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Research article

A novel decision making technique based on spherical hesitant fuzzy Yager aggregation information: application to treat Parkinson's disease

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Abstract: The concept of spherical hesitant fuzzy set is a mathematical tool that have the ability to easily handle imprecise and uncertain information. The method of aggregation plays a great role in decision-making problems, particularly when there are more conflicting criteria. The purpose of this article is to present novel operational laws based on the Yager t-norm and t-conorm under spherical hesitant fuzzy information. Furthermore, based on the Yager operational laws, we develop the list of Yager weighted averaging and Yager weighted geometric aggregation operators. The basic fundamental properties of the proposed operators are given in detail. We design an algorithm to address the uncertainty and ambiguity information in multi-criteria group decision making (MCGDM) problems. Finally, a numerical example related to Parkinson disease is presented for the proposed model. To show the supremacy of the proposed algorithms, a comparative analysis of the proposed techniques with some existing approaches and with validity test is presented.

Keywords: spherical fuzzy set; spherical hesitant fuzzy sets; Yager aggregation operators; decision making
Mathematics Subject Classification: 03B52, 03E72

1. Introduction

Many of the routine tasks, such as entering the bedroom, employing technology, reading the newspaper, or typing text, necessitate action. We never notice how the brain functions, and we cannot think twice before doing an action that goes through the brain and allows us to move; nevertheless, some of us suffer from movement disorders as a result of anything happening to the brain's deep parts, the basal ganglia and the substantia nigra, and you lose control over the motor system. Changes of speech and movement, as well as depression and anxiety, are all symptoms of deficiency. The most well-known movement disorder is Parkinson's disease (PD) [31]. Worryingly, the global prevalence of Parkinson's disease is rising. According to the American Parkinson Disease Association (APDA), about 1 million Americans and 10 million people worldwide suffer from Parkinson's disease, a progressive neurological condition. Muhammad Ali, Michael J. Fox, and Janet Reno are only a few of the popular patients-turned-advocates. We looked at the research on complementary and alternative medicine (CAM) therapy for Parkinson's disease, with an emphasis on mind-body approaches and natural products [12].

The multi-criteria group decision making (MCGDM) process, which depicts a method for choosing the best option to a group of decision makers (DMs) and circumstances, is an important and evolving subject. In this approach, there are two major objectives. The first goal is to create an atmosphere in which the values of a few key attributes can be easily evaluated, while the second is to analyze the information. However, as the systems become more complex to manage, it is becoming increasingly difficult to collect data from records, properties, and practitioners in a simple and easy format. As a result, the concept of fuzzy sets [51] was developed by Zadeh to convey knowledge more amenably. The most basic property of a fuzzy sets is membership degree only. After a large number of applications of FS theory, Atanassov discovered a number of flaws in this concept. As a result, he developed the intuitionistic fuzzy set (IFS) model [11]. IFS stands for FS in its entirety. Each factor of the IFS is expressed by an ordered pair with positive and negative membership grades. The sum of grades value is atmost 1. Yager [47] recently proposed the concept of Pythagorean fuzzy sets (PFSs), whose membership values are ordered pairs with square sums of membership and non-membership less than or equal to unity. More researchers are working on improving the capabilities of PFSs. For example, Yager [48] gave Pythagorean membership grades in MADM. Zhang and Xu [53] suggested in extensions of the method for order preference by similarity to an ideal solution (TOPSIS) to MADM with Pythagorean and HFSs. Zhang [54] proposed a new technique for Pythagorean fuzzy MGDM based on a similarity measure. Ren et al. [38] presented a Pythagorean fuzzy TODIM (an acronym in Portuguese for Interative Multi-criteria Decision Making) method to MCGDM. Peng and Yang [32] established a Pythagorean fuzzy Choquet integral based MABAC (multi-attributive border approximation area comparison) technique for MCGDM. Zhang [55] introduced a QUALIFLEX (qualitative flexible multiple criteria method) method that was hierarchical. Pythagorean fuzzy information measures were studied by Peng et al. [34]. Generalized Pythagorean fuzzy Bonferroni mean aggregation operations were proposed by Zhang et al. [56]. TOPSIS was extended to hesitant Pythagorean fuzzy sets by Liang and Xu [26]. It should be observed that both IFSs and PFSs have still some limitations, although they have been seen to be efficient when dealing with complex fuzzy information in some practical applications. Yager [49] thereafter introduced the concept q-rung orthopair fuzzy (q-ROF) set as an extension to the conventional IFS set. The limitation of q-ROF set is that the sum of the qth-power of membership and non-membership is less than or equal to 1. Xu et al. [43] discussed some improved q-rung orthopair fuzzy aggregation operators and their applications to multi attribute group decision-making. Riaz et al. [37] developed q-Rung orthopair fuzzy geometric aggregation operators based on generalized and group-generalized parameters for water loss management. A q-rung orthopair fuzzy MCGDM technique based on a new distance measure for supplier selection were discussed by Pinar et al. [33]. For more study of decision making methods we refer to [15, 18, 25, 27, 44, 45, 50, 57].

Mahmood [28] presented the generalized structure of bipolar soft sets and discussed their application in DMPs. The role of aggregation operators is essential throughout the assimilation procedure for the accumulation of all administrative criteria. Xu [40] presented the weighted averaging operator for IFSs. Xu [41] also presented weighted geometric and hybrid geometric operators for IFSs. The IFS has received more importance after its appearance [14, 46]. The definition of IFSs for t-norms and t-conorms was extensively defined by Deschrijver [17]. When the summation of positive and negative membership grades is greater than one, in this case, DMs have not enough space for their decisions. As a result, in those circumstances, IFSs is unable to produce any appropriate results. Yager [47, 48] proposed the Pythagorean fuzzy set (PyFS), which is a general type of IFS, to deal with such a situation. Ashraf et al. [1] further upgraded the notion of PyFS to the proposed idea of spherical fuzzy sets (SFSs), which is a complete setup of all current fuzzy set structures in the literature. In decision-making issues, SFS deals with uncertainty more effectively and proficiently. Researchers using spherical fuzzy (SF) information to contribute the SFS notion by establishing a number of decision-making techniques. The theory of the SF Dombi aggregation operators under the SF information was formulated by Ashraf et al. [8]. Rafiq et al. [36] proposed a DM approach based on cosine similarity measures to tackle the uncertain information in the form of SF setting. Ashraf et al. [9] developed a DM technique focused on spherical distance measure based aggregation operator (AGOp). Ashraf et al. [10] adapted the SFS representations of SF t-norm and t-conorm and debated the TOPSIS [13] under SF environment. Jin et al. [21] proposed SF logarithmic AGOp based on entropy measures and discussed their applicability in DMPs. Jin et al. [22] established the linguistic SF AGOp and highlighted their applicability for addressing uncertain information under SF settings. Zeng et al. [52] proposed the SF rough set on the basis of TOPSIS method. For detail study, we recommended [2–7].

Torra [52] introduced the concept of hesitant FS to encourage the creation of a fuzzy set with a collection of values rather than a single value in the form of membership. Hesitant FS is a useful method for addressing the uncertain and ambiguous SF information in DMPs. Xu and Xia [42] established the novel distance and similarity measures under hesitant FSs. Qian et al. [35] presented a novel concept of generalized hesitant FSs and discussed their applicability in DMPs. Farhadinia [19] proposed the information measures under hesitant FSs and presented the application in real life decision making problems. Chen [16] established the novel correlation coefficients under HFSs and presented the application in clustering analysis. Guan [20] introduced the Grey relational analysis under hesitant FSs. Mahmood et al. [29] established the Jaccard and Dice similarity measures under complex dual hesitant FSs and discussed their applicability in DMPs. Furthermore, Khan et al. [23] established the generalized hesitant fuzzy set, so-called spherical hesitant fuzzy set, by considering the positive, neutral, and negative membership grades in the form of hesitant fuzzy sets. Also additionally, we proposed novel operational laws based on the logarithmic function under spherical hesitant fuzzy settings. Naeem et al. [30] established the sine function based spherical hesitant fuzzy AGOp and discussed their application in decision making problems.

Due to the motivation of the above discussion, in this study, we propose novel spherical hesitant fuzzy Yager aggregation operators to address the uncertain and incomplete information under spherical hesitant fuzzy information. Spherical hesitant fuzzy Yager aggregation operators have namely as: spherical hesitant fuzzy Yager weighted averaging (SHFYWA), spherical hesitant fuzzy Yager ordered weighted averaging (SHFYOWA), spherical hesitant fuzzy Yager hybrid weighted averaging (SHFYHWA), spherical hesitant fuzzy Yager weighted geometric (SHFYWG), spherical hesitant fuzzy Yager ordered weighted geometric (SHFYOWG) and spherical hesitant fuzzy Yager hybrid weighted geometric (SHFYHWG). The main contribution of this study is followed as:

(1) To construct the list of novel Yager norm-based basic operational laws.

(2) To develop a list of aggregation operators based on Yager t-norm and t-conorm and to discuss the related properties in detail.

(3) To develop a decision making methodology using the proposed aggregation operators to aggregate the uncertain information in real word decision making problems.

(4) To provide a numerical case study concerning to tackle Parkinson's disease, to show the applicability and reliability of the developed methodology.

(5) To provide "Validity and Reliability" test to show the validation of the proposed decision making methodology.

The rest of this manuscript is organized as follows: Section 2 briefly recalls some basic concepts. A novel operational laws are established using Yager t-norm and t-conorm in Section 3. Section 4 highlights a list of Yager aggregation operators based on the yager operational laws and their basic properties. Section 5 is devoted to a decision making methodology based on the developed aggregation operators. Section 6 presents the numerical illustration concerning to tackle the Parkinson's disease problem. Section 7 establishes the validity and reliability test to validate the proposed aggregation operator based multi-attribute decision making methodology. Section 8 concludes this manuscript.

2. Preliminaries

Definition 1. *[51] A fuzzy set (FS) V on the ground set $\mathbb{Z}$ is a structure having the following form*

$$V = \{\langle e, A(e)\rangle \,| e \in \mathbb{Z}\},$$

where $A(e) \in [0, 1]$ known to be membership grade.

Definition 2. *[39] A hesitant FS (HFS) V on the ground set $\mathbb{Z}$ is a structure having the following form*

$$V = \{\langle e, h(e)\rangle \,| e \in \mathbb{Z}\},$$

where $h(e)$ be the set of some possible membership degrees in $[0, 1]$.

Definition 3. *[1] A spherical FS (SFS) V on the ground set $\mathbb{Z}$ is a structure having the following form*

$$V = \{\langle e, A(e), B(e), C(e))\rangle \,| e \in \mathbb{Z}\},$$

where $A(e) \in [0, 1]$, $B(e) \in [0, 1]$ and $C(e) \in [0, 1]$ known to be positive, neutral and negative membership grades, subject to the condition $A^2(e) + B^2(e) + C^2(e) \leq 1$, for all $e \in \mathbb{Z}$.

The space of the spherical fuzzy set is presented in Figure 1:

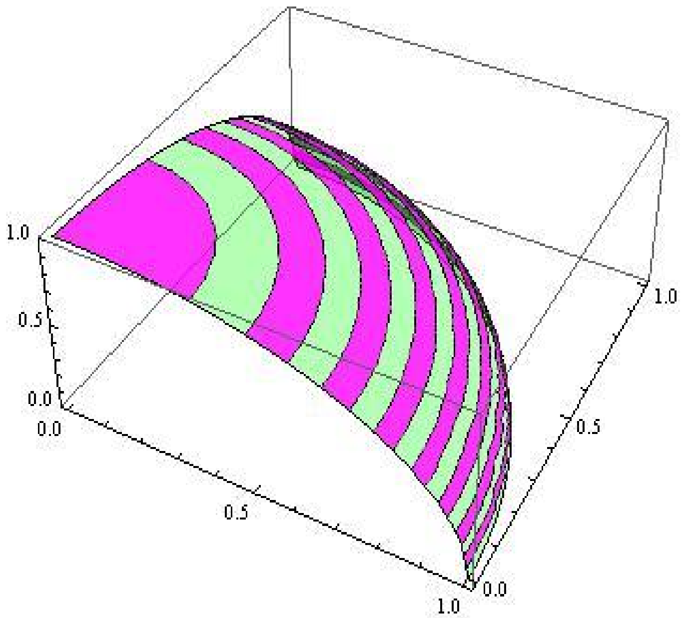

Figure 1. Spherical fuzzy set space.

