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Research article

The random convolution sampling stability in multiply generated shift invariant subspace of weighted mixed Lebesgue space

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Abstract: In this paper, we mainly investigate the random convolution sampling stability for signals in multiply generated shift invariant subspace of weighted mixed Lebesgue space. Under some restricted conditions for the generators and the convolution function, we conclude that the defined multiply generated shift invariant subspace could be approximated by a finite dimensional subspace. Furthermore, with overwhelming probability, the random convolution sampling stability holds for signals in some subset of the defined multiply generated shift invariant subspace when the sampling size is large enough.

Keywords: random convolution sampling; multiply generated shift invariant subspace; weighted mixed Lebesgue space
Mathematics Subject Classification: 94A20

1. Introduction

Random sampling problems widely arise in compressed sensing [5, 7], learning theory [17] and image processing [6]. Recently, many random sampling results for signals in different subspaces of the classical Lebesgue space have already been presented such as bandlimited space [2, 3], shift invariant subspace [19], multiply generated shift invariant subspace [18] and reproducing kernel subspace [14, 16]. However, an obvious shortcoming of the classical Lebesgue space is that it imposed the same control over all the variables of a function [12]. Thus, when considering the random sampling problems for some time-varying signals which depend on independent quantities with different properties, the mixed Lebesgue space, which could realize the separate integrability for each variable, seems to be a much more suitable tool to model those signals.

Up to date, abundant sampling results for many subspaces of a mixed Lebesgue space have already been presented [10–13]. While, we should notice that most of those sampling results are all obtained based on a pre-given relatively separated set whose sampling gap satisfies some restricted conditions.

As for the random sampling results, which are based on a randomly selected sampling set, there are very few results. Furthermore, an obvious precondition for these existing sampling results is that those signals are all integrable in the corresponding spaces, which is impossible for some non-decaying or infinitely growing signals. Based on these two facts and the properties of the moderated weight functions, which could control the growth or decay of the signals [1, 9], we consider to use weighted mixed Lebesgue space to model those non-decaying or infinitely growing signals such that the corresponding random sampling problem could be well solved .

Generally, the sampling problem mainly consists of the following two aspects. Firstly, finding out the proper conditions which ensure the given sampling set satisfies sampling stability. Secondly, designing an efficient reconstruction algorithm to restore the signals. In this paper, we mainly focus on the investigation of sampling stability. As for the research about the reconstruction algorithm, it will be the goal of our future work. In the following, some essential definitions or properties are presented such that the random convolution sampling problem is well understood.

The weighted mixed Lebesgue space $L^{p,q}_\nu(\mathbb{R}^{d+1})$ consists of all measurable functions $f = f(x,y)$ defined on $\mathbb{R}\times\mathbb{R}^d$ such that

$$\|f\|_{L^{p,q}_\nu(\mathbb{R}^{d+1})} = \Big\| \|f(x,y)\nu(x,y)\|_{L^q_y(\mathbb{R}^d)} \Big\|_{L^p_x(\mathbb{R})} < \infty, \quad 1 \le p,q \le \infty. \tag{1.1}$$

The corresponding weighted sequence space $\ell^{p,q}_\nu(\mathbb{Z}^{d+1})$ is defined by

$$\ell^{p,q}_\nu(\mathbb{Z}^{d+1}) = \Big\{ c : \|c\|_{\ell^{p,q}_\nu(\mathbb{Z}^{d+1})} = \Big\| \|c(k_1,k_2)\nu(k_1,k_2)\|_{\ell^q_{k_2}(\mathbb{Z}^d)} \Big\|_{\ell^p_{k_1}(\mathbb{Z})} < \infty \Big\}, \quad 1 \le p,q \le \infty. \tag{1.2}$$

In this paper, the weight function $\nu(x,y)$ is assumed to be continuous, positive, symmetric and moderated with respect to the weight function $\omega(x,y)$, i.e., there exists a constant $C > 0$, such that,

$$0 < \nu(x+x', y+y') \le C\nu(x,y)\omega(x',y'), \quad (x,y),(x',y') \in \mathbb{R}\times\mathbb{R}^d. \tag{1.3}$$

Unless otherwise specified, the constant C without any subscript or superscript in this paper all stands for the above mentioned constant in (1.3). Furthermore, we also make the assumption that the weight function $\omega(x,y)$ is continuous, positive, symmetric and submultiplicative, that is, for all $(x,y),(x',y') \in \mathbb{R}\times\mathbb{R}^d$,

$$0 < \omega(x+x', y+y') \le \omega(x,y)\omega(x',y'), \quad (x,y),(x',y') \in \mathbb{R}\times\mathbb{R}^d.$$

The classical example about the above weight function is

$$m(x) = e^{a|x|^b}(1+|x|)^s(log(e+|x|))^t, \quad x \in \mathbb{R}^{d+1}.$$

If $a,s,t \ge 0$ and $0 \le b \le 1$, then the weight function $m(x)$ is submultiplicative. If $a,s,t \in \mathbb{R}$ and $0 \le b \le 1$, then the weight function $m(x)$ is moderated. For further information about weight function, please refer to [8].

In addition, the sampling set $X = \{(x_j,y_k)\}_{j=1,\cdots,m;k=1,\cdots,n}$ in this paper is assumed to consist of sampling points which are randomly selected in C_{R_1,R_2} with the density function $\rho(x,y)$ satisfying

$$0 < C_{\rho,l} \le \rho(x,y) \le C_{\rho,u}, \quad (x,y) \in C_{R_1,R_2}, \tag{1.4}$$

where $C_{R_1,R_2} := [-R_1,R_1]\times[-R_2,R_2]^d$ with $R_1,R_2 > 0$. Meanwhile, due to the limitation of the sampling devices, the obtained sampling value is not the exact value of signal at each sampling point but is the local average value near the corresponding sampling location. Thus, we assume that the corresponding sampling values are obtained by the following convolution version

$$\{(f * \psi)(x_j, y_k), \quad (x_j, y_k) \in X\},$$

where the convolution function ψ satisfies

$$\psi \in L^{1,1}_{\omega}(\mathbb{R}^{d+1}), \quad supp\psi \subset C_{R_1,R_2}. \tag{1.5}$$

In this paper, we mainly consider the random convolution sampling stability for signals in multiply generated shift-invariant subspace of weighted mixed Lebesgue space, which has the form

$$V^{p,q}_{\nu} := \Big\{ \sum_{i=1}^{r} \sum_{k_1 \in \mathbb{Z}} \sum_{k_2 \in \mathbb{Z}^d} c_i(k_1, k_2)\phi_i(x - k_1, y - k_2), c_i \in \ell^{p,q}_{\nu}(\mathbb{Z}^{d+1}) \Big\}, 1 \le p, q \le \infty, \tag{1.6}$$

where the generators ϕ_i, $i = 1, \cdots, r$ satisfy the following conditions:

- For any $(x, y) \in \mathbb{R} \times \mathbb{R}^d$,

$$|(\phi_i \omega)(x, y)| \le \frac{1}{(1 + |x|)^{n_1}(1 + |y|)^{n_2}}, \quad n_1 > d + 1, n_2 > d + 1, \tag{1.7}$$

  where $|\cdot|$ means the traditional Euclidean norm.
- There exist constants $c_{p,q}, C_{p,q} > 0$ such that

$$\begin{aligned} c_{p,q} \sum_{i=1}^{r} \|c_i\|_{\ell^{p,q}_{\nu}(\mathbb{Z}^{d+1})} &\le \Big\| \sum_{i=1}^{r} \sum_{k_1 \in \mathbb{Z}} \sum_{k_2 \in \mathbb{Z}^d} c_i(k_1, k_2)\phi_i(x - k_1, y - k_2) \Big\|_{L^{p,q}_{\nu}(\mathbb{R}^{d+1})} \\ &\le C_{p,q} \sum_{i=1}^{r} \|c_i\|_{\ell^{p,q}_{\nu}(\mathbb{Z}^{d+1})}. \end{aligned} \tag{1.8}$$

- There exists a constant $0 < \beta < 1$ such that

$$\beta \|\psi\|_{L^{1,1}_{\nu}(C_{R_1,R_2})} \|f\|_{L^{p,q}_{\nu}(\mathbb{R}^{d+1})} \le \|f * \psi\|_{L^{p,q}_{\nu}(C_{R_1,R_2})}. \tag{1.9}$$

Besides these, for a well defined function f in the practice, the values of the sampling points which are located in a very distant place may not be significant. Thus, we only consider the following subset

$$V^{p,q}_{\nu,R_1,R_2} := \Big\{ f \in V^{p,q}_{\nu} : (1 - \delta)\|f\|_{L^{p,q}_{\nu}(\mathbb{R}^{d+1})} \le \|f\|_{L^{p,q}_{\nu}(C_{R_1,R_2})} \Big\} \tag{1.10}$$

or its normalization

$$V^{p,q,*}_{\nu,R_1,R_2} := \Big\{ f \in V^{p,q}_{\nu,R_1,R_2} : \|f\|_{L^{p,q}_{\nu}(\mathbb{R}^{d+1})} = 1 \Big\}, \tag{1.11}$$

where $0 \le \delta < 1$. Obviously, the sets $V^{p,q}_{\nu,R_1,R_2}$ and $V^{p,q,*}_{\nu,R_1,R_2}$ consist of those functions whose energy is mainly concentrated in C_{R_1,R_2}.

This paper is organized as follows. In Section 2, we prove that the defined multiply generated shift invariant subspace could be approximated by a finite dimensional subspace. In Section 3, we present some essential results which will contribute to the proof of the sampling stability. In Section 4, we prove that with overwhelming probability, the sampling stability holds for the signals in some subset of the defined multiply generated shift invariant subspace when the sampling size is large enough.

