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Research article

A general form for precise asymptotics for complete convergence under sublinear expectation

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Abstract: Let $\{X_n, n \geq 1\}$ be a sequence of independent and identically distributed random variables in a sublinear expectation $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$ with a capacity $\mathbb{V}$ under $\widehat{\mathbb{E}}$. In this paper, under some suitable conditions, I show that a general form of precise asymptotics for complete convergence holds under sublinear expectation. It can describe the relations among the boundary function, weighted function, convergence rate and limit value in studies of complete convergence. The results extend some precise asymptotics for complete convergence theorems from the traditional probability space to the sublinear expectation space. The results also generalize the known results obtained by Xu and Cheng [34].

Keywords: complete convergence; precise asymptotics; sublinear expectation
Mathematics Subject Classification: 60F15, 60F05

1. Introduction

Recently, limit theorems for sublinear expectations have raised a large number of issues of interest, because that the sublinear expectation space has advantages of modelling the uncertainty of probability and distribution. Classical limit theorems only hold in the case of model certainty. However, in practice, such model certainty assumption is not realistic in many areas of applications because the uncertainty phenomena cannot be modeled using model certainty. Motivated by modelling uncertainty in practice, Peng [1] introduced a new notion of sublinear expectation. As an alternative to the traditional probability/expectation, capacity/sublinear expectation has been studied in many fields such as statistics, finance, economics, and measures of risk (see Denis and Martini [2], Gilboa [3], Marinacci [4], Peng [5] etc.). Peng [1, 6, 7] introduced the reasonable framework of the sublinear expectation of random variables in a general function space by relaxing the linear property of the classical linear expectation to the subadditivity and positive homogeneity. And sublinear expectation is a natural extension of the classical linear expectation. Later on, more and more limit theorems under sublinear expectation space have been established, which generalize the corresponding fundamental, important limit theorems in probability and statistics. Zhang [8–11] proved the central limit theorem and Donskers invariance principle, the exponential inequalities , Rosenthals inequalities and Lindeberg's central limit theorems for martingale like sequences under sublinear expectation. Chen [12] proved strong laws of large numbers for sublinear expectation. Wu and Jiang [13] obtained a strong law of large numbers and Chovers law of the iterated logarithm under sublinear expectation. Xu and Zhang [14, 15] studied three series theorem and the law of logarithm for arrays of random variables under sublinear expectation. Song [16] obtained normal approximation by Stein's method under sublinear expectation. Liu and Zhang [17, 18] established the central limit theorem and the law of iterated logarithm for linear processes generated by independent identically distributed random variables under sublinear expectation. For more results about limit theorems under sublinear expectation, the interested reader could refer to the studies of Chen et al. [19], Wu et al. [20], Feng [21], Fang et al. [22], Zhang [23], Kuczmaszewska [24], Feng et al. [25], Guo and Li [26], and references therein.

Let $\{X, X_n, n \geq 1\}$ be a sequence of identically distributed random variables with $EX = 0$ and $EX^2 < \infty$ in a traditional probability space $(\Omega, \mathcal{F}, P)$ and define the partial sums $S_n = \sum_{i=1}^{n} X_i$ for $n \geq 1$. Hsu and Robbins [27] introduced the concept of complete convergence, since then there have been extensions in several directions. One of them is to discuss the precise rate and limit value of $\sum\limits_{n=1}^{\infty} \varphi(n) P\{|S_n| \geq \varepsilon g(n)\}$ as $\varepsilon \downarrow a$, $a \geq 0$, where $\varphi(x)$ and $g(x)$ are the positive functions defined on $[0, \infty)$. We call $\varphi(x)$ and $g(x)$ weighted function and boundary function. A first result in this direction was Heyde [28], who proved that

$$\lim_{\varepsilon \downarrow 0} \varepsilon^2 \sum_{n=1}^{\infty} P\{|S_n| \geq \varepsilon n\} = EX^2, \tag{1.1}$$

where $EX = 0$ and $EX^2 < \infty$. For analogous results in more general case, see Spătaru [29], Gut and Spătaru [30, 31]. The research in this field are called the precise asymptotics. Recently, some results on precise asymptotics under sublinear expectation have been obtained. Wu [32] established precise asymptotics for complete integral convergence under sublinear expectation. Zhang [33] established the Heyde's theorem under the sublinear expectation. Xu and Cheng [34] obtained the precise asymptotics in the law of the iterated logarithm under sublinear expectations.

The purpose of this paper is to establish the general form of precise asymptotics for complete convergence under sublinear expectation. The paper is organized as follows: In Section 2, some basic concepts and related lemmas under sublinear expectation which are used in this paper are given. In Section 3, the main result of this paper is sated. The proofs of main results are presented in Sections 4 and 5. The conclusion part is listed in Section 6.

Throughout the paper, C denotes a positive constant, which may take different values whenever it appears in different expressions, $a_n \sim b_n$ stands for $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = 1$, $[x]$ denotes the integer part of x, $\log x = \ln\{\max\{e, x\}\}$, $\log\log x = \ln\ln\{\max\{e^e, x\}\}$.

2. Preliminaries

Let us recall some notations on sublinear expectation space. More detailed information are referred to Peng [1, 6, 7]. Let $(\Omega, \mathcal{F})$ be a given measurable space. Let $\mathcal{H}$ be a linear space of real functions defined on $(\Omega, \mathcal{F})$ such that if $X_1, X_2, ..., X_n \in \mathcal{H}$ then $\varphi(X_1, X_2, ..., X_n) \in \mathcal{H}$ for each $\varphi \in C_{l,Lip}(\mathbb{R}^n)$ where $\varphi \in C_{l,Lip}(\mathbb{R}^n)$ denotes the linear space of local Lipschitz continuous functions φ satisfying

$$|\varphi(x) - \varphi(y)| \leq c(1 + |x|^m + |y|^m)|x - y|, \quad \forall x, y \in \mathbb{R}^n,$$

for some $c > 0$, $m \in \mathbb{N}$ depending on φ. $\mathcal{H}$ contains all I_A where $A \in \mathcal{F}$. I also denote $\varphi \in C_{b,Lip}(\mathbb{R}^n)$ as the linear space of bounded Lipschitz continuous functions φ satisfying

$$|\varphi(x) - \varphi(y)| \leq c|x - y|, \quad \forall x, y \in \mathbb{R}^n,$$

for some $c > 0$.