Definition 4. *[30] A spherical hesitant FS (SHFS) V on the ground set $\mathbb{Z}$ is a structure having the following form*

$$V = \{\langle e, A(e), B(e), C(e)\rangle\rangle \, | e \in \mathbb{Z}\},$$

where

$$A(e) = \{a|a \in [0,1]\}, \; B(e) = \{b|b \in [0,1]\} \text{ and } C(e) = \{c|c \in [0,1]\},$$

known to be positive, neutral and negative membership grades, subject to the condition $0 \leq (a^{+})^2 + (b^{+})^2 + (c^{+})^2 \leq 1, \forall \; e \in \mathbb{Z}$*, such that*

$$a^{+} = \bigcup_{a \in A(e)} \max\{a\}, \; b^{+} = \bigcup_{b \in B(e)} \max\{b\}, \text{ and } c^{+} = \bigcup_{c \in C(e)} \max\{c\}.$$

Definition 5. *[24] Let us consider* $V = \{A_e, B_e, C_e\} \in SHFS \; (e \in \mathbb{N})$. *The elementary operational laws are given below;*

(1) $(V_1)^c = \bigcup_{(a_1,b_1,c_1) \in (A_1,B_1,C_1)} \{c_1, b_1, a_1\}$;

(2) $V_1 \cup V_2 = \bigcup_{(a_e,b_e,c_e) \in (A_e,B_e,C_e)(e=1,2)} \{\max(a_e), \min(b_e), \min(c_e)\}$;

(3) $V_1 \cap V_2 = \bigcup_{(a_1,b_1,c_1) \in (A_1,B_1,C_1)} \{\min(a_e), \min(b_e), \max(c_e)\}$;

Definition 6. *[24] Let we have* $V = \{A, B, C\}$, $V_1 = \{A_1, B_1, C_1\}$, *and* $V_2 = \{A_2, B_2, C_2\}$ *three sets of SHFNs with* $\delta > 0$. *Then, the operations of SHFEs are shown below:*

(1) $V_1 \oplus V_2 = \bigcup_{\substack{a \in A_1, b_1 \in B_1, c_1 \in C_1 \\ a_2 \in A_2, b_2 \in B, c_2 \in C_1}} \left\{ \sqrt{a_1^2 + a_2^2 - a_1^2 a_2^2}, b_1 b_2, c_1 c_2 \right\}$;

(2) $V_1 \otimes V_2 = \bigcup_{\substack{a \in A_1, b_1 \in B_1, c_1 \in C_1 \\ a_2 \in A_2, b_2 \in B, c_2 \in C_1}} \left\{ \{a_1 a_2, b_1 b_2, \sqrt{c_1^2 + c_2^2 - c_1^2 c_2^2} \right\}$;

(3) $\delta V = \bigcup_{a\in A, b\in B, c\in C} \left\{ \sqrt{1-(1-a^2)^\delta}, (b)^\delta, (c)^\delta \right\}$;

(4) $V^\delta = \bigcup_{a\in A, b\in B, c\in C} \left\{ (a)^\delta, (b)^\delta, \sqrt{1-(1-c^2)^\delta} \right\}$.

Example 1. *Let we have two SHFNs,* $V_1 = \{\{0.2, 0.5\}, \{0.3\}, \{0.3, 0.4\}\}$ *and* $V_2 = \{\{0.2\}, \{0.3, 0.6\}, \{0.1, 0.2\}\}$ *with* $\delta > 2$ *then the operations of SHFEs are shown below:*

(1) $V_1 \oplus V_2 = \bigcup_{\substack{a\in A_1, b_1\in B_1, c_1\in C_1 \\ a_2\in A_2, b_2\in B, c_2\in C_1}} \left\{ \sqrt{a_1^2+a_2^2-a_1^2a_2^2}, b_1b_2, c_1c_2 \right\}$

$$\begin{aligned} &\{\{0.2, 0.5\}, \{0.3\}, \{0.3, 0.4\}\} \oplus \{\{0.2\}, \{0.3, 0.6\}, \{0.1, 0.2\}\} \\ =& \left\{ \begin{array}{c} \left\{ \sqrt{(0.2)^2+(0.5)^2-(0.2)^2(0.5)^2}, \sqrt{(0.2)^2+(0.2)^2-(0.2)^2(0.2)^2} \right\}, \\ \{(0.3)(0.3), (0.3)(0.6)\}, \\ \{(0.3)(0.1), (0.3)(0.2), (0.4)(0.1), (0.4)(0.2)\} \end{array} \right\} \\ =& \{\{0.28, 0.529\}, \{0.09, 0.18\}, \{0.03, 0.06, 0.04, 0.08\}\}. \end{aligned}$$

(2) $V_1 \otimes V_2 = \bigcup_{\substack{a\in A_1, b_1\in B_1, c_1\in C_1 \\ a_2\in A_2, b_2\in B, c_2\in C_1}} \left\{ \{a_1a_2, b_1b_2, \sqrt{c_1^2+c_2^2-c_1^2c_2^2} \right\}$

$$\begin{aligned} &\{\{0.2, 0.5\}, \{0.3\}, \{0.3, 0.4\}\} \otimes \{\{0.2\}, \{0.3, 0.6\}, \{0.1, 0.2\}\} \\ =& \left\{ \begin{array}{c} \{(0.2)(0.2), (0.5)(0.2)\}, \{(0.3)(0.3), (0.3)(0.6)\}, \\ \left\{ \begin{array}{c} \sqrt{(0.3)^2+(0.1)^2-(0.3)^2(0.1)^2}, \sqrt{(0.3)^2+(0.2)^2-(0.3)^2(0.2)^2}, \\ \sqrt{(0.4)^2+(0.1)^2-(0.4)^2(0.1)^2}, \sqrt{(0.4)^2+(0.2)^2-(0.4)^2(0.2)^2} \end{array} \right\} \end{array} \right\} \\ =& \{\{0.04, 0.10\}, \{0.09, 0.18\}, \{0.31, 0.35, 0.41, 0.44\}\}. \end{aligned}$$

(3) $\delta \cdot V_1 = \bigcup_{a\in A_1, b_1\in B_1, c_1\in C_1} \left\{ \sqrt{1-(1-a_1^2)^\delta}, (b_1)^\delta, (c_1)^\delta \right\}$

$$\begin{aligned} \delta \cdot V_1 &= 2 \cdot \{\{0.2, 0.5\}, \{0.3\}, \{0.3, 0.4\}\} \\ &= \left\{ \begin{array}{c} \left\{ \sqrt{1-\left(1-(0.2)^2\right)^2}, \sqrt{1-\left(1-(0.5)^2\right)^2} \right\}, \\ \left\{(0.3)^2\right\}, \left\{(0.3)^2, (0.4)^2\right\} \end{array} \right\} \\ &= \{\{0.28, 0.66\}, \{0.09\}, \{0.09, 0.16\}\}. \end{aligned}$$

(4) $(V_1)^\delta = \bigcup_{a\in A_1, b_1\in B_1, c_1\in C_1} \left\{ (a_1)^\delta, (b_1)^\delta, \sqrt{1-(1-c_1^2)^\delta} \right\}$

$$\begin{aligned} (V_1)^\delta &= (\{\{0.2, 0.5\}, \{0.3\}, \{0.3, 0.4\}\})^2 \\ &= \left\{ \begin{array}{c} \left\{(0.2)^2, (0.5)^2\right\}, \left\{(0.3)^2\right\}, \\ \left\{ \sqrt{1-\left(1-(0.3)^2\right)^2}, \sqrt{1-\left(1-(0.4)^2\right)^2} \right\} \end{array} \right\} \\ &= \{\{0.04, 0.25\}, \{0.09\}, \{0.41, 0.54\}\}. \end{aligned}$$

Definition 7. *[24] Suppose* $\mathbb{Z}_g = \left\{M_g, L_g, K_g\right\} = \left\{ \begin{array}{c} \left\{\kappa_g^1, \kappa_g^2, \kappa_g^3, ..., \kappa_g^{l(M)}\right\}, \left\{\delta_g^1, \delta_g^2, \delta_g^3, ..., \delta_g^{l(L)}\right\}, \\ \left\{\partial_g^1, \partial_g^2, \partial_g^3, ..., \partial_g^{l(K)}\right\} \end{array} \right\} \in SHFS(\mathbb{R})\ (g \in \mathbb{N})$, *where l represent the possible number of elements in* M_g, L_g *and* K_g *respectively.*

The score (Sc) *and accuracy* (Ac) *functions are defined as follows;*
(1) $Sc(\mathbb{Z}_g) = \frac{2+\sum_{g=1}^{n}\kappa_g-\sum_{g=1}^{n}\delta_g-\sum_{g=1}^{n}\partial_g}{3}$;
(2) $Ac(\mathbb{Z}_g) = \frac{\sum_{g=1}^{n}\kappa_g+\sum_{g=1}^{n}\delta_g+\sum_{g=1}^{n}\partial_g}{3}$.

Definition 8. *Suppose* $\mathbb{Z}_g = \left\{M_g, L_g, K_g\right\} \in SHFS\ (\mathbb{R})\ (g \in \mathbb{N})$. *Then,*
(1) *If* $Sc(\mathbb{Z}_1) > Sc(\mathbb{Z}_2)$, *then* $\mathbb{Z}_1 > \mathbb{Z}_2$;
(2) *If* $Sc(\mathbb{Z}_1) = Sc(\mathbb{Z}_2)$, *then;*
(a) *If* $Ac(\mathbb{Z}_1) > Ac(\mathbb{Z}_2)$, *then* $\mathbb{Z}_1 > \mathbb{Z}_2$;
(b) *If* $Ac(\mathbb{Z}_1) = Ac(\mathbb{Z}_2)$, *then* $\mathbb{Z}_1 = \mathbb{Z}_2$.

Yager operational laws for spherical HFS.

Definition 9. *Let us consider for any real numbers m and n, Yager t-norm and t-conorm are described as*
(1) $T(m,n) = 1 - min(1, ((1-m)^{\mu} + (1-n)^{\mu})^{\frac{1}{\mu}})$;
(2) $S(m,n) = min(1, (m^{\mu} - n^{\mu})^{\frac{1}{\mu}}), \mu \in (0,\infty)$.

Definition 10. *Let we have* $V_1 = \{A_1, B_1, C_1\}$ *and* $V_2 = \{A_2, B_2, C_2\}$ *the two sets of SHFNs with* $\omega, \mu > 0$. *Then, Yager norm based operating laws* $(YOLs)$ *are fallows below:*

$$(1)\ V_1 \oplus V_2 = \bigcup_{\substack{a_1\in A_1, b_1\in B_1, c_1\in C_1\\ a_2\in A_2, b_2\in B_2, c_2\in C_2}} \left\{ \begin{array}{c} \sqrt{\min\left(1, a_1^{2\mu} + a_2^{2\mu}\right)}, \\ \sqrt{1 - \min\left(1, \left((1-b_1^2)^{\mu} + (1-b_2^2)^{\mu}\right)\right)^{\frac{1}{\mu}}}, \\ \sqrt{1 - \min\left(1, \left((1-c_1^2)^{\mu} + (1-c_2^2)^{\mu}\right)\right)^{\frac{1}{\mu}}} \end{array} \right\};$$

$$(2)\ V_1 \otimes V_2 = \bigcup_{\substack{a_1\in A_1, b_1\in B_1, c_1\in C_1\\ a_2\in A_2, b_2\in B_2, c_2\in C_2}} \left\{ \begin{array}{c} \sqrt{1 - \min\left(1, \left((1-a_1^2)^{\mu} + (1-a_2^2)^{\mu}\right)\right)^{\frac{1}{\mu}}}, \\ \sqrt{1 - \min\left(1, \left((1-b_1^2)^{\mu} + (1-b_2^2)^{\mu}\right)\right)^{\frac{1}{\mu}}}, \\ \sqrt{\min\left(1, c_1^{2\mu} + c_2^{2\mu}\right)} \end{array} \right\};$$

$$(3)\ \omega.V_1 = \bigcup_{a_1\in A_1, b_1\in B_1, c_1\in C_1} \left\{ \begin{array}{c} \sqrt{\min\left(1, \left(\omega a_1^2\right)^{\mu}\right)^{\frac{1}{\mu}}}, \sqrt{1 - \min\left(1, \left(\omega(1-b_1^2)^{\mu}\right)\right)^{\frac{1}{\mu}}}, \\ \sqrt{1 - \min\left(1, \left(\omega(1-c_1^2)^{\mu}\right)\right)^{\frac{1}{\mu}}} \end{array} \right\};$$

$$(4)\ (V_1)^{\omega} = \bigcup_{a_1\in A_1, b_1\in B_1, c_1\in C_1} \left\{ \begin{array}{c} \sqrt{1 - \min\left(1, \left(\omega(1-a_1^2)^{\mu}\right)\right)^{\frac{1}{\mu}}}, \sqrt{1 - \min\left(1, \left(\omega(1-b_1^2)^{\mu}\right)\right)^{\frac{1}{\mu}}}, \\ \sqrt{\min\left(1, \left(\omega c_1^2\right)^{\mu}\right)^{\frac{1}{\mu}}} \end{array} \right\}.$$

Theorem 1. *For any two SHFSs* $V_1 = \{A_1, B_1, C_1\}$ *&* $V_2 = \{A_2, B_2, C_2\}$ *with* $\omega_1, \omega_2 > 0$. *Then*
(1) $V_1 \oplus V_2 = V_2 \oplus V_1$;
(2) $V_1 \otimes V_2 = V_2 \otimes V_1$;
(3) $\omega_1 (V_1 \oplus V_2) = \omega_1 V_1 \oplus \omega_1 V_2$;
(4) $(\omega_1 \oplus \omega_2) V_1 = \omega_1 V_1 \oplus \omega_2 V_1$;
(5) $(V_1 \otimes V_2)^{\omega_1} = (V_1)^{\omega_1} \otimes (V_2)^{\omega_1}$;
(6) $(V_1)^{\omega_1} \otimes (V_1)^{\omega_2} = (V_1)^{\omega_1+\omega_2}$.