2. Approximation

Define a finite dimensional subspace by

$$V_{\nu,N}^{p,q} := \Big\{ \sum_{i=1}^{r} \sum_{|k_1|\le N} \sum_{|k_2|\le N} c_i(k_1,k_2)\phi_i(x-k_1, y-k_2), c_i \in \ell_\nu^{p,q}([-N,N]^{d+1})\Big\}, 1 \le p,q \le \infty \tag{2.1}$$

and its normalization

$$V_{\nu,N}^{p,q,*} := \Big\{ f \in V_{\nu,N}^{p,q} : \quad \|f\|_{L_\nu^{p,q}(\mathbb{R}^{d+1})} = 1\Big\}, 1 \le p,q \le \infty. \tag{2.2}$$

Lemma 2.1. *[15] Let $\alpha > 0$, $x \in \mathbb{R}^d$, $I_1(M,\alpha) = \int_{|x|\ge M} |x|^{-d-\alpha}dx$, then*

$$I_1(M,\alpha) = 2\pi^{\frac{d}{2}}\Gamma(\frac{d}{2})^{-1}\frac{1}{\alpha}M^{-\alpha}. \tag{2.3}$$

Lemma 2.2. *Let $1 < p,q < \infty$, $R = \max\{R_1, \sqrt{d}R_2\}$ and $s = \min\{n_1-1-d, n_2-1-d\}$. Assume that the function $f \in V_\nu^{p,q}$ with $\|f\|_{L_\nu^{p,q}(\mathbb{R}^{d+1})} = 1$, then for the given $\varepsilon_1, \varepsilon_2 > 0$, there exists a function $f_N \in V_{\nu,N}^{p,q}$ such that*

$$\|f - f_N\|_{L_\nu^{p,q}(C_{2R_1,2R_2})} \le \varepsilon_1, \tag{2.4}$$

if

$$\begin{aligned} N \ge 2R + 1 + \Big\{\frac{C(4R_1)^{\frac{1}{p}}(4R_2)^{\frac{d}{q}}}{c_{p,q}\varepsilon_1}\Big[\frac{8C_d\pi^{\frac{d}{2}}\Gamma(\frac{d}{2})^{-1}(1+\sqrt{d}R_2)}{(n_2-d)} + \frac{2^{2d+1}C_1(1+R_1)^d}{(n_1-1)} \\ + \frac{4C_1C_d\pi^{\frac{d}{2}}\Gamma(\frac{d}{2})^{-1}}{(n_1-1)(n_2-d)}\Big]\Big\}^{1/s} := 2R + 1 + N_1\{\varepsilon_1\} \end{aligned} \tag{2.5}$$

and

$$\|f - f_N\|_{L_\nu^{\infty,\infty}(C_{2R_1,2R_2})} \le \varepsilon_2, \tag{2.6}$$

if

$$\begin{aligned} N \ge 2R + 1 + \Big\{\frac{C}{c_{p,q}\varepsilon_2}\Big[\frac{8C_d\pi^{\frac{d}{2}}\Gamma(\frac{d}{2})^{-1}(1+\sqrt{d}R_2)}{(n_2-d)} + \frac{2^{2d+1}C_1(1+R_1)^d}{(n_1-1)} \\ + \frac{4C_1C_d\pi^{\frac{d}{2}}\Gamma(\frac{d}{2})^{-1}}{(n_1-1)(n_2-d)}\Big]\Big\}^{1/s} := 2R + 1 + N_2\{\varepsilon_2\}, \end{aligned} \tag{2.7}$$

where C_1 and C_d are the constants which depend on the dimensional of the corresponding space.

Proof. By the definitions of $V_\nu^{p,q}$ and $V_{\nu,N}^{p,q}$ in (1.6) and (2.1),

$$\begin{aligned} f - f_N = &\sum_{i=1}^{r} \sum_{|k_1|\le N} \sum_{|k_2|> N} c_i(k_1,k_2)\phi_i(x-k_1,y-k_2) + \sum_{i=1}^{r} \sum_{|k_1|> N} \sum_{|k_2|\le N} c_i(k_1,k_2)\phi_i(x-k_1,y-k_2) \\ &+ \sum_{i=1}^{r} \sum_{|k_1|> N} \sum_{|k_2|> N} c_i(k_1,k_2)\phi_i(x-k_1,y-k_2) := I_1 + I_2 + I_3. \end{aligned} \tag{2.8}$$

Next, we will separately estimate $\|I_1\|_{L^{p,q}_\nu(C_{2R_1,2R_2})}$, $\|I_2\|_{L^{p,q}_\nu(C_{2R_1,2R_2})}$ and $\|I_3\|_{L^{p,q}_\nu(C_{2R_1,2R_2})}$.

Firstly, by (1.3), (1.7) and (1.8), we could obtain

$$\begin{aligned}
&\|I_1\|_{L^{p,q}_\nu(C_{2R_1,2R_2})} \\
&\leq C\sum_{i=1}^{r}\Big\|\sum_{|k_1|\leq N}\sum_{|k_2|>N}|(c_i\nu)(k_1,k_2)||(\phi_i\omega)(x-k_1,y-k_2)|\Big\|_{L^{p,q}(C_{2R_1,2R_2})} \\
&\leq C\sum_{i=1}^{r}\|c_i\|_{\ell^{p,q}_\nu(\mathbb{Z}^{d+1})}\Big\|\sum_{|k_1|\leq N}\sum_{|k_2|>N}|(\phi_i\omega)(x-k_1,y-k_2)|\Big\|_{L^{p,q}(C_{2R_1,2R_2})} \\
&\leq C\sum_{i=1}^{r}\|c_i\|_{\ell^{p,q}_\nu(\mathbb{Z}^{d+1})}\Big\|\sum_{|k_1|\leq N}\sum_{|k_2|>N}\frac{1}{(1+|x-k_1|)^{n_1}(1+|y-k_2|)^{n_2}}\Big\|_{L^{p,q}(C_{2R_1,2R_2})} \\
&\leq \frac{C}{c_{p,q}}\Big\|\sum_{|k_1|\leq N}\sum_{|k_2|>N}\frac{1}{(1+|x-k_1|)^{n_1}(1+|y-k_2|)^{n_2}}\Big\|_{L^{p,q}(C_{2R_1,2R_2})}.
\end{aligned} \tag{2.9}$$

With the help of Lemma 2.1 and the fact $(a+b)^d \leq a^d(1+b)^d$ for $a \geq 1$ and $b > 0$,

$$\begin{aligned}
&\sum_{|k_1|\leq N}\sum_{|k_2|>N}\frac{1}{(1+|x-k_1|)^{n_1}(1+|y-k_2|)^{n_2}} \\
&= \Big(\sum_{|k_1|\leq N}\frac{1}{(1+|x-k_1|)^{n_1}}\Big)\Big(\sum_{|k_2|>N}\frac{1}{(1+|y-k_2|)^{n_2}}\Big) \\
&\leq \Big(\sum_{|k_1|\leq N}\frac{1}{(1+|x-k_1|)^{n_1}}\Big)\Big(\sum_{|k_2-y|\geq N-|y|}\frac{1}{(1+|y-k_2|)^{n_2}}\Big) \\
&\leq (2N+1)C_d\int_{|u-y|\geq N-|y|}\frac{1}{|u-y|^{n_2}}du \\
&= (2N+1)C_d\int_{|u-y|\geq N-|y|}|u-y|^{-d-(n_2-d)}du \\
&\leq \frac{2C_d(N-|y|+|y|+1)2\pi^{\frac{d}{2}}\Gamma(\frac{d}{2})^{-1}}{(n_2-d)(N-|y|)^{n_2-d}} \\
&\leq \frac{2C_d(N-|y|)(|y|+2)2\pi^{\frac{d}{2}}\Gamma(\frac{d}{2})^{-1}}{(n_2-d)(N-|y|)^{n_2-d}} \\
&\leq \frac{2C_d(2\sqrt{d}R_2+2)2\pi^{\frac{d}{2}}\Gamma(\frac{d}{2})^{-1}}{(n_2-d)(N-2\sqrt{d}R_2)^{n_2-d-1}}.
\end{aligned} \tag{2.10}$$

Combining the results of (2.9) and (2.10),

$$\|I_1\|_{L^{p,q}_\nu(C_{2R_1,2R_2})} \leq \frac{8CC_d(4R_1)^{\frac{1}{p}}(4R_2)^{\frac{d}{q}}\pi^{\frac{d}{2}}\Gamma(\frac{d}{2})^{-1}(1+\sqrt{d}R_2)}{c_{p,q}(n_2-d)(N-2\sqrt{d}R_2)^{n_2-d-1}}. \tag{2.11}$$

By the similar method, we could obtain

$$\|I_2\|_{L^{p,q}_\nu(C_{2R_1,2R_2})} \leq \frac{2^{2d+1}CC_1(4R_1)^{\frac{1}{p}}(4R_2)^{\frac{d}{q}}(1+R_1)^d}{c_{p,q}(n_1-1)(N-2R_1)^{n_1-d-1}}, \tag{2.12}$$

$$\|I_3\|_{L^{p,q}_\nu(C_{2R_1,2R_2})} \le \frac{4CC_1C_d\pi^{\frac{d}{2}}\Gamma(\frac{d}{2})^{-1}(4R_1)^{\frac{1}{p}}(4R_2)^{\frac{d}{q}}}{c_{p,q}(n_1-1)(n_2-d)(N-2R)^{n_1+n_2-d-1}}. \tag{2.13}$$

Thus, the result (2.4) is followed by (2.11)–(2.13).

When $p = q = \infty$,

$$\|I_1\|_{L^{\infty,\infty}_\nu(C_{2R_1,2R_2})} \le \frac{8CC_d\pi^{\frac{d}{2}}\Gamma(\frac{d}{2})^{-1}(1+\sqrt{d}R_2)}{c_{p,q}(n_2-d)(N-2\sqrt{d}R_2)^{n_2-d-1}}, \tag{2.14}$$

$$\|I_2\|_{L^{\infty,\infty}_\nu(C_{2R_1,2R_2})} \le \frac{2^{2d+1}CC_1(1+R_1)^d}{c_{p,q}(n_1-1)(N-2R_1)^{n_1-d-1}}, \tag{2.15}$$

$$\|I_3\|_{L^{\infty,\infty}_\nu(C_{2R_1,2R_2})} \le \frac{4CC_1C_d\pi^{\frac{d}{2}}\Gamma(\frac{d}{2})^{-1}}{c_{p,q}(n_1-1)(n_2-d)(N-2R)^{n_1+n_2-d-1}}. \tag{2.16}$$

Thus, the result (2.6) is followed by (2.14)–(2.16). □

3. Some lemmas

For dealing well with the random convolution sampling stability for signals in multiply generated shift invariant subspace of weighted mixed Lebesgue space, some essential conclusions are presented in the following.

