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Research article

Localized modes in time-fractional modified coupled Korteweg-de Vries equation with singular and non-singular kernels

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Abstract: In this article, the modified coupled Korteweg-de Vries equation with Caputo and Caputo-Fabrizio time-fractional derivatives are considered. The system is studied by applying the modified double Laplace transform decomposition method which is a very effective tool for solving nonlinear coupled systems. The proposed method is a composition of the double Laplace and decomposition method. The results of the problems are obtained in the form of a series solution for $0 < \alpha \leq 1$, which is approaching to the exact solutions when $\alpha = 1$. The precision and effectiveness of the considered method on the proposed model are confirmed by illustrated with examples. It is observed that the proposed model describes the nonlinear evolution of the waves suffered by the weak dispersion effects. It is also observed that the coupled system forms the wave solution which reveals the evolution of the shock waves because of the steeping effect to temporal evolutions. The error analysis is performed, which is comparatively very small between the exact and approximate solutions, which signifies the importance of the proposed method.

Keywords: modified coupled KdV equation; Caputo and Caputo-Fabrizio operators; double Laplace transform; decomposition method
Mathematics Subject Classification: 35Bxx, 35Qxx, 37Mxx, 41Axx, 65Mxx

1. Introduction

The time-fractional singular and non-singular operator are widely used in modern sciences and technology to study the behavior and applications of nonlinear ordinary and partial differential equations [1–4]. Fractional-order models are useful to study numerous real-world problems for longtime memory, and chaotic behaviour [5]. Due to these characteristics, the nonlinear models having fractional-order derivatives have been widely attracted in many areas of fractional calculus including image processing and signal, mechanics, biophysics and bioengineering, electrical engineering, biology, viscoelasticity, rheology, and control theory [6–8].

The Korteweg-de Vries (KdV) equation has extensively studied for nonlinear models describing the evaluation in time of long, unidirectional shallow water waves [9]. It was mainly presented by Boussinesq in 1877 and then retrieved by Diederik Korteweg and Gustav-de Vries in 1895 [10]. The existence solution to the a KdV equation can be seen in [11, 12]. The KdV system having time-fractional derivatives attained remarkable attention in plasma physics especially in electrons acoustic waves (EAWs) propagation, due to its important role in studying diverse forms of mutual developments experimentally [13, 14]. It has observed that the time-fractional operators in the system dramatically change the soliton amplitudes of the electron-acoustic solitary waves. This effects have been particularly equated with the structures of the broadband electrostatic noise experienced in the dayside auroral zone [15]. The system of coupled KdV system plays a leading part in various areas of sciences and engineering, particularly in water waves, quantum field theory, hydrodynamics and plasma physics [16, 17]. It also represents the relations in extended waves with altered dispersion relations and describes iterations of water waves [18, 19].

Here, we consider the mCKdV in the form [20]

$$\begin{aligned} &\frac{\partial^\alpha \phi}{\partial t^\alpha} + 3\phi^2 \frac{\partial \phi}{\partial x} - \frac{3}{2}\frac{\partial^2 \psi}{\partial x^2} - 3\phi \frac{\partial \psi}{\partial x} - 3\psi \frac{\partial \phi}{\partial x} + 3\lambda \frac{\partial \phi}{\partial x} - \frac{1}{2}\frac{\partial^3 \phi}{\partial x^3} = 0, \\ &\frac{\partial^\alpha \psi}{\partial t^\alpha} - 3\phi^2 \frac{\partial \psi}{\partial x} + 3\psi \frac{\partial \psi}{\partial x} + 3\frac{\partial \phi}{\partial x}\frac{\partial \psi}{\partial x} - 3\lambda \frac{\partial \psi}{\partial x} + \frac{\partial^3 \psi}{\partial x^3} = 0, \qquad 0 < \alpha \leq 1, \quad t > 0 \end{aligned} \tag{1.1}$$

with subsidiary conditions

$$\phi(x, 0) = f(x), \quad \psi(x, 0) = g(x),$$

where λ is a real number to be chosen accordingly.

The proposed mCKdV Eq (1.1) has more interesting features than classical, because the operators will be defined by an integral which play a vital role in modern technology, engineering, plasma physics, hydrodynamics and quantum theory [21]. The nonlinear differential equations contain numerous fractional differential singular and non-singular operators such as Hilfer, Caputo, Caputo-Fabrizio, Riemann-Liouville, Antagana-Balenau in Caputo's sense [22–24]. These operators can be reduced in Caputo's form after some parametric addition. One can assume that the fractional operator could provide a power-law estimate of the local conduct of non-differentiable functions [25]. The Caputo operator possesses a power-law kernel and has restrictions to apply in modeling physical phenomena.

A modified laplace decomposition method (MLDM) is applied to Schrodinger-KdV equation in the sense of Atangana-Baleanu derivative in [26]. To deal successfully in such a situation, an alternative fractional operator possess a kernel with exponential decay has been introduced [27]. This novel operator is called the Caputo-Fabrizio (CF) operator which has a non-singular kernel. This operator is broadly applicable for modeling particular type of physical problems which follows the exponential decay law. Currently, mathematical and physical models having the CF operator have a remarkable development. The characteristics and applications of the above derivatives has been extensively studied (see [28–35] and the reference therein).

There are many analytical methods that offer approximate solutions to nonlinear models, for example, perturbation methods [36–38], homotopy perturbation [39–42], the Adomian decomposition methods (ADM) [43, 44] and Laplace Adomian decomposition methods (LADM) [45–47]. The modified coupled KdV model has been analyzed numerically by applying the q-homotopy approach with Caputo operator by using Petrov-Galerkin method and product approximation technique [20, 48]. We will study the mCKdV equations with Caputo and CF derivatives by applying the modified double Laplace transform decomposing method (MDLDM) [49]. The decomposition method has extensively applied for many fractional models in physics and applied mathematics [50–52].

Some resent contribution of the time fractional order KdV equations have been studied by using different techniques [53–57]. The proposed method is an essential and effective approach to finding approximate solutions of the nonlinear models having time-fractional derivatives. In the non-linear model (1.1), the time fractional derivative has been taken in the form of Caputo's and Caputo-Fabrizio operator form which have a remarkable advantages in such physical models. The model is solved by using an affective method called modified double Laplace decomposition method (MDLDM) to obtain an approximate solution. The numerical solution of the model has been discuss in the form tables follow by its plots.

The rest of the paper is organized as follows: In Section 2, some main definitions, remarks, important results and a brief discussion of the proposed method is included. The convergence and uniqueness of the proposed method with the help of Banach contraction principle theorems are also studied in Section 3. In Section 4, the problem is discussed in Caputo's and Fabrizio forms and its solutions are obtained in a series form. We consider two numerical examples in Caputo's and Fabrizio form with the application of the proposed method, approximate solutions are obtained in the same section. This section also includes numerical discussion and figures for both the examples. Section 5 concludes the article followed by the Section 6 contain the future work in the manuscript, and Appendix contains the numerical values of the examples in the form of tables and also includes some parameters values.

2. Materials and methods

2.1. Some basic definitions

In this section, we provide some basic definitions, lemmas and remarks regarding to the proposed method. Here also given some basics rules and definition related to double Laplace transform and decomposition method.

Definition 1. [5,35,58] Caputo's fractional derivative of positive real order $\alpha > 0$ of a function $\phi(x,t)$ is given by function $\phi : (0,\infty) \to \mathbb{R}$ is defined by

$$^{c}D^{\alpha}\phi(x,t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1}\phi(x,t)^{n}(x,s)ds, \quad n-1 < \alpha \le n, \tag{2.11}$$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of a real number α, provided the right-hand side is point-wise defined on $(0,\infty)$.

Definition 2. [35, 59] Let $\phi(x,t) \in H^1(a,b)$, with $b > a$ and $\alpha \in (0,1]$, then the fractional order in sense of Caputo-Fabrizio is define as

$$^{CF}D^{\alpha}\phi(x,t) = \frac{M(\alpha)(2-\alpha)}{1-\alpha)} \int_0^t \phi(x,t)^n(x,s)exp\left(\frac{-\alpha t - s}{1-\alpha}\right)ds, \tag{2.12}$$

where $^{CF}D^{\alpha}\phi(x,t)$ is a fractional operator with Mittage-Leffler kernel in Caputo sense. $M(\alpha)$ is called normalization function with properties $M(0) = M(1) = 1$.

Definition 3. [60] For a $\phi(x,t)$, where $x, t > 0$ lies in xt-plane, the double Laplace transform of the function $\phi(x,t)$ is defined by

$$\mathscr{L}_x\mathscr{L}_t\,[\phi(x,t)] = \int_0^{\infty} e^{-px} \int_0^{\infty} e^{-st}\,\phi(x,t)\,dt\,dx, \tag{2.13}$$

where p and s are complex numbers.