Proof. For any V_1, V_2 be any two SHFSs with $\omega_1, \omega_2 > 0$. We have

$$\begin{aligned}
& V_1 \oplus V_2 \\
= & \bigcup_{\substack{a_1\in A_1, b_1\in B_1, c_1\in C_1\\ a_2\in A_2, b_2\in B_2, c_2\in C_2}} \left\{ \begin{array}{c} \sqrt{\min\left(1, \left(a_1^{2\mu} + a_2^{2\mu}\right)\right)}, \\ \sqrt{1-\min\left(1, \left((1-b_1^2)^{\mu} + (1-b_2^2)^{\mu}\right)\right)^{\frac{1}{\mu}}}, \\ \sqrt{1-\min\left(1, \left((1-c_1^2)^{\mu} + (1-c_2^2)^{\mu}\right)\right)^{\frac{1}{\mu}}} \end{array} \right\} \\
= & \bigcup_{\substack{a_1\in A_1, b_1\in B_1, c_1\in C_1\\ a_2\in A_2, b_2\in B, c_2\in C_2}} \left\{ \begin{array}{c} \sqrt{\min\left(1, \left(a_2^{2\mu} + a_1^{2\mu}\right)\right)}, \\ \sqrt{1-\min\left(1, \left((1-b_2^2)^{\mu} + (1-b_1^2)^{\mu}\right)\right)^{\frac{1}{\mu}}}, \\ \sqrt{1-\min\left(1, \left((1-c_2^2)^{\mu} + (1-c_1^2)^{\mu}\right)\right)^{\frac{1}{\mu}}} \end{array} \right\} \\
= & V_2 \oplus V_1
\end{aligned}$$

$$\begin{aligned}
& V_1 \otimes V_2 \\
= & \bigcup_{\substack{a_1\in A_1, b_1\in B_1, c_1\in C_1\\ a_2\in A_2, b_2\in B, c_2\in C_2}} \left\{ \begin{array}{c} \sqrt{1-\min\left(1, \left((1-a_1^2)^{\mu} + (1-a_2^2)^{\mu}\right)\right)^{\frac{1}{\mu}}}, \\ \sqrt{1-\min\left(1, \left((1-b_1^2)^{\mu} + (1-b_2^2)^{\mu}\right)\right)^{\frac{1}{\mu}}}, \\ \sqrt{\min\left(1, \left(c_1^{2\mu} + c_2^{2\mu}\right)\right)} \end{array} \right\}; \\
= & \bigcup_{\substack{a_1\in A_1, b_1\in B_1, c_1\in C_1\\ a_2\in A_2, b_2\in B, c_2\in C_2}} \left\{ \begin{array}{c} \sqrt{1-\min\left(1, \left((1-a_2^2)^{\mu} + (1-a_1^2)^{\mu}\right)\right)^{\frac{1}{\mu}}}, \\ \sqrt{1-\min\left(1, \left((1-b_2^2)^{\mu} + (1-b_1^2)^{\mu}\right)\right)^{\frac{1}{\mu}}}, \\ \sqrt{\min\left(1, \left(c_2^{2\mu} + c_1^{2\mu}\right)\right)} \end{array} \right\}; \\
= & V_1 \otimes V_2
\end{aligned}$$

$$\begin{aligned}
\omega_1 (V_1 \oplus V_2) = & \bigcup_{\substack{a_1\in A_1, b_1\in B_1, c_1\in C_1\\ a_2\in A_2, b_2\in B_2, c_2\in C_2}} \left\{ \omega_1 \left(\begin{array}{c} \sqrt{\min\left(1, \left(a_1^{2\mu} + a_2^{2\mu}\right)\right)}, \\ \sqrt{1-\min\left(1, \left((1-b_1^2)^{\mu} + (1-b_2^2)^{\mu}\right)\right)^{\frac{1}{\mu}}}, \\ \sqrt{1-\min\left(1, \left((1-c_1^2)^{\mu} + (1-c_2^2)^{\mu}\right)\right)^{\frac{1}{\mu}}} \end{array} \right) \right\} \\
= & \bigcup_{\substack{a_1\in A_1, b_1\in B_1, c_1\in C_1\\ a_2\in A_2, b_2\in B_2, c_2\in C_2}} \left\{ \left(\begin{array}{c} \sqrt{\min\left(1, \left(\omega_1 a_1^{2\mu} + \omega_1 a_2^{2\mu}\right)\right)}, \\ \sqrt{1-\min\left(1, \left(\omega_1(1-b_1^2)^{\mu} + \omega_1(1-b_2^2)^{\mu}\right)\right)^{\frac{1}{\mu}}}, \\ \sqrt{1-\min\left(1, \left(\omega_1(1-c_1^2)^{\mu} + \omega_1(1-c_2^2)^{\mu}\right)\right)^{\frac{1}{\mu}}} \end{array} \right) \right\}
\end{aligned}$$

$$\omega_1 V_1 \oplus \omega_1 V_2 = \bigcup_{\substack{a_1\in A_1, b_1\in B_1, c_1\in C_1 \\ a_2\in A_2, b_2\in B, c_2\in C_2}} \left\{ \begin{array}{c} \left(\begin{array}{c} \sqrt{\min\left(1, \left(\omega_1 a_1^{2\mu}\right)\right)^{\frac{1}{\mu}}}, \\ \sqrt{1-\min\left(1, \left(\omega_1 (1-b_1^2)^{\mu}\right)\right)^{\frac{1}{\mu}}}, \\ \sqrt{1-\min\left(1, \left(\omega_1 (1-c_1^2)^{\mu}\right)\right)^{\frac{1}{\mu}}} \end{array} \right) \\ \oplus \left(\begin{array}{c} \sqrt{\min\left(1, \left(\omega_1 a_2^{2\mu}\right)\right)^{\frac{1}{\mu}}}, \\ \sqrt{1-\min\left(1, \left(\omega_1 (1-b_2^2)^{\mu}\right)\right)^{\frac{1}{\mu}}}, \\ \sqrt{1-\min\left(1, \left(\omega_1 (1-c_2^2)^{\mu}\right)\right)^{\frac{1}{\mu}}} \end{array} \right) \end{array} \right\}$$

$$= \bigcup_{\substack{a_1\in A_1, b_1\in B_1, c_1\in C_1 \\ a_2\in A_2, b_2\in B, c_2\in C_2}} \left\{ \left(\begin{array}{c} \sqrt{\min\left(1, \left(\omega_1 a_1^{2\mu} + \omega_1 a_2^{2\mu}\right)\right)^{\frac{1}{\mu}}}, \\ \sqrt{1-\min\left(1, \left(\omega_1 (1-b_1^2)^{\mu} + \omega_1 (1-b_2^2)^{\mu}\right)\right)^{\frac{1}{\mu}}}, \\ \sqrt{1-\min\left(1, \left(\omega_1 (1-c_1^2)^{\mu} + \omega_1 (1-c_2^2)^{\mu}\right)\right)^{\frac{1}{\mu}}} \end{array} \right) \right\}$$

$$= \omega_1 \left(V_1 \oplus V_2\right)$$

Hence

$$\omega \left(V_1 \oplus V_2\right) = \omega V_1 \oplus \omega V_2$$

Proof of (4), (5) and (6) are similarly as above. □

3. Spherical hesitant fuzzy Yager aggregation operators

In this portion, we introduce some SHF AGOp based on the Yager t-norm and t-conorm.

3.1. Weighted averaging aggregation operators for Yager's norms

Definition 11. *Suppose $V_e = \{A_e, B_e, C_e\}$ is a SHFS $(e \in \mathbb{N})$. Then, Yager weighted averaging aggregation operators for SHFNs are stated as below;*

$$\begin{aligned} SHFYWA\left(V_1, V_2, ...V_k\right) &= \delta_1 V_1 \oplus \delta_2 V_2 \oplus ... \oplus \delta_k V_k \\ &= \sum_{e=1}^{k} \delta_e V_e \end{aligned} \tag{3.1}$$

where $\delta = (\delta_1, \delta_2, ...\delta_k)^T$ denote the weighted values of $(V_1, V_2, ...V_k)$ such that $\delta_e \geq 0$; $\sum_{e=1}^{k} \delta_e = 1$.

Theorem 2. *Suppose $V_e = \{A_e, B_e, C_e\}$ is a SHFS $(e \in \mathbb{N})$, $(e = 1, 2, ...k)$ and $\delta = (\delta_1, \delta_2, ...\delta_k)^T$ denote the weighted values of $(V_1, V_2, ...V_k)$ such that $\delta_e \geq 0$; $\sum_{e=1}^{k} \delta_e = 1$. Then, SHFYWA aggregation operator is a mapping $\alpha^k \to \alpha$ such that;*

$$SHFYWA\left(V_1, V_2, ...V_k\right) = \sum_{e=1}^{k} \delta_e V_e$$

$$= \bigcup_{a_e \in A_e, b_e \in B_e, c_e \in C_e} \left\{ \begin{array}{c} \sqrt{\min\left(1, \left(\sum_{e=1}^{k} \delta_e a_e^{2\mu}\right)\right)^{\frac{1}{\mu}}}, \\ \sqrt{1 - \min\left(1, \left(\sum_{e=1}^{k} \delta_e (1 - b_e^2)^{\mu}\right)\right)^{\frac{1}{\mu}}}, \\ \sqrt{1 - \min\left(1, \left(\sum_{e=1}^{k} \delta_e (1 - c_e^2)^{\mu}\right)\right)^{\frac{1}{\mu}}} \end{array} \right\} \tag{3.2}$$

Proof. To prove this theorem, we apply mathematical induction on k.

Take $k = 2$

$$\begin{aligned} SHFYWA(V_1, V_2) &= \delta_1 V_1 \oplus \delta_2 V_2 \\ &= \bigcup_{\substack{a_1 \in A_1, b_1 \in B_1, c_1 \in C_1 \\ a_2 \in A_2, b_2 \in B, c_2 \in C_2}} \left\{ \begin{array}{c} \left(\begin{array}{c} \sqrt{\min\left(1, \left(\delta_1 a_1^{2\mu}\right)\right)^{\frac{1}{\mu}}}, \sqrt{1 - \min\left(1, \left(\delta_1 (1 - b_1^2)^{\mu}\right)\right)^{\frac{1}{\mu}}}, \\ \sqrt{1 - \min\left(1, \left(\delta_1 (1 - c_1^2)^{\mu}\right)\right)^{\frac{1}{\mu}}} \end{array} \right) \oplus \\ \left(\begin{array}{c} \sqrt{\min\left(1, \left(\delta_2 a_2^{2\mu}\right)\right)^{\frac{1}{\mu}}}, \sqrt{1 - \min\left(1, \left(\delta_2 (1 - b_2^2)^{\mu}\right)\right)^{\frac{1}{\mu}}}, \\ \sqrt{1 - \min\left(1, \left(\delta_2 (1 - c_2^2)^{\mu}\right)\right)^{\frac{1}{\mu}}} \end{array} \right) \end{array} \right\} \end{aligned}$$

$$= \bigcup_{(a_e, b_e, c_e) \in (A_e, B_e, C_e)(e=1,2)} \left\{ \left(\begin{array}{c} \sqrt{\min\left(1, \left(\sum_{e=1}^{2} \delta_e a_e^{2\mu}\right)\right)^{\frac{1}{\mu}}}, \sqrt{1 - \min\left(1, \left(\sum_{e=1}^{2} \delta_e (1 - b_e^2)^{\mu}\right)\right)^{\frac{1}{\mu}}}, \\ \sqrt{1 - \min\left(1, \left(\sum_{e=1}^{2} \delta_e (1 - c_e^2)^{\mu}\right)\right)^{\frac{1}{\mu}}} \end{array} \right) \right\}$$

Let Eq (3.2) is true for $k = m$;

$$\begin{aligned} & SHFYWA(V_1, V_2, ...V_k) \\ = & \bigcup_{a_e \in A_e, b_e \in B_e, c_e \in C_e} \left\{ \begin{array}{c} \sqrt{\min\left(1, \left(\sum_{e=1}^{m} \delta_e a_e^{2\mu}\right)\right)^{\frac{1}{\mu}}}, \sqrt{1 - \min\left(1, \left(\sum_{e=1}^{m} \delta_e (1 - b_e^2)^{\mu}\right)\right)^{\frac{1}{\mu}}}, \\ \sqrt{1 - \min\left(1, \left(\sum_{e=1}^{m} \delta_e (1 - c_e^2)^{\mu}\right)\right)^{\frac{1}{\mu}}} \end{array} \right\} \end{aligned}$$

To show Eq (3.2) is true for $k = m + 1$;

$$\begin{aligned} SHFYWA(V_1, V_2, ...V_k) &= \sum_{e=1}^{k} \delta_e V_e \oplus \delta_{k+1} V_{k+1} \\ &= \bigcup_{a_e \in A_e, b_e \in B_e, c_e \in C_e} \left\{ \begin{array}{c} \left(\begin{array}{c} \sqrt{\min\left(1, \left(\sum_{e=1}^{m} \delta_e a_e^{2\mu}\right)\right)^{\frac{1}{\mu}}}, \\ \sqrt{1 - \min\left(1, \left(\sum_{e=1}^{m} \delta_e (1 - b_e^2)^{\mu}\right)\right)^{\frac{1}{\mu}}}, \\ \sqrt{1 - \min\left(1, \left(\sum_{e=1}^{m} \delta_e (1 - c_e^2)^{\mu}\right)\right)^{\frac{1}{\mu}}} \end{array} \right) \oplus \\ \left(\begin{array}{c} \sqrt{\min\left(1, \left(\delta_{e+1} a_{e+1}^{2\mu}\right)\right)^{\frac{1}{\mu}}}, \\ \sqrt{1 - \min\left(1, \left(\delta_{e+1} (1 - b_{e+1}^2)^{\mu}\right)\right)^{\frac{1}{\mu}}}, \\ \sqrt{1 - \min\left(1, \left(\delta_{e+1} (1 - c_{e+1}^2)^{\mu}\right)\right)^{\frac{1}{\mu}}} \end{array} \right) \end{array} \right\} \end{aligned}$$

$$= \bigcup_{a_e\in A_e, b_e\in B_e, c_e\in C_e} \left\{ \begin{array}{c} \sqrt{\min\left(1, \left(\sum_{e=1}^{m+1} \delta_e a_e^{2\mu}\right)\right)^{\frac{1}{\mu}}}, \\ \sqrt{1-\min\left(1, \left(\sum_{e=1}^{m+1} \delta_e (1-b_e^2)^{\mu}\right)\right)^{\frac{1}{\mu}}}, \\ \sqrt{1-\min\left(1, \left(\sum_{e=1}^{m+1} \delta_e (1-c_e^2)^{\mu}\right)\right)^{\frac{1}{\mu}}} \end{array} \right\}$$

Thus Eq (3.2) is true for $k = m + 1$. Hence it is true $\forall$ +ve integers. □