Lemma 3.1. *For $f \in L^{p,q}_\nu(\mathbb{R}^{d+1})$ and $h \in L^{1,1}_\omega(\mathbb{R}^{d+1})$,*

$$\|f*h\|_{L^{p,q}_\nu(\mathbb{R}^{d+1})} \le C\|f\|_{L^{p,q}_\nu(\mathbb{R}^{d+1})}\|h\|_{L^{1,1}_\omega(\mathbb{R}^{d+1})}. \tag{3.1}$$

Proof. For any $f \in L^{p,q}_\nu(\mathbb{R}^{d+1})$ and $h \in L^{1,1}_\omega(\mathbb{R}^{d+1})$,

$$\begin{aligned} &|(f*h)(x,y)|\nu(x,y) \\ &\le \int_{\mathbb{R}}\int_{\mathbb{R}^d} |f(t,s)h(x-t,y-s)|dtds\nu(x,y) \\ &\le C\int_{\mathbb{R}}\int_{\mathbb{R}^d} (|f|\nu)(t,s)(|h|\omega)(x-t,y-s)dtds \\ &= C(|f|\nu * |h|\omega)(x,y). \end{aligned} \tag{3.2}$$

By the result in [4],

$$\|f*h\|_{L^{p,q}_\nu(\mathbb{R}^{d+1})} \le C\Big\|(|f|\nu)*(|h|\omega)\Big\|_{L^{p,q}(\mathbb{R}^{d+1})} \le C\|f\|_{L^{p,q}_\nu(\mathbb{R}^{d+1})}\|h\|_{L^{1,1}_\omega(\mathbb{R}^{d+1})}. \tag{3.3}$$

□

Furthermore, we also need the help of the covering number which is a powerful tool in estimating the probability error or the number of samples required for a given confidence and error bound [20].

Lemma 3.2. *Let $V_{\nu,N}^{p,q,*}$ be defined by* (2.2)*. Then for any $\eta > 0$, the covering number of $V_{\nu,N}^{p,q,*}$ with respect to the norm $\|\cdot\|_{L_\nu^{p,q}(\mathbb{R}^{d+1})}$ is bounded by*

$$\mathcal{N}(V_{\nu,N}^{p,q,*},\eta) \le \exp\Big(r(2N+1)^{d+1}\ln\Big(\frac{2}{\eta}+1\Big)\Big). \tag{3.4}$$

The proof could refer to [10]. There will omit it.

Lemma 3.3. *Let $V_{\nu,N}^{p,q,*}$ be defined by* (2.2)*. Then for any $\eta > 0$, the covering number of $V_{\nu,N}^{p,q,*}$ with respect to the norm $\|\cdot\|_{L_\nu^{\infty,\infty}(\mathbb{R}^{d+1})}$ is bounded by*

$$\mathcal{N}(V_{\nu,N}^{p,q,*},\eta) \le \exp\Big(r(2N+1)^{d+1}\ln\Big(\frac{2C^*}{\eta}+1\Big)\Big), \tag{3.5}$$

where

$$C^* := \frac{C}{c_{p,q}}\Big(\sum_{k_1\in\mathbb{Z}}\frac{2}{(1+|k_1|)^{n_1}}\Big)\Big(\sum_{k_2\in\mathbb{Z}^d}\frac{2}{(1+|k_2|)^{n_2}}\Big). \tag{3.6}$$

Proof. For any $f \in V_{\nu,N}^{p,q,*}$,

$$\begin{aligned}
\|f\|_{L_\nu^{\infty,\infty}(\mathbb{R}^{d+1})} &= \Big\|\sum_{i=1}^{r}\sum_{|k_1|\le N}\sum_{|k_2|\le N}c_i(k_1,k_2)\phi_i(x-k_1,y-k_2)\Big\|_{L_\nu^{\infty,\infty}(\mathbb{R}^{d+1})}\\
&\le C\Big\|\sum_{i=1}^{r}\sum_{|k_1|\le N}\sum_{|k_2|\le N}(c_i\nu)(k_1,k_2)(\phi_i\omega)(x-k_1,y-k_2)\Big\|_{L^{\infty,\infty}(\mathbb{R}^{d+1})}\\
&\le C\Big\|\sum_{i=1}^{r}\sum_{|k_1|\le N}\sum_{|k_2|\le N}\frac{|(c_i\nu)(k_1,k_2)|}{(1+|x-k_1|)^{n_1}(1+|y-k_2|)^{n_2}}\Big\|_{L^{\infty,\infty}(\mathbb{R}^{d+1})}\\
&\le C\sum_{i=1}^{r}\|c_i\|_{\ell_\nu^{p,q}([-N,N]^{d+1})}\Big\|\sum_{|k_1|\le N}\sum_{|k_2|\le N}\frac{1}{(1+|x-k_1|)^{n_1}(1+|y-k_2|)^{n_2}}\Big\|_{L^{\infty,\infty}(\mathbb{R}^{d+1})}\\
&= C\sum_{i=1}^{r}\|c_i\|_{\ell_\nu^{p,q}([-N,N]^{d+1})}\Big\|\Big(\sum_{|k_1|\le N}\frac{1}{(1+|x-k_1|)^{n_1}}\Big)\Big(\sum_{|k_2|\le N}\frac{1}{(1+|y-k_2|)^{n_2}}\Big)\Big\|_{L^{\infty,\infty}(\mathbb{R}^{d+1})}\\
&\le C\sum_{i=1}^{r}\|c_i\|_{\ell_\nu^{p,q}([-N,N]^{d+1})}\Big(\sum_{k_1\in\mathbb{Z}}\frac{2}{(1+|k_1|)^{n_1}}\Big)\Big(\sum_{k_2\in\mathbb{Z}^d}\frac{2}{(1+|k_2|)^{n_2}}\Big)\\
&\le \frac{C\|f\|_{L_\nu^{p,q}(\mathbb{R}^{d+1})}}{c_{p,q}}\Big(\sum_{k_1\in\mathbb{Z}}\frac{2}{(1+|k_1|)^{n_1}}\Big)\Big(\sum_{k_2\in\mathbb{Z}^d}\frac{2}{(1+|k_2|)^{n_2}}\Big) = C^*.
\end{aligned} \tag{3.7}$$

Let $\mathcal{F}$ be the corresponding $\frac{\eta}{C^*}-$ net for the space $V_{\nu,N}^{p,q,*}$ with respect to the norm $\|\cdot\|_{L_\nu^{p,q}(\mathbb{R}^{d+1})}$. Then for any $f \in V_{\nu,N}^{p,q,*}$, there exists a function $\widetilde{f} \in \mathcal{F}$ such that $\|f-\widetilde{f}\|_{L_\nu^{p,q}(\mathbb{R}^{d+1})} \le \frac{\eta}{C^*}$. Furthermore,

$$\|f-\widetilde{f}\|_{L_\nu^{\infty,\infty}(\mathbb{R}^{d+1})} \le C^*\|f-\widetilde{f}\|_{L_\nu^{p,q}(\mathbb{R}^{d+1})} \le \eta. \tag{3.8}$$

Thus, $\mathcal{F}$ is also a $\eta-$ net of $V_{\nu,N}^{p,q,*}$ with respect to the norm $\|\cdot\|_{L_\nu^{\infty,\infty}(\mathbb{R}^{d+1})}$ and the cardinality of $\mathcal{F}$ is at most (3.5). □

Based on the function $f \in V_\nu^{p,q}$, we will introduce the random variable

$$Z_{j,k}(f) = |(f * \psi)(x_j, y_k)|\nu(x_j, y_k) - \int_{-R_1}^{R_1}\int_{[-R_2,R_2]^d} \rho(x,y)|(f*\psi)(x,y)|\nu(x,y)dxdy, \tag{3.9}$$

where the sampling set $\{(x_j, y_k)\}_{j=1,\cdots,m;k=1,\cdots,n}$ is a sequence of independent random variables which are drawn from a general probability distribution over C_{R_1,R_2} with the density function ρ satisfying (1.4). Obviously, $\{Z_{j,k}(f), j = 1,\cdots,m; k = 1,\cdots,n\}$ is a sequence of independent random variables with $\mathbb{E}[Z_{j,k}(f)] = 0$. Regarding to the other properties, we will elaborate on them in the following section.

Lemma 3.4. *Let the density function ρ satisfy the condition* (1.4) *and the convolution function ψ satisfy* (1.5). *Then for any $f, g \in V_\nu^{p,q}$,*

(1) $\|Z_{j,k}(f)\|_{\ell^{\infty,\infty}} \le C\|f\|_{L_\nu^{\infty,\infty}(C_{2R_1,2R_2})}\|\psi\|_{L_\omega^{1,1}(C_{R_1,R_2})}$;

(2) $\|Z_{j,k}(f) - Z_{j,k}(g)\|_{\ell^{\infty,\infty}} \le 2C\|f-g\|_{L_\nu^{\infty,\infty}(C_{2R_1,2R_2})}\|\psi\|_{L_\omega^{1,1}(C_{R_1,R_2})}$;

(3) $Var(Z_{j,k}(f)) \le C^2\|f\|^2_{L_\nu^{\infty,\infty}(C_{2R_1,2R_2})}\|\psi\|^2_{L_\omega^{1,1}(C_{R_1,R_2})}$;

(4) $Var(Z_{j,k}(f) - Z_{j,k}(g)) \le C^2\|\psi\|^2_{L_\omega^{1,1}(C_{R_1,R_2})}\|f-g\|_{L_\nu^{\infty,\infty}(C_{2R_1,2R_2})}\Big(\|f\|_{L_\nu^{\infty,\infty}(C_{2R_1,2R_2})} + \|g\|_{L_\nu^{\infty,\infty}(C_{2R_1,2R_2})}\Big)$.

Proof.