Definition 2.1. *A function $\widehat{\mathbb{E}} : \mathcal{H} \to [-\infty, +\infty]$ is said to be a sublinear expectation if it satisfies for $\forall X, Y \in \mathcal{H}$,*
(1) Monotonicity: $X \geq Y$ implies $\widehat{\mathbb{E}}[X] \geq \widehat{\mathbb{E}}[Y]$.
(2) Constant preserving: $\widehat{\mathbb{E}}[c] = c$, $\forall c \in \mathbb{R}$.
(3) Subadditivity: $\widehat{\mathbb{E}}[X + Y] \leq \widehat{\mathbb{E}}[X] + \widehat{\mathbb{E}}[Y]$.
(4) Positive homogeneity: $\widehat{\mathbb{E}}[\lambda X] = \lambda\widehat{\mathbb{E}}[X]$, $\forall \lambda \geq 0$.

The triple $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$ is called a sublinear expectation space. Give a sublinear expectation $\widehat{\mathbb{E}}$, let us denote the conjugate expectation $\widehat{\mathcal{E}}$ of $\widehat{\mathbb{E}}$ by $\widehat{\mathcal{E}}[X] := -\widehat{\mathbb{E}}[-X]$, $\forall X \in \mathcal{H}$.

Remark 2.2. *(i) The sublinear expectation $\widehat{\mathbb{E}}[\cdot]$ satisfies translation invariance: $\widehat{\mathbb{E}}[X + c] = \widehat{\mathbb{E}}[X] + c$, $\forall c \in \mathbb{R}$.*
(ii) From the definition, it is easily shown that $\widehat{\mathcal{E}}[X] \leq \widehat{\mathbb{E}}[X]$ and $\widehat{\mathbb{E}}[X - Y] \geq \widehat{\mathbb{E}}[X] - \widehat{\mathbb{E}}[Y]$, $\forall X, Y \in \mathcal{H}$ with $\widehat{\mathbb{E}}[Y]$ being finite.

Next, I introduce the capacities corresponding to the sublinear expectation.

Definition 2.3. *A set function V: $\mathcal{F} \to [0, 1]$ is called a capacity, if*
(1) $V(\emptyset) = 0$, $V(\Omega) = 1$.
(2) $V(A) \leq V(B)$, $\forall A \subset B$, A, $B \subset \mathcal{F}$.
It is called to be subadditive if $V(A \cup B) \leq V(A) + V(B)$ for all A, $B \in \mathcal{F}$ with $A \cup B \in \mathcal{F}$.
A sublinear expectation $\widehat{\mathbb{E}}$ could generate a pair $(\mathbb{V}, \mathcal{V})$ of capacity denoted by
$\mathbb{V}(A) := \inf\{\widehat{\mathbb{E}}[\xi] : I_A \leq \xi, \xi \in \mathcal{H}\}$, $\mathcal{V}(A) = 1 - \mathbb{V}(A^c)$, $\forall A \in \mathcal{F}$,
where A^c is the complement set of A. Then

$$\mathbb{V}(A) := \widehat{\mathbb{E}}[I_A],$$
$$\mathcal{V}(A) := \widehat{\mathcal{E}}[I_A], \quad if I_A \in \mathcal{H},$$

$$\widehat{\mathbb{E}}[f] \leq \mathbb{V}(A) \leq \widehat{\mathbb{E}}[g], \quad \widehat{\mathcal{E}}[f] \leq \mathcal{V}(A) \leq \widehat{\mathcal{E}}[g], \quad \text{if } f \leq I_A \leq g, \quad f, g \in \mathcal{H}. \tag{2.1}$$

In addition, a pair $(C_{\mathbb{V}}, C_{\mathcal{V}})$ of the Choquet integrals/expecations denoted by

$$C_V[X] = \int_0^\infty V(X \geq t)dt + \int_{-\infty}^0 [V(X \geq t) - 1]dt,$$

with V being replaced by $\mathbb{V}$ and $\mathcal{V}$, respectively. If $\widehat{\mathbb{E}}$ is countably subadditive or

$$\widehat{\mathbb{E}}[|X|^p] = \lim_{c\to\infty} \widehat{\mathbb{E}}[(|X| \wedge c)^p]$$

then

$$\widehat{\mathbb{E}}[|X|^p] \leq C_V(|X|^p) < \infty$$

for all $p > 0$ (See Lemma 4.5 (iii) of Zhang [9]).

Definition 2.4. *(a) A sublinear expectation* $\widehat{\mathbb{E}} : \mathcal{H} \to [-\infty, +\infty]$ *is called to be countably subadditive if it satisfies*

$$\widehat{\mathbb{E}}[X] \leq \Sigma_{n=1}^{\infty} \widehat{\mathbb{E}}[X_n],$$

whenever $X \leq \Sigma_{n=1}^{\infty} X_n,\ X,\ X_n \in \mathcal{H}$ *and* $X \geq 0,\ X_n \geq 0,\ n = 1, 2, ...$

(b) It is called to be continuous if it satisfies

b1. Continuity from below: $\widehat{\mathbb{E}}[X_n] \uparrow \widehat{\mathbb{E}}[X]$, *if* $X_n \uparrow X$, *where* $X_n,\ X \in \mathcal{H}$.

b2. Continuity from above: $\widehat{\mathbb{E}}[X_n] \downarrow \widehat{\mathbb{E}}[X]$, *if* $X_n \downarrow X$, *where* $X_n,\ X \in \mathcal{H}$.

(c) A function V*:* $\mathcal{F} \to [0, 1]$ *is called to be countably subadditive if*

$$V\Big(\bigcup_{n=1}^{\infty} A_n\Big) \leq \Sigma_{n=1}^{\infty} V(A_n), \quad \forall A_n \in \mathcal{F}.$$

(d) A capacity V*:* $\mathcal{F} \to [0, 1]$ *is called a continuous capacity if it satisfies*

d1. Continuity from below: $V(A_n) \uparrow V(A)$, *if* $A_n \uparrow A$, *where* $A_n,\ A \in \mathcal{F}$.

d2. Continuity from above: $V(A_n) \downarrow V(A)$, *if* $A_n \downarrow A$, *where* $A_n,\ A \in \mathcal{F}$.

It is obvious that a continuous subadditive capacity V is countably subadditive.

Peng [7] introduced the concept of independent and identically distributed (IID) random variable and G-normal distribution under sublinear expectation. The definitions are as follows.

Definition 2.5. *(i) (Identical distribution) Let* X_1 *and* X_2 *be two n-dimensional random vectors defined respectively in sublinear expectation spaces* $(\Omega_1, \mathcal{H}_1, \widehat{\mathbb{E}}_1)$ *and* $(\Omega_2, \mathcal{H}_2, \widehat{\mathbb{E}}_2)$*. They are called identically distributed, denoted by* $X_1 \overset{d}{=} X_2$*, if*

$$\widehat{\mathbb{E}}_1[\varphi(X_1)] = \widehat{\mathbb{E}}_2[\varphi(X_2)],\ \forall \varphi \in C_{l,Lip}(\mathbb{R}^n),$$

whenever the sub-expectations are finite. A sequence of random variables $\{X_n, n \geq 1\}$ *is said to be identically distributed if* $X_i \overset{d}{=} X_1$ *for each* $i \geq 1$.