Theorem 3. *Suppose* $V_e = \{A_e, B_e, C_e\}$ *is a SHFS* $(e \in \mathbb{N})$, $(e = 1, 2, ...k)$ *such that* $V_e = V$. *Then*

$$SHFYWA\,(V_1, V_2, ...V_k) = V \tag{3.3}$$

Proof. Since $V_e = V$ $(e = 1, 2, ...k)$, then by Theorem (2), we have

$$\begin{aligned} & SHFYWA\,(V_1, V_2, ...V_k) \\ = & \bigcup_{a_e\in A_e, b_e\in B_e, c_e\in C_e} \left\{ \begin{array}{c} \sqrt{\min\left(1, \left(\sum_{e=1}^{k} \delta_e a_e^{2\mu}\right)\right)^{\frac{1}{\mu}}}, \\ \sqrt{1-\min\left(1, \left(\sum_{e=1}^{k} \delta_e (1-b_e^2)^{\mu}\right)\right)^{\frac{1}{\mu}}}, \\ \sqrt{1-\min\left(1, \left(\sum_{e=1}^{k} \delta_e (1-c_e^2)^{\mu}\right)\right)^{\frac{1}{\mu}}} \end{array} \right\} \\ = & \bigcup_{a_e\in A_e, b_e\in B_e, c_e\in C_e} \left\{ \begin{array}{c} \sqrt{\min\left(1, \left(\sum_{e=1}^{k} \delta_e a^{2\mu}\right)\right)^{\frac{1}{\mu}}}, \\ \sqrt{1-\min\left(1, \left(\sum_{e=1}^{k} \delta_e (1-b^2)^{\mu}\right)\right)^{\frac{1}{\mu}}}, \\ \sqrt{1-\min\left(1, \left(\sum_{e=1}^{k} \delta_e (1-c^2)^{\mu}\right)\right)^{\frac{1}{\mu}}} \end{array} \right\} \\ = & \bigcup_{a_e\in A_e, b_e\in B_e, c_e\in C_e} \left\{ \begin{array}{c} \sqrt{\min\left(1, (a^{2\mu})\right)^{\frac{1}{\mu}}}, \\ \sqrt{1-\min\left(1, ((1-b^2)^{\mu})\right)^{\frac{1}{\mu}}}, \\ \sqrt{1-\min\left(1, ((1-c^2)^{\mu})\right)^{\frac{1}{\mu}}} \end{array} \right\} \\ = & (a, b, c) \\ = & V. \end{aligned}$$

□

Theorem 4. *Suppose* $V_e = \{A_e, B_e, C_e\}$ *is a SHFS* $(e \in \mathbb{N})$, $(e = 1, 2, ...k)$ *and* $V_e^- = \{\min(A_e),\ \min(B_e),\ \max(C_e)\}$, $V_e^+ = \{\max(A_e),\ \min(B_e),\ \min(C_e)\}$. *Then we have,*

$$V_e^- \leq SHFYWA\,(V_1, V_2, ...V_n) \leq V_e^+.$$

Proof. Since,

$$SHFYWA\,(V_1, V_2, ...V_k) = \sum_{e=1}^{k} \delta_e V_e$$

$$= \bigcup_{a_e\in A_e, b_e\in B_e, c_e\in C_e} \left\{ \begin{array}{c} \sqrt{\min\left(1, \sum_{e=1}^{k} \delta_e a_e^{2\mu}\right)^{\frac{1}{\mu}}}, \\ \sqrt{1-\min\left(1, \left(\sum_{e=1}^{k} \delta_e (1-b_e^2)^{\mu}\right)\right)^{\frac{1}{\mu}}}, \\ \sqrt{1-\min\left(1, \left(\sum_{e=1}^{k} \delta_e (1-c_e^2)^{\mu}\right)\right)^{\frac{1}{\mu}}} \end{array} \right\}$$

We have $V_e = \{A_e, B_e, C_e\}$ is a SHFS $(e \in \mathbb{N})$, $(e = 1, 2, ...k)$ and $V_e^- = \{\min(A_e^-),\ \min(B_e^-),\ \max(C_e^-)\}$, $V_e^+ = \{\max(A_e^+),\ \min(B_e^+),\ \min(C_e^+)\}$.

Since

$$a_e^- \leq a_e \leq a_e^+$$

Therefore, we have

$$\sqrt{\min\left(1, \sum_{e=1}^{k} \delta_e a_e^{-2\mu}\right)^{\frac{1}{\mu}}} \leq \sqrt{\min\left(1, \sum_{e=1}^{k} \delta_e a_e^{2\mu}\right)^{\frac{1}{\mu}}} \leq \sqrt{\min\left(1, \sum_{e=1}^{k} \delta_e a_e^{+2\mu}\right)^{\frac{1}{\mu}}}$$

Also,

$$b_e^- \leq b_e \leq b_e^+$$

Similarly we obtaine

$$\begin{aligned} \sqrt{1-\min\left(1, \left(\sum_{e=1}^{k} \delta_e (1-(b_e^-)^2)^{\mu}\right)\right)^{\frac{1}{\mu}}} &\leq \sqrt{1-\min\left(1, \left(\sum_{e=1}^{k} \delta_e (1-(b_e)^2)^{\mu}\right)\right)^{\frac{1}{\mu}}} \\ &\leq \sqrt{1-\min\left(1, \left(\sum_{e=1}^{k} \delta_e (1-(b_e^+)^2)^{\mu}\right)\right)^{\frac{1}{\mu}}} \end{aligned}$$

and

$$\begin{aligned} \sqrt{1-\min\left(1, \left(\sum_{e=1}^{k} \delta_e (1-(c_e^-)^2)^{\mu}\right)\right)^{\frac{1}{\mu}}} &\leq \sqrt{1-\min\left(1, \left(\sum_{e=1}^{k} \delta_e (1-(c_e)^2)^{\mu}\right)\right)^{\frac{1}{\mu}}} \\ &\leq \sqrt{1-\min\left(1, \left(\sum_{e=1}^{k} \delta_e (1-(c_e^+)^2)^{\mu}\right)\right)^{\frac{1}{\mu}}} \end{aligned}$$

Hence,

$$V_e^- \leq SHFYWA(V_1, V_2, ...V_n) \leq V_e^+.$$

□

Theorem 5. *Suppose $V_e = \{A_e, B_e, C_e\}$ & $V_e^* = \{A_e^*, B_e^*, C_e^*\} \in SHFS$ $(e \in \mathbb{N})$. If $A_e \geq A_e^*$, $B_e \leq B_e^*$ and $C_e \leq C_e^*$, then*

$$SHFYWA(V_1^*, V_2^*, ...V_n^*) \leq SHFYWA(V_1, V_2, ...V_k)$$

Proof. Suppose that $SHFYWA\left(V_1^*, V_2^*, ...V_n^*\right) = (A_e^*, B_e^*, C_e^*)$ and $SHFYWA\,(V_1, V_2, ...V_k) = (A_e, B_e, C_e)$. First of all, we are to show that $(A_e, B_e, C_e) \geq (A_e^*, B_e^*, C_e^*)$. As given that $a_e^* \leq a_e$. Therefore, we have

$$\left(\sum_{i=1}^{\acute{s}}\left(\alpha_i u_i^{*\delta}\right)\right)^{\frac{1}{\delta}} \leq \left(\sum_{i=1}^{\acute{s}}\left(\alpha_i u_i^{\delta}\right)\right)^{\frac{1}{\delta}}$$

$$\sqrt{\min\left(1, \sum_{e=1}^{k} \delta_e a_e^{*2\mu}\right)^{\frac{1}{\mu}}} \leq \sqrt{\min\left(1, \sum_{e=1}^{k} \delta_e a_e^{2\mu}\right)^{\frac{1}{\mu}}}$$

Similarly, we obtained

$$\sqrt{1-\min\left(1, \left(\sum_{e=1}^{k} \delta_e(1-(b_e^*)^2)^{\mu}\right)\right)^{\frac{1}{\mu}}} \leq \sqrt{1-\min\left(1, \left(\sum_{e=1}^{k} \delta_e(1-(b_e)^2)^{\mu}\right)\right)^{\frac{1}{\mu}}}$$

and

$$\sqrt{1-\min\left(1, \left(\sum_{e=1}^{k} \delta_e(1-(c_e^*)^2)^{\mu}\right)\right)^{\frac{1}{\mu}}} \leq \sqrt{1-\min\left(1, \left(\sum_{e=1}^{k} \delta_e(1-(c_e)^2)^{\mu}\right)\right)^{\frac{1}{\mu}}}$$

Hence,

$$SHFYWA\,(V_1^*, V_2^*, ...V_n^*) \leq SHFYWA\,(V_1, V_2, ...V_k)\,.$$

□

Definition 12. *Suppose $V_e = \{A_e, B_e, C_e\} \in SHFS\ (e \in \mathbb{N})$. Then, Yager order weighted averaging aggregation operators for SHFNs are stated as below;*

$$\begin{aligned} SHFYOWA\,(V_1, V_2, ...V_k) &= \delta_1 V_{r(1)} \oplus \delta_2 V_{r(2)} \oplus ... \oplus \delta_n V_{r(k)} \\ &= \sum_{e=1}^{k} \delta_e V_{r(e)} \end{aligned}$$

where $r(e)$ is represented the order according to $(r(1), r(2), r(3), ..., r(k))$ and $(\delta_1, \delta_2, ...\delta_k)^T$ denote the weighted values of $(V_1, V_2, ...V_k)$ such that $\delta_e \geq 0$; $\sum_{e=1}^{k} \delta_e = 1$.

Theorem 6. *Suppose $V_e = \{A_e, B_e, C_e\}$ is a SHFS $(e \in \mathbb{N})$, $(e = 1, 2, ...k)$ and $\delta = (\delta_1, \delta_2, ...\delta_k)^T$ denote the weighted values of $V_1, V_2, ...V_k$ such that $\delta_e \geq 0$; $\sum_{e=1}^{k} \delta_e = 1$. Then, SHFYOWA aggregation operator is a mapping $\alpha^k \to \alpha$ such that;*

$$\begin{aligned} SHFYOWA\,(V_1, V_2, ...V_k) &= \sum_{e=1}^{k} \delta_e V_{r(e)} \\ &= \bigcup_{a_e \in A_e, b_e \in B_e, c_e \in C_e} \left\{ \begin{array}{c} \sqrt{\min\left(1, \left(\sum_{e=1}^{k} \delta_e a_{r(e)}^{2\mu}\right)\right)^{\frac{1}{\mu}}}, \\ \sqrt{1-\min\left(1, \left(\sum_{e=1}^{k} \delta_e (1-b_{r(e)}^2)^{\mu}\right)\right)^{\frac{1}{\mu}}}, \\ \sqrt{1-\min\left(1, \left(\sum_{e=1}^{k} \delta_e (1-c_{r(e)}^2)^{\mu}\right)\right)^{\frac{1}{\mu}}} \end{array} \right\} \end{aligned} \tag{3.4}$$

Theorem 7. *(1) Suppose $V_e = \{A_e, B_e, C_e\}$ is a SHFS $(e \in \mathbb{N})$, $(e = 1, 2, ...k)$ such that $V_e = V$. Then*

$$SHFYOWA(V_1, V_2, ...V_k) = V.$$

(2) Suppose $V_e = \{A_e, B_e, C_e\}$ is a SHFS $(e \in \mathbb{N})$, $(e = 1, 2, ...k)$ and $V_e^- = \{\min(A_e), \min(B_e), \max(C_e)\}$, $V_e^+ = \{\max(A_e), \min(B_e), \min(C_e)\}$. Then we have,

$$V_e^- \leq SHFYOWA(V_1, V_2, ...V_n) \leq V_e^+.$$

(3)Suppose $V_e = \{A_e, B_e, C_e\}$ & $V_e^* = \{A_e^*, B_e^*, C_e^*\} \in SHFS\ (e \in \mathbb{N})$. If $A_e \geq A_e^*$, $B_e \leq B_e^*$ and $C_e \leq C_e^*$, then

$$SHFYOWA(V_1, V_2, ...V_k) \leq SHFYOWA(V_1^*, V_2^*, ...V_k^*).$$

Definition 13. *Suppose $V_e = \{A_e, B_e, C_e\} \in SHFS\ (e \in \mathbb{N})$. Then, Yager hybrid weighted averaging aggregation operators for SHFNs are stated as below;*

$$\begin{aligned} SHFYHWA(V_1, V_2, ...V_k) &= \delta_1 V'_{r(1)} \oplus \delta_2 V'_{r(2)} \oplus ... \oplus \delta_k V'_{r(k)} \\ &= \sum_{e=1}^{k} \delta_e V'_{r(e)} \end{aligned} \tag{3.5}$$

where $r(e)$ is represented the order according to $(r(1), r(2), r(3), ..., r(k))$, such that $V'_{r(e)} \left(V'_{r(e)} = k\delta_e V_{r(e)} : e \in \mathbb{N}\right)$ and $(\delta_1, \delta_2, ...\delta_k)^T$ is the associated weight information respectively of $(V_1, V_2, ...V_k)$ such that $\delta_e \geq 0; \sum_{e=1}^{k} \delta_e = 1$.

3.2. Weighted geometric aggregation operators for Yager's norms

Definition 14. *Suppose $V_e = \{A_e, B_e, C_e\} \in SHFS\ (e \in \mathbb{N})$. Then, Yager weighted geometric aggregation operators for SHFNs are stated as below;*

$$\begin{aligned} SHFYWG(V_1, V_2, ...V_k) &= (V_1)^{\delta_1} \otimes (V_2)^{\delta_2} \otimes ... \otimes (V_n)^{\delta_k} \\ &= \prod_{e=1}^{k} (V_e)^{\delta_e} \end{aligned} \tag{3.6}$$

where $(\delta_1, \delta_2, ...\delta_k)^T$ denote the weighted values of $(V_1, V_2, ...V_n)$ such that $\delta_e \geq 0$; $\sum_{e=1}^{k} \delta_e = 1$.