Definition 4. [61, 62] Applying the definition of double Laplace transform on fractional order derivative with respect t and x in Caputo sense is given by

$$\mathscr{L}_x\mathscr{L}_t\left\{^{C}D_x^{\alpha}\,\phi(x,t)\right\} = p^{\alpha}\,\bar{\phi}(p,s) - \sum_{k=0}^{n-1} p^{\alpha-1-k}\mathscr{L}_t\left\{\frac{\partial^k u(0,t)}{\partial x^k}\right\}, \tag{2.14}$$

$$\mathscr{L}_x\mathscr{L}_t\left\{^{C}D_t^{\beta}\,\phi(x,t)\right\} = s^{\beta}\bar{\phi}(p,s) - \sum_{k=0}^{m-1} s^{\beta-1-k}\mathscr{L}_x\{\frac{\partial^k\phi(x,0)}{\partial t^k}\}, \tag{2.15}$$

where $n = [\alpha] + 1$, $m = [\beta] + 1$.

Definition 5. [61, 62] Applying the definition of double Laplace transform on fractional order derivative with respect t and x in Caputo-Fabrizio sense is given by

$$\mathscr{L}_x\mathscr{L}_t\left\{^{CF}D_x^{\alpha+r}\,\phi(x,t)\right\} = \frac{M(\alpha)}{P+\alpha(1-p)}\left[p^{r+1}\,\mathscr{L}_x\mathscr{L}_t\phi(p,s) - \sum_{k=1}^{n} p^{r-k}\mathscr{L}_t\left\{\frac{\partial^k\phi(0,t)}{\partial x^k}\right\}\right], \tag{2.16}$$

$$\mathscr{L}_x\mathscr{L}_t\left\{^{CF}D_t^{\beta+r}\,\phi(x,t)\right\} = \frac{M(\beta)}{s+\beta(1-s)}\left[s^{r+1}\,\mathscr{L}_x\mathscr{L}_t\phi(p,s) - \sum_{k=1}^{m} s^{r-k}\mathscr{L}_x\left\{\frac{\partial^k\phi(x,0)}{\partial t^k}\right\}\right], \tag{2.17}$$

where $n = [\alpha] + 1$, $m = [\beta] + 1$ for $r = 0, 1$. For Eq (2.17), with normalization property hold, we can write

$$\mathscr{L}_x\mathscr{L}_t\left\{^{CF}D_t^{\alpha}\,\phi(x,t)\right\} = \frac{1}{s+\alpha(1-s)}\left[s\,\mathscr{L}_x\mathscr{L}_t\phi(x,t) - \mathscr{L}_x\phi(x,0)\right],$$

$$\mathscr{L}_x\mathscr{L}_t\left\{^{CF}D_t^{\beta}\,\phi(x,t)\right\} = \frac{1}{s+\beta(1-s)}\left[s\,\mathscr{L}_x\mathscr{L}_t\phi(x,t) - s\mathscr{L}_t\phi(x,0) - \mathscr{L}_x\frac{\partial}{\partial t}\phi(x,0)\right].$$

From the above definitions, we can conclude that

$$\mathscr{L}_x\mathscr{L}_t\phi(x,t)\,\psi(x,t) = \bar{\phi}(p,s)\bar{\psi}(p,s) = \mathscr{L}_x\phi(x,t)\mathscr{L}_t\psi(x,t). \tag{2.18}$$

The inverse double Laplace transform $\mathscr{L}_x^{-1}\mathscr{L}_t^{-1}\{\bar{\phi}(x,t)\} = \phi(x,t)$, is represented by a complex double integral formula

$$\mathscr{L}_x^{-1}\mathscr{L}_t^{-1}\Big\{\bar{\phi}(x,t)\Big\} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}\int_{d-i\infty}^{d+i\infty} e^{px}\bar{\phi}(p,s)\,dp\,ds, \tag{2.19}$$

where $\bar{\phi}\,(p,s)$ is an analytic function $\forall p$ and s defined in the region by the inequalities $Re\,(p) \geq c$ and $Re\,(s) \geq d$, where $c, d \in \mathbb{R}$ to be considered accordingly.

2.2. Series solutions of mCKdV using the proposed method

The proposed method MDLDM is hybrid method used widely for non-linear ordinary or partial differential equations. It is the combination of double Laplace and Adomian decomposition method to obtain an approximate solution for problems in hand. To discuss the analysis of the proposed method, we consider the following coupled non-linear problems:

$$\begin{aligned} &L\,\phi(x,t) + R\,\phi(x,t)) + N_1\,(\phi,\psi)) = f_1(x,t),\\ &L\,\psi(x,t) + R\,\psi(x,t)) + N_1\,(\phi,\psi)) = f_2(x,t), \qquad \forall \quad t \in \mathbb{R}, \end{aligned} \tag{2.21}$$

where $L =^c D_t^{\alpha}(.)$ is the fractional-time derivative in Caputo's form, R_1, R_2 are the operators contains the linear terms of Eq (1.1), N_1, N_2 are the non-linear operators and $f_1(x,t)$, $f_2(x,t)$ are some external functions. Applying double Laplace on both sides to Eq (2.21), we obtain

$$\begin{aligned} &\mathscr{L}_x\mathscr{L}_t\{^cD_t^{\alpha}\,\phi\} + \mathscr{L}_x\mathscr{L}_t\{R_1\phi\} + \mathscr{L}_x\mathscr{L}_t\{N_1(\phi,\psi)\} = \mathscr{L}_x\mathscr{L}_t\{f_1(x,t)\},\\ &\mathscr{L}_x\mathscr{L}_t\{^cD_t^{\alpha}\,\psi)\} + \mathscr{L}_x\mathscr{L}_t\{R_2\psi)\} + \mathscr{L}_x\mathscr{L}_t\{N_2(\phi,\psi)\} = \mathscr{L}_x\mathscr{L}_t\{f_2(x,t)\}. \end{aligned} \tag{2.22}$$

Using the scheme defined above of double Laplace on the n^{th}-derivative in Caputo's form, we obtain

$$\begin{aligned} \mathscr{L}_x\mathscr{L}_t\{\phi(x,t))\} &= G_1(p,s) - \frac{1}{s^{\alpha}}\mathscr{L}_x\mathscr{L}_t\{R_1\phi(x,t))\} - \frac{1}{s^{\alpha}}\mathscr{L}_x\mathscr{L}_t\{N_1(\phi,\psi)\},\\ \mathscr{L}_x\mathscr{L}_t\{\psi(x,t))\} &= G_2(p,s) - \frac{1}{s^{\alpha}}\mathscr{L}_x\mathscr{L}_t\{R_1\psi(x,t))\} - \frac{1}{s^{\alpha}}\mathscr{L}_x\mathscr{L}_t\{N_2(\phi,\psi)\}, \end{aligned} \tag{2.23}$$

where

$$\begin{aligned} G_1(p,s) &= s^{\alpha}\mathscr{L}_x\mathscr{L}_t\{\phi(x,t))\} - \sum_{k=0}^{n-1} s^{\alpha-1-k}\mathscr{L}_x\Big\{\frac{\partial^k\phi(x,0)}{\partial t^k}\Big\} + f_1(x,t),\\ G_2(p,s) &= s^{\alpha}\mathscr{L}_x\mathscr{L}_t\{\psi(x,t))\}) - \sum_{k=0}^{n-1} s^{\alpha-1-k}\mathscr{L}_x\Big\{\frac{\partial^k\psi(x,0)}{\partial t^k}\Big\} + f_2(x,t). \end{aligned} \tag{2.24}$$

Consider the series solution of the form

$$\phi(x,t)) = \sum_{n=0}^{\infty}\phi_n(x,t), \quad \psi(x,t)) = \sum_{n=0}^{\infty}\psi_n(x,t), \tag{2.25}$$

let N_1 and N_2 be the non-linear operators in the model are defined by

$$N_1(\phi,\psi) = \sum_{n=0}^{\infty} A_n, \quad N_2(\phi,\psi) = \sum_{n=0}^{\infty} B_n, \tag{2.26}$$

where A_n and B_n are well known Adomian polynomials [50] of the functions $\phi_0, \phi_1, \phi_2, \dots$ and $\psi_0, \psi_1, \psi_2, \dots$ respectively. For convince consider A_n can be described by the following formula:

$$A_n(\phi_0, \dots, \psi_0, \dots) = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N_1 \sum_{k=0}^{n} \lambda^k \phi_k(x,t) \sum_{k=0}^{n} \lambda^k \psi_k(x,t)\right]_{\lambda=0}. \tag{2.27}$$

