof is complete. □

Theorem 3.4. *If* $[A_1]$, $[A_2]$, $[A_4]$ *hold, and* $\lambda u+\ln\|B\|<0$*, then* $\{W(t,s), t>s\ge 0\}$ *is exponentially stable.*

Proof. Note $[A_2]$ via [9, Theorem 4.3], we have

$$\lim_{t-s\to\infty}\frac{i(t,s)}{t-s}=\frac{m}{\omega}:=\sigma<\infty.$$

Then for an arbitrary small $\varepsilon>0$,

$$\left|\frac{i(t,s)}{t-s}-\sigma\right|<\varepsilon,\quad t-s>0,$$

that is,

$$(\sigma-\varepsilon)(t-s)\le i(t,s)\le(\sigma+\varepsilon)(t-s). \tag{3.5}$$

Since $\lambda u+\ln\|B\|<0$, for any $0<\varepsilon<\sigma$, by (3.2) and (3.5), we have

$$\begin{aligned}\|W(t,s)\| &\le M\exp[i(t,s)(\lambda u+\ln\|B\|)+|\lambda|u]\\ &\le M\exp(|\lambda|u)\exp[i(t,s)(\lambda u+\ln\|B\|)]\\ &\le M\exp(|\lambda|u)\exp[(\sigma-\varepsilon)(\lambda u+\ln\|B\|)(t-s)]\\ &:= K_1\exp(-\gamma(t-s)),\end{aligned}$$

where $K_1=M\exp(|\lambda|u)>0$ and $\gamma=-(\sigma-\varepsilon)(\lambda u+\ln\|B\|)>0$. The proof is complete. □

Theorem 3.5. *Assume* $[A_1]$, $[A_2]$, $[A_4]$ *hold. If there exists a constant* $\alpha>0$, *such that*

$$\lambda+\frac{1}{u}\ln\|B\|\le-\alpha<0, \tag{3.6}$$

where

$$u=\begin{cases}u_1, & \alpha+\lambda<0,\\ u_2, & \alpha+\lambda\ge 0,\end{cases}$$

then

$$\|W(t,s)\|\le M\exp\{u|\alpha+\lambda|+u_1\alpha-\alpha u_1 i(s,t)\},$$

which is exponentially stable.

Proof. Set

$$\Delta := (t - s_{i(t,0)})^{+} - (s - s_{i(s,0)})^{+} + \sum_{k=i(s,0)}^{i(t,0)-1} (t_{k+1} - s_k).$$

Note $u_1 \leq u_2$, where $u_1,\ u_2$ are the same as in Lemma 3.2. We have

$$(i(t,s) - 1)u_1 \leq \Delta \leq (i(t,s) + 1)u_2, \tag{3.7}$$

which implies that

$$\frac{\Delta}{u_2} - 1 \leq i(t,s) \leq \frac{\Delta}{u_1} + 1. \tag{3.8}$$

If $\alpha + \lambda < 0$, then $-u(\alpha + \lambda) = -u_1(\alpha + \lambda) > 0$, from the right hand side of (3.8), we have

$$-u_1(\alpha + \lambda)i(t,s) \leq -(\alpha + \lambda)\Delta - u_1(\alpha + \lambda).$$

If $\alpha + \lambda \geq 0$, then $-u(\alpha + \lambda) = -u_2(\alpha + \lambda) \leq 0$, from the left hand side of (3.8), we obtain

$$-u_2(\alpha + \lambda)i(t,s) \leq -(\alpha + \lambda)\Delta + u_2(\alpha + \lambda).$$

So,

$$-u(\alpha + \lambda)i(t,s) \leq -(\alpha + \lambda)\Delta + u|\alpha + \lambda|. \tag{3.9}$$

By (3.6), we have

$$-u(\alpha + \lambda)i(t,s) \geq i(t,s)\ln\|B\|, \tag{3.10}$$

then

$$\begin{aligned} \lambda\Delta + i(t,s)\ln\|B\| &\leq \lambda\Delta - u(\alpha + \lambda)i(t,s) \quad (where\ \ use\ \ (3.10)) \\ &\leq \lambda\Delta - (\alpha + \lambda)\Delta + u|\alpha + \lambda| \quad (where\ \ use\ \ (3.9)) \\ &\leq -\alpha\Delta + u|\alpha + \lambda| \\ &\leq -\alpha(i(t,s) - 1)u_1 + u|\alpha + \lambda| \quad (where\ \ use\ \ (3.7)). \end{aligned} \tag{3.11}$$