(1) $\|Z_{j,k}(f)\|_{\ell^{\infty,\infty}}$

$$\begin{aligned}
&= \sup_{x_j\in[-R_1,R_1]}\sup_{y_k\in[-R_2,R_2]^d}\Big||(f*\psi)(x_j,y_k)|\nu(x_j,y_k) - \int_{-R_1}^{R_1}\int_{[-R_2,R_2]^d}\rho(x,y)|(f*\psi)(x,y)|\nu(x,y)dxdy\Big| \\
&\le \max\Big\{\sup_{x\in[-R_1,R_1]}\sup_{y\in[-R_2,R_2]^d}|(f*\psi)(x,y)|\nu(x,y), \int_{-R_1}^{R_1}\int_{[-R_2,R_2]^d}\rho(x,y)|(f*\psi)(x,y)|\nu(x,y)dxdy\Big\} \\
&\le \sup_{x\in[-R_1,R_1]}\sup_{y\in[-R_2,R_2]^d}|(f*\psi)(x,y)|\nu(x,y) \\
&\le C\sup_{x\in[-R_1,R_1]}\sup_{y\in[-R_2,R_2]^d}\int_{-R_1}^{R_1}\int_{[-R_2,R_2]^d}(|f|\nu)(x-t,y-s)(|\psi|\omega)(t,s)dtds \\
&\le C\|f\|_{L_\nu^{\infty,\infty}(C_{2R_1,2R_2})}\|\psi\|_{L_\omega^{1,1}(C_{R_1,R_2})}.
\end{aligned}$$

(2) $\|Z_{j,k}(f) - Z_{j,k}(g)\|_{\ell^{\infty,\infty}}$

$$\begin{aligned}
&= \sup_{x_j\in[-R_1,R_1]}\sup_{y_k\in[-R_2,R_2]^d}\Big||(f*\psi)(x_j,y_k)|\nu(x_j,y_k) - |(g*\psi)(x_j,y_k)|\nu(x_j,y_k) \\
&\quad - \int_{-R_1}^{R_1}\int_{[-R_2,R_2]^d}\rho(x,y)\Big(|(f*\psi)(x,y)|\nu(x,y) - |(g*\psi)(x,y)|\nu(x,y)\Big)dxdy\Big| \\
&\le \sup_{x\in[-R_1,R_1]}\sup_{y\in[-R_2,R_2]^d}\Big|((f-g)*\psi)(x,y)\Big|\nu(x,y) + \int_{-R_1}^{R_1}\int_{[-R_2,R_2]^d}\rho(x,y)\Big|((f-g)*\psi)(x,y)\Big|\nu(x,y)dxdy \\
&\le 2\sup_{x\in[-R_1,R_1]}\sup_{y\in[-R_2,R_2]^d}\Big(\Big|((f-g)*\psi)(x,y)\Big|\nu(x,y)\Big)
\end{aligned}$$

$$\leq 2C \sup_{x\in[-R_1,R_1]} \sup_{y\in[-R_2,R_2]^d} \int_{-R_1}^{R_1}\int_{[-R_2,R_2]^d} (|f-g|\nu)(x-t,y-s)(|\psi|\omega)(t,s)dtds$$
$$\leq 2C\|f-g\|_{L^{\infty,\infty}_\nu(C_{2R_1,2R_2})}\|\psi\|_{L^{1,1}_\omega(C_{R_1,R_2})}.$$

(3) $Var(Z_{j,k}(f)) = \mathbb{E}[Z_{j,k}(f)]^2 - (\mathbb{E}[Z_{j,k}(f)])^2 = \mathbb{E}[Z_{j,k}(f)]^2$

$$= \int_{-R_1}^{R_1}\int_{[-R_2,R_2]^d} \rho(t,s)\Big(|(f*\psi)(t,s)|\nu(t,s) - \int_{-R_1}^{R_1}\int_{[-R_2,R_2]^d}\rho(x,y)|(f*\psi)(x,y)|\nu(x,y)dxdy\Big)^2 dtds$$
$$\leq \int_{-R_1}^{R_1}\int_{[-R_2,R_2]^d} \rho(t,s)\Big(|(f*\psi)(t,s)|\nu(t,s)\Big)^2 dtds$$
$$\leq C^2\int_{-R_1}^{R_1}\int_{[-R_2,R_2]^d} \rho(t,s)\Big(\int_{-R_1}^{R_1}\int_{[-R_2,R_2]^d}(|f|\nu)(t-x,s-y)(|\psi|\omega)(x,y)dxdy\Big)^2 dtds$$
$$\leq C^2\|f\|^2_{L^{\infty,\infty}_\nu(C_{2R_1,2R_2})}\|\psi\|^2_{L^{1,1}_\omega(C_{R_1,R_2})}.$$

(4) $Var(Z_{j,k}(f)-Z_{j,k}(g)) = \mathbb{E}[Z_{j,k}(f)-Z_{j,k}(g)]^2 - (\mathbb{E}[Z_{j,k}(f)-Z_{j,k}(g)])^2 = \mathbb{E}[Z_{j,k}(f)-Z_{j,k}(g)]^2$

$$= \int_{-R_1}^{R_1}\int_{[-R_2,R_2]^d} \rho(t,s)\Big[\Big(|(f*\psi)(t,s)| - |(g*\psi)(t,s)|\Big)\nu(t,s)$$
$$- \int_{-R_1}^{R_1}\int_{[-R_2,R_2]^d}\rho(x,y)\Big(|(f*\psi)(x,y)| - |(g*\psi)(x,y)|\Big)\nu(x,y)dxdy\Big]^2 dtds$$
$$\leq \int_{-R_1}^{R_1}\int_{[-R_2,R_2]^d} \rho(t,s)\Big[\Big(|(f*\psi)(t,s)| - |(g*\psi)(t,s)|\Big)\nu(t,s)\Big]^2 dtds$$
$$\leq \int_{-R_1}^{R_1}\int_{[-R_2,R_2]^d} \rho(t,s)\Big(\Big|((f-g)*\psi)(t,s)\Big|\nu(t,s)\Big)\Big(\Big|((f+g)*\psi)(t,s)\Big|\nu(t,s)\Big)dtds$$
$$\leq C^2\|f-g\|_{L^{\infty,\infty}_\nu(C_{2R_1,2R_2})}\Big(\|f\|_{L^{\infty,\infty}_\nu(C_{2R_1,2R_2})} + \|g\|_{L^{\infty,\infty}_\nu(C_{2R_1,2R_2})}\Big)\|\psi\|^2_{L^{1,1}_\omega(C_{R_1,R_2})}.$$

□

Lemma 3.5. *[10] Let $Z_{j,k}$ be independent random variables with expected values $\mathbb{E}[Z_{j,k}] = 0$, $Var(Z_{j,k}) \leq \sigma^2$ and $|Z_{j,k}| \leq M$ for all $j = 1,\cdots,m$ and $k = 1,\cdots,n$. Then for any $\gamma \geq 0$,*

$$Prob\Big(\Big|\sum_{j=1}^{m}\sum_{k=1}^{n} Z_{j,k}\Big| \geq \gamma\Big) \leq 2\exp\Big(-\frac{\gamma^2}{2mn\sigma^2 + \frac{2}{3}M\gamma}\Big). \tag{3.10}$$

Lemma 3.6. *Assume that $\{(x_j, y_k)\}_{j=1,\cdots,m;k=1,\cdots,n}$ is a sequence of independent random variables which are drawn from a general probability distribution over C_{R_1,R_2} with the density function ρ satisfying* (1.4)*. Then for any $m,n \in \mathbb{N}$, there exist positive constants $A, B > 0$ such that*

$$Prob\Big(\sup_{f\in V^{p,q,*}_{\nu,N}}\Big|\sum_{j=1}^{m}\sum_{k=1}^{n} Z_{j,k}(f)\Big| \geq \gamma\Big) \leq A\exp\Big(-B\frac{\gamma^2}{12mnC^*D + 2\gamma}\Big), \tag{3.11}$$

where A is of order $\exp(C'(2N+1)^{d+1})$*, $C' = (5r)\ln 2 + 2r\ln(C^*+1) + r\ln(4C^*+1)$, $B = \min\{\frac{3}{2C^*D}, \frac{\sqrt{2}}{1296D}\}$ and $D = C\|\psi\|_{L^{1,1}_\omega(C_{R_1,R_2})}$.*

Proof. For given $l \in \mathbb{N}$, we construct a 2^{-l}-covering for $V^{p,q,*}_{\nu,N}$ with respect to the norm $\|\cdot\|_{L^{\infty,\infty}_{\nu}(\mathbb{R}^{d+1})}$. Let $\mathcal{A}(2^{-l})$ be the corresponding 2^{-l}-net for $l = 1, 2, \cdots$. By Lemma 3.3, $\mathcal{A}(2^{-l})$ has cardinality at most $\mathcal{N}(V^{p,q,*}_{\nu,N}, 2^{-l})$. Suppose that f_l is the function in $\mathcal{A}(2^{-l})$ that is close to $f \in V^{p,q,*}_{\nu,N}$ with respect to the norm $\|\cdot\|_{L^{\infty,\infty}_{\nu}(\mathbb{R}^{d+1})}$, then $\|f - f_l\|_{L^{\infty,\infty}_{\nu}(\mathbb{R}^{d+1})} \le 2^{-l} \to 0$ as $l \to \infty$. Define

$$Z_{j,k}(f) = Z_{j,k}(f_1) + \sum_{l=2}^{\infty}\Big(Z_{j,k}(f_l) - Z_{j,k}(f_{l-1})\Big).$$

By Lemma 3.4, the random variable $Z_{j,k}(f)$ is well defined.