Applying an inverse double Laplace on both sides to Eq (2.23):

$$\sum_{n=0}^{\infty} \phi_n(x,t) = \mathscr{L}_x^{-1}\mathscr{L}_t^{-1} G_1(p,s) - \mathscr{L}_x^{-1}\mathscr{L}_t^{-1}\left\{\frac{1}{s^\alpha}\mathscr{L}_x\mathscr{L}_t\{R_1\phi(x,t))\}\right\} - \mathscr{L}_x^{-1}\mathscr{L}_t^{-1}\left\{\frac{1}{s^\alpha}\mathscr{L}_x\mathscr{L}_t \sum_{n=0}^{\infty} A_n\right\},$$
$$\sum_{n=0}^{\infty} \psi_n(x,t) = \mathscr{L}_x^{-1}\mathscr{L}_t^{-1} G_2(p,s) - \mathscr{L}_x^{-1}\mathscr{L}_t^{-1}\left\{\frac{1}{s^\alpha}\mathscr{L}_x\mathscr{L}_t\{R_2\psi(x,t))\}\right\} - \mathscr{L}_x^{-1}\mathscr{L}_t^{-1}\left\{\frac{1}{s^\alpha}\mathscr{L}_x\mathscr{L}_t \sum_{n=0}^{\infty} B_n\right\},$$

equating terms on both sides, we obtain

$$\begin{aligned}
\phi_0 &= \mathscr{L}_x^{-1}\mathscr{L}_t^{-1}[G_1(p,s)], \\
\psi_0 &= \mathscr{L}_x^{-1}\mathscr{L}_t^{-1}[G_2(p,s)], \\
\phi_1 &= -\mathscr{L}_x^{-1}\mathscr{L}_t^{-1}\left\{\frac{1}{s^\alpha}\mathscr{L}_x\mathscr{L}_t\{R_1\phi_0\}\right\} - \mathscr{L}_x^{-1}\mathscr{L}_t^{-1}\left\{\frac{1}{s^\alpha}\mathscr{L}_x\mathscr{L}_t \sum_{n=0}^{\infty} A_0\right\}, \\
\psi_1 &= -\mathscr{L}_x^{-1}\mathscr{L}_t^{-1}\left\{\frac{1}{s^\alpha}\mathscr{L}_x\mathscr{L}_t\{R_1\psi_0\}\right\} - \mathscr{L}_x^{-1}\mathscr{L}_t^{-1}\left\{\frac{1}{s^\alpha}\mathscr{L}_x\mathscr{L}_t \sum_{n=0}^{\infty} B_0\right\}, \\
\phi_2 &= -\mathscr{L}_x^{-1}\mathscr{L}_t^{-1}\left\{\frac{1}{s^\alpha}\mathscr{L}_x\mathscr{L}_t\{R_1\phi_1\}\right\} - \mathscr{L}_x^{-1}\mathscr{L}_t^{-1}\left\{\frac{1}{s^\alpha}\mathscr{L}_x\mathscr{L}_t \sum_{n=0}^{\infty} A_1\right\}, \\
\psi_2 &= -\mathscr{L}_x^{-1}\mathscr{L}_t^{-1}\left\{\frac{1}{s^\alpha}\mathscr{L}_x\mathscr{L}_t\{R_1\psi_1\}\right\} - \mathscr{L}_x^{-1}\mathscr{L}_t^{-1}\left\{\frac{1}{s^\alpha}\mathscr{L}_x\mathscr{L}_t \sum_{n=0}^{\infty} B_1\right\}, \\
\phi_3 &= -\mathscr{L}_x^{-1}\mathscr{L}_t^{-1}\left\{\frac{1}{s^\alpha}\mathscr{L}_x\mathscr{L}_t\{R_1\phi_2\}\right\} - \mathscr{L}_x^{-1}\mathscr{L}_t^{-1}\left\{\frac{1}{s^\alpha}\mathscr{L}_x\mathscr{L}_t \sum_{n=0}^{\infty} A_2\right\}, \\
\psi_2 &= -\mathscr{L}_x^{-1}\mathscr{L}_t^{-1}\left\{\frac{1}{s^\alpha}\mathscr{L}_x\mathscr{L}_t\{R_1\psi_2\}\right\} - \mathscr{L}_x^{-1}\mathscr{L}_t^{-1}\left\{\frac{1}{s^\alpha}\mathscr{L}_x\mathscr{L}_t \sum_{n=0}^{\infty} B_2\right\}.
\end{aligned}$$

In general, the following recursive formulas can be obtained, the final solutions can be obtained as

$$\phi_{n+1} = \phi(x,0) - \mathscr{L}_x^{-1}\mathscr{L}_t^{-1}\left\{\frac{1}{s^\alpha}\mathscr{L}_x\mathscr{L}_t\{R_1\phi_n\}\right\} - \mathscr{L}_x^{-1}\mathscr{L}_t^{-1}\left\{\frac{1}{s^\alpha}\mathscr{L}_x\mathscr{L}_t \sum_{n=0}^{\infty} A_n\right\}, \tag{2.29}$$

$$\psi_{n+1} = \psi(x,0) - \mathscr{L}_x^{-1}\mathscr{L}_t^{-1}\left\{\frac{1}{s^\alpha}\mathscr{L}_x\mathscr{L}_t\{R_1\psi_n\}\right\} - \mathscr{L}_x^{-1}\mathscr{L}_t^{-1}\left\{\frac{1}{s^\alpha}\mathscr{L}_x\mathscr{L}_t \sum_{n=0}^{\infty} B_n\right\}, \tag{2.30}$$

for numerical purposes we take $\alpha = 1$, the exact solution can be obtained as

$$\lim_{n\to\infty}(\phi_n, \psi_n) = (\phi, \psi). \tag{2.31}$$

Similar procedures can be used to a problem having Caputo-Fabrizio fractional-time derivative to obtain a recursive relation.

3. Convergence analysis of the proposed method

In this section, we are discussing the existence and uniqueness of the solutions (2.29) and (2.30) of the general coupled time fractional non-linear partial differential equation (2.21). The following theorems are taken from [63].

Theorem 1. *(Uniqueness theorem) For* $0 < \sigma < 1$*, the solution (2.29) of Eq (2.21) is a unique solution, where* $\sigma = \frac{(\kappa_1+\kappa_2+\kappa_3)t^{\alpha+1}}{\Gamma(\alpha)}$.

Proof. We create a mapping T : B→ B where $B = (C[J], \|.\|)$ is the Banach space of all continuous functions on $J = [0, T]$ with the norm $\|.\|$. We can write Eq (2.29) in the following form:

$$\phi_{n+1} = \phi(x,0) + \mathscr{L}_x^{-1}\mathscr{L}_t^{-1}\left\{\frac{1}{s^\alpha}\mathscr{L}_x\mathscr{L}_t\Big(L\phi_n + M\phi_n + N\phi_n\Big)\right\},$$

where $L = -\frac{3}{2}\frac{\partial^2\psi}{\partial x^2} + 3\lambda\frac{\partial^2\phi}{\partial x^2}$, $M = -\frac{\partial^3\phi}{\partial x^3}$ and $N = 3\phi^2\frac{\partial^2\psi}{\partial x^2} - 3\psi\frac{\partial\phi}{\partial x} - 3\phi\frac{\partial\psi}{\partial x}$. Let $L\phi$, $M\phi$ and $N\phi$ are also Lipschitzian with $\left|L\phi - L\bar{\phi}\right| < \kappa_1\left|\phi - \bar{\phi}\right|$, $\left|M\phi - M\bar{\phi}\right| < \kappa_2\left|\phi - \bar{\phi}\right|$ and $\left|N\phi - N\bar{\phi}\right| < \kappa_3\left|\phi - \bar{\phi}\right|$ where κ_1, κ_2 and κ_3 are Lipschitz constants, ϕ and $\bar{\phi}$ are the function's distinct values. Now we proceed as follow:

$$\begin{aligned}
\left\|T\phi - T\bar{\phi}\right\| &= \max_{t\in J}\left|\mathscr{L}_x^{-1}\mathscr{L}_t^{-1}\left\{\frac{1}{s^\alpha}\mathscr{L}_x\mathscr{L}_t\Big(L\phi + M\phi + N\phi\Big)\right\} - \mathscr{L}_x^{-1}\mathscr{L}_t^{-1}\left\{\frac{1}{s^\alpha}\mathscr{L}_x\mathscr{L}_t\Big(L\bar{\phi} + M\bar{\phi} + N\bar{\phi}\Big)\right\}\right| \\
&\leq \max_{t\in J}\left|\mathcal{N}^{-1}\left\{\frac{1}{s^\alpha}\mathscr{L}_x\mathscr{L}_t\Big(L\phi - L\bar{\phi}\Big)\right\} + \mathscr{L}_x\mathscr{L}_t^{-1}\left\{\frac{1}{s^\alpha}\mathscr{L}_x\mathscr{L}_t\Big(M\phi - M\bar{\phi}\Big)\right\}\right. \\
&\left. + \mathscr{L}_x\mathscr{L}_t^{-1}\left\{\frac{1}{s^\alpha}\mathscr{L}_x\mathscr{L}_t\Big(N\phi - N\bar{\phi}\Big)\right\}\right| \\
&\leq \max_{t\in J}\left|\kappa_1\mathscr{L}_x^{-1}\mathscr{L}_t^{-1}\left\{\frac{1}{s^\alpha}\mathscr{L}_x\mathscr{L}_t\Big(\phi - \bar{\phi}\Big)\right\} + \kappa_2\mathscr{L}_x^{-1}\mathscr{L}_t^{-1}\left\{\frac{1}{s^\alpha}\mathscr{L}_x\mathscr{L}_t\Big(\phi - \bar{\phi}\Big)\right\}\right. \\
&\left. + \kappa_3\mathscr{L}_x^{-1}\mathscr{L}_t^{-1}\left\{\frac{1}{s^\alpha}\mathscr{L}_x\mathscr{L}_t\Big(N\phi - \bar{\phi}\Big)\right\}\right| \\
&\leq \max_{t\in J}(\kappa_1 + \kappa_2 + \kappa_3)\left|\mathscr{L}_x^{-1}\mathscr{L}_t^{-1}\left\{\frac{1}{s^\alpha}\mathscr{L}_x\mathscr{L}_t\Big(\phi - \bar{\phi}\Big)\right\}\right| = \frac{(\kappa_1 + \kappa_2 + \kappa_3)t^{(\alpha-1)}}{\Gamma(\alpha)}\left\|\phi - \bar{\phi}\right\|.
\end{aligned}$$

The mapping is contraction under the condition $0 < \sigma < 1$. As a result of the Banach fixed point theorem for contraction, Eq (2.21) has a unique solution. This marks the end of the proof. □

Next we discuss convergence analysis of the problem.