By the Lemma 3.1 and (3.11), we have

$$\begin{aligned} \|W(t,s)\| &\leq M\exp\left\{\lambda[(t - s_{i(t,0)})^{+} - (s - s_{i(s,0)})^{+} + \sum_{k=i(s,0)}^{i(t,0)-1}(t_{k+1} - s_k)] + i(t,s)\ln\|B\|\right\} \\ &\leq M\exp\{-\alpha(i(t,s) - 1)u_1 + u|\alpha + \lambda|\} \\ &= M\exp\{u|\alpha + \lambda| + u_1\alpha - \alpha u_1 i(s,t)\}. \end{aligned}$$

Let $0 < \varepsilon < \sigma$ and $t - s > 0$, by (3.5), we have

$$\begin{aligned} \|W(t,s)\| &\leq M\exp\{u|\alpha + \lambda| + u_1\alpha - \alpha u_1 i(s,t)\} \\ &\leq M\exp\{u|\alpha + \lambda| + u_1\alpha - \alpha u_1(\sigma - \varepsilon)(t - s)\} \\ &= M\exp\{u|\alpha + \lambda| + u_1\alpha\}\exp\{-\alpha u_1(\sigma - \varepsilon)(t - s)\} \\ &= K\exp\{-\gamma(t - s)\}, \end{aligned}$$

where $K = M\exp\{u|\alpha + \lambda| + u_1\alpha\}$ and $\gamma = \alpha u_1(\sigma - \varepsilon) > 0$. This proof is finish. □

Theorem 3.6. *If* $[A_1]$, $[A_2]$, $[A_4]$, $[A_7]$ *hold and* $-ku_1 + \ln\|B\| < 0$, *then* $\{W(t,s), t > s \geq 0\}$ *is exponentially stable.*

Proof. Note that (1.3) via $[A_7]$, similar to the proof of Theorem 3.4, we obtain

$$\begin{aligned}
\|W(t,s)\| &\leq \tilde{K}\exp\Big[-k[(t-s_{i(t,0)})^+ - (s-s_{i(s,0)})^+ + \sum_{k=i(s,0)}^{i(t,0)-1}(t_{k+1}-s_k)] + i(t,s)\ln\|B\|\Big] \\
&\leq \tilde{K}\exp[-k(i(t,s)-1)u_1 + i(t,s)\ln\|B\|] \\
&= \tilde{K}\exp(ku_1)\exp[i(t,s)(-ku_1+\ln\|B\|)].
\end{aligned}$$

Since $-ku_1 + \ln\|B\| < 0$, for any $0 < \varepsilon < \sigma$,

$$\|W(t,s)\| \leq M\exp(ku_1)\exp[(\sigma-\varepsilon)(-ku_1+\ln\|B\|)(t-s)].$$

The proof is finished. □

By [32, p.109] and [31, p.44], for any $\varepsilon > 0$, there exists a $\tilde{K}_\varepsilon \geq 1$ such that

$$\|W(t,s)\| \leq \tilde{K}_\varepsilon \exp\Big((\alpha(A)+\varepsilon)[(t-s_{i(t,0)})^+ - (s-s_{i(s,0)})^+ + \sum_{k=i(s,0)}^{i(t,0)-1}(t_{k+1}-s_k)]\Big)(\rho(B)+\varepsilon)^{i(t,0)-i(s,0)}. \quad (3.12)$$

Using (3.12), similar to the proof of Theorem 3.4, we obtain

Theorem 3.7. *If* $[A_1]$, $[A_2]$ *hold, and* $\alpha(A) + \frac{1}{u}\rho(B) < 0$, *then* $\{W(t,s), t > s \geq 0\}$ *is exponentially stable.*

From above we can formulate the following exponentially stability result.

Theorem 3.8. *If the conditions of the Theorem 3.3, or Theorem 3.4, or Theorem 3.5, or Theorem 3.6, or Theorem 3.9 holds, then* (1.1) *is exponential stable.*

To end this section, an example is illustrated to demonstrate the above theoretically results.