If $\sup_{f\in V^{p,q,*}_{\nu,N}} \Big|\sum_{j=1}^{m}\sum_{k=1}^{n} Z_{j,k}(f)\Big| \ge \gamma$, then the event ω_l must hold for some $l \ge 1$, where

$$\omega_1 = \Big\{\text{there exists } f_1 \in \mathcal{A}(2^{-1}) \text{ such that } \Big|\sum_{j=1}^{m}\sum_{k=1}^{n} Z_{j,k}(f_1)\Big| \ge \gamma/2\Big\}$$

and for $l \ge 2$,

$$\begin{aligned}\omega_l =&\Big\{\text{there exist } f_l \in \mathcal{A}(2^{-l}), f_{l-1} \in \mathcal{A}(2^{-(l-1)}) \text{ with } \|f_l - f_{l-1}\|_{L^{\infty,\infty}_{\nu}(\mathbb{R}^{d+1})} \le 3\cdot 2^{-l},\\ &\text{such that } \Big|\sum_{j=1}^{m}\sum_{k=1}^{n}\Big(Z_{j,k}(f_l) - Z_{j,k}(f_{l-1})\Big)\Big| \ge \frac{\gamma}{2l^2}\Big\}.\end{aligned}$$

If this were not the case, then with $f_0 = 0$, we have

$$\Big|\sum_{j=1}^{m}\sum_{k=1}^{n} Z_{j,k}(f)\Big| \le \sum_{l=1}^{\infty}\Big|\sum_{j=1}^{m}\sum_{k=1}^{n}\Big(Z_{j,k}(f_l) - Z_{j,k}(f_{l-1})\Big)\Big| \le \sum_{l=1}^{\infty}\frac{\gamma}{2l^2} = \frac{\pi^2\gamma}{12} < \gamma.$$

In the following, we will estimate the probability of ω_1. By Lemma 3.5, for each fixed function $f \in \mathcal{A}(2^{-1})$,

$$\begin{aligned}Prob\Big(\Big|\sum_{j=1}^{m}\sum_{k=1}^{n} Z_{j,k}(f)\Big| \ge \frac{\gamma}{2}\Big) \le& 2\exp\Big(-\frac{(\gamma/2)^2}{2mnVar(Z_{j,k}(f)) + 2/3\cdot\gamma/2\cdot\|Z_{j,k}(f)\|_{\ell^{\infty,\infty}}}\Big)\\ \le& 2\exp\Big(-\frac{3\gamma^2}{2(12mn(C^*D)^2 + 2\gamma C^*D)}\Big)\\ =& 2\exp\Big(-\frac{3\gamma^2}{2C^*D(12mnC^*D + 2\gamma)}\Big).\end{aligned}$$

Moreover, by the result of Lemma 3.3, there are at most

$$\mathcal{N}(V^{p,q,*}_{\nu,N}, \frac{1}{2}) \le \exp\Big(r(2N+1)^{d+1}\ln(4C^*+1)\Big)$$

functions in $\mathcal{A}(2^{-1})$. Therefore, the probability of ω_1 is bounded by

$$Prob(\omega_1) \le 2\exp\Big(r(2N+1)^{d+1}\ln(4C^*+1)\Big)\exp\Big(-\frac{3\gamma^2}{2C^*D(12mnC^*D + 2\gamma)}\Big). \tag{3.12}$$

By the similar method, we could obtain the following estimations about the probabilities of ω_l, $l \geq 2$. In fact, for $f_l \in \mathcal{A}(2^{-l})$, $f_{l-1} \in \mathcal{A}(2^{-(l-1)})$ and $\|f_l - f_{l-1}\|_{L_\nu^{\infty,\infty}(\mathbb{R}^{d+1})} \leq 3 \cdot 2^{-l}$,

$$\begin{aligned}
&Prob\Big(\Big|\sum_{j=1}^{m}\sum_{k=1}^{n}\big(Z_{j,k}(f_l) - Z_{j,k}(f_{l-1})\big)\Big| \geq \frac{\gamma}{2l^2}\Big)\\
&\leq 2\exp\Big(-\frac{(\frac{\gamma}{2l^2})^2}{2mnVar(Z_{j,k}(f_l) - Z_{j,k}(f_{l-1})) + \frac{2}{3}\|Z_{j,k}(f_l) - Z_{j,k}(f_{l-1})\|_{\ell^{\infty,\infty}}\frac{\gamma}{2l^2}}\Big)\\
&\leq 2\exp\Big(-\frac{(\frac{\gamma}{2l^2})^2}{2mn\cdot 3\cdot 2^{-l}\cdot D^2\cdot 2C^* + \frac{2}{3}\cdot 2D\cdot 3\cdot 2^{-l}\cdot\frac{\gamma}{2l^2}}\Big)\\
&\leq 2\exp\Big(-\frac{\vartheta 2^l}{l^4}\Big), \qquad (3.13)
\end{aligned}$$

where $\vartheta := \frac{\gamma^2}{4D(12mnDC^*+2\gamma)}$. There are at most $\mathcal{N}(V_{\nu,N}^{p,q,*}, 2^{-l})$ functions in $\mathcal{A}(2^{-l})$ and $\mathcal{N}(V_{\nu,N}^{p,q,*}, 2^{-l+1})$ functions in $\mathcal{A}(2^{-(l-1)})$. Therefore, we have

$$\begin{aligned}
&Prob\Big(\bigcup_{l=2}^{\infty}\omega_l\Big) \leq \sum_{l=2}^{\infty}\mathcal{N}(V_{\nu,N}^{p,q,*}, 2^{-l})\mathcal{N}(V_{\nu,N}^{p,q,*}, 2^{-l+1})2\exp\Big(-\frac{\vartheta 2^l}{l^4}\Big)\\
&\leq \sum_{l=2}^{\infty}2\exp\Big(r(2N+1)^{d+1}\ln(2^{l+1}C^*+1)\Big)\exp\Big(r(2N+1)^{d+1}\ln(2^lC^*+1)\Big)\exp\Big(-\frac{\vartheta 2^l}{l^4}\Big)\\
&\leq \sum_{l=2}^{\infty}2\exp\Big(2r(2N+1)^{d+1}\Big[(l+1)\ln 2 + \ln(C^*+1)\Big] - \frac{\vartheta 2^l}{l^4}\Big)\\
&= \sum_{l=2}^{\infty}2\exp\Big(\Big[(2r\ln 2)(2N+1)^{d+1}\Big]l + (2r\ln 2)(2N+1)^{d+1} + 2r(2N+1)^{d+1}\ln(C^*+1) - \frac{\vartheta 2^l}{l^4}\Big)\\
&= C_1\sum_{l=2}^{\infty}\exp\Big(C_2 l - \frac{\vartheta 2^l}{l^4}\Big)\\
&= C_1\sum_{l=2}^{\infty}\exp\Big(-\vartheta 2^{\frac{l}{2}}\Big(\frac{2^{\frac{l}{2}}}{l^4} - \frac{C_2 l}{\vartheta 2^{\frac{l}{2}}}\Big)\Big),
\end{aligned}$$

where $C_1 = 2\exp\Big((2r\ln 2)(2N+1)^{d+1} + 2r(2N+1)^{d+1}\ln(C^*+1)\Big)$ and $C_2 = (2r\ln 2)(2N+1)^{d+1}$.

Notice that

$$\min_{l\geq 2}\frac{2^{\frac{l}{2}}}{l^4} = \frac{1}{324}, \qquad \max_{l\geq 2}\frac{l}{2^{\frac{l}{2}}} = \frac{3\sqrt{2}}{4},$$

then

$$\frac{2^{\frac{l}{2}}}{l^4} - \frac{lC_2}{2^{\frac{l}{2}}\vartheta} \geq \frac{1}{324} - \frac{(6\sqrt{2}\ln 2)Dr(2N+1)^{d+1}(12mnDC^*+2\gamma)}{\gamma^2}.$$

We first consider the case that

$$\frac{1}{324} - \frac{(6\sqrt{2}\ln 2)Dr(2N+1)^{d+1}(12mnDC^*+2\gamma)}{\gamma^2} > 0. \qquad (3.14)$$

Notice that $s, a > 0$, one has $\sum_{l=2}^{\infty} e^{-sa^l} \leq \frac{e^{-as}}{sa\ln a}$, see [17] and let

$$s = \vartheta\Big(\frac{2^{\frac{l}{2}}}{l^4} - \frac{lC_2}{2^{\frac{l}{2}}\vartheta}\Big), \quad a = 2^{\frac{1}{2}},$$

we can obtain

$$\begin{aligned}
Prob\Big(\bigcup_{l=2}^{\infty}\omega_l\Big) &\leq \frac{C_1\exp\Big(-\sqrt{2}\vartheta\Big(\frac{1}{324} - \frac{(6\sqrt{2}\ln 2)Dr(2N+1)^{d+1}(12mnDC^*+2\gamma)}{\gamma^2}\Big)\Big)}{\Big(\sqrt{2}\ln\sqrt{2}\Big)\vartheta\Big(\frac{1}{324} - \frac{(6\sqrt{2}\ln 2)Dr(2N+1)^{d+1}(12mnDC^*+2\gamma)}{\gamma^2}\Big)} \\
&= \frac{C_1}{\Big(\sqrt{2}\ln\sqrt{2}\Big)\vartheta\Big(\frac{1}{324} - \frac{(6\sqrt{2}\ln 2)Dr(2N+1)^{d+1}(12mnDC^*+2\gamma)}{\gamma^2}\Big)} \\
&\times\exp\Big(-\sqrt{2}\vartheta\Big(\frac{1}{324} - \frac{(6\sqrt{2}\ln 2)Dr(2N+1)^{d+1}(12mnDC^*+2\gamma)}{\gamma^2}\Big)\Big) \\
&= \frac{C_1\exp\Big(\frac{\sqrt{2}\vartheta(6\sqrt{2}\ln 2)Dr(2N+1)^{d+1}(12mnDC^*+2\gamma)}{\gamma^2}\Big)}{\Big(\sqrt{2}\ln\sqrt{2}\Big)\vartheta\Big(\frac{1}{324} - \frac{(6\sqrt{2}\ln 2)Dr(2N+1)^{d+1}(12mnDC^*+2\gamma)}{\gamma^2}\Big)}\exp\Big(-\frac{\sqrt{2}\vartheta}{324}\Big) \\
&= \frac{2\exp\Big(\Big(2r\ln 2 + 2r\ln(C^*+1) + 3r\ln 2\Big)(2N+1)^{d+1}\Big)}{\Big(\sqrt{2}\ln\sqrt{2}\Big)\vartheta\Big(\frac{1}{324} - \frac{(6\sqrt{2}\ln 2)Dr(2N+1)^{d+1}(12mnDC^*+2\gamma)}{\gamma^2}\Big)}\exp\Big(-\frac{\sqrt{2}\vartheta}{324}\Big) \\
&= \frac{2\exp\Big(\Big((5r)\ln 2 + 2r\ln(C^*+1)\Big)(2N+1)^{d+1}\Big)}{\Big(\sqrt{2}\ln\sqrt{2}\Big)\vartheta\Big(\frac{1}{324} - \frac{(6\sqrt{2}\ln 2)Dr(2N+1)^{d+1}(12mnDC^*+2\gamma)}{\gamma^2}\Big)}\exp\Big(-\frac{\sqrt{2}\gamma^2}{1296D(12mnDC^*+2\gamma)}\Big). \qquad (3.15)
\end{aligned}$$

Combining the results of (3.12) and (3.15), we can obtain that

$$Prob\Big(\sup_{f\in V_{\nu,N}^{p,q,*}}\Big|\sum_{j=1}^{m}\sum_{k=1}^{n}Z_{j,k}(f)\Big| \geq \gamma\Big) \leq A\exp\Big(-B\frac{\gamma^2}{12mnDC^*+2\gamma}\Big), \qquad (3.16)$$

where A is of order $\exp(C'(2N+1)^{d+1})$, $C' = (5r)\ln 2 + 2r\ln(C^*+1) + r\ln(4C^*+1)$ and $B = \min\{\frac{3}{2C^*D}, \frac{\sqrt{2}}{1296D}\}$.