Theorem 8. *Suppose $V_e = \{A_e, B_e, C_e\}$ is a SHFS $(e \in \mathbb{N})$, $(e = 1, 2, ...k)$ and $\delta = (\delta_1, \delta_2, ...\delta_k)^T$ denote the weighted values of $(V_1, V_2, ...V_k)$ such that $\delta_e \geq 0$; $\sum_{e=1}^{k} \delta_e = 1$. Then, SHFYWG aggregation operator is a mapping $\alpha^k \to \alpha$ such that;*

$$\begin{aligned} SHFYWG(V_1, V_2, ...V_k) &= \prod_{e=1}^{k} (V_e)^{\delta_e} \\ &= \bigcup_{a_e \in A_e, b_e \in B_e, c_e \in C_e} \left\{ \begin{array}{c} \sqrt{1 - \min\left(1, \left(\sum_{e=1}^{k} \delta_e (1 - a_e^2)^{\mu}\right)\right)^{\frac{1}{\mu}}}, \\ \sqrt{1 - \min\left(1, \left(\sum_{e=1}^{k} \delta_e (1 - b_e^2)^{\mu}\right)\right)^{\frac{1}{\mu}}}, \\ \sqrt{\min\left(1, \left(\sum_{e=1}^{k} \delta_e c_e^{2\mu}\right)\right)^{\frac{1}{\mu}}} \end{array} \right\} \end{aligned} \tag{3.7}$$

Proof. We prove by using mathematical induction

For $k = 2$, we have

$$SHFYWG(V_1, V_2) = V_1^{\delta_1} \otimes V_2^{\delta_2}$$

$$= \bigcup_{a_e \in A_e, b_e \in B_e, c_e \in C_e (e=1,2)} \left\{ \begin{pmatrix} \sqrt{\sqrt{1 - \min\left(1, \left(\delta_1 (1 - a_1^2)^{\mu}\right)\right)^{\frac{1}{\mu}}}}, \\ \sqrt{1 - \min\left(1, \left(\delta_1 (1 - b_1^2)^{\mu}\right)\right)^{\frac{1}{\mu}}}, \\ \min\left(1, \left(\delta_1 c_1^{2\mu}\right)\right)^{\frac{1}{\mu}} \end{pmatrix} \otimes \begin{pmatrix} \sqrt{\sqrt{1 - \min\left(1, \left(\delta_2 (1 - a_2^2)^{\mu}\right)\right)^{\frac{1}{\mu}}}}, \\ \sqrt{1 - \min\left(1, \left(\delta_2 (1 - b_2^2)^{\mu}\right)\right)^{\frac{1}{\mu}}}, \\ \min\left(1, \left(\delta_2 c_2^{2\mu}\right)\right)^{\frac{1}{\mu}} \end{pmatrix} \right\}$$

$$= \bigcup_{a_e \in A_e, b_e \in B_e, c_e \in C_e (e=1,2)} \left\{ \begin{array}{c} \sqrt{\sqrt{1 - \min\left(1, \left(\sum_{e=1}^{2} \delta_e (1 - a_e^2)^{\mu}\right)\right)^{\frac{1}{\mu}}}}, \\ \sqrt{1 - \min\left(1, \left(\sum_{e=1}^{2} \delta_e (1 - b_e^2)^{\mu}\right)\right)^{\frac{1}{\mu}}}, \\ \min\left(1, \left(\sum_{e=1}^{2} \delta_e c_e^{2\mu}\right)\right)^{\frac{1}{\mu}} \end{array} \right\}$$

Let Eq (3.7) is true for $k = m$;

$$SHFYWG(V_1, V_2, ...V_m)$$
$$= \bigcup_{a_e \in A_e, b_e \in B_e, c_e \in C_e} \left\{ \begin{array}{c} \sqrt{\sqrt{1 - \min\left(1, \left(\sum_{e=1}^{m} \delta_e (1 - a_e^2)^{\mu}\right)\right)^{\frac{1}{\mu}}}}, \\ \sqrt{1 - \min\left(1, \left(\sum_{e=1}^{m} \delta_e (1 - b_e^2)^{\mu}\right)\right)^{\frac{1}{\mu}}}, \\ \min\left(1, \left(\sum_{e=1}^{m} \delta_e c_e^{2\mu}\right)\right)^{\frac{1}{\mu}} \end{array} \right\}$$

To show Eq (3.7) is true for $k = m + 1$;

$$SHFYWG(V_1, V_2, ...V_k) = \prod_{e=1}^{m} (V_e)^{\delta_e} \otimes (V_{m+1})^{\delta_{m+1}}$$

$$SHFYWG(V_1, V_2, ...V_{m+1})$$

$$= \bigcup_{a_e \in A_e, b_e \in B_e, c_e \in C_e} \left\{ \begin{array}{c} \left(\begin{array}{c} \sqrt{1-\min\left(1, \left(\sum_{e=1}^{m} \delta_e (1-a_e^2)^{\mu}\right)\right)^{\frac{1}{\mu}}}, \\ \sqrt{1-\min\left(1, \left(\sum_{e=1}^{m} \delta_e (1-b_e^2)^{\mu}\right)\right)^{\frac{1}{\mu}}}, \\ \sqrt{\min\left(1, \left(\sum_{e=1}^{m} \delta_e c_e^{2\mu}\right)\right)^{\frac{1}{\mu}}} \end{array} \right) \otimes \\ \left(\begin{array}{c} \sqrt{1-\min\left(1, \left(\delta_{m+1} (1-a_{m+1}^2)^{\mu}\right)\right)^{\frac{1}{\mu}}}, \\ \sqrt{1-\min\left(1, \left(\delta_{m+1} (1-b_{m+1}^2)^{\mu}\right)\right)^{\frac{1}{\mu}}}, \\ \sqrt{\min\left(1, \left(\delta_{m+1} c_{m+1}^{2\mu}\right)\right)^{\frac{1}{\mu}}} \end{array} \right) \end{array} \right\}$$

$$= \bigcup_{a_e \in A_e, b_e \in B_e, c_e \in C_e} \left\{ \begin{array}{c} \sqrt{1-\min\left(1, \left(\sum_{e=1}^{m+1} \delta_e (1-a_e^2)^{\mu}\right)\right)^{\frac{1}{\mu}}}, \\ \sqrt{1-\min\left(1, \left(\sum_{e=1}^{m+1} \delta_e (1-b_e^2)^{\mu}\right)\right)^{\frac{1}{\mu}}}, \\ \sqrt{\min\left(1, \left(\sum_{e=1}^{m+1} \delta_e c_e^{2\mu}\right)\right)^{\frac{1}{\mu}}} \end{array} \right\}$$

So, the given result is true for $k = m + 1$.

Hence it is true ∀ positive integers. □

Theorem 9. *Suppose $V_e = \{A_e, B_e, C_e\}$ is a SHFS $(e \in \mathbb{N})$, $(e = 1, 2, ...k)$ such that $V_e = V$. Then*

$$SHFYWG(V_1, V_2, ...V_k) = V$$

Proof. Since $V_e = V$ $(e = 1, 2, ...k)$, then by Theorem (10), we have

$$\begin{aligned} & SHFYWG(V_1, V_2, ...V_k) \\ = & \bigcup_{a_e \in A_e, b_e \in B_e, c_e \in C_e} \left\{ \begin{array}{c} \sqrt{1-\min\left(1, \left(\sum_{e=1}^{k} \delta_e (1-a^2)^{\mu}\right)\right)^{\frac{1}{\mu}}}, \\ \sqrt{1-\min\left(1, \left(\sum_{e=1}^{k} \delta_e (1-b^2)^{\mu}\right)\right)^{\frac{1}{\mu}}}, \\ \sqrt{\min\left(1, \left(\sum_{e=1}^{k} \delta_e c^{2\mu}\right)\right)^{\frac{1}{\mu}}} \end{array} \right\} \\ = & \bigcup_{a_e \in A_e, b_e \in B_e, c_e \in C_e} \left\{ \begin{array}{c} \sqrt{1-\min(1, (1-a^2))}, \\ \sqrt{1-\min(1, (1-a^2))}, \\ \min\left(1, (c^2\right)) \end{array} \right\} \\ = & (a, b, c) \\ = & V. \end{aligned}$$

□

Theorem 10. *(1) Suppose $V_e = \{A_e, B_e, C_e\}$ is a SHFS $(e \in \mathbb{N})$, $(e = 1, 2, ...k)$ and $V_e^- = \{\min(A_e),\ \min(B_e),\ \max(C_e)\}$, $V_e^+ = \{\max(A_e),\ \min(B_e),\ \min(C_e)\}$. Then we have,*

$$V_e^- \leq SHFYWA(V_1, V_2, ...V_n) \leq V_e^+.$$

(2) Suppose $V_e = \{A_e, B_e, C_e\}$ & $V_e^* = \{A_e^*, B_e^*, C_e^*\} \in SHFS\ (e \in \mathbb{N})$. If $A_e \geq A_e^*$, $B_e \leq B_e^*$ and $C_e \leq C_e^*$, then

$$SHFYWA(V_1, V_2, ...V_k) \leq SHFYWA(V_1^*, V_2^*, ...V_n^*).$$

Definition 15. *Suppose $V_e = \{A_e, B_e, C_e\} \in SHFS\ (e \in \mathbb{N})$. Then, Yager order weighted averaging aggregation operators for SHFNs are stated as below;*

$$\begin{aligned} SHFYOWG(V_1, V_2, ...V_k) &= (V_{r(1)})^{\delta_1} \oplus (V_{r(2)})^{\delta_2} \oplus ... \oplus (V_{r(1)})^{\delta_1} \\ &= \prod_{e=1}^{k} (V_{r(e)})^{\delta_e} \end{aligned}$$

where $r(e)$ is represented the order according to $(r(1), r(2), r(3), ..., r(k))$ and $(\delta_1, \delta_2, ...\delta_k)^T$ denote the weighted values of $(V_1, V_2, ...V_k)$ such that $\delta_e \geq 0$; $\sum_{e=1}^{k} \delta_e = 1$.

Theorem 11. *Suppose $V_e = \{A_e, B_e, C_e\}$ is a SHFS $(e \in \mathbb{N})$, $(e = 1, 2, ...k)$ and $\delta = (\delta_1, \delta_2, ...\delta_k)^T$ is weight information of $(V_1, V_2, ...V_k)$ such that $\delta_e \geq 0$; $\sum_{e=1}^{k} \delta_e = 1$. Then, SHFYOWG aggregation operator is a mapping $\alpha^k \to \alpha$ such that;*

$$\begin{aligned} SHFYOWG(V_1, V_2, ...V_k) &= \prod_{e=1}^{k} (V_{r(e)})^{\delta_e} \\ &= \bigcup_{a_e \in A_e, b_e \in B_e, c_e \in C_e} \left\{ \begin{array}{c} \sqrt{1 - \min\left(1, \left(\sum_{e=1}^{k} \delta_e (1 - a_{r(e)}^2)^{\mu}\right)\right)^{\frac{1}{\mu}}}, \\ \sqrt{1 - \min\left(1, \left(\sum_{e=1}^{k} \delta_e (1 - b_{r(e)}^2)^{\mu}\right)\right)^{\frac{1}{\mu}}}, \\ \sqrt{\min\left(1, \left(\sum_{e=1}^{k} \delta_e c_{r(e)}^{2\mu}\right)\right)^{\frac{1}{\mu}}} \end{array} \right\} \end{aligned} \quad (3.8)$$

Theorem 12. *(1) Suppose $V_e = \{A_e, B_e, C_e\}$ is a SHFS $(e \in \mathbb{N})$, $(e = 1, 2, ...k)$ such that $V_e = V$. Then*

$$SHFYOWG(V_1, V_2, ...V_k) = V.$$

(2) Suppose $V_e = \{A_e, B_e, C_e\}$ is a SHFS $(e \in \mathbb{N})$, $(e = 1, 2, ...k)$ and $V_e^- = \{\min(A_e),\ \min(B_e),\ \max(C_e)\}$, $V_e^+ = \{\max(A_e),\ \min(B_e),\ \min(C_e)\}$. Then we have,

$$V_e^- \leq SHFYOWG(V_1, V_2, ...V_n) \leq V_e^+.$$

(3) Suppose $V_e = \{A_e, B_e, C_e\}$ & $V_e^* = \{A_e^*, B_e^*, C_e^*\} \in SHFS\ (e \in \mathbb{N})$. If $A_e \geq A_e^*$, $B_e \leq B_e^*$ and $C_e \leq C_e^*$, then

$$SHFYOWG(V_1, V_2, ...V_k) \leq SHFYOWG(V_1^*, V_2^*, ...V_k^*).$$

Definition 16. *Suppose $V_e = \{A_e, B_e, C_e\} \in SHFS\ (e \in \mathbb{N})$. Then, Yager hybrid weighted geometric aggregation operators for SHFNs are stated as below;*

$$SHFYHWG(V_1, V_2, ...V_k) = \left(V'_{r(1)}\right)^{\delta_1} \otimes \left(V'_{r(1)}\right)^{\delta_2} \otimes ... \left(V'_{r(k)}\right)^{\delta_k} \quad (3.9)$$

$$= \prod_{e=1}^{k} \left(V'_{r(e)}\right)^{\delta_e}$$

where $r(e)$ *is represented the order according to* $(r(1), r(2), r(3), ..., r(k))$, *such that* $V'_{r(e)}\left(V'_{r(e)} = k\delta_e V_{r(e)} : e \in \mathbb{N}\right)$ *and* $(\delta_1, \delta_2, ...\delta_k)^t$ *is weight, associated weight information respectively of* $(V_1, V_2, ...V_k)$ *such that* $\delta_e \geq 0; \sum_{e=1}^{k} \delta_e = 1$.