Example 3.9. *Consider the following linear non-instantaneous impulsive system*

$$\begin{cases} x'(t) = Ax(t),\ t \in (s_i, t_{i+1}],\ i = 0,1,2,\cdots, \\ x(t_i^+) = Bx(t_i^-),\ i = 1,2,\cdots, \\ x(t) = Bx(t_i^-),\ t \in (t_i, s_i],\ i = 1,2,\cdots, \\ x(s_i^+) = x(s_i^-),\ i = 1,2,\cdots, \end{cases} \quad (3.13)$$

where $t_i = \frac{2i-1}{2}$, $s_i = i$ *and*

$$x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix},\quad A = \begin{pmatrix} -\frac{1}{6} & -\frac{1}{72} \\ 0 & -\frac{1}{8} \end{pmatrix},\quad B = \begin{pmatrix} \frac{1}{16} & \frac{1}{32} \\ 0 & -\frac{1}{32} \end{pmatrix}.$$

Clearly, $AB = BA$, $\alpha(A) = -\frac{1}{6}$, $\rho(B) = \|B\| = \frac{1}{16}$, $t_{i+1} = \frac{2i+1}{2} = \frac{2i-1}{2} + 1 = t_i + 1$, $s_{i+1} = i + 1 = s_i + 1$, *for all* $i \in \mathbb{N}$, *so,* $m = 1$, $u = u_1 = \frac{1}{2}$, *then* $[A_1]$ *and* $[A_2]$ *hold. By elementary calculations, we obtain* $\sigma(A) = \{-\frac{1}{6}, -\frac{1}{8}\}$, *set* $\lambda = -\frac{1}{8}$, *so* $[A_4]$ *is verified. Set* $k = \lambda_0 = \frac{1}{8}$.

Note

$$\prod_{k=1}^{m} \exp\{\lambda_0(s_k - t_k) + \ln \|B\|\} = \exp\{\frac{1}{8} + \ln \frac{1}{16}\} < \exp(-2.64) < 1.$$

By Theorem 3.3 and (3.13) *are exponential stable.*

Next, $\lambda u + \ln\|B\| = -\frac{1}{16} + \ln\frac{1}{16} < -2.71 < 0$*, by Theorems 3.4, 3.5, 3.6 and* (3.13) *are exponential stable.*

Finally, $\alpha(A) + \frac{1}{u}\rho(B) = -\frac{1}{6} + 2\frac{1}{16} = -\frac{1}{24} < 0$*, by Theorem 3.7 and* (3.13) *are exponential stable.*

4. Exponential and asymptotical stability of (ω, c)-periodic solution of (1.2)

Theorem 4.1. *Assume that* $[A_1]$, $[A_2]$, $[A_3]$, $[A_5]$, $[A_8]$ *hold. If* $\{W(t,s), t > s \geq 0\}$ *is exponentially stable, then the* (ω, c)*-periodic solution of* (1.2) *exists, which is exponentially stable.*

Proof. By [15, Lemma 3.1], (ω, c)-periodic solution of (1.2) exists, which has the following form

$$x(t; 0, x_0) = \int_0^{\omega} Y(t)H(t,s)g(s, x(s; 0, x_0))ds, \ x(\omega) = cx(0), \tag{4.1}$$

where

$$Y(t) = \begin{cases} I, \ t \in (s_i, t_{i+1}], \ i = 0, 1, 2, \cdots, \\ 0, \ t \in (t_i, s_i], \ i = 1, 2, \cdots, \end{cases} \tag{4.2}$$

and

$$H(t,s) = \begin{cases} c(cI - W(\omega, 0))^{-1}W(t,s), & 0 < s < t, \\ W(t,0)(cI - W(\omega,0))^{-1}W(\omega, s), & t \leq s < \omega. \end{cases} \tag{4.3}$$

By (4.1) via the exponential stability of $W(t,s)$, we have

$$\begin{aligned} \|x(t;0,x_0)\| &\leq \int_0^{\omega} \|Y(t)H(t,s)\|\|g(s, x(s;0,x_0))\|ds \\ &\leq \int_0^t |c|\|(cI - W(\omega,0))^{-1}\|\|W(t,s)\|\|g(s,x(s;0,x_0))\|ds \\ &\quad + \int_t^{\omega} \|W(t,0)\|\|(cI - W(\omega,0))^{-1}\|\|W(\omega,s)\|\|g(s,x(s;0,x_0))\|ds \\ &\leq |c|\|(cI - W(\omega,0))^{-1}\|KL_g \int_0^t \exp(-\gamma(t-s))\|x(s;0,x_0)\|ds \\ &\quad + \|(cI - W(\omega,0))^{-1}\|K^2L_g \int_t^{\omega} \exp(-\gamma(t+\omega-s))\|x(s;0,x_0)\|ds. \end{aligned}$$