If

$$\frac{1}{324} - \frac{(6\sqrt{2}\ln 2)Dr(2N+1)^{d+1}(12mnDC^*+2\gamma)}{\gamma^2} \leq 0 \qquad (3.17)$$

then we could choose $C' \geq 324B(6\sqrt{2}\ln 2)Dr$ such that $A\exp(-B\frac{\gamma^2}{12mnDC^*+2\gamma}) \geq 1$. □

4. Sampling stability

In order to obtain the random convolution sampling stability for signals in multiply generated subspaces of weighted mixed Lebesgue space, we also need the help of the corresponding sampling stability for some subset of $V_{\nu,N}^{p,q}$.

Theorem 4.1. *Assume that* $\{(x_j, y_k)\}_{j=1,\cdots,m;k=1,\cdots,n}$ *is a sequence of independent random variables which are drawn from a general probability distribution over* C_{R_1,R_2} *with the density function* ρ *satisfying* (1.4) *and the convolution function* ψ *satisfies* (1.5). *Then for any* $m, n \in \mathbb{N}$, *there exist positive constants* $0 < \theta, \gamma < 1$ *such that*

$$A_1\{\theta\} := m^{\frac{1}{p}} n^{\frac{1}{q}} (1-\gamma)(C_{\rho,l}\theta G^{-1})^{pq} > 0,$$

$$B_1\{\theta\} := mn\Big(\gamma(C_{\rho,l}\theta G^{-1})^{pq} + CC^*\|\psi\|_{L^{1,1}_\omega(C_{R_1,R_2})}\Big) > 0,$$

where

$$G := (2R_1)^{\frac{q-1}{pq}} (2R_2)^{\frac{d(p-1)}{pq}} \Big(CC_{\rho,u}C^*\|\psi\|_{L^{1,1}_\omega(C_{R_1,R_2})}\Big)^{\frac{pq-1}{pq}}.$$

Furthermore, the random convolution sampling stability

$$A_1\{\theta\}\|f\|_{L^{p,q}_\nu(\mathbb{R}^{d+1})} \le \Big\|\Big\{(f*\psi)(x_j, y_k)\Big\}_{j=1,\cdots,m;k=1,\cdots,n}\Big\|_{\ell^{p,q}_\nu} \le B_1\{\theta\}\|f\|_{L^{p,q}_\nu(\mathbb{R}^{d+1})} \tag{4.1}$$

holds for all $f \in V^{p,q,\diamond}_{\nu,N} := \{f \in V^{p,q}_{\nu,N} : \|f*\psi\|_{L^{p,q}_\nu(C_{R_1,R_2})} \ge \theta\}$ *with the probability at least*

$$1 - A\exp\Big(-B\frac{\Big(\gamma mn(C_{\rho,l}\theta G^{-1})^{pq}\Big)^2}{12mnC^*C\|\psi\|_{L^{1,1}_\omega(C_{R_1,R_2})} + 2\gamma mn(C_{\rho,l}\theta G^{-1})^{pq}}\Big),$$

where A, B *are defined as in Lemma 3.6.*

Proof. Obviously, every function $f \in V^{p,q,\diamond}_{\nu,N}$ satisfies the random convolution sampling stability (4.1) if and only if $f/\|f\|_{L^{p,q}_\nu(\mathbb{R}^{d+1})}$ also does. Thus we assume that $f \in V^{p,q,\diamond,*}_{\nu,N} := \{f \in V^{p,q,\diamond}_{\nu,N} : \|f\|_{L^{p,q}_\nu(\mathbb{R}^{d+1})} = 1\}$.

Define the event

$$\mathcal{H} = \Big\{\sup_{f\in V^{p,q,\diamond,*}_{\nu,N}} \Big|\sum_{j=1}^{m}\sum_{k=1}^{n} Z_{j,k}(f)\Big| \ge \gamma mn(C_{\rho,l}\theta G^{-1})^{pq}\Big\}. \tag{4.2}$$

Its complement is

$$\begin{aligned}\widetilde{\mathcal{H}} = \Big\{ &-\gamma mn(C_{\rho,l}\theta G^{-1})^{pq} + mn\int_{-R_1}^{R_1}\int_{[-R_2,R_2]^d} \rho(x,y)\Big|(f*\psi)(x,y)\Big|\nu(x,y)dxdy \\ &\le \sum_{j=1}^{m}\sum_{k=1}^{n}\Big|(f*\psi)(x_j,y_k)\Big|\nu(x_j,y_k) \\ &\le \gamma mn(C_{\rho,l}\theta G^{-1})^{pq} + mn\int_{-R_1}^{R_1}\int_{[-R_2,R_2]^d} \rho(x,y)\Big|(f*\psi)(x,y)\Big|\nu(x,y)dxdy, \quad f \in V^{p,q,\diamond,*}_{\nu,N}\Big\}. \end{aligned} \tag{4.3}$$

Write $g(x,y) := \rho(x,y)|(f*\psi)(x,y)|\nu(x,y)$ with $f \in V^{p,q,\diamond,*}_{\nu,N}$, we can obtain

$$\begin{aligned}&\int_{-R_1}^{R_1}\int_{[-R_2,R_2]^d} |g(x,y)|dxdy \\ &= \int_{-R_1}^{R_1}\int_{[-R_2,R_2]^d} \Big|\rho(x,y)\Big[\Big|\int_{-R_1}^{R_1}\int_{[-R_2,R_2]^d} f(x-t,y-s)\psi(t,s)dtds\Big|\Big]\nu(x,y)\Big|dxdy \\ &\le C\int_{-R_1}^{R_1}\int_{[-R_2,R_2]^d} \Big(\rho(x,y)\int_{-R_1}^{R_1}\int_{[-R_2,R_2]^d} |(f\nu)(x-t,y-s)||(\psi\omega)(t,s)|dtds\Big)dxdy\end{aligned}$$

$$
\begin{aligned}
&\leq C\|f\|_{L_\nu^{\infty,\infty}(C_{2R_1,2R_2})}\|\psi\|_{L_\omega^{1,1}(C_{R_1,R_2})}\\
&\leq C\|f\|_{L_\nu^{\infty,\infty}(\mathbb{R}^{d+1})}\|\psi\|_{L_\omega^{1,1}(C_{R_1,R_2})}\\
&\leq CC^*\|\psi\|_{L_\omega^{1,1}(C_{R_1,R_2})}. \qquad (4.4)
\end{aligned}
$$

Furthermore,

$$
\left\|\left\{(f*\psi)(x_j,y_k)\right\}_{j=1,\cdots,m;k=1,\cdots,n}\right\|_{\ell_\nu^{p,q}} \leq \left\|\left\{(f*\psi)(x_j,y_k)\right\}_{j=1,\cdots,m;k=1,\cdots,n}\right\|_{\ell_\nu^{1,1}}. \qquad (4.5)
$$

Combining the results of (4.4) and (4.5), we can obtain

$$
\left\|\left\{(f*\psi)(x_j,y_k)\right\}_{j=1,\cdots,m;k=1,\cdots,n}\right\|_{\ell_\nu^{p,q}} \leq mn\Big(\gamma(C_{\rho,l}\theta G^{-1})^{pq} + CC^*\|\psi\|_{L_\omega^{1,1}(C_{R_1,R_2})}\Big). \qquad (4.6)
$$