Theorem 13. *Suppose* $V_e = \{A_e, B_e, C_e\}$ *is a SHFS* $(e \in \mathbb{N})$, $(e = 1, 2, ...k)$ *and* $\delta = (\delta_1, \delta_2, ...\delta_k)^T$ *denote the weighted values of* $(V_1, V_2, ...V_k)$ *such that* $\delta_e \geq 0$; $\sum_{e=1}^{k} \delta_e = 1$. *Then, SHFYHWG aggregation operator is a mapping* $\alpha^k \to \alpha$ *such that;*

$$\begin{aligned} SHFYHWG(V_1, V_2, ...V_k) &= \prod_{e=1}^{k} \left(V'_{r(e)}\right)^{\delta_e} \\ &= \bigcup_{a_e \in A_e, b_e \in B_e, c_e \in C_e} \left\{ \begin{array}{c} \sqrt{1 - \min\left(1, \left(\sum_{e=1}^{k} \delta_e (1 - a^{/2}_{r(e)})^{\mu}\right)\right)^{\frac{1}{\mu}}}, \\ \sqrt{1 - \min\left(1, \left(\sum_{e=1}^{k} \delta_e (1 - b^{/2}_{r(e)})^{\mu}\right)\right)^{\frac{1}{\mu}}}, \\ \sqrt{\min\left(1, \left(\sum_{e=1}^{k} \delta_e c^{/2\mu}_{r(e)}\right)\right)^{\frac{1}{\mu}}} \end{array} \right\} \end{aligned} \quad (3.10)$$

Theorem 14. *(1) Suppose* $V_e = \{A_e, B_e, C_e\}$ *is a SHFS* $(e \in \mathbb{N})$, $(e = 1, 2, ...k)$ *such that* $V_e = V$. *Then*

$$SHFYHWG(V_1, V_2, ...V_k) = V.$$

(2) Suppose $V_e = \{A_e, B_e, C_e\}$ *is a SHFS* $(e \in \mathbb{N})$, $(e = 1, 2, ...k)$ *and* $V_e^- = \{\min(A_e), \min(B_e), \max(C_e)\}$, $V_e^+ = \{\max(A_e), \min(B_e), \min(C_e)\}$. *Then we have,*

$$V_e^- \leq SHFYHWG(V_1, V_2, ...V_n) \leq V_e^+.$$

(3) Suppose $V_e = \{A_e, B_e, C_e\}$ & $V_e^* = \{A_e^*, B_e^*, C_e^*\} \in SHFS\ (e \in \mathbb{N})$. *If* $A_e \geq A_e^*$, $B_e \leq B_e^*$ *and* $C_e \leq C_e^*$, *then*

$$SHFYHWG(V_1, V_2, ...V_k) \leq SHFYHWG(V_1^*, V_2^*, ...V_k^*).$$

4. Algorithm for decision making problems

We develop a technique for dealing with uncertainty in decision making problems under the spherical hesitant fuzzy environment. Suppose a DM problem with g possible alternatives $\left\{V_1, V_2, ..., V_g\right\}$ and $\{T_1, T_2, ..., T_h\}$ denote the set of attributes with weighted values $(\delta_1, \delta_2, ...\delta_h)^t$ such that $\delta_h \in [0, 1]$, $\sum_{e=1}^{h} \delta_e = 1$. Let $\left\{\mathring{D}_1, \mathring{D}_2, ..., \mathring{D}_{\hat{j}}\right\}$ be a set of decision makers (DMs) and $(\eta_1, \eta_2, ..., \eta_{\hat{j}})^T$ be DMs weights such that $\eta_s \in [0, 1]$, $\sum_{s=1}^{\hat{j}} \eta_s = 1$. The expert assessment matrix is defined as follows:

$$\begin{bmatrix} (A_{11}(e), B_{11}(e), C_{11}(e)) & (A_{12}(e), B_{12}(e), C_{12}(e)) & \cdots & (A_{1h}(e), B_{1h}(e), C_{1h}(e)) \\ (A_{21}(e), B_{21}(e), C_{21}(e)) & (A_{22}(e), B_{22}(e), C_{22}(e)) & \cdots & (A_{2h}(e), B_{2h}(e), C_{2h}(e)) \\ (A_{31}(e), B_{31}(e), C_{31}(e)) & (A_{32}(e), B_{32}(e), C_{32}(e)) & \cdots & (A_{3h}(e), B_{3h}(e), C_{3h}(e)) \\ \vdots & \vdots & \ddots & \vdots \\ \left(A_{g1}(e), B_{g1}(e), C_{g1}(e)\right) & \left(A_{g2}(e), B_{g2}(e), C_{g2}(e)\right) & \cdots & \left(A_{gh}(e), B_{gh}(e), C_{gh}(e)\right) \end{bmatrix}$$

where $\left(A_{gh}(e), B_{gh}(e), C_{gh}(e)\right) \in [0,1]$, denoted the P_MD, Ne_MD and N_MD subject the condition

$$0 \leq (a^{+})^2 + (b^{+})^2 + (c^{+})^2 \leq 1, \forall e \in \mathbb{N},$$

such that

$$a^{+} = \bigcup_{a \in A(e)} \max\{a\},\ b^{+} = \bigcup_{b \in B(e)} \max\{b\},\ \text{and } c^{+} = \bigcup_{c \in C(e)} \max\{c\}.$$

Step-1 Construct the expert evaluation matrix $(R)^{\hat{j}}$

$$\begin{bmatrix} \left(A^{\hat{j}}_{11}(e), B^{\hat{j}}_{11}(e), C^{\hat{j}}_{11}(e)\right) & \left(A^{\hat{j}}_{12}(e), B^{\hat{j}}_{12}(e), C^{\hat{j}}_{12}(e)\right) & \cdots & \left(A^{\hat{j}}_{1h}(e), B^{\hat{j}}_{1h}(e), C^{\hat{j}}_{1h}(e)\right) \\ \left(A^{\hat{j}}_{21}(e), B^{\hat{j}}_{21}(e), C^{\hat{j}}_{21}(e)\right) & \left(A^{\hat{j}}_{22}(e), B^{\hat{j}}_{22}(e), C^{\hat{j}}_{22}(e)\right) & \cdots & \left(A^{\hat{j}}_{2h}(e), B^{\hat{j}}_{2h}(e), C^{\hat{j}}_{2h}(e)\right) \\ \left(A^{\hat{j}}_{31}(e), B^{\hat{j}}_{31}(e), C^{\hat{j}}_{31}(e)\right) & \left(A^{\hat{j}}_{32}(e), B^{\hat{j}}_{32}(e), C^{\hat{j}}_{32}(e)\right) & \cdots & \left(A^{\hat{j}}_{3h}(e), B^{\hat{j}}_{3h}(e), C^{\hat{j}}_{3h}(e)\right) \\ \vdots & \vdots & \ddots & \vdots \\ \left(A^{\hat{j}}_{g1}(e), B^{\hat{j}}_{g1}(e), C^{\hat{j}}_{g1}(e)\right) & \left(A^{\hat{j}}_{g2}(e), B^{\hat{j}}_{g2}(e), C^{\hat{j}}_{g2}(e)\right) & \cdots & \left(A^{\hat{j}}_{gh}(e), B^{\hat{j}}_{gh}(e), C^{\hat{j}}_{gh}(e)\right) \end{bmatrix}$$

where $\hat{j}$ denotes the number of experts.

Step-2 Make the decision matrix normalized $(L)^{\hat{j}}$. Where

$$(L)^{\hat{j}} = \begin{cases} \left(A^{\hat{j}}_{gh}(q), B^{\hat{j}}_{gh}(q), C^{\hat{j}}_{gh}(q)\right) & if \quad \text{Benefit type criteria} \\ \left(C^{\hat{j}}_{gh}(q), B^{\hat{j}}_{gh}(q), A^{\hat{j}}_{gh}(q)\right) & if \quad \text{Cost type criteria} \end{cases}$$

Step-3 To create the aggregate matrix, combine the individual decision matrices using the spherical hesitant fuzzy aggregation operators;

$$\begin{bmatrix} (A_{11}(e), B_{11}(e), C_{11}(e)) & (A_{12}(e), B_{12}(e), C_{12}(e)) & \cdots & (A_{1h}(e), B_{1h}(e), C_{1h}(e)) \\ (A_{21}(e), B_{21}(qe), C_{21}(e)) & (A_{22}(e), B_{22}(e), C_{22}(e)) & \cdots & (A_{2h}(e), B_{2h}(e), C_{2h}(e)) \\ (A_{31}(e), B_{31}(e), C_{31}(e)) & (A_{32}(e), B_{32}(e), C_{32}(e)) & \cdots & (A_{3h}(e), B_{3h}(e), C_{3h}(e)) \\ \vdots & \vdots & \ddots & \vdots \\ \left(A_{g1}(e), B_{g1}(e), C_{g1}(e)\right) & \left(A_{g2}(e), B_{g2}(e), C_{g2}(e)\right) & \cdots & \left(A_{gh}(e), B_{gh}(e), C_{gh}(e)\right) \end{bmatrix}$$

Step-4 Using the developed spherical hesitant fuzzy Yager aggregation operators, we find out the aggregated spherical hesitant fuzzy decision matrix.

Step-5 Calculate the score of all the aggregated values (according to Definition 7).

Step-6 Rank the alternatives V_e $(e = 1, 2, ...g)$ and choose the best one with the highest score value.

4.1. Illustrative example

In this part, we look at the results of the established MAGDM technique using a mathematical example and compare them to one of the existing MAGDM techniques.

Parkinson's disease (PD) is a multi-system neurodegenerative disease that damages the brain steadily. symptoms include muscle weakness, limb tremor, and poor balance, which all worsen over time. The available therapies are aimed at improving the quality of life by addressing the symptoms.

Apart from medication, people with PD can improve their health and well-being, maintain physical function, reduce symptoms, and improve their quality of life in a variety of ways. Regular exercise, a balanced diet, keeping hydrated, and having enough sleep are among the most important of these. But what are alternative therapies? Integrative treatments, such as yoga, massage, dietary supplements, and multiple movement exercises, have sparked years of study to see whether they can help with PD care. Consider the following four integrative therapies $V = \{V_1, V_2, V_3, V_4\}$.

Tai Chi (V_1):

Since this type of exercise improves balance and coordination, it makes sense that it will be beneficial to PD patients. Tai chi improved balance and flexibility in people with mild Parkinson's disease, according to a 2012 study of three types of exercise: resistance training, stretching, and tai chi.

Yoga (V_2) :

Yoga has been shown to help people with PD improve their flexibility and balance. According to a 2012 report, yoga can improve mobility, balance, strength, and endurance in people with movement disorders like Parkinson's disease if it's tailored to their needs. It can also help you sleep better and improve your mood.

Nutritional Supplements (V_3) :

It's possible you've heard that the antioxidant coenzyme Q10, or Co-Q10, will help with Parkinson's disease. The National Institute of Neurological Disorders and Stroke, on the other hand, halted a review into the efficacy of Co-Q10 in 2011 when it became apparent that the reported preventive effects were no different from a placebo.

Calcium is one supplement that may be beneficial for people with PD, owing to the fact that certain calcium-rich foods (such as dairy products) are often full of protein, which may interfere with medication absorption.

Acupuncture (V_4) :

Acupuncture is a common practice in traditional Chinese medicine, and its basic premise is that simulating points along the body's meridians, or energy channels, will relieve pain and provide other benefits. As a result, it's widely used in China and other countries to treat PD.

On the basis of above four alternatives the health expert will select the best integrative therapy for PD patients. Let the set of four alternatives is $V = \{V_1, V_2, V_3, V_4\}$. Through consulting experts, the following four attributes are taken into account for reliable modelling the properties of alternatives, which are displayed as: $T = \{T_1, T_2, T_3, T_4\}$. The weights of the attributes specified by the professionals are $(0.2, 0.1, 0.3, 0.4)^t$ with $\mu = 2$.

Step-1 The skilled assessment data in the form of SHYFSs contains in Table 1:

Table 1. Expert evaluation information.

	T_1	T_2
V_1	$\{(0.29, 0.36, 0.39), (0.14, 0.12, 0.41,)\}$	$\{(0.31, 0.30, 0.46)\}$
V_2	$\{(0.21, 0.19, 0.35)\}$	$\{(0.47, 0.37, 0.56), (0.23, 0.16, 0.32)\}$
V_3	$\{(0.27, 0.15, 0.59)\}$	$\{(0.30, 0.25, 0.35)\}$
V_4	$\{(0.41, 0.27, 0.52)\}$	$\{(0.47, 0.23, 0.52)\}$
	T_3	T_4
V_1	$\{(0.42, 0.29, 0.65)\}$	$\{(0.30, 0.29, 0.45)\}$
V_2	$\{(0.40, 0.25, 0.45)\}$	$\{(0.46, 0.19, 0.65)\}$
V_3	$\{(0.37, 0.27, 0.47), (0.57, 0.22, 0.74)\}$	$\{(0.39, 0.36, 0.48)\}$
V_4	$\{(0.41, 0.22, 0.49)\}$	$\{(0.38, 0.14, 0.52), (0.16, 0.21, 0.46)\}$

Step-2 Normalized spherical hesitant fuzzy yager decision matrix are calculated as shown in Table 2:

Table 2. Normalized spherical hesitant fuzzy Yager decision matrix.