Theorem 2. *(Convergence theorem) The solution of Eq (2.21) in general forum will be convergence.*

Proof. The Banach space of all continuous functions on the interval J with the norm $\|\phi\|=\max_{t\in J}\|\phi\|$ is denoted as $(C[J], \|.\|)$. Define the $\{Sn\}$ sequence of partial sums i.e $S_n = \sum_{j=0}^{n}$. With $n \geq m$, let S_n and S_m be arbitrary partial sums. In this Banach space, we will show that S_n is a Cauchy sequence. We obtain by employing a new formulation of Adomian polynomials.

$$P(S_n) = \bar{A}_n + \sum_{r1=0}^{n-1} \bar{A}_{r_1}, \quad Q(S_n) = \bar{A}_n + \sum_{r2=0}^{n-1} \bar{A}_{r_2},$$

now

$$\begin{aligned}
\|S_n - S_m\| &= \left|\sum_{j=0}^{n} \phi_j - \sum_{k=0}^{m} \phi_k\right| = \max_{t\in J}\left|\sum_{j=m+1}^{n} \phi_j\right| \leq \max_{t\in J}\left|\mathscr{L}_x^{-1}\mathscr{L}_t^{-1}\left\{\frac{1}{s^\alpha}\mathscr{L}_x\mathscr{L}_t\Big(\sum_{j=m+1}^{n} L\phi_{j-1}\Big)\right\}\right.\\
&\quad \left. + \mathscr{L}_x^{-1}\mathscr{L}_t^{-1}\left\{\frac{1}{s^\alpha}\mathscr{L}_x\mathscr{L}_t\Big(\sum_{j=m+1}^{n} M\phi_{j-1}\Big)\right\} + \mathscr{L}_x^{-1}\mathscr{L}_t^{-1}\left\{\frac{1}{s^\alpha}\mathscr{L}_x\mathscr{L}_t\Big(\sum_{j=m+1}^{n} A_{j-1}\Big)\right\}\right|\\
&= \max_{t\in J}\left|\mathscr{L}_x^{-1}\mathscr{L}_t^{-1}\left\{\frac{1}{s^\alpha}\mathscr{L}_x\mathscr{L}_t\Big(\sum_{j=m}^{n-1} L\phi_j\Big)\right\} + \mathscr{L}_x^{-1}\mathscr{L}_t^{-1}\left\{\frac{1}{s^\alpha}\mathscr{L}_x\mathscr{L}_t\Big(\sum_{j=m}^{n-1} M\phi_j\Big)\right\}\right.\\
&\quad \left. + \mathscr{L}_x^{-1}\mathscr{L}_t^{-1}\left\{\frac{1}{s^\alpha}\mathscr{L}_x\mathscr{L}_t\Big(\sum_{j=m}^{n-1}(A_j)\Big)\right\}\right|\\
&\leq \max_{t\in J}\left|\mathscr{L}_x^{-1}\mathscr{L}_t^{-1}\left\{\frac{1}{s^\alpha}\mathscr{L}_x\mathscr{L}_t\Big(\sum_{j=m}^{n-1} LS_{n-1} - LS_{m-1}\Big)\right\}\right.\\
&\quad \left. + \mathscr{L}_x^{-1}\mathscr{L}_t^{-1}\left\{\frac{1}{s^\alpha}\mathscr{L}_x\mathscr{L}_t\Big(\sum_{j=m}^{n-1} MS_{n-1} - MS_{m-1}\Big)\right\} + \mathscr{L}_x^{-1}\mathscr{L}_t^{-1}\left\{\frac{1}{s^\alpha}\mathscr{L}_x\mathscr{L}_t\Big(\sum_{j=m}^{n-1} NS_{n-1} - NS_{m-1}\Big)\right\}\right|\\
&\leq \kappa_1 \max_{t\in J}\left|\mathscr{L}_x^{-1}\mathscr{L}_t^{-1}\left\{\frac{1}{s^\alpha}\mathscr{L}_x\mathscr{L}_t\Big(S_{n-1} - S_{m-1}\Big)\right\}\right| + \kappa_2 \max_{t\in J}\left|\mathscr{L}_x^{-1}\mathscr{L}_t^{-1}\left\{\frac{1}{s^\alpha}\mathscr{L}_x\mathscr{L}_t\Big(S_{n-1} - S_{m-1}\Big)\right\}\right|\\
&\quad + \kappa_3 \max_{t\in J}\left|\mathscr{L}_x^{-1}\mathscr{L}_t^{-1}\left\{\frac{1}{s^\alpha}\mathscr{L}_x\mathscr{L}_t\Big(S_{n-1} - S_{m-1}\Big)\right\}\right| = \frac{(\kappa_1+\kappa_2+\kappa_3)t^{\alpha-1}}{\Gamma(\alpha)}\|S_{n-1} - S_{m-1}\|.
\end{aligned}$$

Choosing $n = m + 1$, then

$$\|S_{m+1} - S_m\| \leq \sigma\|S_m - S_{m-1}\| \leq \sigma^2\|S_{m-1} - S_{m-2}\| \leq \cdots\cdots \leq \sigma^m\|S_1 - S_0\|,$$

with $\sigma = \frac{(\kappa_1+\kappa_2+\kappa_3)t^{\alpha-1}}{\Gamma(\alpha)}$, by using the following triangular inequality

$$\begin{aligned}
\|S_n - S_m\| &\leq \|S_m - S_{m+1}\| \leq \|S_{m+1} - S_{m+2}\| \leq \cdots\cdots \leq \sigma^m\|S_1 - S_0\|\\
&\leq \left(\sigma^m + \sigma^{m+1} + \cdots + \sigma^n\right)\|S_1 - S_0\|\\
&\leq \sigma^m\left(1 + \sigma + \sigma^2 + \cdots + \sigma^{n-m-1}\right)\|S_1 - S_0\| \leq \sigma^m\left(\frac{1-\sigma^{n-m}}{1-\sigma}\right)\|\phi_1\|.
\end{aligned}$$

Now by definition $0 < \sigma < 1$, we have $1 - \sigma^{n-m} < 1$, thus we have

$$\|S_n - S_m\| \leq \frac{\sigma^m}{1-\sigma}\max_{t\in J}\|\phi_1\|, \tag{3.1}$$

and also as $|\phi| < \infty$ (ϕ is bounded), therefore, $\|S_n - S_m\| \to 0$, hence S_n is a Cauchy sequence in the Banach space B, so the series $\sum_{j=0}^{n} \phi_j$ is convergent. □

Remark 1. *Theorems 1 and 2 can also be applicable to the solution (2.30) for its uniqueness and convergent.*

4. Applications of the proposed method

In this section, we illustrate some examples on Eq (1.1) with time-fractional derivatives in the form of Caputo and Caputo-Fabrizio operators in each case, and apply the proposed method discussed in this section with $(f_1, f_2) = 0$.

Example 1. *Consider the following (mCKdV) [20] in Caputo's form*

$$\begin{aligned}
&{}^{c}D_t^{\alpha}\phi + 3\phi^2\frac{\partial\phi}{\partial x} - \frac{3}{2}\frac{\partial^2\psi}{\partial x^2} - 3\phi\frac{\partial\psi}{\partial x} - 3\psi\frac{\partial\phi}{\partial x} + 3\lambda\frac{\partial\phi}{\partial x} - \frac{1}{2}\frac{\partial^3\phi}{\partial x^3} = 0,\\
&{}^{c}D_t^{\alpha}\psi + 3\psi\frac{\partial\psi}{\partial x} + 3\frac{\partial\phi}{\partial x}\frac{\partial\psi}{\partial x} - 3\phi^2\frac{\partial\psi}{\partial x} - 3\lambda\frac{\partial\psi}{\partial x} + \frac{\partial^3\psi}{\partial x^3} = 0, \qquad 0 < \alpha \le 1, \quad t > 0,
\end{aligned} \tag{4.1}$$

with initial conditions

$$\phi(x,0) = \frac{r_1}{2k} + k\tanh(kx), \quad \psi(x,0) = \frac{\lambda(r_1 + k)}{2r_1} + r_1\tanh(kx). \tag{4.2}$$

The exact solution of Eq (4.1) for $\alpha = 1$ is [20]

$$\phi(x,t) = \frac{r_1}{2k} + k\tanh(\xi), \quad \psi(x,t) = \frac{\lambda(r_1 + k}{2r_1} + r_1\tanh(\xi), \tag{4.3}$$

where

$$\xi = kx + \frac{k}{4}\left(-6\lambda - 4k^2 - \frac{k\lambda}{r_1} + \frac{3r_1^2}{k^2}\right)t.$$