(ii) (Independence) In a sublinear expectation space $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$*, a random vector* $Y = (Y_1, ..., Y_n)(Y_i \in \mathcal{H})$ *is said to be independent to another random vector* $X = (X_1, ..., X_m)(X_i \in \mathcal{H})$ *under* $\widehat{\mathbb{E}}$ *if for each test function* $\varphi \in C_{l,Lip}(\mathbb{R}^m \times \mathbb{R}^n)$ *we have*

$$\widehat{\mathbb{E}}[\varphi(X, Y)] = \widehat{\mathbb{E}}[\widehat{\mathbb{E}}[\varphi(x, Y)]|_{x=X}],$$

whenever $\overline{\varphi}(x) := \widehat{\mathbb{E}}[|\varphi(x, Y)|] < \infty$ *for all* x *and* $\widehat{\mathbb{E}}[|\overline{\varphi}(x)|] < \infty$.

(iii) (IID random variables) A sequence of random variables $\{X_n, n \geq 1\}$ *is said to be independent and identically distributed (IID), if* $X_i \overset{d}{=} X_1$ *and* X_{i+1} *is independent to* $(X_1, ..., X_i)$ *for each* $i \geq 1$.

Definition 2.6. *(G-normal distribution) A random variable $\xi \in \mathcal{H}$ under sublinear expectation $\widehat{\mathbb{E}}$ with $\overline{\sigma}^2 = \widehat{\mathbb{E}}[\xi^2]$, $\underline{\sigma}^2 = -\widehat{\mathbb{E}}[-\xi^2]$ is called G-normal distribution, denoted by $\mathcal{N}(0; [\underline{\sigma}^2, \overline{\sigma}^2])$, if for any function $\varphi \in C_{l,Lip}(\mathbb{R})$, $u(t.x) := \widehat{\mathbb{E}}[\varphi(x + \sqrt{t}\xi)], (t, x) \in [0, \infty) \times \mathbb{R}$, then u is the unique viscosity solution of PDE:*

$$\begin{cases} \partial_t u - G(\partial_{xx} u) = 0, \\ u|_{t=0} = \varphi, \end{cases}$$

where $G(\alpha) = \frac{1}{2}(\overline{\sigma}^2\alpha^+ - \underline{\sigma}^2\alpha^-)$ and $\alpha^+ := max(\alpha, 0)$, $\alpha^- := (-\alpha)^+$.

In the following, some useful lemmas are given. Lemma 2.7 (Markov inequality) in sublinear expectation space $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$ was established by Zhang [9].

Lemma 2.7. *(Markov inequality) Let $\{X_n,\ n \geq 1\}$ be a sequence of independent random variables on the sublinear expectation space $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$, and denote $S_k = X_1 + X_2 + ... + X_k$, $S_0 = 0$. If both the upper expectation $\widehat{\mathbb{E}}[X_k]$ and the lower expectation $\widehat{\mathcal{E}}[X_k]$ are zeros, $k = 1, 2, \dots$, then*

$$\mathbb{V}(|S_n| \geq x) \leq C \frac{\sum_{k=1}^n \widehat{\mathbb{E}}[|X_k|^2]}{x^2}, \qquad \forall\, x > 0.$$

The last lemma obtained by Wu [32] shows the uniform convergence rate of Berry-Esseen ineqality.

Lemma 2.8. *Assume that $\{X_n, n \geq 1\}$ is a sequence of independent and identically distributed random variables with $\widehat{\mathbb{E}}[X_1] = \widehat{\mathbb{E}}[-X_1] = 0$ and $\lim_{c\to\infty} \widehat{\mathbb{E}}[(X_1^2 - c)^+] = 0$. Denote $S_n = \sum_{k=1}^n X_k$, $\overline{\sigma}^2 = \widehat{\mathbb{E}}[X_1^2]$, $\underline{\sigma}^2 = \widehat{\mathcal{E}}[X_1^2]$. Suppose that $\widehat{\mathbb{E}}$ is continuous and set*

$$\Delta_n(x) = \mathbb{V}(\frac{|S_n|}{\sqrt{n}} \geq x) - \mathbb{V}(|\xi| \geq x), \ \ \xi \sim \mathcal{N}(0; [\underline{\sigma}^2, \overline{\sigma}^2]) \ \ under \ \ \widehat{\mathbb{E}}.$$

Then

$$\Delta_n = \sup_{x \geq 0} |\Delta_n(x)| \to 0, \ \ as \ \ n \to \infty.$$

3. Main results

At first, I give the following assumptions on boundary functions and weighted functions:

(A1) Let $g(x)$ be a positive and differentiable function defined on $[n_0, \infty)$, which is strictly increasing to ∞.

(A2) $\rho(x) = g'(x)/g^t(x)$ is monotone for $t < 1$, and if $\rho(x)$ is monotone nondecreasing, we assume $\lim_{n\to\infty} \rho(n + 1)/\rho(n) = 1$.

(A3) $\varphi(x) = g'(x)/g(x)$ is monotone, and if $\varphi(x)$ is monotone nondecreasing, we assume $\lim_{n\to\infty} \varphi(n + 1)/\varphi(n) = 1$.

The following are main results.

Theorem 3.1. *Let $\{X, X_n, n \geq 1\}$ be a sequence of independent and identically distributed random variables in a sublinear expectation space $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$ with $\widehat{\mathbb{E}}[X] = \widehat{\mathbb{E}}[-X] = 0$ and $\overline{\sigma}^2 = \widehat{\mathbb{E}}[X^2] < \infty$, $\underline{\sigma}^2 = \widehat{\mathcal{E}}[X^2]$, $S_n = \sum_{k=1}^n X_k$. Suppose that $\widehat{\mathbb{E}}$ is continuous and $\lim_{c\to\infty} \widehat{\mathbb{E}}[(X^2 - c)^+] = 0$ and $C_{\mathbb{V}}(X^2) < \infty$. Assume that (A1), (A3) hold. Then for any $s > 0$,*

$$\lim_{\varepsilon \downarrow 0} \frac{1}{-\log \varepsilon} \sum_{n=n_0}^{\infty} \varphi(n) \mathbb{V}\{\frac{|S_n|}{\sqrt{n}} \geq \varepsilon g^s(n)\} = \frac{1}{s}, \tag{3.1}$$

here and later, $\xi \sim \mathcal{N}(0; [\underline{\sigma}^2, \overline{\sigma}^2])$.