Let $\tilde{u}(t) = \exp(\gamma t)\|x(t;0,x_0)\|$, we obtain

$$\begin{aligned} \tilde{u}(t) &\leq |c|\|(cI - W(\omega,0))^{-1}\|KL_g \int_0^t \tilde{u}(s)ds \\ &\quad + \|(cI - W(\omega,0))^{-1}\|K^2L_g \int_t^{\omega} \exp(-\gamma\omega)\tilde{u}(s)ds \end{aligned}$$

$$
\begin{aligned}
&\leq |c|\|(cI - W(\omega,0))^{-1}\|KL_g \int_0^t \tilde{u}(s)ds \\
&\quad +\|(cI - W(\omega,0))^{-1}\|K^2L_g \exp(-\gamma\omega) \int_t^\omega \tilde{u}(s)ds \\
&\leq \|(cI - W(\omega,0))^{-1}\|KL_g \max\{|c|, K\exp(-\gamma\omega)\} \int_0^\omega \tilde{u}(s)ds := M_{\gamma\omega} > 0.
\end{aligned}
$$

This implies that $\tilde{u}(t) = \exp(\gamma t)\|x(t;x_0)\| \leq M_{\gamma\omega}$, i.e. $\|x(t;0,x_0)\| \leq M_{\gamma\omega}\exp(-\gamma t)$. The proof is finished. □

Theorem 4.2. *Assume that* $[A_1]$, $[A_2]$, $[A_3]$, $[A_5]$, $[A_6]$ *hold. If* $\{W(t,s), t > s \geq 0\}$ *is exponentially stable, then any nontrivial* (ω,c)*-periodic solution of* (1.2) *is asymptotically stable.*

Proof. Let $x(t;0,x_0)$ be a nontrivial (ω,c)-periodic solution of (1.2) and $x(t;0,y_0)$ be another nontrivial solution of (1.2). By [15, Lemma 3.1], for any (ω,c)-periodic solution and nontrivial solution of (1.2) has the following form

$$
\begin{aligned}
x(t;0,x_0) &= \int_0^\omega Y(t)H(t,s)g(s,x(s;0,x_0))ds,\ x(\omega) = cx(0), \\
x(t;0,y_0) &= \int_0^\omega Y(t)H(t,s)g(s,x(s;0,y_0))ds,
\end{aligned} \tag{4.4}
$$

where $Y(t)$ and $H(t,s)$ are defined in (4.2) and (4.3).

By (4.4), we have

$$
\begin{aligned}
&\|x(t;0,x_0) - x(t;0,y_0)\| \\
\leq &\int_0^t |c|\|(cI - W(\omega,0))^{-1}\|\|W(t,s)\|\|g(s,x(s;0,x_0)) - g(s,x(s;0,y_0))\|ds \\
&+ \int_t^\omega \|W(t,0)\|\|(cI - W(\omega,0))^{-1}\|\|W(\omega,s)\|\|g(s,x(s;0,x_0)) - g(s,x(s;0,y_0))\|ds \\
\leq &|c|\|(cI - W(\omega,0))^{-1}\|KL \int_0^t \exp(-\gamma(t-s))\|x(s;0,x_0) - x(s;0,y_0)\|ds \\
&+\|(cI - W(\omega,0))^{-1}\|K^2L \int_t^\omega \exp(-\gamma(t+\omega-s))\|x(s;0,x_0) - x(s;0,y_0)\|ds.
\end{aligned}
$$

Let $u_2(t) = \exp(\gamma t)\|x(t;0,x_0) - x(t;0,y_0)\|$, we obtain

$$
\begin{aligned}
u_2(t) &\leq |c|\|(cI - W(\omega,0))^{-1}\|KL \int_0^t u_2(s)ds \\
&\quad +\|(cI - W(\omega,0))^{-1}\|K^2L \int_t^\omega \exp(-\gamma\omega)u_2(s)ds \\
&\leq KL\|(cI - W(\omega,0))^{-1}\| \max\{|c|, K\exp(-\gamma\omega)\} \int_0^\omega u_2(s)ds := K_{\omega\gamma}.
\end{aligned}
$$