In addition,

$$
\begin{aligned}
&\|g\|_{L^{p,q}(C_{R_1,R_2})}\\
&\leq (2R_1)^{\frac{q-1}{pq}}\Big[\Big(\int_{-R_1}^{R_1}\Big(\int_{[-R_2,R_2]^d}|g(x,y)|^q dy\Big)^p dx\Big)^{\frac{1}{q}}\Big]^{\frac{1}{p}}\\
&\leq (2R_1)^{\frac{q-1}{pq}}\Big(\|g\|_{L^{\infty,\infty}(C_{R_1,R_2})}\Big)^{\frac{q-1}{q}}\Big[\Big(\int_{-R_1}^{R_1}\Big(\int_{[-R_2,R_2]^d}|g(x,y)|dy\Big)^p dx\Big)^{\frac{1}{q}}\Big]^{\frac{1}{p}}\\
&\leq (2R_1)^{\frac{q-1}{pq}}\Big(\|g\|_{L^{\infty,\infty}(C_{R_1,R_2})}\Big)^{\frac{q-1}{q}}\Big(\int_{[-R_2,R_2]^d}\Big(\int_{-R_1}^{R_1}|g(x,y)|^p dx\Big)^{\frac{1}{p}}dy\Big)^{\frac{1}{q}}\\
&\leq (2R_1)^{\frac{q-1}{pq}}(2R_2)^{\frac{d(p-1)}{pq}}\Big(\|g\|_{L^{\infty,\infty}(C_{R_1,R_2})}\Big)^{\frac{q-1}{q}}\Big(\int_{[-R_2,R_2]^d}\Big(\int_{-R_1}^{R_1}|g(x,y)|^p dx\Big)dy\Big)^{\frac{1}{pq}}\\
&\leq (2R_1)^{\frac{q-1}{pq}}(2R_2)^{\frac{d(p-1)}{pq}}\Big(\|g\|_{L^{\infty,\infty}(C_{R_1,R_2})}\Big)^{\frac{q-1}{q}}\Big(\|g\|_{L^{\infty,\infty}(C_{R_1,R_2})}\Big)^{\frac{p-1}{pq}}\|g\|_{L^{1,1}(C_{R_1,R_2})}^{\frac{1}{pq}}\\
&\leq (2R_1)^{\frac{q-1}{pq}}(2R_2)^{\frac{d(p-1)}{pq}}\Big(CC_{\rho,u}\|f\|_{L_\nu^{\infty,\infty}(C_{2R_1,2R_2})}\|\psi\|_{L_\omega^{1,1}(C_{R_1,R_2})}\Big)^{\frac{pq-1}{pq}}\|g\|_{L^{1,1}(C_{R_1,R_2})}^{\frac{1}{pq}}\\
&\leq (2R_1)^{\frac{q-1}{pq}}(2R_2)^{\frac{d(p-1)}{pq}}\Big(CC_{\rho,u}C^*\|\psi\|_{L_\omega^{1,1}(C_{R_1,R_2})}\Big)^{\frac{pq-1}{pq}}\|g\|_{L^{1,1}(C_{R_1,R_2})}^{\frac{1}{pq}}\\
&= G\|g\|_{L^{1,1}(C_{R_1,R_2})}^{\frac{1}{pq}}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\|g\|_{L^{1,1}(C_{R_1,R_2})} &\geq \frac{\|g\|_{L^{p,q}(C_{R_1,R_2})}^{pq}}{G^{pq}}\\
&\geq \frac{\Big[C_{\rho,l}\Big(\int_{[-R_1,R_1]}\Big(\int_{[-R_2,R_2]^d}\Big(|(f*\psi)(x,y)|\nu(x,y)\Big)^q dy\Big)^{\frac{p}{q}}dx\Big)^{\frac{1}{p}}\Big]^{pq}}{G^{pq}}\\
&= \frac{\Big[C_{\rho,l}\|(f*\psi)\|_{L_\nu^{p,q}(C_{R_1,R_2})}\Big]^{pq}}{G^{pq}} \geq (C_{\rho,l}\theta G^{-1})^{pq}. \qquad (4.7)
\end{aligned}
$$

By Hölder inequality, we have

$$
\left\|\left\{(f*\psi)(x_j,y_k)\right\}_{j=1,\cdots,m;k=1\cdots,n}\right\|_{\ell_\nu^{1,1}} \leq m^{\frac{p-1}{p}}n^{\frac{q-1}{q}}\left\|\left\{(f*\psi)(x_j,y_k)\right\}_{j=1,\cdots,m;k=1\cdots,n}\right\|_{\ell_\nu^{p,q}}. \qquad (4.8)
$$

Combining the results of (4.7) and (4.8), we can obtain

$$m^{-\frac{p-1}{p}}n^{-\frac{q-1}{q}}\Big[-\gamma mn(C_{\rho,l}\theta G^{-1})^{pq}+mn(C_{\rho,l}\theta G^{-1})^{pq}\Big]\leq\Big\|\Big\{(f*\psi)(x_j,y_k)\Big\}_{j=1,\cdots,m;k=1\cdots,n}\Big\|_{\ell_\nu^{p,q}}. \tag{4.9}$$

Followed by the results of (4.3), (4.6) and (4.9), the event

$$\begin{aligned}\overline{\mathcal{H}}&=\Big\{m^{-\frac{p-1}{p}}n^{-\frac{q-1}{q}}\Big[mn(C_{\rho,l}\theta G^{-1})^{pq}-\gamma mn(C_{\rho,l}\theta G^{-1})^{pq}\Big]\\&\quad\leq\Big\|\Big\{(f*\psi)(x_j,y_k)\Big\}_{j=1,\cdots,m;k=1,\cdots,n}\Big\|_{\ell_\nu^{p,q}}\\&\quad\leq\Big[\gamma mn(C_{\rho,l}\theta G^{-1})^{pq}+mnCC^*\|\psi\|_{L_\omega^{1,1}(C_{R_1,R_2})}\Big],\quad f\in V_{\nu,N}^{p,q,\diamond,*}\Big\}\\&=\Big\{m^{-\frac{p-1}{p}}n^{-\frac{q-1}{q}}\Big[mn(C_{\rho,l}\theta G^{-1})^{pq}-\gamma mn(C_{\rho,l}\theta G^{-1})^{pq}\Big]\|f\|_{L_\nu^{p,q}(\mathbb{R}^{d+1})}\\&\quad\leq\Big\|\Big\{(f*\psi)(x_j,y_k)\Big\}_{j=1,\cdots,m;k=1,\cdots,n}\Big\|_{\ell_\nu^{p,q}}\\&\quad\leq\Big[\gamma mn(C_{\rho,l}\theta G^{-1})^{pq}+mnCC^*\|\psi\|_{L_\omega^{1,1}(C_{R_1,R_2})}\Big]\|f\|_{L_\nu^{p,q}(\mathbb{R}^{d+1})},\quad f\in V_{\nu,N}^{p,q,\diamond}\Big\}\end{aligned} \tag{4.10}$$

contains the event $\widetilde{\mathcal{H}}$.

By Lemma 3.6, we can obtain

$$\begin{aligned}Prob(\overline{\mathcal{H}})&\geq Prob(\widetilde{\mathcal{H}})\geq 1-Prob(\mathcal{H})\\&\geq 1-Prob\Big(\sup_{f\in V_{\nu,N}^{p,q,\diamond,*}}\Big|\sum_{j=1}^{m}\sum_{k=1}^{n}Z_{j,k}(f)\Big|\geq\gamma mn(C_{\rho,l}\theta G^{-1})^{pq}\Big)\\&\geq 1-Prob\Big(\sup_{f\in V_{\nu,N}^{p,q,*}}\Big|\sum_{j=1}^{m}\sum_{k=1}^{n}Z_{j,k}(f)\Big|\geq\gamma mn(C_{\rho,l}\theta G^{-1})^{pq}\Big)\\&\geq 1-A\exp\Big(-B\frac{\Big(\gamma mn(C_{\rho,l}\theta G^{-1})^{pq}\Big)^2}{12mnC^*C\|\psi\|_{L^{1,1}(C_{R_1,R_2})}+2\gamma mn(C_{\rho,l}\theta G^{-1})^{pq}}\Big).\end{aligned}$$

□

Theorem 4.2. *Assume that* $\{(x_j,y_k)\}_{j=1,\cdots,m;k=1,\cdots,n}$ *is a sequence of independent random variables which are drawn from a general probability distribution over* C_{R_1,R_2} *with the density function* ρ *satisfying* (1.4) *and the convolution function* ψ *satisfies* (1.5)*. Then for any* $m,n\in\mathbb{N}$*, there exist positive constants* $0<\gamma,\varepsilon<1$ *such that*

$$A_2:=A_1\Big\{(\beta-C\varepsilon)\|\psi\|_{L_\omega^{1,1}(C_{R_1,R_2})}\Big\}(1-\delta-\varepsilon)-\frac{C\varepsilon mn\|\psi\|_{L_\omega^{1,1}(C_{R_1,R_2})}}{(4R_1)^{\frac{1}{p}}(4R_2)^{\frac{d}{q}}}>0$$

and

$$B_2:=B_1\Big\{(\beta-C\varepsilon)\|\psi\|_{L_\omega^{1,1}(C_{R_1,R_2})}\Big\}\frac{C_{p,q}}{c_{p,q}}+\frac{C\varepsilon mn\|\psi\|_{L_\omega^{1,1}(C_{R_1,R_2})}}{(4R_1)^{\frac{1}{p}}(4R_2)^{\frac{d}{q}}}>0,$$

where A_1,B_1 *are defined as in Theorem 4.1. Furthermore, the random convolution sampling stability*

$$A_2\|f\|_{L_\nu^{p,q}(\mathbb{R}^{d+1})}\leq\Big\|\Big\{(f*\psi)(x_j,y_k)\Big\}_{j=1,\cdots,m;k=1,\cdots,n}\Big\|_{\ell_\nu^{p,q}}\leq B_2\|f\|_{L_\nu^{p,q}(\mathbb{R}^{d+1})} \tag{4.11}$$

holds for any $f \in V^{p,q}_{\nu,R_1,R_2}$ *with the probability at least*

$$1 - A\exp\Big(-B\frac{\Big(\gamma mn\Big[C_{\rho,l}(\beta - C\varepsilon)\|\psi\|_{L^{1,1}_{\omega}(C_{R_1,R_2})}G^{-1}\Big]^{pq}\Big)^2}{12mnC^{*}C\|\psi\|_{L^{1,1}_{\omega}(C_{R_1,R_2})} + 2\gamma mn\Big[C_{\rho,l}(\beta - C\varepsilon)\|\psi\|_{L^{1,1}_{\omega}(C_{R_1,R_2})}G^{-1}\Big]^{pq}}\Big), \tag{4.12}$$

where the constants A, B, G *are defined as in Theorem 4.1 with*

$$N \geq \max\{N_1(\varepsilon) + 2R + 1, N_2(\frac{\varepsilon}{(4R_1)^{\frac{1}{p}}(4R_2)^{\frac{d}{q}}}) + 2R + 1\}, \quad R = \max\{R_1, \sqrt{d}R_2\}.$$

Proof. It is obvious that every function $f \in V^{p,q}_{\nu,R_1,R_2}$ satisfies the random convolution sampling stability (4.11) if and only if $f/\|f\|_{L^{p,q}_{\nu}(\mathbb{R}^{d+1})}$ also does. Thus, we assume that $f \in V^{p,q,*}_{\nu,R_1,R_2}$.

By Lemma 2.2, for any $f \in V^{p,q,*}_{\nu,R_1,R_2}$ and $\varepsilon > 0$,

$$\|f - f_N\|_{L^{\infty,\infty}_{\nu}(C_{2R_1,2R_2})} \leq \frac{\varepsilon}{(4R_1)^{\frac{1}{p}}(4R_2)^{\frac{d}{q}}}, \tag{4.13}$$

$$\|f - f_N\|_{L^{p,q}_{\nu}(C_{2R_1,2R_2})} \leq \varepsilon, \tag{4.14}$$

if $N \geq \max\{N_1(\varepsilon) + 2R + 1, N_2(\frac{\varepsilon}{(4R_1)^{\frac{1}{p}}(4R_2)^{\frac{d}{q}}}) + 2R + 1\}$ and $R = \max\{R_1, \sqrt{d}R_2\}$.