Theorem 3.2. *Let* $\{X, X_n, n \geq 1\}$ *be a sequence of independent and identically distributed random variables in a sublinear expectation space* $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$ *with* $\widehat{\mathbb{E}}[X] = \widehat{\mathbb{E}}[-X] = 0$ *and* $\overline{\sigma}^2 = \widehat{\mathbb{E}}[X^2] < \infty$, $\underline{\sigma}^2 = \widehat{\mathcal{E}}[X^2]$, $S_n = \sum_{k=1}^n X_k$. *Suppose that* $\widehat{\mathbb{E}}$ *is continuous and* $\lim_{c\to\infty} \widehat{\mathbb{E}}[(X^2 - c)^+] = 0$ *and* $C_{\mathbb{V}}(X^2) < \infty$. *Assume that (A1), (A2) hold. Then for any* $s > (1-t)/2$, *where* $t < 1$,

$$\lim_{\varepsilon\downarrow 0} \varepsilon^{\frac{1-t}{s}} \sum_{n=n_0}^{\infty} \rho(n)\mathbb{V}\{\frac{|S_n|}{\sqrt{n}} \geq \varepsilon g^s(n)\} = \frac{1}{1-t} C_{\mathbb{V}}(|\xi|^{\frac{1-t}{s}}), \tag{3.2}$$

here and later, $\xi \sim \mathcal{N}(0; [\underline{\sigma}^2, \overline{\sigma}^2])$.

Remark 3.3. *Assumptions (A1)–(A3) are all mild conditions.* $g(x) = x^{\alpha}$, $(\log x)^{\beta}, (\log\log x)^{\gamma}$ *with some suitable conditions of* $\alpha > 0$, $\beta > 0$, $\gamma > 0$ *and some others all satisfy these conditions. In the following, some typical examples are given.*

If taking $g(n) = n^{\frac{2-p}{2p}}$, $s = 1$ in Theorem 3.1 with $1 \leq p < 2$, then

Corollary 3.4. *For* $1 \leq p < 2$,

$$\lim_{\varepsilon\downarrow 0} \frac{1}{-\log\varepsilon} \sum_{n=1}^{\infty} \frac{1}{n}\mathbb{V}\{\frac{|S_n|}{\sqrt{n}} \geq n^{\frac{2-p}{2p}}\} = \frac{2p}{2-p}.$$

If taking $g(n) = \log n$, $s = 1/2$ in Theorem 3.1, then

Corollary 3.5.

$$\lim_{\varepsilon\downarrow 0} \frac{1}{-\log\varepsilon} \sum_{n=1}^{\infty} \frac{1}{n\log n}\mathbb{V}\{\frac{|S_n|}{\sqrt{n}} \geq \varepsilon\sqrt{\log n}\} = 2.$$

If taking $g(n) = (\log\log n)^2$, $s = d/2$ in Theorem 3.1 with $d > 0$, then

Corollary 3.6. *For* $d > 0$,

$$\lim_{\varepsilon\downarrow 0} \frac{1}{-\log\varepsilon} \sum_{n=3}^{\infty} \frac{1}{n\log n\log\log n}\mathbb{V}\{\frac{|S_n|}{\sqrt{n}} \geq \varepsilon(\log\log n)^d\} = \frac{1}{d}.$$

If taking $g(n) = n$, $s = \frac{2-p}{2p}$, $t = \frac{2p-r}{p}$ in Theorem 3.2 with $1 \leq p < r < 2$, then

Corollary 3.7. *For* $1 \leq p < r < 2$,

$$\lim_{\varepsilon\downarrow 0} \varepsilon^{\frac{2(r-p)}{2-p}} \sum_{n=1}^{\infty} n^{\frac{r-2p}{p}}\mathbb{V}\{\frac{|S_n|}{\sqrt{n}} \geq \varepsilon n^{\frac{2-p}{2p}}\} = \frac{p}{r-p} C_{\mathbb{V}}(|\xi|^{\frac{2(r-p)}{2-p}}).$$

If taking $g(n) = \log n$, $s = 1/2$, $t = -\delta$ in Theorem 3.2 with $-1 < \delta < 0$, then

Corollary 3.8. *For* $-1 < \delta < 0$,

$$\lim_{\varepsilon\downarrow 0} \varepsilon^{2(1+\delta)} \sum_{n=3}^{\infty} \frac{(\log n)^{\delta}}{n}\mathbb{V}\{\frac{|S_n|}{\sqrt{n}} \geq \varepsilon\sqrt{\log n}\} = \frac{1}{1+\delta} C_{\mathbb{V}}(|\xi|^{2(1+\delta)}).$$

If taking $g(n) = \log\log n$, $s = d$, $t = 1 - b$ in Theorem 3.2 with $b > 0$, $d > 0$, then

Corollary 3.9. *For $b > 0$, $d > 0$,*

$$\lim_{\varepsilon\downarrow 0} \varepsilon^{\frac{b}{d}} \sum_{n=3}^{\infty} \frac{(\log\log n)^{b-1}}{n\log n} \mathbb{V}\{\frac{|S_n|}{\sqrt{n}} \geq \varepsilon(\log\log n)^d\} = \frac{1}{b} C_{\mathbb{V}}(|\xi|^{\frac{b}{d}}).$$

Remark 3.10. *In fact, Corollary 3.6 and Corollary 3.9 are the Theorem 2 and Theorem 1 from Xu and Cheng [34] respectively, therefore our results extend the known results.*

4. Proof of Theorem 3.1

Set $b(\varepsilon) = [g^{-1}(\varepsilon^{-r})]$, where $g^{-1}(x)$ is the inverse function of $g(x)$ and $r > 1/s$.

Proposition 4.1. *Under the conditions of Theorem 3.1, one has*

$$\lim_{\varepsilon\downarrow 0} \frac{1}{-\log\varepsilon} \sum_{n=n_0}^{\infty} \varphi(n)\mathbb{V}\{|\xi| \geq \varepsilon g^s(n)\} = \frac{1}{s}. \tag{4.1}$$