Then

$$
\|x(t;0,x_0) - x(t;0,y_0)\| \leq K_{\omega\gamma}\exp(-\gamma t).
$$

The proof is complete. □

Theorem 4.3. *Assume that* $[A_1]$, $[A_2]$, $[A_4]$, $[A_5]$, $[A_6]$ *hold. If* $\gamma - NK > 0$ *and* $\|W(\omega, 0)\| \leq c$, *then the* (ω, c)*-periodic solution of* (1.2) *is exponentially stable.*

Proof. Note that $\|W(\omega, 0)\| \leq c$ implies $(I - \frac{1}{c}W(\omega, 0))^{-1}$ exists, which is equivalent to $(cI - W(\omega, 0))^{-1}$ exists. By Theorem 4.2, one can complete the proof. □

Theorem 4.4. *Assume that* $[A_1]$, $[A_2]$, $[A_3]$, $[A_5]$, $[A_6]$ *and* $[A_9]$ *hold. Then* (1.2) *has a* (ω, c)*-periodic solution.*

Proof. Consider the operator $T : PC([0, \omega], \mathbb{R}^n) \to PC([0, \omega], \mathbb{R}^n)$ on B_r, given by

$$Tx(t; 0, x_0) = \int_0^\omega Y(t)H(t, s)g(s, x(s; 0, x_0))ds. \tag{4.5}$$

where $Y(t)$ and $H(t)$ are defined in (4.2) and (4.3), $B_r := \{x \in PC([0, \omega] \mid \|x\| \leq (\frac{r}{NK_\lambda})^{\frac{1}{\varrho}}$ and $r > 0\}$. For any $0 \leq t \leq \omega$ and $x \in B_r$, using [15, Lemma 3.6], we have $\|H(t, s)\| \leq K_\lambda$, then

$$\begin{aligned} \|Tx(t; 0, x_0)\| &\leq \int_0^\omega \|Y(t)H(t, s)\|\|g(s, x(s; 0, x_0))\|ds \\ &\leq N \int_0^\omega \|Y(t)H(t, s)\|\|x(s; 0, x_0)\|^\varrho ds \\ &\leq NK_\lambda \|x\|^\varrho \leq r, \end{aligned}$$

Thus $T(B_r) \subset B_r$. Next, T is continuous and $T(B_r)$ is pre-compact. From Schauder's fixed point Theorem, (1.2) has at least one (ω, c)-periodic solution. □

Theorem 4.5. *Assume that* $[A_1]$, $[A_2]$, $[A_3]$, $[A_4]$, $[A_5]$, $[A_9]$ *hold. If* $\gamma - NK > 0$ *and* $\{W(t, s), t > s \geq 0\}$ *is exponentially stable. Then* (ω, c)*-periodic solution of* (1.2) *is exponentially stable.*

Proof. By [15, Lemma 3.1] and Theorem 4.4, any (ω, c)-periodic solution of (1.2) has the following form

$$\begin{aligned} x(t; 0, x_0) &= \int_0^\omega Y(t)W(t, 0)(cI - W(\omega, 0))^{-1}W(\omega, \theta)g(\theta, x(\theta, x(\theta; 0, x_0))d\theta \\ &\quad + \int_0^t Y(t)W(t, \theta)g(\theta, x(\theta; 0, x_0))d\theta. \end{aligned} \tag{4.6}$$

Set $a := \|(cI - W(\omega, 0))^{-1}\| = \frac{1}{c}\frac{1}{1 - c\|W(\omega, 0)\|}$. By (4.6), we have

$$\begin{aligned} \|x(t; 0, x_0)\| &\leq \int_0^\omega \|Y(t)\|\|W(t, 0)\|\|(cI - W(\omega, 0)^{-1})\|\|W(\omega, \theta)\|\|g(\theta, x(\theta; 0, x_0))\|d\theta \\ &\quad + \int_0^t \|Y(t)\|\|W(t, \theta)\|\|g(\theta, x(\theta; 0, x_0))\|d\theta \\ &\leq aNK^2 \exp(-\gamma t) \int_0^\omega \exp(-\gamma(\omega - \theta))\|x(\theta; 0, x_0)\|^\varrho d\theta \\ &\quad + NK \int_0^t \exp(-\gamma(t - \theta))\|x(\theta; 0, x_0)\|^\varrho d\theta, \end{aligned}$$