Apply the modified double Laplace transform decomposition (MDLDM) scheme discussed in this section to Eq (4.1) and single Laplace transform to Eq (4.2), we obtain the following recurrence relations:

$$\begin{aligned}
&\phi_0 = G_1(p,s), \quad \phi_{n+1} = \frac{t^{(n+1)\alpha}}{\Gamma((n+1)\alpha+1)}\left[-3A_n + \frac{3}{2}\psi_{nxx} + 3B_n + 3C_n - 3\lambda\phi_{nx} + \frac{1}{2}\phi_{nxxx}\right],\\
&\psi_0 = G_2(p,s), \quad \psi_{n+1} = \frac{t^{(n+1)\alpha}}{\Gamma((n+1)\alpha+1)}\left[-3D_n - 3E_n + 3F_n + 3\lambda\psi_{nx} - \psi_{nxxx}\right],
\end{aligned} \tag{4.4}$$

for $n \ge 0$. The Adomian polynomials $A_n, \cdots F_n$, $n = 1,2,3$, for nonlinearity appears in Eq (4.4) are calculated by using the general rules shown in Eq (2.27)

$$\begin{aligned}
&A_0 = \phi_0^2\phi_{0x}, \quad A_1 = 2\phi_0\phi_1\phi_{0x} + \phi_0^2\phi_{1x}, \quad A_2 = 2\phi_0\phi_2\phi_{0x} + \phi_1^2\phi_{0x} + 2\phi_0\phi_1\phi_{1x} + \phi_0^2\phi_{2x},\\
&B_0 = \phi_0\psi_{0x}, \quad B_1 = \phi_1\psi_{0x} + \phi_0\psi_{1x}, \quad B_2 = \phi_2\psi_{0x} + \phi_1\psi_{1x} + \phi_0\psi_{2x},\\
&C_0 = \psi_0\phi_{0x}, \quad C_1 = \psi_1\phi_{0x} + \psi_0\phi_{1x}, \quad C_2 = \psi_2\phi_{0x} + \psi_1\phi_{1x} + \psi_0\phi_{2x},\\
&D_0 = \psi_0\psi_{0x}, \quad D_1 = \psi_1\psi_{0x} + \psi_0\psi_{1x}, \quad D_2 = \psi_2\psi_{0x} + \psi_1\psi_{1x} + \psi_0\psi_{2x},\\
&E_0 = \phi_{0x}\psi_{0x}, \quad E_1 = \phi_{1x}\psi_{0x} + \phi_{0x}\psi_{1x}, \quad E_2 = \phi_{2x}\psi_{0x} + \phi_{1x}\psi_{1x} + \phi_{0x}\psi_{2x},\\
&F_0 = \phi_0^2\psi_{0x}, \quad F_1 = 2\phi_0\phi_1\psi_{0x} + \phi_0^2\psi_{1x}, \quad F_2 = 2\phi_0\phi_2\psi ox + \phi_1^2\psi_{0x} + 2\phi_0\phi_1\psi_{1x} + \phi_0^2\psi_{2x}.
\end{aligned} \tag{4.5}$$

For more details, one can see [51, 52, 64]. Plugging Eq (4.5) in Eq (4.4), we obtain

$$
\begin{aligned}
\phi_0 &= \frac{r_1}{2k} + k\tanh(kx), \quad \psi_0 = \frac{\lambda(r_1+k)}{2r_1} + r_1\tanh(kx),\\
\phi_1 &= \frac{t^{\alpha}}{4r_1\Gamma(\alpha+1)}\left[\left(3r_1^3 - 4r_1k^4 - 6r_1\lambda k^2 + 6\lambda k^3\right)\operatorname{sech}^2(kx)\right],\\
\psi_1 &= \frac{t^{\alpha}}{4k\Gamma(\alpha+1)}\left[\left(-4k^4r_1 - 6\lambda k^3 + 6\lambda k^2 r_1 + 3r_1^3\right)\operatorname{sech}^2(kx)\right],\\
\phi_2 &= -\frac{t^{2\alpha}}{(8r_1^2k)\Gamma(2\alpha+1)}\Big[\eta_1\tanh(kx)\operatorname{sech}^2(kx)\Big],\\
\psi_2 &= \frac{t^{2\alpha}}{(16r_1k^2)\Gamma(2\alpha+1)}\Big[\Big(a_1\tanh(kx)(a_2\cosh(2kx)) + a_3\tanh(kx))\Big)\operatorname{sech}^4(kx)\Big],\\
\phi_3 &= \frac{t^{3\alpha}}{(32r_1^3k)\Gamma(3\alpha+1)}\Big[\left(2b_0 + 3\operatorname{sech}^2(kx)\left(2r_1\tanh(kx)\left(b_3\operatorname{sech}^2(kx)\right) + b_2\operatorname{sech}^2(kx)\right)\right)\operatorname{sech}^2(kx)\Big],\\
\psi_3 &= \frac{t^{3\alpha}}{(32r_1^2k^3)\Gamma(3\alpha+1)}\Big[3c_1c_2\tanh(kx)\operatorname{sech}^4(kx)(2c_0 + 3\operatorname{sech}^2(kx))(c_3\operatorname{sech}^2(kx) - \operatorname{sech}^2(kx))\Big],
\end{aligned}
$$

where the coefficients given above can be seen in Appendix. It should be noted that, other terms can be calculated in the similar way. The final solutions can be obtained

$$
\phi(x,t) = \sum_{n=0}^{\infty}\phi(x,t), \quad \psi(x,t) = \sum_{n=0}^{\infty}\psi_n(x,t). \tag{4.6}
$$

For numerical purposes we take $\alpha = 1$, the exact solution can be obtained as

$$
\lim_{n\to\infty}(\phi_n, \psi_n) = (\phi, \psi).
$$

For the numerical illustrations, Figure 1 depicts obtained solutions ϕ and ψ in Eq (4.6) associated with the modified coupled Korteweg-de Vries (mCKdV) Eq (4.1). Thus the waves solutions $\phi(x,t)$ and $\psi(x,t)$ in Figure 1(a),(b) reveals the evolution of the shock excitations. The three-dimensional profiles of $\phi(x,t)$ and $\psi(x,t)$ in Figure 1(c),(d) manifest sudden changes in the potential fields for the spatial variables $-10 \le x \le 10$. We observe that the mCKdV equations describe nonlinear evolution of the waves, suffered by weak dispersion effect in an inviscid fluid. Thus the nonlinear steeping attributed to temporal evolution excites the shocks. For the impact of the time fraction coefficient α on the waves characteristics, we have displayed solutions (4.6) versus x at $\alpha = 0.1, 0.2, 0.4, 1$, see Figure 2(a),(b). Notice that a degree enhancement in α modifies the steeping effect and therefore rise the waves amplitudes. Similarly Figure 2(c),(d) show the nature of the MDLDM method in Caputo's sense for different values of α and time when spatial variable x is kept constant of the Eq (4.6) in Example 1. The wave solutions are revealing the propagation of monotonic shocks. Figure 3(a),(b) show the error plots of the Table 1 (see Appendix).

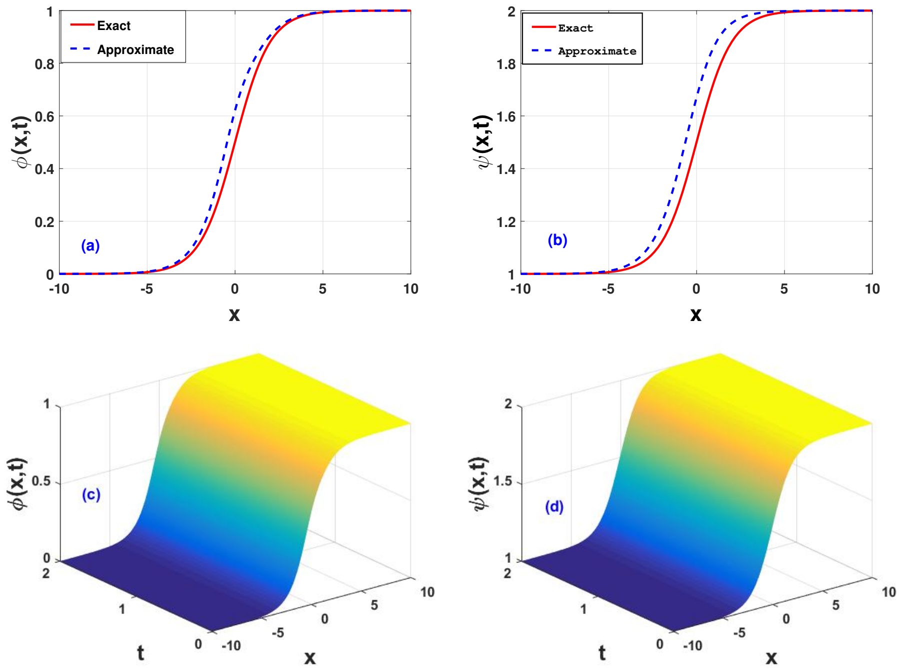

Figure 1. (a) and (b): Comparison of exact (Solid curve for $\alpha = 1$) with approximate (ϕ, ψ)-solution (Dashed curves for $\alpha = 0.5$) by MDLDM, (c) and (d): 3D plots for exact and approximate (ϕ, ψ)-solution via MDLDM for Example 1.