Proof. At first I discuss the relations between the integral and the series. If $\varphi(y)$ is nonincreasing, then $\varphi(y)\mathbb{V}\{|\xi| \geq \varepsilon g^s(y)\}$ is also nonincreasing, thus one can get

$$\int_{n_0+1}^{\infty} \varphi(y)\mathbb{V}\{|\xi| \geq \varepsilon g^s(y)\}dy \leq \sum_{n=n_0+1}^{\infty} \varphi(n)\mathbb{V}\{|\xi| \geq \varepsilon g^s(n)\} \leq \int_{n_0}^{\infty} \varphi(y)\mathbb{V}\{|\xi| \geq \varepsilon g^s(y)\}dy,$$

therefore, by L'Hospital's rule, one can get

$$\begin{aligned}
& \lim_{\varepsilon\downarrow 0} \frac{1}{-\log\varepsilon} \sum_{n=n_0}^{\infty} \varphi(n)\mathbb{V}\{|\xi| \geq \varepsilon g^s(n)\} = \lim_{\varepsilon\downarrow 0} \frac{1}{-\log\varepsilon} \int_{n_0}^{\infty} \frac{g'(y)}{g(y)}\mathbb{V}\{|\xi| \geq \varepsilon g^s(y)\}dy \\
= & \lim_{\varepsilon\downarrow 0} \frac{1}{-\log\varepsilon} \int_{g(n_0)}^{\infty} \frac{1}{t}\mathbb{V}\{|\xi| \geq \varepsilon t^s\}dt = \lim_{\varepsilon\downarrow 0} \frac{1}{s} \frac{1}{-\log\varepsilon} \int_{\varepsilon g^s(n_0)}^{\infty} \frac{1}{x}\mathbb{V}\{|\xi| \geq x\}dx \\
= & \lim_{\varepsilon\downarrow 0} \frac{-g^s(n_0)\cdot\frac{1}{\varepsilon g^s(n_0)}\mathbb{V}\{|\xi| \geq \varepsilon g^s(n_0)\}}{s} \frac{1}{-\frac{1}{\varepsilon}} = \lim_{\varepsilon\downarrow 0} \frac{1}{s}\mathbb{V}\{|\xi| \geq \varepsilon g^s(n_0)\} \\
= & \frac{1}{s}\mathbb{V}\{|\xi| \geq 0\} = \frac{1}{s}.
\end{aligned} \tag{4.2}$$

If $\varphi(y)$ is nondecreasing, noting $\lim_{n\to\infty} \varphi(n+1)/\varphi(n) = 1$, then for any $0 < \delta < 1$, there exists $n_1 = n_1(\delta)$, such that $\varphi(n+1)/\varphi(n) < 1+\delta$ and $\varphi(n)/\varphi(n+1) > 1-\delta$ for $n \geq n_1$. Then one can conclude

$$\begin{aligned}
& \lim_{\varepsilon\downarrow 0} \frac{1}{-\log\varepsilon}(1+\delta)^{-1} \int_{n_1+1}^{\infty} \varphi(y)\mathbb{V}\{|\xi| \geq \varepsilon g^s(y)\}dy \\
\leq & \lim_{\varepsilon\downarrow 0} \frac{1}{-\log\varepsilon} \sum_{n=n_0}^{\infty} \varphi(n)\mathbb{V}\{|\xi| \geq \varepsilon g^s(n)\} \\
\leq & \lim_{\varepsilon\downarrow 0} \frac{1}{-\log\varepsilon}(1-\delta)^{-1} \int_{n_1}^{\infty} \varphi(y)\mathbb{V}\{|\xi| \geq \varepsilon g^s(y)\}dy.
\end{aligned}$$

And then by (4.2), one can get

$$\frac{1}{s}(1+\delta)^{-1} \leq \lim_{\varepsilon\downarrow 0} \frac{1}{-\log\varepsilon} \sum_{n=n_0}^{\infty} \varphi(n)\mathbb{V}\{|\xi| \geq \varepsilon g^s(n)\} \leq \frac{1}{s}(1-\delta)^{-1}.$$

Thus (4.1) follows by letting $\delta \downarrow 0$. □

Remark 4.2. *In the following, without loss of generality, one can assume that $\varphi(x)$ is nonincreasing. For the other case, the discussion process is similar to that of Proposition 4.1.*

Proposition 4.3. *Under the conditions of Theorem 3.1, one has*

$$\lim_{\varepsilon\downarrow 0} \frac{1}{-\log\varepsilon} \sum_{n=n_0}^{b(\varepsilon)} \varphi(n) \mid \mathbb{V}\{\frac{|S_n|}{\sqrt{n}} \geq \varepsilon g^s(n)\} - \mathbb{V}\{|\xi| \geq \varepsilon g^s(n)\} \mid= 0.$$

Proof. Noting $\sum\limits_{n=n_0}^{b(\varepsilon)} \varphi(n) \sim -r\log\varepsilon$, then by Lemma 2.8 and Toeplitz's lemma, one can get

$$\lim_{\varepsilon\downarrow 0} \frac{1}{-\log\varepsilon} \sum_{n=n_0}^{b(\varepsilon)} \varphi(n) \mid \mathbb{V}\{\frac{|S_n|}{\sqrt{n}} \geq \varepsilon g^s(n)\} - \mathbb{V}\{|\xi| \geq \varepsilon g^s(n)\}| \leq \lim_{\varepsilon\downarrow 0} \frac{1}{-\log\varepsilon} \sum_{n=n_0}^{b(\varepsilon)} \varphi(n)\Delta_n = 0.$$

Thus the proof is completed. □

Proposition 4.4. *Under the conditions of Theorem 3.1, one has*

$$\lim_{\varepsilon\downarrow 0} \frac{1}{-\log\varepsilon} \sum_{n=b(\varepsilon)+1}^{\infty} \varphi(n)\mathbb{V}\{|\xi| \geq \varepsilon g^s(n)\} = 0.$$

Proof. Since $n > b(\varepsilon)$ implies $\varepsilon g^s(n) > \varepsilon^{1-rs}$. Then by the same argument in Proposition 4.1, using L'Hospital's rule and note that $r > 1/s$,

$$\begin{aligned}
&\lim_{\varepsilon\downarrow 0} \frac{1}{-\log\varepsilon} \sum_{n=b(\varepsilon)+1}^{\infty} \varphi(n)\mathbb{V}\{|\xi| \geq \varepsilon g^s(n)\} \\
\leq &\lim_{\varepsilon\downarrow 0} \frac{C}{-\log\varepsilon} \int_{\varepsilon g^s(b(\varepsilon))}^{\infty} \frac{1}{x}\mathbb{V}\{|\xi| \geq x\}dx \\
\leq &\lim_{\varepsilon\downarrow 0} \frac{C}{-\log\varepsilon} \int_{\varepsilon^{1-rs}}^{\infty} \frac{1}{x}\mathbb{V}\{|\xi| \geq x\}dx \\
= &\lim_{\varepsilon\downarrow 0} \frac{-C(1-rs)\varepsilon^{-rs}}{\varepsilon^{1-rs}}\mathbb{V}\{|\xi| \geq \varepsilon^{1-rs}\}\frac{1}{-\frac{1}{\varepsilon}} \\
= &\lim_{\varepsilon\downarrow 0} C(1-rs)\mathbb{V}\{|\xi| \geq \varepsilon^{1-rs}\} = 0.
\end{aligned}$$