	T_1	T_2
V_1	$\{(0.39, 0.36, 0.29), (0.41, 0.12, 0.14)\}$	$\{(0.46, 0.30, 0.31)\}$
V_2	$\{(0.35, 0.19, 0.21)\}$	$\{(0.56, 0.37, 0.47)\}, (0.32, 0.16, 0.23)$
V_3	$\{(0.59, 0.15, 0.27)\}$	$\{(0.35, 0.25, 0.30)\}$
V_4	$\{(0.52, 0.27, 0.41)\}$	$\{(0.52, 0.23, 0.47)\}$
	T_3	T_4
V_1	$\{(0.42, 0.29, 0.65)\}$	$\{(0.45, 0.29, 0.30)\}$
V_2	$\{(0.40, 0.25, 0.45)\}$	$\{(0.65, 0.19, 0.46)\}$
V_3	$\{(0.37, 0.27, 0.47), (0.57, 0.22, 0.74)\}$	$\{(0.48, 0.36, 0.39)\}$
V_4	$\{(0.41, 0.22, 0.49)\}$	$\{(0.52, 0.14, 0.38), (0.46, 0.21, 0.16)\}$

Step-3(a) In this step, we use the suggested list of SHF Yager AOs to measure the aggregate overall preference values of every option when the criteria weight value is $(0.2,0.1,0.3,0.4)^{\mathrm{T}}$. $\mu = 2$

Case-1: Using $SHFYWA$ aggregation operator

$$SHFYWA(V_1, V_2, ...V_k) = \bigcup_{a_e \in A_e, b_e \in B_e, c_e \in C_e} \left\{ \begin{array}{c} \sqrt{\min\left(1, \left(\sum_{e=1}^{k} \delta_e a_e^{2\mu}\right)\right)^{\frac{1}{\mu}}}, \\ \sqrt{1 - \min\left(1, \left(\sum_{e=1}^{k} \delta_e (1 - b_e^2)^{\mu}\right)\right)^{\frac{1}{\mu}}}, \\ \sqrt{1 - \min\left(1, \left(\sum_{e=1}^{k} \delta_e (1 - c_e^2)^{\mu}\right)\right)^{\frac{1}{\mu}}} \end{array} \right\}$$

Using the SHFYWA aggregation operator, the total preference values of each option are enclosed in Table 3:

Table 3. Total preference value $(SHFYWA)$.

V_1	$\{(0.5302, 0.3059, 0.2912), (0.5320, 0.2651, 0.2676)\}$
V_2	$\{(0.5574, 0.2315, 0.4014), (0.5443, 0.2071, 0.3795)\}$
V_3	$\{(0.4982, 0.2891, 0.3534), (0.4982, 0.2759, 0.4195)\}$
V_4	$\{(0.5116, 0.2049, 0.4045), (0.4889, 0.2280, 0.3365)\}$

$$S_c(V) = \bigcup_{a \in A, b \in B, c \in C} \left\{ \frac{1}{l(A)} \sum a_V - \frac{1}{l(B)} \sum b_V - \frac{1}{l(C)} \sum c_V \right\}$$

Step-4(a) Now, every option's score of combined total preference values is contained in Table 4:

Table 4. Score values.

Operators	$Sc(V_1)$	$Sc(V_2)$	$Sc(V_3)$	$Sc(V_4)$
$SHFYWA$	−0.0338	−0.0589	−0.1707	−0.0867

Step-5(a) Rank the alternatives $V_q(q = 1, 2, 3, 4)$ is enclosed in Table 5:

Table 5. Ranking of the alternatives.

Operators	Score	Best Alternative
$SHFYWA$	$Sc(V_1) > Sc(V_2) > Sc(V_4) > Sc(V_3)$	V_1

Case-2: For $SHFYOWA$ aggregation operator

Using the $SHFYOWA$ aggregation operator, the total preference values of each option are enclosed in Table 6:

Table 6. Total preference value $(SHFYOWA)$.

V_1	$\{(0.4791, 0.3089, 0.3156), (0.4814, 0.2683, 0.3297)\}$
V_2	$\{(0.5369, 0.2757, 0.4188), (0.4881, 0.2083, 0.3524)\}$
V_3	$\{(0.4839, 0.2855, 0.3399), (0.5318, 0.2811, 0.3633)\}$
V_4	$\{(0.5172, 0.2272, 0.4288), (0.5067, 0.2381, 0.3971)\}$

Step-4(b) Now, every option's score of combined total preference values is contained in Table 7:

Table 7. Score values.

Operators	$Sc(V_1)$	$Sc(V_2)$	$Sc(V_3)$	$Sc(V_4)$
$SHFYOWA$	0.1987	−0.1151	−0.1245	−0.1336

Step-5(b) Rank the alternatives $V_q(q = 1, 2, 3, 4)$ is enclosed in Table 8:

Table 8. Ranking of the alternatives.

Operators	Score	Best Alternative
$SHFYOWA$	$Sc(V_1) > Sc(V_2) > Sc(V_3) > Sc(V_4)$	V_1

Case-3: For $SHFYHWA$ aggregation operator

Using the $SHFYHWA$ aggregation operator, the total preference values of each option are enclosed in Table 9:

Table 9. Total preference value $(SHFYHWA)$.

V_1	$\{0.5111, 0.3015, 0.3294\}, \{0.5120, 0.2814, 0.3189\}$
V_2	$\{0.5459, 0.2886, 0.4258\}, \{0.4814, 0.1992, 0.3366\}$
V_3	$\{0.4753, 0.2737, 0.3306\}, \{0.5253, 0.2691, 0.3545\}$
V_4	$\{0.5116, 0.2285, 0.4317\}, \{0.5062, 0.2337, 0.4161\}$

Step-4(c) Now, every option's score of combined total preference values is contained in Table 10:

Table 10. Score values.

Operators	$Sc(V_1)$	$Sc(V_2)$	$Sc(V_3)$	$Sc(V_4)$
$SHFYHWA$	−0.1040	−0.1114	−0.1136	−0.1461

Step-5(c) Rank the alternatives $V_q (q = 1, 2, 3, 4)$ is enclosed in Table 11:

Table 11. Ranking of the alternatives.

Operators	Score	Best Alternative
$SHFYHWA$	$Sc(V_1) > Sc(V_2) > Sc(V_3) > Sc(V_4)$	V_1

5. Comparative analysis

In this section, we compared the suggested Yager aggregation operators-based decision making approach to the current system of SHF aggregation operators based on sine trigonometric. For this, we apply the SHF intelligence system. Naeem et al. [30] in Table 12. The attribute weight information is $(0.2, 0.4, 0.1, 0.3)^T$.

Step-1 The expert evaluation information [30] in the form of SHFSs is enclosed in Table 12:

Table 12. Expert evaluation data.

	V_1	V_2	V_3	V_4
G_1	$\{(0.3, 0.2, 0.4)\}$	$\{(0.2, 0.6, 0.5)\}$	$\{(0.1, 0.5, 0.3)\}$	$\left\{\begin{array}{c}(0.1, 0.5, 0.6), \\ (0.3, 0.4, 0.5)\end{array}\right\}$
G_2	$\{(0.1, 0.5, 0.2)\}$	$\{(0.2, 0.3, 0.4)\}$	$\left\{\begin{array}{c}(0.1, 0.1, 0.6), \\ (0.3, 0.1, 0.4)\end{array}\right\}$	$\{(0.1, 0.4, 0.2)\}$
G_3	$\{(0.4, 0.1, 0.5)\}$	$\left\{\begin{array}{c}(0.1, 0.1, 0.6), \\ (0.3, 0.2, 0.4)\end{array}\right\}$	$\{(0.4, 0.2, 0.5)\}$	$\{(0.4, 0.2, 0.5)\}$
G_4	$\{(0.2, 0.2, 0.3)\}$	$\{(0.1, 0.2, 0.3)\}$	$\{(0.2, 0.4, 0.3), (0.4, 0.4, 0.6)\}$	$\{(0.2, 0.4, 0.3)\}$

Step-2 The enclosed expert assessment data has been normalized in Table 13:

Table 13. Normalized expert evaluation information.

	V_1	V_2	V_3	V_4
G_1	$\{(0.4, 0.2, 0.3)\}$	$\{(0.5, 0.6, 0.2)\}$	$\{(0.3, 0.5, 0.1)\}$	$\left\{\begin{array}{c}(0.6, 0.5, 0.1), \\ (0.5, 0.4, 0.3)\end{array}\right\}$
G_2	$\{(0.2, 0.5, 0.1)\}$	$\{(0.4, 0.3, 0.2)\}$	$\left\{\begin{array}{c}(0.6, 0.1, 0.1), \\ (0.4, 0.1, 0.3)\end{array}\right\}$	$\{(0.2, 0.4, 0.1)\}$
G_3	$\{(0.5, 0.1, 0.4)\}$	$\left\{\begin{array}{c}(0.6, 0.1, 0.1), \\ (0.4, 0.2, 0.3)\end{array}\right\}$	$\{(0.5, 0.2, 0.4)\}$	$\{(0.5, 0.2, 0.4)\}$
G_4	$\{(0.3, 0.2, 0.2)\}$	$\{(0.3, 0.2, 0.1)\}$	$\{(0.3, 0.4, 0.2), (0.6, 0.4, 0.4)\}$	$\{(0.3, 0.4, 0.2)\}$

Step-3 This task makes use of the SHF information form.

Step-4 The weight values of the experts are given in this case study are $(0.2, 0.4, 0.1, 0.3)^T$.

Step-5 Now, using SHFYWA aggregation operators, under weight knowledge, we calculate the aggregated values of each alternative as follows:

$$SHFYWA(V_1, V_2, ...V_k) = \bigcup_{a_e \in A_e, b_e \in B_e, c_e \in C_e} \left\{ \begin{array}{c} \sqrt{\min\left(1, \left(\sum_{e=1}^{k} \delta_e a_e^{2\mu}\right)\right)^{\frac{1}{\mu}}}, \\ \sqrt{1 - \min\left(1, \left(\sum_{e=1}^{k} \delta_e (1 - b_e^2)^{\mu}\right)\right)^{\frac{1}{\mu}}}, \\ \sqrt{1 - \min\left(1, \left(\sum_{e=1}^{k} \delta_e (1 - c_e^2)^{\mu}\right)\right)^{\frac{1}{\mu}}} \end{array} \right\}$$

collective overall preference values of each alternative using $SHFYWA$ aggregation operator is enclosed in Table 14:

Table 14. Total preference value $(SHFYWA)$.

V_1	$\{0.5140, 0.4928, 0.1939\}, \{0.4721, 0.4638, 0.2482\}$
V_2	$\{0.3936, 0.3633, 0.1480\}, \{0.3415, 0.3633, 0.1723\}$
V_3	$\{0.5467, 0.1480, 0.3114\}, \{0.4674, 0.1841, 0.3624\}$
V_4	$\{0.3000, 0.2934, 0.1670\}, \{0.3772, 0.2934, 0.1977\}$

Step-6 Ranking result is enclosed in Table 15:

Table 15. Score values.

Operators	$Sc(V_1)$	$Sc(V_2)$	$Sc(V_3)$	$Sc(V_4)$	Ranking of the alternatives
$ST - SHFWA$ [30]	0.4505	0.2463	0.4983	0.2332	$Sc(V_3) > Sc(V_1) > Sc(V_2) > Sc(V_4)$
$SHFYWA$	−0.2063	−0.1559	0.0041	−0.1371	$Sc(V_3) > Sc(V_4) > Sc(V_2) > Sc(V_1)$

5.1. Discussion

We compared proven Yager function-based aggregation operators to the known sine trigonometric SHF aggregation operators presented in [30], demonstrating their ability to manage uncertainty in real-world DMPs. Because of its generalized form, this approach covers the valuation spaces of PyHFSs, PFSs, and SFSs, which is an amazing feature. The findings in Table 15 show that the suggested decision-making strategy is accurate and effective in addressing ambiguity in decision-making problems. Because of its generalized form, this approach covers the valuation spaces of PyHFSs, PFSs, and SFSs, which is an amazing feature. The findings in Table 15 show that the suggested decision-making strategy is accurate and effective in addressing ambiguity in decision-making problems. We may apply our approach in a variety of situations; in this case, we're using it to find the finest hotel. The suggested DM technique is simple and basic, and it may be applied to a variety of outcomes with ease.

6. Conclusions

This research presents a robust decision-supporting model that incorporates social data to aid Emergency Response Systems. Unlike traditional decision support modelling techniques, the established technique completely implements social data, such as online reviews and social interactions, and it considers the interconnection among parameters by using innovative SHFYAOs. Furthermore, the proposed research introduced a list of new operation laws generated by utilizing Yager function to create a list of SHFYAOs to address ambiguity in real-world DMPs. To solve multi-attribute DMPs, a specialized DM algorithm is created. The suggested SHFYAOs are used in this research to look at a case study of an Emergency Response System. In the comparison analysis, the proposed Emergency Program Selection strategy was found to be more suitable and effective than the comparative methods. We used the Yager t-norms and conorms in this paper and identified six aggregation operators for SHFSs, including SHFYWA, SHFYOWA, SHFYHWA, SHFYWG, SHFYOWG, and SHFYHWG.

We have spoken about how to solve realistic MADM problems in the Emergency Response System based on various attributes. To compile the relevant data for each alternative, we used the SHFYWAA, SHFYOWA, and SHFYHWA operators in this problem. The related results were then obtained using score functions. We were able to compare the value of each alternative according to such operators. We discovered that using the operators SHFYWAA, SHFYOWA, and SHFYHWA yielded the same results. As a result, our existing models for dealing with spherical hesitant fuzzy MCGDM problems are more common and versatile than other current methods.

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Conflict of interest

The authors declare that they have no conflicts of interest.

References

1. S. Ashraf, S. Abdullah, T. Mahmood, GRA method based on spherical linguistic fuzzy Choquet integral environment and its application in multi-attribute decision-making problems, *Math. Sci.*, **12** (2018), 263–275. doi: 10.1007/s40096-018-0266-0.