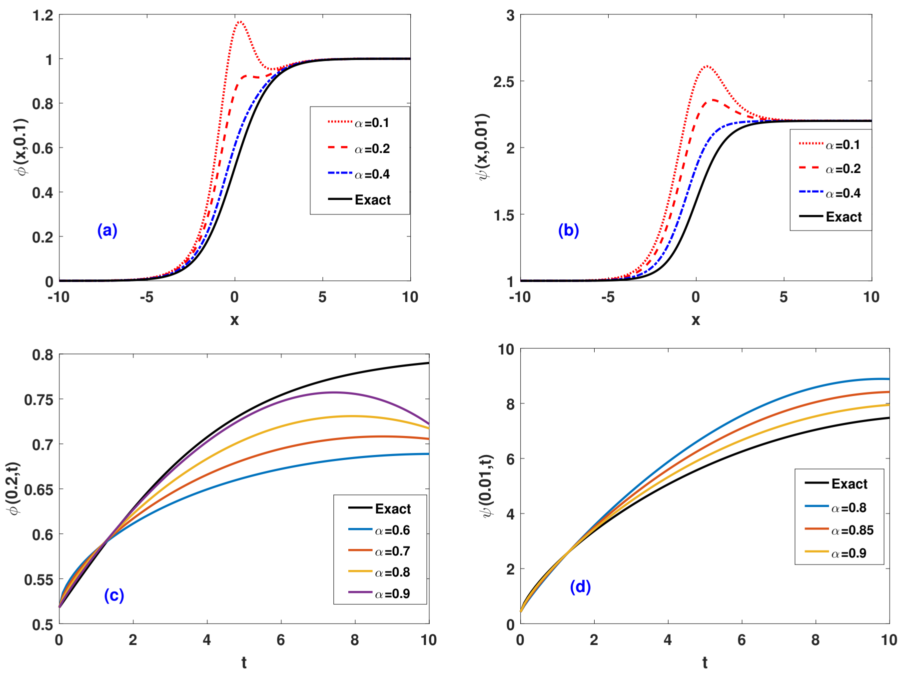

Figure 2. (a) and (b): Comparison between approximate and exact solutions when α values approaching 1, (c) and (d): MDLDM Caputo solution when time and α are changing with fixed x-values (x=0.2) for b=k=0.3, $\lambda = 1.5$ of Example 1.

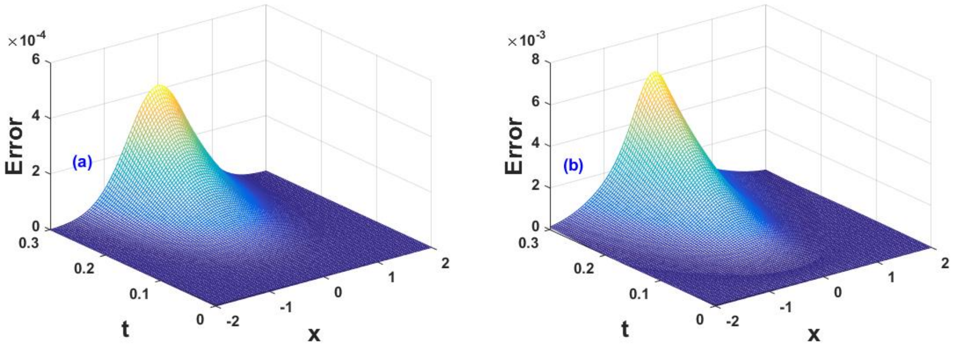

Figure 3. (a) and (b): The absolute error plots of the Table 1.

Example 2. *Consider the following example having time fractional derivative in Caputo-Fabrizio (CF) form*

$$\begin{aligned} &{}^{CF}D_t^{\alpha}\phi + 3\phi^2\frac{\partial\phi}{\partial x} - \frac{3}{2}\frac{\partial^2\psi}{\partial x^2} - 3\phi\frac{\partial\psi}{\partial x} - 3\psi\frac{\partial\phi}{\partial x} + 3\lambda\frac{\partial\phi}{\partial x} - \frac{1}{2}\frac{\partial^3\phi}{\partial x^3} = 0, \\ &{}^{CF}D_t^{\alpha}\psi + 3\psi\frac{\partial\psi}{\partial x} + 3\frac{\partial\phi}{\partial x}\frac{\partial\psi}{\partial x} - 3\phi^2\frac{\partial\psi}{\partial x} - 3\lambda\frac{\partial\psi}{\partial x} + \frac{\partial^3\psi}{\partial x^3} = 0, \qquad 0 < \alpha \le 1, \quad t > 0. \end{aligned} \tag{4.7}$$

With the help of Eq (2.17) and the procedure used in Example 1, we can obtain the following series solutions in Caputo-Fabrizio form

$$\begin{aligned}
\phi_0 &= \frac{r_1}{2k} + k\tanh(kx), \quad \psi_0 = \frac{\lambda(r_1+k)}{2r_1} + r_1\tanh(kx), \\
\phi_1 &= \frac{1+\alpha(t-1)}{4r_1}\left[\left(3r_1^3 - 4r_1k^4 - 6r_1\lambda k^2 + 6\lambda k^3\right)\operatorname{sech}^2(kx)\right], \\
\psi_1 &= \frac{1+\alpha(t-1)}{4k}\left[\left(-4k^4r_1 - 6\lambda k^3 + 6\lambda k^2 r_1 + 3r_1^3\right)\operatorname{sech}^2(kx)\right], \\
\phi_2 &= -\frac{h_1(t)}{8r_1^2k}\left[\eta_1\tanh(kx)\operatorname{sech}^2(kx)\right], \\
\psi_2 &= \frac{h_1(t)}{(16r_1k^2)}\left[\left(a_1\tanh(kx)(a_2\cosh(2kx)) + a_3\tanh(kx))\right)\operatorname{sech}^4(kx)\right], \\
\phi_3 &= \frac{h_2(t)}{(32r_1^3k)}\left[\left(2b_0 + 3\operatorname{sech}^2(kx)\left(2r_1\tanh(kx)\left(b_3\operatorname{sech}^2(kx)\right) + b_2\operatorname{sech}^2(kx)\right)\right)\operatorname{sech}^2(kx)\right], \\
\psi_3 &= \frac{h_2(t)}{(32r_1^2k^3)}\left[(2c_0 + 3\operatorname{sech}^2(kx)(c_1\tanh(kx)(c_3\operatorname{sech}^2(kx) - \operatorname{sech}^2(kx)\left(3c_2\operatorname{sech}^2(kx)\right))))\operatorname{sech}^2(kx)\right],
\end{aligned}$$

where

$$\begin{aligned}
h_1(t) &= 1/2\,\alpha^2t^2 + \alpha^2 - 2\,\alpha + 1 - 2\,\alpha^2t + 2\,\alpha\,t, \\
h_2(t) &= 3/2\,\alpha^3t^2 - 3/2\,\alpha^2t^2 + \alpha^3 - 3\,\alpha^2 + 3\,\alpha - 1 - 3\,\alpha^3t + 6\,\alpha^2t - 3\,\alpha\,t - 1/6\,\alpha^3t^3.
\end{aligned}$$

Other terms can be calculated in the similar way. The general solutions can be obtained in the form

$$\phi(x,t) = \sum_{n=0}^{\infty}\phi_n(x,t), \quad \psi(x,t) = \sum_{n=0}^{\infty}\psi_n(x,t). \tag{4.8}$$

For numerical purposes we take $\alpha = 1$, the exact solution can be obtained as

$$\lim_{n\to\infty}(\phi_n, \psi_n) = (\phi, \psi).$$

The shock wave solutions $\phi(x, t)$ and $\psi(x, t)$ in Eq (4.8) for the modified coupled Korteweg-de Vries (mCKdV) Eq 4.1 with Caputo's Fabrizio operators (CF) are depicted in Figure 4(a),(b) against x. Obviously, the approximate Fabrizio solution (red stared dotted and blue dashed, black dotted curves) in both the plots (a) and (b) by MDLDM method is approaching to the exact solution (black solid curve) of the Eq (4.8) in Example 2, the corresponding plots in three dimension are shown in Figure 4(c),(d).

Figure 5(a),(b) show the nature of the MDLDM method in Fabrizio's sense for different values of α and time when spatial variable x is kept constant of the Eq (4.8) in Example 2. For the purpose of error analysis, we have depicted the absolute of the difference of exact and approximate solutions $|Exact - (\phi(x, t)\,\psi(x, t))|$ and for mCKdV equation, see Figure 6(a),(b). Notice that the difference of the wave solutions from the exact solutions is much small. It signifies the importance of the MDLDM for the approximate solutions. The observed errors are inserted in Table 2 (see Appendix) and their related plots are shown in Figure 6(a),(b) which shows the effectiveness of our proposed method to such non-linear modified coupled Korteweg-de Vries (mCKdV) equations. A comparative analysing of the problem in Tables 3–6 (see Appendix) between the two methods (MDLDM and q-HATM [20]) are shown. By analysing these tables, one can conclude easily the validation and accuracy of the proposed method over the q-HATM. Figure 7(a)–(d) are also show a remarkable results for the presented method.