□

Proposition 4.5. *Under the conditions of Theorem 3.1, one has*

$$\lim_{\varepsilon\downarrow 0} \frac{1}{-\log\varepsilon} \sum_{n=b(\varepsilon)+1}^{\infty} \varphi(n)\mathbb{V}\{\frac{|S_n|}{\sqrt{n}} \geq \varepsilon g^s(n)\} = 0.$$

Proof. By the same argument in Proposition 4.1, Lemma 2.7 (Markov's inequality), $\overline{\sigma}^2 = \widehat{\mathbb{E}}[X^2] < \infty$ and note that $r > 1/s > 0$, then

$$\begin{aligned}
&\lim_{\varepsilon\downarrow 0} \frac{1}{-\log\varepsilon} \sum_{n=b(\varepsilon)+1}^{\infty} \varphi(n)\mathbb{V}\{\frac{|S_n|}{\sqrt{n}} \geq \varepsilon g^s(n)\} \\
\leq &\lim_{\varepsilon\downarrow 0} \frac{1}{-\log\varepsilon} \sum_{n=b(\varepsilon)+1}^{\infty} \varphi(n) \frac{\sum_{k=1}^{n} \widehat{\mathbb{E}}[|X_k|^2]}{n\cdot\varepsilon^2 g^{2s}(n)} \\
\leq &\lim_{\varepsilon\downarrow 0} \frac{C}{-\log\varepsilon} \sum_{n=b(\varepsilon)+1}^{\infty} \frac{g'(n)}{g(n)} \frac{1}{\varepsilon^2 g^{2s}(n)} \\
\leq &\lim_{\varepsilon\downarrow 0} \frac{C\varepsilon^{-2}}{-\log\varepsilon} \int_{b(\varepsilon)}^{\infty} \frac{g'(x)}{g^{1+2s}(x)} dx \\
= &\lim_{\varepsilon\downarrow 0} \frac{C\varepsilon^{-2}}{-\log\varepsilon} \int_{g(b(\varepsilon))}^{\infty} \frac{1}{y^{1+2s}} dy \\
\leq &\lim_{\varepsilon\downarrow 0} \frac{C\varepsilon^{2rs-2}}{-\log\varepsilon} = 0.
\end{aligned}$$

□

Proof of Theorem 3.1. Theorem 3.1 will be proved by the Propositions 4.1, 4.3–4.5 and the triangular inequality directly. □

5. Proof of Theorem 3.2

Set $d(\varepsilon) = [g^{-1}(M\varepsilon^{-1/s})]$, where $g^{-1}(x)$ is the inverse function of $g(x)$, $M \geq 1$.

Proposition 5.1. *Under the conditions of Theorem 3.2, one has*

$$\lim_{\varepsilon\downarrow 0} \varepsilon^{1/s-t/s} \sum_{n=n_0}^{\infty} \rho(n)\mathbb{V}\{|\xi| \geq \varepsilon g^s(n)\} = \frac{1}{1-t} C_{\mathbb{V}}(|\xi|^{1/s-t/s}).$$

Proof. At first I discuss the relations between the integral and the series. If $\rho(y)$ is nonincreasing, then $\rho(y)\mathbb{V}\{|\xi| \geq \varepsilon g^s(y)\}$ is nonincreasing, hence one has

$$\int_{n_0+1}^{\infty} \rho(y)\mathbb{V}\{|\xi| \geq \varepsilon g^s(y)\}dy \leq \sum_{n=n_0+1}^{\infty} \rho(n)\mathbb{V}\{|\xi| \geq \varepsilon g^s(n)\} \leq \int_{n_0}^{\infty} \rho(y)\mathbb{V}\{|\xi| \geq \varepsilon g^s(y)\}dy,$$

then one can get

$$\begin{aligned}
&\lim_{\varepsilon\downarrow 0} \varepsilon^{1/s-t/s} \sum_{n=n_0}^{\infty} \rho(n)\mathbb{V}\{|\xi| \geq \varepsilon g^s(n)\} \\
= &\lim_{\varepsilon\downarrow 0} \varepsilon^{1/s-t/s} \int_{n_0}^{\infty} \rho(y)\mathbb{V}\{|\xi| \geq \varepsilon g^s(y)\}dy
\end{aligned}$$

$$
\begin{aligned}
&= \lim_{\varepsilon\downarrow 0} \varepsilon^{1/s-t/s} \int_{g(n_0)}^{\infty} \frac{1}{y^t} \mathbb{V}\{|\xi| \geq \varepsilon y^s\} dy \\
&= \lim_{\varepsilon\downarrow 0} \frac{1}{s} \int_{\varepsilon g^s(n_0)}^{\infty} \frac{1}{x^{t/s-1/s+1}} \mathbb{V}\{|\xi| \geq x\} dx \\
&= \frac{1}{s} \int_{0}^{\infty} \frac{1}{x^{t/s-1/s+1}} \mathbb{V}\{|\xi| \geq x\} dx \\
&= \frac{1}{1-t} C_{\mathbb{V}}(|\xi|^{1/s-t/s}).
\end{aligned}
$$

If $\rho(y)$ is nondecreasing, then by $\lim_{n\to\infty} \rho(n+1)/\rho(n) = 1$, the proof is similar to that of Proposition 4.1. Thus Proposition 5.1 is obtained by above steps. □

Proposition 5.2. *Under the conditions of Theorem 3.2, one can get*

$$
\lim_{\varepsilon\downarrow 0} \varepsilon^{1/s-t/s} \sum_{n=n_0}^{d(\varepsilon)} \rho(n) \mid \mathbb{V}\{\frac{|S_n|}{\sqrt{n}} \geq \varepsilon g^s(n)\} - \mathbb{V}\{|\xi| \geq \varepsilon g^s(n)\} \mid = 0.
$$

Proof. Noting $\sum_{n=n_0}^{d(\varepsilon)} \rho(n) \sim -\frac{M^{1-t}}{1-t} \varepsilon^{-\frac{1-t}{s}}$, then by Lemma 2.8 and Toeplitz's lemma, the proof is similar to that of Proposition 4.3, so we omit it here. □

Proposition 5.3. *Under the conditions of Theorem 3.2, one has*

$$
\lim_{M\to\infty} \varepsilon^{1/s-t/s} \sum_{n=d(\varepsilon)+1}^{\infty} \rho(n) \mathbb{V}\{|\xi| \geq \varepsilon g^s(n)\} = 0.
$$

Proof. By the proof of Proposition 5.1, one can get

$$
\frac{1}{s} \int_{0}^{\infty} \frac{1}{x^{t/s-1/s+1}} \mathbb{V}\{|\xi| \geq x\} dx = \frac{1}{1-t} C_{\mathbb{V}}(|\xi|^{1/s-t/s}) < \infty.
$$