then

$$\begin{aligned}&\exp(\gamma t)\|x(t;0,x_0)\|^{\varrho} \le \exp(\gamma t)\|x(t;0,x_0)\|\\ \le\ & aNK^2\exp(-\gamma\omega)\int_0^{\omega}\exp(\gamma\theta)\|x(\theta;0,x_0)\|^{\varrho}d\theta + NK\int_0^{t}\exp(\gamma\theta)\|x(\theta;0,x_0)\|^{\varrho}d\theta\\ \le\ & \tilde{N} + NK\int_0^{t}\exp(\gamma\theta)\|x(\theta;0,x_0)\|^{\varrho}d\theta\end{aligned}$$

where $\tilde{N}$ is calculated as follows

$$\begin{aligned}&aNK^2\exp(-\gamma\omega)\int_0^{\omega}\exp(\gamma\theta)\|x(\theta;0,x_0)\|^{\varrho}d\theta\\ \le\ & aNK^2\exp(-\gamma\omega)\exp(\gamma\omega)\int_0^{\omega}\|x(\theta;0,x_0)\|^{\varrho}d\theta\\ \le\ & aNK^2\exp(-\gamma\omega)\exp(\gamma\omega)\omega\|x\|_B^{\varrho} := \tilde{N},\end{aligned}$$

where $\|x\|_B = \sup_{0\le s\le\theta}\|x(s)\|$.

Let $u_3(t) = \exp(\gamma t)\|x(t;0,x_0)\|^{\varrho}$, we obtain

$$u_3(t) \le \tilde{N} + NK\int_0^{t}u_3(\theta)d\theta.$$

By [32, Lemma 1, p.12], we have

$$u_3(t) = \exp(\gamma t)\|x(t;0,x_0)\|^{\varrho} \le \tilde{N}\exp(NKt),$$

this imply

$$\|x(t;0,x_0)\| \le \tilde{N}^{\frac{1}{\varrho}}\exp\Big(-\frac{\gamma - NK}{\varrho}t\Big).$$

The proof is complete. □

Example 4.6. *Consider the following nonlinear non-instantaneous impulsive system*

$$\begin{cases} x'(t) = Ax(t) + g(t,x(t)),\ t\in(s_i,t_{i+1}],\ i = 0,1,2,\cdots,\\ x(t_i^+) = Bx(t_i^-),\ i = 1,2,\cdots,\\ x(t) = Bx(t_i^-),\ t\in(t_i,s_i],\ i = 1,2,\cdots,\\ x(s_i^+) = x(s_i^-),\ i = 1,2,\cdots,\end{cases} \tag{4.7}$$

where

$$x(t) = \begin{pmatrix} x_1(t)\\ x_2(t)\end{pmatrix},\ A = \begin{pmatrix} -3 & \frac{1}{2}\\ 0 & -2\end{pmatrix},\ B = \begin{pmatrix} 1 & -1\\ 0 & -1\end{pmatrix},$$

$$g(t,x(t)) = (ax(t)\sin(7^{-t}x(t)),0)^{\top},\ a\in\mathbb{R},\ \ t_i = \frac{2i-1}{4},\ s_i = \frac{i}{2}.$$

Let $\omega = 1$, $c = 7$, and by a simple calculation, we have $AB = BA$, $t_{i+2} = \frac{2i+3}{4} = \frac{2i-1}{4} + 1 = t_i + 1$, $s_{i+2} = \frac{(i+2)}{2} = \frac{i}{2} + 1 = s_i + 1$, $b_{i+2} = b_i$ for all $i \in \mathbb{N}$. Then $m = 2$, $[A_1]$ and $[A_2]$ hold. By elementary calculations, we obtain $\sigma(A) = \{-2, -3\}$, and

$$e^{At} = \begin{pmatrix} e^{-3t} & \frac{1}{2}(e^{-2t} - e^{-3t}) \\ 0 & e^{-2t} \end{pmatrix},$$

and

$$\begin{aligned} W(\omega, 0) &= W(1,0) = B^{i(\omega,0)} e^{A[(\omega - s_{i(\omega,0)})^+ + \sum_{k=0}^{i(\omega,0)-1} (t_{k+1} - s_k)]} \\ &= B^2 e^{A[t_1 - s_0 + t_2 - s_1]} \\ &= B^2 e^{A\frac{1}{2}} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} e^{-\frac{3}{2}} & \frac{1}{2}(e^{-1} - e^{-\frac{3}{2}}) \\ 0 & e^{-1} \end{pmatrix} \\ &= \begin{pmatrix} e^{-\frac{3}{2}} & \frac{1}{2}(e^{-1} - e^{-\frac{3}{2}}) \\ 0 & e^{-1} \end{pmatrix}. \end{aligned}$$