Moreover, by the result of Lemma 3.1,

$$\begin{aligned}
&\|f * \psi - f_N * \psi\|_{L^{p,q}_{\nu}(C_{R_1,R_2})}\\
&\leq C\|f - f_N\|_{L^{p,q}_{\nu}(C_{R_1,R_2})}\|\psi\|_{L^{1,1}_{\omega}(C_{R_1,R_2})}\\
&\leq C\|f - f_N\|_{L^{p,q}_{\nu}(C_{2R_1,2R_2})}\|\psi\|_{L^{1,1}_{\omega}(C_{R_1,R_2})}\\
&\leq C\varepsilon\|\psi\|_{L^{1,1}_{\omega}(C_{R_1,R_2})}.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\|f_N * \psi\|_{L^{p,q}_{\nu}(C_{R_1,R_2})}\\
&\leq C\varepsilon\|\psi\|_{L^{1,1}_{\omega}(C_{R_1,R_2})} + \|f * \psi\|_{L^{p,q}_{\nu}(C_{R_1,R_2})}\\
&\leq C\varepsilon\|\psi\|_{L^{1,1}_{\omega}(C_{R_1,R_2})} + \|f * \psi\|_{L^{p,q}_{\nu}(\mathbb{R}^{d+1})}\\
&\leq C\varepsilon\|\psi\|_{L^{1,1}_{\omega}(C_{R_1,R_2})} + C\|f\|_{L^{p,q}_{\nu}(\mathbb{R}^{d+1})}\|\psi\|_{L^{1,1}_{\omega}(C_{R_1,R_2})}.
\end{aligned} \tag{4.15}$$

Furthermore, by inequality (1.9),

$$\begin{aligned}
&\|f_N * \psi\|_{L^{p,q}_{\nu}(C_{R_1,R_2})}\\
&\geq -C\varepsilon\|\psi\|_{L^{1,1}_{\omega}(C_{R_1,R_2})} + \|f * \psi\|_{L^{p,q}_{\nu}(C_{R_1,R_2})}\\
&\geq -C\varepsilon\|\psi\|_{L^{1,1}_{\omega}(C_{R_1,R_2})} + \beta\|\psi\|_{L^{1,1}_{\omega}(C_{R_1,R_2})}\|f\|_{L^{p,q}_{\nu}(\mathbb{R}^{d+1})}.
\end{aligned} \tag{4.16}$$

Combining the results of (4.15) and (4.16), we can obtain for any function $f \in V^{p,q,*}_{\nu,R_1,R_2}$,

$$(\beta - C\varepsilon)\|\psi\|_{L^{1,1}_{\omega}(C_{R_1,R_2})} \leq \|f_N * \psi\|_{L^{p,q}_{\nu}(C_{R_1,R_2})} \leq C\varepsilon\|\psi\|_{L^{1,1}_{\omega}(C_{R_1,R_2})} + C\|\psi\|_{L^{1,1}_{\omega}(C_{R_1,R_2})}. \tag{4.17}$$

By Theorem 4.1, the inequality

$$A_1\Big\{(\beta - C\varepsilon)\|\psi\|_{L^{1,1}_\omega(C_{R_1,R_2})}\Big\}\|f_N\|_{L^{p,q}_\nu(\mathbb{R}^{d+1})} \le \Big\|\Big\{(f_N * \psi)(x_j, y_k)\Big\}_{j=1,\cdots,m;k=1,\cdots,n}\Big\|_{\ell^{p,q}_\nu}$$
$$\le B_1\Big\{(\beta - C\varepsilon)\|\psi\|_{L^{1,1}_\omega(C_{R_1,R_2})}\Big\}\|f_N\|_{L^{p,q}_\nu(\mathbb{R}^{d+1})}$$

holds with the probability at least

$$1 - A\exp\Big(-B\frac{\Big(\gamma mn\Big[C_{\rho,l}(\beta - C\varepsilon)\|\psi\|_{L^{1,1}_\omega(C_{R_1,R_2})}G^{-1}\Big]^{pq}\Big)^2}{12mnC^*C\|\psi\|_{L^{1,1}_\omega(C_{R_1,R_2})} + 2\gamma mn\Big[C_{\rho,l}(\beta - C\varepsilon)\|\psi\|_{L^{1,1}_\omega(C_{R_1,R_2})}G^{-1}\Big]^{pq}}\Big), \tag{4.18}$$

At the same time, by inequality (4.13),

$$\begin{aligned}
&\Big\|\Big\{(f * \psi)(x_j, y_k) - (f_N * \psi)(x_j, y_k)\Big\}_{j=1,\cdots,m;k=1,\cdots,n}\Big\|_{\ell^{p,q}_\nu}\\
&\le \Big\|\Big\{(|f - f_N| * |\psi|)(x_j, y_k)\Big\}_{j=1,\cdots,m;k=1,\cdots,n}\Big\|_{\ell^{p,q}_\nu}\\
&= \Big(\sum_{j=1}^{m}\Big(\sum_{k=1}^{n}\Big|(|f - f_N| * |\psi|)(x_j, y_k)\nu(x_j, y_k)\Big|^q\Big)^{\frac{p}{q}}\Big)^{1/p}\\
&\le \sum_{j=1}^{m}\sum_{k=1}^{n}\Big|(|f - f_N| * |\psi|)(x_j, y_k)\nu(x_j, y_k)\Big|\\
&\le \frac{C\varepsilon}{(4R_1)^{\frac{1}{p}}(4R_2)^{\frac{d}{q}}}\sum_{j=1}^{m}\sum_{k=1}^{n}\Big|\int_{-R_1}^{R_1}\int_{[-R_2,R_2]^d}|\psi(t, s)|\omega(t, s)dtds\Big|\\
&= \frac{C\varepsilon mn\|\psi\|_{L^{1,1}_\omega(C_{R_1,R_2})}}{(4R_1)^{\frac{1}{p}}(4R_2)^{\frac{d}{q}}},
\end{aligned} \tag{4.19}$$

which is equivalent to

$$\begin{aligned}
&\Big\|\Big\{(f_N * \psi)(x_j, y_k)\Big\}_{j=1,\cdots,m;k=1,\cdots,n}\Big\|_{\ell^{p,q}_\nu} - \frac{C\varepsilon mn\|\psi\|_{L^{1,1}_\omega(C_{R_1,R_2})}}{(4R_1)^{\frac{1}{p}}(4R_2)^{\frac{d}{q}}}\\
&\le \Big\|\Big\{(f * \psi)(x_j, y_k)\Big\}_{j=1,\cdots,m;k=1,\cdots,n}\Big\|_{\ell^{p,q}_\nu}\\
&\le \Big\|\Big\{(f_N * \psi)(x_j, y_k)\Big\}_{j=1,\cdots,m;k=1,\cdots,n}\Big\|_{\ell^{p,q}_\nu} + \frac{C\varepsilon mn\|\psi\|_{L^{1,1}_\omega(C_{R_1,R_2})}}{(4R_1)^{\frac{1}{p}}(4R_2)^{\frac{d}{q}}}.
\end{aligned}$$

Furthermore,

$$(1-\delta)\|f\|_{L^{p,q}_\nu(\mathbb{R}^{d+1})} - \varepsilon \le \|f\|_{L^{p,q}_\nu(C_{R_1,R_2})} - \varepsilon \le \|f\|_{L^{p,q}_\nu(C_{2R_1,2R_2})} - \varepsilon \le \|f_N\|_{L^{p,q}_\nu(C_{2R_1,2R_2})} \le \|f_N\|_{L^{p,q}_\nu(\mathbb{R}^{d+1})}$$

and

$$\|f_N\|_{L^{p,q}_\nu(\mathbb{R}^{d+1})} \le C_{p,q}\sum_{i=1}^{r}\|c_i\|_{\ell^{p,q}_\nu([-N,N]^{d+1})} \le C_{p,q}\sum_{i=1}^{r}\|c_i\|_{\ell^{p,q}_\nu(\mathbb{Z}^{d+1})} \le \frac{C_{p,q}}{c_{p,q}}\|f\|_{L^{p,q}_\nu(\mathbb{R}^{d+1})}. \tag{4.20}$$

Thus, for any function $f \in V^{p,q,*}_{\nu,R_1,R_2}$,

$$
\begin{aligned}
& A_1\Big\{(\beta - C\varepsilon)\|\psi\|_{L^{1,1}_\omega(C_{R_1,R_2})}\Big\}\Big(1-\delta-\varepsilon\Big) - \frac{C\varepsilon mn\|\psi\|_{L^{1,1}_\omega(C_{R_1,R_2})}}{(4R_1)^{\frac{1}{p}}(4R_2)^{\frac{d}{q}}} \\
& \le \Big\|\Big\{(f*\psi)(x_j,y_k)\Big\}_{j=1,\cdots,m;k=1,\cdots,n}\Big\|_{\ell^{p,q}_\nu} \\
& \le B_1\Big\{(\beta - C\varepsilon)\|\psi\|_{L^{1,1}_\omega(C_{R_1,R_2})}\Big\}\frac{C_{p,q}}{c_{p,q}} + \frac{C\varepsilon mn\|\psi\|_{L^{1,1}_\omega(C_{R_1,R_2})}}{(4R_1)^{\frac{1}{p}}(4R_2)^{\frac{d}{q}}}
\end{aligned}
$$

holds with the probability at least (4.12). Thus the random convolution sampling stability is proved. □

5. Conclusions

This paper is aimed at studying the random convolution sampling stability in multiply generated shift invariant subspace of weighted mixed Lebesgue space. Under some restricted conditions and essential results, we prove that with overwhelming probability, the random convolution sampling stability holds for signals in some subset of the defined multiply generated shift invariant subspace when the sampling size is large enough.

Conflict of interest

The author declares no conflicts of interest in this paper.

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