Then

$$
\begin{aligned}
&\lim_{M\to\infty} \varepsilon^{1/s-t/s} \sum_{n=d(\varepsilon)+1}^{\infty} \rho(n) \mathbb{V}\{|\xi| \geq \varepsilon g^s(n)\} \\
\leq &\lim_{M\to\infty} \varepsilon^{1/s-t/s} \int_{d(\varepsilon)}^{\infty} \rho(y) \mathbb{V}\{|\xi| \geq \varepsilon g^s(y)\} dy \\
\leq &\lim_{M\to\infty} C\varepsilon^{1/s-t/s} \int_{g(d(\varepsilon))}^{\infty} \frac{1}{y^t} \mathbb{V}\{|\xi| \geq \varepsilon y^s\} dy \\
= &\lim_{M\to\infty} C \int_{\varepsilon g^s(d(\varepsilon))}^{\infty} \frac{1}{x^{t/s-1/s+1}} \mathbb{V}\{|\xi| \geq x\} dx \\
\leq &\lim_{M\to\infty} C \int_{M^s}^{\infty} \frac{1}{x^{t/s-1/s+1}} \mathbb{V}\{|\xi| \geq x\} dx = 0.
\end{aligned}
$$

□

Proposition 5.4. *Under the conditions of Theorem 3.2, one has*

$$\lim_{M\to\infty} \varepsilon^{1/s-t/s} \sum_{n=d(\varepsilon)+1}^{\infty} \rho(n)\mathbb{V}\{\frac{|S_n|}{\sqrt{n}} \geq \varepsilon g^s(n)\} = 0. \tag{5.1}$$

Proof. By the same argument in Proposition 4.1, Lemma 2.7 (Markov's inequality), $\overline{\sigma}^2 = \widehat{\mathbb{E}}[X^2] < \infty$ and note that $s > (1-t)/2,\ t < 1$, then

$$\begin{aligned}
&\lim_{M\to\infty} \varepsilon^{1/s-t/s} \sum_{n=d(\varepsilon)+1}^{\infty} \rho(n)\mathbb{V}\{\frac{|S_n|}{\sqrt{n}} \geq \varepsilon g^s(n)\} \\
\leq\ & \lim_{M\to\infty} \varepsilon^{1/s-t/s} \sum_{n=d(\varepsilon)+1}^{\infty} \frac{g'(n)}{g^t(n)} \frac{\sum_{k=1}^{n} \widehat{\mathbb{E}}[|X_k|^2]}{n\cdot\varepsilon^2 g^{2s}(n)} \\
\leq\ & \lim_{M\to\infty} C\varepsilon^{1/s-t/s-2} \int_{d(\varepsilon)}^{\infty} \frac{g'(x)}{g^{2s+t}(x)} dx \\
\leq\ & \lim_{M\to\infty} C\varepsilon^{1/s-t/s-2} \int_{g(d(\varepsilon))}^{\infty} \frac{1}{y^{2s+t}} dy \\
\leq\ & \lim_{M\to\infty} CM^{1-2s-t} = 0.
\end{aligned}$$

□

Proof of Theorem 3.2. Theorem 3.2 will be proved by the Propositions 5.1–5.4 and the triangular inequality. □

6. Conclusions

In this paper, using the Markov's inequality and uniform convergence rate of Berry-Esseen ineqality, the author establish a general form of precise asymptotics for complete convergence holds under sublinear expectation. The results extend some precise asymptotics for complete convergence theorems from the traditional probability space to the sublinear expectation space. The results also generalize the known results obtained by Xu and Cheng [34]. Recently, the research about statistical probability convergence and its application is a new trend in probability and statistics, one can refer to [35–42] and references therein for details, I will consider the statistical probability convergence and its application under expectation space in future.

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Conflict interest

The author declares no conflict of interest in this paper.

References

1. S. G. Peng, G-Expectation, G-Brownian motion and related stochastic calculus of Itô type, In: F. E. Benth, G. Di Nunno, T. Lindstrøm, B. Øksendal, T. Zhang, *Stochastic analysis and applications*, Abel Symposia, **2** (2007), 541–567. doi: 10.1007/978-3-540-70847-6_25.
2. L. Denis, C. Martini, A theoretical framework for the pricing of contingent claims in the presence of model uncertainty, *Ann. Appl. Probab.*, **16** (2006), 827–852. doi: 10.1214/105051606000000169.
3. I. Gilboa, Expected utility with purely subjective non-additive probabilities, *J. Math. Econ.*, **16** (1987), 65–88. doi: 10.1016/0304-4068(87)90022-X.
4. M. Marinacci, Limit laws for non-additive probabilities and their frequentist interpretation, *J. Econom. Theory*, **84** (1999), 145–195. doi: 10.1006/jeth.1998.2479.
5. S. G. Peng, Backward SDE and related g-expectation, In: N. El Karoui, L. Mazliak, *Backward stochastic differential equations*, Pitman Research Notes in Mathematics Serie, Longman, Harlow, **364** (1997), 141–159.
6. S. G. Peng, Survey on normal distributions, central limit theorem, Brownian motion and the related stochastic calculus under sublinear expectations, *Sci. China Ser. A: Math.*, **52** (2009), 1391–1411.
7. S. G. Peng, Nonlinear expectations and stochastic calculus under uncertainty, Springer, Berlin, Heidelberg, 2019. doi: 10.1007/978-3-662-59903-7.
8. L. X. Zhang, Donsker's invariance principle under the sub-linear expectation with an application to chung's law of the iterated logarithm, *Commun. Math. Stat.*, **3** (2015), 187–214. doi: 10.1007/s40304-015-0055-0.
9. L. X. Zhang, Exponential inequalities under the sub-linear expectations with applications to laws of the iterated logarithm, *Sci. China Math.*, **59** (2016), 2503–2526. doi: 10.1007/s11425-016-0079-1.
10. L. X. Zhang, Rosenthal's inequalities for independent and negatively dependent random variables under sub-linear expectations with applications, *Sci. China Math.*, **59** (2016), 751–768. doi: 10.1007/s11425-015-5105-2.
11. L. X. Zhang, Lindeberg's central limit theorems for martingale like sequences under sub-linear expectations, *Sci. China Math.*, **64** (2021), 1263–1290. doi: 10.1007/s11425-018-9556-7.
12. Z. J. Chen, Strong laws of large numbers for sub-linear expectations, *Sci. China Math.*, **59** (2016), 945–954. doi: 10.1007/s11425-015-5095-0.
13. Q. Y. Wu, Y. Y. Jiang, Strong law of large numbers and Chover's law of the iterated logarithm under sub-linear expectations, *J. Math. Anal. Appl.*, **460** (2018), 252–270. doi: 10.1016/j.jmaa.2017.11.053.
14. J. P. Xu, L. X. Zhang, Three series theorem for independent random variables under sub-linear expectations with applications, *Acta Math. Sin.-English Ser.*, **35** (2019), 172–184. doi: 10.1007/s10114-018-7508-9.
15. J. P. Xu, L. X. Zhang, The law of logarithm for arrays of random variables under sub-linear expectations, *Acta Math. Appl. Sin. Engl. Ser.*, **36** (2020), 670–688. doi: 10.1007/s10255-020-0958-8.