Then, $c = 7 \notin \sigma(W(1,0)) = \{-e^{\frac{3}{2}}, \ -e^{-1}\}$, so $[A_3]$ holds. In addition, $\|W(\omega, 0)\| = \frac{3}{2}e^{-1} - \frac{1}{2}e^{\frac{3}{2}} <$ 0.4403 < 7.

Note that $g(t + \omega, cx) = g(t + 1, 7x) = a(7x)\sin(5^{-(t+1)}7x) = 7ax\sin(7^{-t}x) = 7g(t, x) = cg(t, x)$, so $[A_5]$ holds. Next, $\|g(t, x)\| \le |a|\|x\sin(7^{-t}x)\| \le |a|\|x\|$, so $[A_8]$ holds and $L_g = |a|$. Since $\sigma(A) = \{-2, -3\}$, $[A_4]$ is verified for $\lambda = -2$. On the other hand,

$$\begin{aligned} \|W(t, s)\| &= \|B^{i(t,0)-i(s,0)} e^{A[(t - s_{i(t,0)})^+ - (s - s_{i(s,0)})^+ + \sum_{k=i(s,0)}^{i(t,0)-1} (t_{k+1} - s_k)]}\| \\ &\le 2^{t-s} e^{-2[(t - s_{i(t,0)})^+ - (s - s_{i(s,0)})^+ + \sum_{k=i(s,0)}^{i(t,0)-1} (t_{k+1} - s_k)]} \\ &\le 2^{t-s} e^{-2[\frac{t-s}{2} - 1]} \\ &\le e^{-(t-s) + 2 + \ln 2 (t-s)} \\ &\le e^2 e^{(-1 + \ln 2)(t-s)} \\ &\le e^2 e^{-(1 - \ln 2)(t-s)}. \end{aligned}$$

Thus, $\{W(t, s), t > s \ge 0\}$ is exponentially stable by setting $K = e^2, \gamma = 1 - \ln 2$. Next, $\gamma - NK = 1 - \ln 2 - e^2|a| > 0$ if $-\frac{1-\ln 2}{e^2} < a < \frac{1-\ln 2}{e^2}$. By Theorem 4.1 or 4.3, the $(1, 7)$-period solution of (4.7) is exponentially stable.

Example 4.7. *Consider the following nonlinear non-instantaneous impulsive system*

$$\begin{cases} x'(t) = Ax(t) + g(t, x(t)), \ t \in (s_i, t_{i+1}], \ i = 0, 1, 2, \cdots, \\ x(t_i^+) = Bx(t_i^-), \ i = 1, 2, \cdots, \\ x(t) = Bx(t_i^-), \ t \in (t_i, s_i], \ i = 1, 2, \cdots, \\ x(s_i^+) = x(s_i^-), \ i = 1, 2, \cdots, \end{cases} \tag{4.8}$$

Let

$$x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}, \ A = \begin{pmatrix} -3 & \frac{112}{3} \\ 0 & -\frac{1}{5} \end{pmatrix}, \ B = \begin{pmatrix} \frac{1}{8} & -\frac{1}{24} \\ 0 & \frac{1}{10} \end{pmatrix},$$

$$g(t,x)=(a[x\sin(2t)]^{\frac{1}{3}},0)^{\top},\ a\in\mathbb{R},\ \ t_i=\frac{(2i+1)\pi}{2},\ s_i=\frac{i\pi}{2}.$$