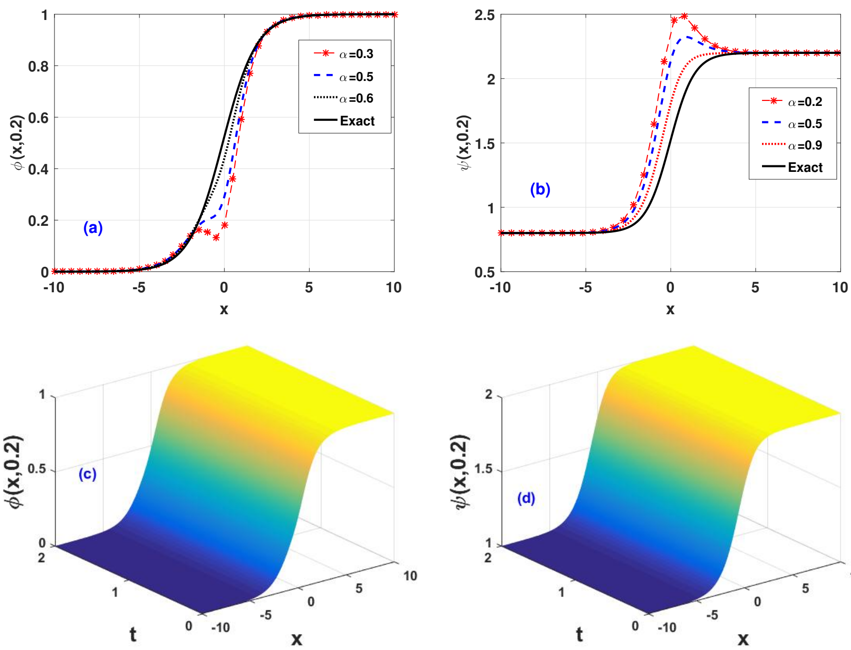

Figure 4. (a) and (b): Comparison between the exact with approximate CF (ϕ, ψ)-solution for k=b=0.5, $\lambda = 1.5$, (c) and (d): 3D plots for exact and approximate (ϕ, ψ)-solution for Example 2.

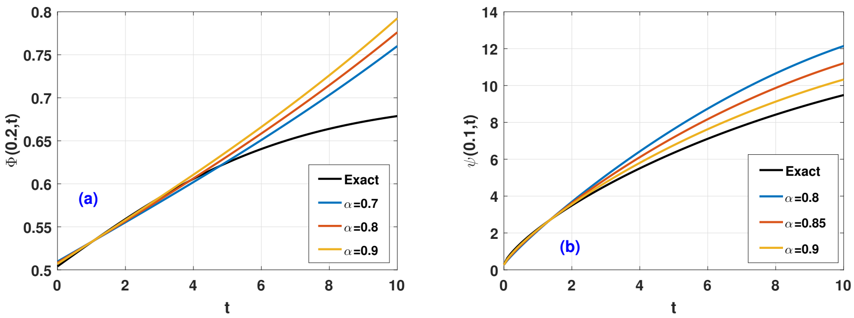

Figure 5. (a) and (b): MDLDM Caputo-Fabrizio solution when time and α are changing with fixed $x = 0.2$ for b=k=0.3, $\lambda = 1.5$ of Example 2.

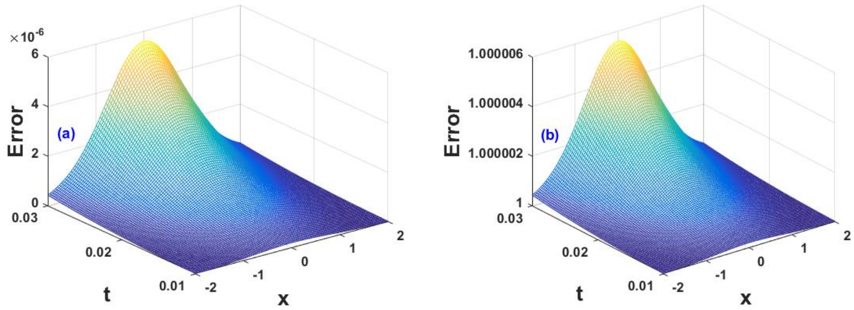

Figure 6. (a) and (b): The absolute error plots of the Table 2.

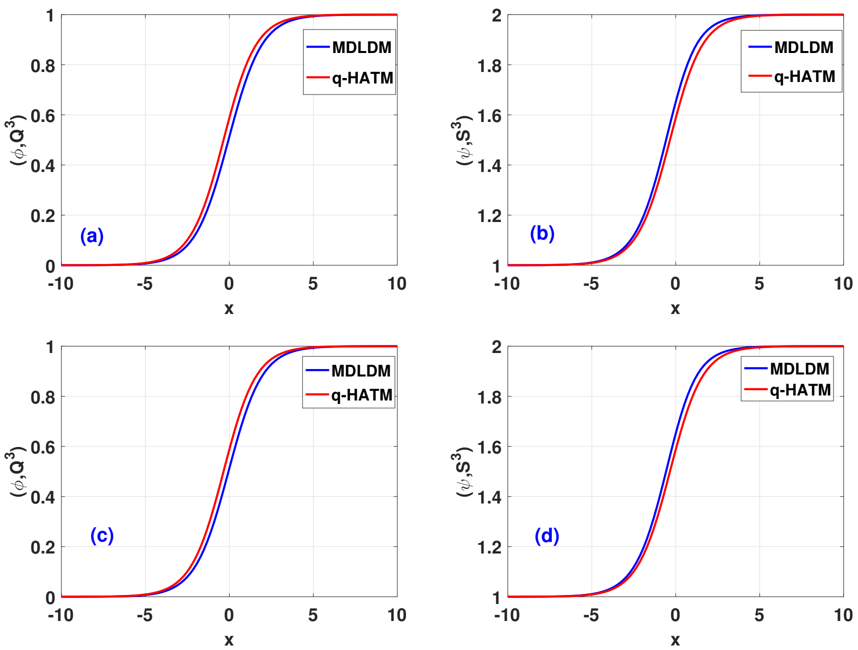

Figure 7. (a) and (b): Comparison between MDLDM and q-HATM [20] for n=3, k=b=0.5, $\lambda = 1.5$, $\alpha = 1$ and h=-1 for Example 1, (c) and (d): comparison between MDLDM and q-HATM [20] with the same parametric data for Example 2.

5. Conclusions

We have studied analytically the mCKdV with Caputo's and Caputo-Fabrizio (CF) derivatives using MDLDM. The proposed method is applied to the mCKdV system and approximate results are obtained in the form of series solutions by considering examples. The error analysis of the proposed method is also discussed. It is observed that the suggested scheme is one of the vigorous tools to investigate nonlinear problems. The main advantage of the suggested method is to analyze analytical solutions of the considered problem without using linearization and discretization. For validation, two examples in Caputo's and CF form are studied numerically and the results are compared with the numerical results with a physical interpretation. When non-integer order was employed in models including non-integer order derivatives, many scientists were less interested. For non-integer order fractional time derivative with singular and non-singular kernels of the mKdV equations, the MDLDM can be used. The MDLDM approach for handling nonlinear evolutionary equations was shown to have a broader applicability in this work. Physical meaning was given to the parameters in the resulting travelling wave solution. For varied values of the parameters, a three-dimensional simulation of wave behaviour is built, which alters the wave profile of the equation.The wave dynamics are discussed with the help of simulation. The results show that this strategy worked. The new wave solution presented in this work can open up new avenues for future research.

6. Future work

A modified coupled time fractional KdV equation with Caputo's and Caputo-Fabrizio operators is considered in this manuscript. The characteristics wave profiles of the solution to the equation show some interesting behaviour especially when altering in the parameters. These interesting behaviours may be more interesting to transform the modified coupled time fractional KdV to modified coupled Schrodinger and sine-Gordon equations with Caputo's and Caputo-Fabrizio operators.

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Conflict of interest

It is declared that all the authors have no conflict of interest regarding this manuscript.