16. Y. S. Song, Normal approximation by Stein's method under sublinear expectations, *Stochastic Process. Appl.*, **130** (2020), 2838–2850. doi: 10.1016/j.spa.2019.08.005.

17. W. Liu, Y. Zhang, Central limit theorem for linear processes generated by IID random variables under the sub-linear expectation, *Appl. Math. J. Chin. Univ. Ser. B*, **36** (2021), 243–255. doi: 10.1007/s11766-021-3882-7.

18. W. Liu, Y. Zhang, The Law of the iterated logarithm for linear processes generated by a sequence of stationary independent random variables under the sub-Linear expectation, *Entropy*, **23** (2021), 1313. doi: 10.3390/e23101313.

19. Z. J. Chen, C. Hu, G. F. Zong, Strong laws of large numbers for sub-linear expectation without independence, *Commun. Stat. Theory Methods*, **46** (2017), 7529–7545. doi: 10.1080/03610926.2016.1154157.

20. Y. Wu, X. J. Wang, L. X. Zhang, On the asymptotic approximation of inverse moment under sub-linear expectations, *J. Math. Anal. Appl.*, **468** (2018), 182–196. doi: 10.1016/j.jmaa.2018.08.010.

21. X. W. Feng, Law of the logarithm for weighted sums of negatively dependent random variables under sublinear expectation, *Stat. Probab. Lett.*, **149** (2019), 132–141. doi: 10.1016/j.spl.2019.01.033.

22. X. Fang, S. G. Peng, Q. M. Shao, Y. S. Song, Limit theorems with rate of convergence under sublinear expectations, *Bernoulli*, **25** (2019), 2564–2596. doi: 10.3150/18-BEJ1063.

23. L. X. Zhang, The convergence of the sums of independent random variables under the sub-linear expectations, *Acta Math. Sin. Engl. Ser.*, **36** (2020), 224–244. doi: 10.1007/s10114-020-8508-0.

24. A. Kuczmaszewska, Complete convergence for widely acceptable random variables under the sublinear expectations, *J. Math. Anal. Appl.*, **484** (2020), 123662. doi: 10.1016/j.jmaa.2019.123662.

25. F. X. Feng, D. C. Wang, Q. Y. Wu, H. W. Huang, Complete and complete moment convergence for weighted sums of arrays of rowwise negatively dependent random variables under the sub-linear expectations, *Commun. Stat. Theory Methods*, **50** (2021), 594–608. doi: 10.1080/03610926.2019.1639747.

26. X. F. Guo, X. P. Li, On the laws of large numbers for pseudo-independent random variables under sublinear expectation, *Stat. Probab. Lett.*, **172** (2021), 109042. doi: 10.1016/j.spl.2021.109042.

27. P. L. Hsu, H. Robbins, Complete convergence and the strong law of large numbers, *Proc. Nat. Acad. Sci. USA*, **33** (1947), 25–31.

28. C. C. Heyde, A supplement to the strong law of large numbers, *J. Appl. Probab.*, **12** (1975), 173–175. doi: 10.2307/3212424.

29. A. Spătaru, Precise asymptotics in Spitzer's law of large numbers, *J. Theoret. Probab.*, **12** (1999), 811–819. doi: 10.1023/A:1021636117551.

30. A. Gut, A. Spătaru, Precise asymptotics in the Baum-Katz and Davis law of large numbers, *J. Math. Anal. Appl.*, **248** (2000), 233–246. doi: 10.1006/jmaa.2000.6892.

31. A. Gut, A. Spătaru, Precise asymptotics in the law of the iterated logarithm, *Ann. Probab.*, **28** (2000), 1870–1883. doi: 10.1214/aop/1019160511.

32. Q. Y. Wu, Precise asymptotics for complete integral convergence under sublinear expectations, *Math. Probl. Eng.*, **2020** (2020), 3145935. doi: 10.1155/2020/3145935.

33. L. X. Zhang, Heyde's theorem under the sub-linear expectations, *Stat. Probab. Lett.*, **170** (2021), 108987. doi: 10.1016/j.spl.2020.108987.

34. M. Z. Xu, K. Cheng, Precise asymptotics in the law of the iterated logarithm under sublinear expectations, *Math. Probl. Eng.*, **2021** (2021), 6691857. doi: 10.1155/2021/6691857.

35. H. M. Srivastava, B. B. Jena, S. K. Paikray, Deferred Cesàro statistical probability convergence and its applications to approximation theorems, *J. Nonlinear Convex Anal.*, **20** (2019), 1777–1792.

36. B. B. Jena, S. K. Paikray, Product of statistical probability convergence and its applications to Korovkin-type theorem, *Miskolc Math. Notes*, **20** (2019), 969–984. doi: 10.18514/MMN.2019.3014.

37. H. M. Srivastava, B. B. Jena, S. K. Paikray, Statistical deferred Nörlund summability and Korovkin-type approximation theorem, *Mathematics*, **8** (2020), 636. doi: 10.3390/math8040636.

38. B. B. Jena, S. K. Paikray, Product of deferred Cesàro and deferred weighted statistical probability convergence and its applications to Korovkin-type theorems, *Univ. Sci.*, **25** (2020), 409–433.

39. B. B. Jena, S. K. Paikray, H. Dutta, On various new concepts of statistical convergence for sequences of random variables via deferred Cesàro mean, *J. Math. Anal. Appl.*, **487** (2020), 123950. doi: 10.1016/j.jmaa.2020.123950.

40. H. M. Srivastava, B. B. Jena, S. K. Paikray, Statistical probability convergence via the deferred Nörlund mean and its applications to approximation theorems, *Rev. Real Acad. Cienc. Exactas Fís. Natur. Ser. A Mat. (RACSAM)*, **114** (2020), 1–14. doi: 10.1007/s13398-020-00875-7.

41. H. M. Srivastava, B. B. Jena, S. K. Paikray, A certain class of statistical probability convergence and its applications to approximation theorems, *Appl. Anal. Discrete Math.*, **14** (2020), 579–598. doi: 10.2298/AADM190220039S.

42. B. B. Jena, S. K. Paikray, H. Dutta, Statistically Riemann integrable and summable sequence of functions via deferred Cesàro mean, *Bull. Iranian Math. Soc.*, 2021, 1–17. doi: 10.1007/s41980-021-00578-8.