Let $\omega=\pi$, $c=-1$, *and by a simple calculation, we have* $AB=BA$, $\|B\|=\frac{17}{120}$, $t_{i+1}=\frac{(2i+3)\pi}{2}=\frac{(2i+1)\pi}{2}+\pi=t_i+\pi$, $s_{i+1}=(i+1)\pi=i\pi+\pi=s_i+\pi$, $b_{i+1}=b_i$ *for all* $i\in\mathbb{N}$. *Then* $m=1$, $[A_1]$ *and* $[A_2]$ *hold. By elementary calculations, we obtain* $\sigma(A)=\{-0.2,-3\}$, *and*

$$e^{At}=\begin{pmatrix} e^{-3t} & \frac{40}{3}(e^{-0.2t}-e^{-3t}) \\ 0 & e^{-0.2t}\end{pmatrix},$$

and

$$\begin{aligned}
W(\omega,0) &= W(\pi,0)=B^{i(\omega,0)}e^{A[(\omega-s_{i(\omega,0)})^{+}+\sum\limits_{k=0}^{i(\omega,0)-1}(t_{k+1}-s_k)]}\\
&= Be^{A[t_1-s_0+\pi-\pi]}\\
&= Be^{A\frac{\pi}{2}}\\
&= \begin{pmatrix} \frac{1}{8} & -\frac{1}{24}\\ 0 & \frac{1}{10}\end{pmatrix}\cdot\begin{pmatrix} e^{-\frac{3\pi}{2}} & \frac{40}{3}(e^{-0.1\pi}-e^{-\frac{3\pi}{2}})\\ 0 & e^{-0.1\pi}\end{pmatrix}\\
&= \begin{pmatrix} \frac{1}{8}e^{-\frac{3\pi}{2}} & \frac{13}{8}e^{-0.1\pi}-\frac{5}{3}e^{-\frac{3\pi}{2}}\\ 0 & \frac{1}{10}e^{-0.1\pi}\end{pmatrix}.
\end{aligned}$$

Then, $c=-1\notin\sigma(W(\pi,0))=\{\frac{1}{8}e^{-\frac{3\pi}{2}},\frac{1}{10}e^{-0.1\pi}\}$, *so* $[A_3]$ *holds.*

Note that $g(t+\omega,cx)=g(t+\pi,-x)=a[-x\sin(2(t+\pi))]^{\frac{1}{3}}=-a[x\sin(2t)]^{\frac{1}{3}}=-g(t,x)=cg(t,x)$, *so* $[A_5]$ *holds. Next,* $\|g(t,x)\|=\|a[x\sin(2t)]^{\frac{1}{3}}\|\le|a|\|x\|^{\frac{1}{3}}\|$, *so* $[A_9]$ *holds and* $N=|a|,\varrho=\frac{1}{3}$. *Since* $\sigma(A)=\{-0.2,-3\}$, $[A_4]$ *is verified for* $\lambda=-0.2$. *On the other hand,*

$$\begin{aligned}
\|W(t,s)\| &= \left\|B^{i(t,0)-i(s,0)}\exp\left\{A[(t-s_{i(t,0)})^{+}-(s-s_{i(s,0)})^{+}+\sum_{k=i(s,0)}^{i(t,0)-1}(t_{k+1}-s_k)]\right\}\right\|\\
&\le \exp\left\{-0.2[(t-s_{i(t,0)})^{+}-(s-s_{i(s,0)})^{+}+\sum_{k=i(s,0)}^{i(t,0)-1}(t_{k+1}-s_k)]\right\}\\
&\le \exp\{-0.2[\frac{t-s}{2}-\frac{\pi}{2}]\}\\
&\le \exp(0.1\pi)\exp(-0.1(t-s)).
\end{aligned}$$

Thus, $\{W(t,s),t>s\ge0\}$ *is exponentially stable by setting* $K=\exp(0.1\pi),\gamma=0.1$. *Next,* $\gamma-NK=0.1-|a|\exp(0.1\pi)>0$ *if* $-0.07304<a<0.07305$. *By Theorem 4.5, the* $(\pi,-1)$*-period solution of* (4.8) *is exponentially stable.*

5. Conclusions

This paper deals with the stability of (ω,c)-periodic solutions of non-instantaneous impulses differential equations. Firstly, some sufficient conditions for exponential stability of linear homogeneous non-instantaneous impulse problems are obtained by using Cauchy matrix. Secondly, by using Gronwall inequality, sufficient conditions are established for exponential stability and asymptotic stability of (ω,c)-periodic solutions of nonlinear problems. Our results can be applied to non-instantaneous impulsive two-parameter equations, and our method can be extended to time-varying differential systems.

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Conflict of interest

The author declare no conflicts of interest in this paper.

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