2. S. Ashraf, S. Abdullah, Spherical aggregation operators and their application in multiattribute group decision-making, *Int. J. Intell. Syst.*, **34** (2019), 493–523. doi: 10.1002/int.22062.

3. S. Ashraf, S. Abdullah, T. Mahmood, F. Ghani, T. Mahmood, Spherical fuzzy sets and their applications in multi-attribute decision making problems, *J. Intell. Fuzzy Syst.*, **36** (2019), 2829–2844. doi: 10.3233/JIFS-172009.

4. S. Ashraf, S. Abdullah, M. Aslam, Symmetric sum based aggregation operators for spherical fuzzy information: Application in multi-attribute group decision making problem, *J. Intell. Fuzzy Syst.*, **38** (2020), 5241–5255. doi: 10.3233/JIFS-191819.

5. S. Ashraf, S. Abdullah, Emergency decision support modeling for COVID-19 based on spherical fuzzy information, *Int. J. Intell. Syst.*, **35** (2020), 1601–1645. doi: 10.1002/int.22262.

6. S. Ashraf, T. Mahmood, S. Abdullah, Q. Khan, Different approaches to multi-criteria group decision making problems for picture fuzzy environment, *Bull. Braz. Math. Soc. New Series*, **50** (2019), 373–397. doi: 10.1007/s00574-018-0103-y.

7. S. Ashraf, S. Abdullah, A. O. Almagrabi, A new emergency response of spherical intelligent fuzzy decision process to diagnose of COVID19, *Soft Comput.*, 2020, in press. doi: 10.1007/s00500-020-05287-8.

8. S. Ashraf, S. Abdullah, T. Mahmood, Spherical fuzzy Dombi aggregation operators and their application in group decision making problems, *J. Ambient Intell. Humaniz Comput.*, **11** (2020), 2731–2749. doi: 10.1007/s12652-019-01333-y.

9. S. Ashraf, S. Abdullah, L. Abdullah, Child development influence environmental factors determined using spherical fuzzy distance measures, *Mathematics*, **7** (2019), 661. doi: 10.3390/math7080661.

10. S. Ashraf, S. Abdullah, S. Aslam, M. Qiyas, M. A. Kutbi, Spherical fuzzy sets and its representation of spherical fuzzy t-norms and t-conorms, *J. Intell. Fuzzy Syst.*, **36** (2019), 6089–6102. doi: 10.3233/JIFS-181941.

11. K. Atanassov, Intuitionistic fuzzy sets, *Fuzzy Sets Syst.*, **20** (1986), 87–96. doi: 10.1016/S0165-0114(86)80034-3.

12. D. Bega, C. Zadikoff, Complementary & alternative management of Parkinson's disease: an evidence-based review of eastern influenced practices, *J. Mov. Disord.*, **7** (2014), 57–66. doi: 10.14802/jmd.14009.

13. O. Barukab, S. Abdullah, S. Ashraf, M. Arif, S. A. Khan, A new approach to fuzzy TOPSIS method based on entropy measure under spherical fuzzy information, *Entropy*, **21** (2019), 1231. doi: 10.3390/e21121231.

14. T. Y. Chen, The inclusion-based TOPSIS method with interval-valued intuitionistic fuzzy sets for multiple criteria group decision making, *Appl. Soft Comput.*, **26** (2015), 57–73. doi: 10.1016/j.asoc.2014.09.015.

15. N. Chen, Z. Xu, M. Xia, Interval-valued hesitant preference relations and their applications to group decision making, *Knowl. Based Syst.*, **37** (2013), 528–540. doi: 10.1016/j.knosys.2012.09.009.

16. N. Chen, Z. Xu, M. Xia, Correlation coefficients of hesitant fuzzy sets and their applications to clustering analysis, *Appl. Math. Model.*, **37** (2013), 2197–2211. doi:10.1016/j.apm.2012.04.031.

17. G. Deschrijver, C. Cornelis, E. Kerre, On the representation of intuitionistic fuzzy t-norms and t-conorms, *IEEE Trans. Fuzzy Syst.*, **12** (2004), 45–61. doi: 10.1109/TFUZZ.2003.822678.

18. J. X. Deng, Y. Deng, Information volume of fuzzy membership function, *Int. J. Comput. Commun. Control*, **16** (2021), jan. 2021. doi: 10.15837/ijccc.2021.1.4106.

19. B. Farhadinia, Information measures for hesitant fuzzy sets and interval-valued hesitant fuzzy sets, *Inf. Sci.*, **240** (2013), 129–144. doi: 10.1016/j.ins.2013.03.034.

20. X. Guan, G. Sun, X. Yi, J. Zhao, Grey relational analysis for hesitant fuzzy sets and its applications to multi attribute decision-making, *Math. Probl. Eng.*, **2018** (2018), 7436054. doi: 10.1155/2018/7436054.

21. Y. Jin, S. Ashraf, S. Abdullah, Spherical fuzzy logarithmic aggregation operators based on entropy and their application in decision support systems, *Entropy*, **21** (2019), 628. doi: 10.3390/e21070628.

22. Y. Jin, S. Ashraf, S. Abdullah, M. Qiyas, M. Bano, S. Zeng, Linguistic spherical fuzzy aggregation operators and their applications in multi-attribute decision making problems, *Mathematics*, **7** (2019), 413. doi: 10.3390/math7050413.

23. A. Khan, S. S. Abosuliman, S. Abdullah, M. Ayaz, A decision support model for hotel recommendation based on the online consumer reviews using logarithmic spherical hesitant fuzzy information, *Entropy*, **23** (2021), 432. doi: 10.3390/e23040432.

24. A. Khan, S. S. Abosuliman, S. Ashraf, S. Abdullah, Hospital admission and care of COVID-19 patients problem based on spherical hesitant fuzzy decision support system, *Int. J. Intell. Syst.*, **36** (2021), 4167–4209. doi: 10.1002/int.22455.

25. G. Kou, O. O. Akdeniz, H. Dinçer, S. Yüksel, Fintech investments in European banks: a hybrid IT2 fuzzy multidimensional decision-making approach, *Financ. Innov.*, **7** (2021), 39. doi: 10.1186/s40854-021-00256-y.

26. D. Liang, Z. Xu, The new extension of TOPSIS method for multiple criteria decision making with hesitant Pythagorean fuzzy sets, *Appl. Soft Comput.*, **60** (2017), 167–179. doi: 10.1016/j.asoc.2017.06.034.

27. G. Li, G. Kou, Y. Peng, Heterogeneous large-scale group decision making using fuzzy cluster analysis and its application to emergency response plan selection, *IEEE Trans. Syst. Man Cybern.*, 2021, 1–13. doi: 10.1109/TSMC.2021.3068759.

28. T. Mahmood, A novel approach towards bipolar soft sets and their applications, *J. Math.*, **2020** (2020), 4690808. doi: 10.1155/2020/4690808.

29. T. Mahmood, U. Ur Rehman, Z. Ali, R. Chinram, Jaccard and Dice similarity measures based on novel complex dual hesitant fuzzy sets and their applications, *Math. Probl. Eng.*, **2020** (2020), 5920432. doi: 10.1155/2020/5920432.

30. M. Naeem, A. Khan, S. Abdullah, S. Ashraf, A. A. A. Khammash, Solid waste collection system selection based on sine trigonometric spherical hesitant fuzzy aggregation information, *Intell. Autom. Soft Comput.*, **28** (2021), 459–476. doi: 10.32604/iasc.2021.016822.

31. A. Naseer, M. Rani, S. Naz, M. I. Razzak, M. Imran, G. Xu, Refining Parkinson's neurological disorder identification through deep transfer learning, *Neural Comput. Applic.*, **32** (2020), 839–854. doi: 10.1007/s00521-019-04069-0.

32. X. Peng, Y. Yang, Pythagorean fuzzy Choquet integral based MABAC method for multiple attribute group decision making, *Int. J. Intell. Syst.*, **31** (2016), 989–1020. doi: 10.1002/int.21814.

33. A. Pinar, F. E. Boran, A q-rung orthopair fuzzy multi-criteria group decision making method for supplier selection based on a novel distance measure, *Int. J. Mach. Learn. Cyber.*, **11** (2020), 1749–1780. doi: 10.1007/s13042-020-01070-1.

34. X. Peng, H. Yuan, Y. Yang, Pythagorean fuzzy information measures and their applications, *Int. J. Intell. Syst.*, **32** (2017), 991–1029. doi: 10.1002/int.21880.

35. G. Qian, H. Wang, X. Feng, Generalized hesitant fuzzy sets and their application in decision support system, *Knowl. Based Syst.*, **37** (2013), 357–365. doi: 10.1016/j.knosys.2012.08.019.
36. M. Rafiq, S. Ashraf, S. Abdullah, T. Mahmood, S. Muhammad, The cosine similarity measures of spherical fuzzy sets and their applications in decision making, *J. Intell. Fuzzy Syst.*, **36** (2019), 6059–6073. doi: 10.3233/JIFS-181922.
37. M. Riaz, A. Razzaq, H. Kalsoom, D. Pamučar, H. M. Athar Farid, Y. M. Chu, q-Rung orthopair fuzzy geometric aggregation operators based on generalized and group-generalized parameters with application to water loss management, *Symmetry*, **12** (2020), 1236. doi: 10.3390/sym12081236.
38. P. Ren, Z. Xu, X. Gou, Pythagorean fuzzy TODIM approach to multi-criteria decision making, *Appl. Soft Comput.*, **42** (2016), 246–259. doi: 10.1016/j.asoc.2015.12.020.
39. V. Torra, Hesitant fuzzy sets, *Int. J. Intell. Syst.*, **25** (2010), 529–539. doi: 10.1002/int.20418.
40. Z. Xu, Intuitionistic fuzzy aggregation operators, *IEEE Trans Fuzzy Syst.*, **15** (2007), 1179–1187. doi: 10.1109/TFUZZ.2006.890678.
41. Z. Xu, R. R. Yager, Some geometric aggregation operators based on intuitionistic fuzzy sets, *Int. J. Gen. Syst.*, **35** (2006), 417–433. doi: 10.1080/03081070600574353.
42. Z. Xu, M. Xia, Distance and similarity measures for hesitant fuzzy sets, *Inf. Sci.*, **181** (2011), 2128–2138. doi: 10.1016/j.ins.2011.01.028.
43. L. Xu, Y. Liu, H. Liu, Some improved q-rung orthopair fuzzy aggregation operators and their applications to multiattribute group decision-making, *Math. Probl. Eng.* **2019** (2019), 2036728. doi: 10.1155/2019/2036728.
44. L. Xu, Y. Liu, H. Liu, On the conjunction of possibility measures under intuitionistic evidence sets, *J. Ambient Intell. Humaniz. Comput.*, **12** (2021), 7827–7836. doi: 10.1007/s12652-020-02508-8.
45. Y. Xue, Y. Deng, Decision making under measure-based granular uncertainty with intuitionistic fuzzy sets, *App. Intell.*, **12** (2021), 1–10. doi: 10.1007/s10489-021-02216-6.
46. G. Wei, X. Zhao, R. Lin, Some induced aggregating operators with fuzzy number intuitionistic fuzzy information and their applications to group decision making, *Int. J. Comput.*, **3** (2010), 84–95. doi: 10.1080/18756891.2010.9727679.
47. R. R. Yager, Pythagorean fuzzy subsets, In: *2013 Joint IFSA World Congress and NAFIPS Annual Meeting (IFSA/NAFIPS)*, 2013, 57–61. doi: 10.1109/IFSA-NAFIPS.2013.6608375.
48. R. R. Yager, Pythagorean membership grades in multicriteria decision making, *IEEE Trans Fuzzy Syst.*, **22** (2014), 958–965. doi: 10.1109/TFUZZ.2013.2278989.
49. R. R. Yager, Generalized orthopair fuzzy sets, *IEEE Trans Fuzzy Syst.*, **25** (2017), 1222–1230. doi: 10.1109/TFUZZ.2016.2604005.
50. D. Yu, G. Kou, Z. Xu, S. Shi, Analysis of collaboration evolution in AHP research: 1982–2018, *Int. J. Inf. Technol. Decis. Mak.*, **20** (2021), 7–36. doi: 10.1142/S0219622020500406.
51. L. A. Zadeh, Fuzzy sets, *Inf. Cont.*, **8** (1965), 338–353. doi: 10.1016/S0019-9958(65)90241-X.
52. S. Zeng, A. Hussain, T. Mahmood, M. Irfan Ali, S. Ashraf, M. Munir, Covering-based spherical fuzzy rough set model hybrid with TOPSIS for multi-attribute decision-making, *Symmetry*, **11** (2019), 547. doi: 10.3390/sym11040547.

53. X. Zhang, Z. Xu, Extension of TOPSIS to multiple criteria decision making with Pythagorean fuzzy sets, *Int. J. Intell. Syst.*, **29** (2014), 1061–1078. doi: 10.1002/int.21676.

54. X. Zhang, A novel approach based on similarity measure for Pythagorean fuzzy multiple criteria group decision making, *Int. J. Intell. Syst.*, **31** (2016), 593–611. doi: 10.1002/int.21796.

55. X. Zhang, Multicriteria Pythagorean fuzzy decision analysis: A hierarchical QUALIFLEX approach with the closeness index-based ranking methods, *Inf. Sci.*, **330** (2016), 104–124. doi: 10.1016/j.ins.2015.10.012.

56. R. Zhang, J. Wang, X. Zhu, M. Xia, M. Yu, Some generalized Pythagorean fuzzy Bonferroni mean aggregation operators with their application to multiattribute group decision-making, *Complexity,* **2017** (2017), 5937376. doi: 10.1155/2017/5937376.

57. J. Zhang, G. Kou, Y. Peng, Y. Zhang, Estimating priorities from relative deviations in pairwise comparison matrices, *Inf. Sci.*, **552** (2021), 310–327. doi: 10.1016/j.ins.2020.12.008.