Appendix

$$
\begin{aligned}
\eta_1 &= (9r_1^6 + r_1^4\left(36k^2\lambda - 24k^4\right) - 36r_1^3k^3\lambda + 4r_1^2k^4(2k^2+3\lambda)^2 - 24r_1k^5\lambda\left(2k^2+3\lambda\right) + 36k^6\lambda^2),\\
a_1 &= -\left(3r_1^3 - 4r_1k^4 + 6r_1\lambda k^2 - 6\lambda k^3\right)),\\
a_2 &= \left(\left(3r_1^3 - 4r_1k^4 + 6r_1\lambda k^2 - 6\lambda k^3\right) + 3r_1^3 + 4r_1k^4 + 6r_1\lambda k^2 - 6\lambda k^3\right),
\end{aligned}
$$

$$
\begin{aligned}
a_3 &= 8r_1k^4\left(\left(3r_1^3-4r_1k^4-66r_1\lambda k^2+66\lambda k^3\right)+18r_1\lambda(k-r_1)\right),\\
b_0 &= \left(3r_1^3-4r_1k^4\right)^3-216r_1\lambda^2k^4(r_1-k)^2\left(r_1^2+4k^4\right)+36r_1^2\lambda k^2(r_1-k)\left(3r_1^4-24r_1^2k^4-16k^8\right)\\
&\quad -432\lambda^3k^6(r_1-k)^3,\\
b_2 &= 36r_1\lambda^2k^4(r_1-k)^2\left(3r_1^2+14k^4\right)-6r_1^2\lambda k^2(r_1-k)\left(9r_1^4-60r_1^2k^4-64k^8\right)\\
&\quad -3r_1^3\left(r_1^2-2k^4\right)\left(3r_1^2-4k^4\right)^2+216\lambda^3k^6(r_1-k)^3,\\
b_3 &= 2r_1\Big((3r_1^3-4r_1k^4)^2+60r_1\lambda k^2(r_1-k)(3r_1^2-20k^4)+2592r_1\lambda k^6(r_1-k)+180\lambda^2k^4(r_1-k)^2\Big),\\
c_0 &= \left(3r_1^3-4r_1k^4+6r_1\lambda k^2-6\lambda k^3\right)^3,\\
c_1 &= -36r_1\lambda^2k^4(r_1-k)^2\left(9r_1^2-124k^4\right)-18r_1^2\lambda k^2(r_1-k)(9r_1^4-152r_1^2k^4+1296k^8)\\
&\quad -r_1^3\left(3r_1^2-4k^4\right)^2\left(3r_1^2-20k^4\right)-216\lambda^3k^6(r_1-k)^3+6r_1k^4,\\
c_2 &= \left(3r_1^3-4r_1k^4\right)^2-4r_1\lambda k^2(r_1-k)\left(105r_1^2-3692k^4\right)-11520r_1\lambda k^6(r_1-k)-780\lambda^2k^4(r_1-k)^2,\\
c_3 &= 8r_1r_1-k\lambda(9r_1^3+216r_1k^4-116r_1k^4+18\lambda k^2(r_1-k)).
\end{aligned}
$$

Error analysis tables:
Please notes that, for error analysis, we consider $\alpha = 1$, $b = k = 0.5$, $\lambda = 1.5$, n=3 and h=-1.

Table 1. Comparison of Caputo's solution with exact solution for Example 1.

t	x	Exact	ϕ	$\vert$ Exact$-\phi\vert$	Exact	ψ	$\vert$ Exact$-\psi\vert$
0.01	-10	4.7725E(-05)	4.7720E(-05)	1.1932E(-11)	1.0000E(-01)	1.0000E(-01)	2.5353E(-05)
0.02	.	5.0172E(-05)	5.0172E(-05)	1.9286E(-10)	1.0019E(-01)	1.0001E(-01)	5.0706E(-05)
0.03	.	5.2744E(-05)	5.2743E(-05)	9.8635E(-10)	1.0001E(-01)	1.0015E(-01)	7.6059E(-05)
0.01	-5	7.0336E(-03)	7.0336E(-03)	1.4449E(-09)	1.0070E(-01)	1.0107E(-01)	3.7126E(-03)
0.02	.	7.3915E(-03)	7.3915E(-03)	1.3553E(-09)	1.0074E(-01)	1.0148E(-01)	7.4247E(-03)
0.03	.	7.7676E(-03)	7.7675E(-03)	4.8942E(-08)	1.0078E(-01)	1.0189E(-01)	1.1136E(-02)
0.01	0	5.1250E(-01)	5.1266E(-01)	1.6117E(-04)	1.5125E(-01)	1.6518E(-01)	1.3931E(-01)
0.02	.	2.4980E(-01)	5.2627E(-01)	1.2893E(-04)	1.5250E(-01)	1.8030E(-01)	2.7798E(-01)
0.03	.	5.3740E(-01)	5.4178E(-01)	4.3515E(-03)	1.5374E(-01)	1.9549E(-01)	4.1601E(-01)
0.01	5	9.9360E(-01)	9.9363E(-01)	4.5887E(-09)	1.9936E(-01)	1.9973E(-01)	3.7128E(-03)
0.02	.	9.9394E(-01)	9.9394E(-01)	4.8954E(-08)	1.9939E(-01)	2.0014E(-01)	7.4254E(-03)
0.03	.	9.9423E(-01)	9.9423E(-01)	2.0582E(-07)	1.9942E(-01)	2.0054E(-01)	1.1138E(-02)
0.01	10	9.9996E(-01)	9.9996E(-01)	1.1698E(-11)	2.0000 E(-01)	2.0000E(-01)	2.5353E(-05)
0.02	.	9.9996E(-01)	9.9996E(-01)	1.8532E(-10)	2.0000E(-01)	2.0000E(-01)	5.0707E(-05)
0.03	.	9.9996E(-01)	9.9996E(-01)	9.2900E(-10)	2.0000E(-01)	2.0000E(-01)	7.6061E(-05)

Table 2. Comparison of Caputo-Fabrizio solution with exact solution for Example 2.

t	x	Exact	ϕ	$\mid$ Exact$-\phi\mid$	Exact	ψ	$\mid$ Exact$-\psi\mid$
0.01	-10	6.7725E(-04)	4.7520E(-05)	1.4932E(-11)	1.0004E(-01)	1.0040E(-01)	2.5356E(-05)
0.02	.	4.3172E(-04)	5.0152E(-05)	1.9276E(-10)	1.0019E(-01)	1.0001E(-01)	5.0706E(-05)
0.03	.	6.7440E(-04)	5.2543E(-05)	9.8637E(-10)	1.0001E(-01)	1.0015E(-01)	7.6056E(-05)
0.01	-5	7.0346E(-04)	7.5336E(-03)	1.4479E(-09)	1.0070E(-01)	1.0107E(-01)	3.7126E(-03)
0.02	.	7.3945E(-03)	7.3515E(-03)	1.3557E(-09)	1.0743E(-01)	1.0148E(-01)	7.4246E(-03)
0.03	.	7.7655E(-03)	7.7575E(-03)	4.8946E(-08)	1.0078E(-01)	1.0189E(-01)	1.1166E(-02)
0.01	0	5.1550E(-01)	5.5266E(-01)	1.6116E(-04)	1.5125E(-01)	1.6518E(-01)	1.6931E(-01)
0.02	.	2.4980E(-01)	5.2527E(-01)	1.2896E(-04)	1.5260E(-01)	1.8030E(-01)	2.7768E(-01)
0.03	.	5.3740E(-01)	5.4678E(-01)	4.3516E(-03)	1.5374E(-01)	1.9534E(-01)	4.1606E(-01)
0.01	5	9.9360E(-01)	9.6363E(-01)	4.5867E(-09)	1.9936E(-01)	1.9973E(-01)	3.6128E(-03)
0.02	.	9.9394E(-01)	9.6394E(-01)	4.8964E(-08)	1.9939E(-01)	2.0014E(-01)	7.4654E(-03)
0.03	.	9.9423E(-01)	9.6423E(-01)	2.0586E(-07)	1.9942E(-01)	2.0054E(-01)	1.1136E(-02)
0.01	10	9.9996E(-01)	9.6996E(-01)	1.1696E(-11)	2.0000 E(-01)	2.0000E(-01)	2.5653E(-05)
0.02	.	9.9996E(-01)	9.6996E(-01)	1.8562E(-10)	2.0000E(-01)	2.0000E(-01)	4.0707E(-05)
0.03	.	9.9996E(-01)	9.96996E(-01)	9.2906E(-10)	2.0000E(-01)	2.0000E(-01)	7.6555E(-05)

Table 3. Comparison between $\phi(x,t)$ solution with q-HATM [20] $Q^3(x,t)$ for Example 1.

t	x	Exact [18]	q-HATM[20]	$\mid Exact - q - HATM\mid$	Exactϕ	MDLDM	$\mid Exact - \phi\mid$
0.1	5.0	0.99360	0.98290	1.0700×10^{-03}	1.00000	1.00000	4.6000×10^{-09}
0.2		0.99390	0.97230	2.1600×10^{-02}	1.00000	1.000000	4.9600×10^{-09}
0.3		0.99420	0.96160	4.0000×10^{-03}	1.00000	1.00000	2.0580×10^{-09}
0.1	6.0	0.99760	0.99370	8.0000×10^{-03}	0.99760	0.99760	1.0970×10^{-08}
0.2		0.99780	0.98970	1.2100×10^{-02}	0.99780	0.99780	5.3000×10^{-08}
0.3		0.99790	0.98580	3.3660×10^{-05}	0.99790	0.99790	7.7000×10^{-10}
0.1	7.0	0.99910	0.99770	1.5000×10^{-03}	0.99910	0.99910	2.4000×10^{-10}
0.2		0.99920	0.99620	3.0000×10^{-03}	0.99920	0.99920	3.7300×10^{-09}
0.3		0.99920	0.99470	4.5000×10^{-03}	0.99920	0.99920	1.8620×10^{-08}
0.1	8.0	0.99970	0.9991	5.0000×10^{-04}	0.99970	0.99970	8.6000×10^{-11}
0.2		0.99970	0.99860	1.1000×10^{-03}	0.99970	0.99970	1.3660×10^{-09}
0.3		0.99970	0.99810	1.6000×10^{-03}	0.99990	0.99970	6.8440×10^{-09}
0.1	9.0	0.99990	9.99700	1.9900×10^{-04}	0.99990	0.99990	3.2000×10^{-11}
0.2		0.99990	9.99500	4.0100×10^{-04}	0.99990	0.99990	5.0300×10^{-10}
0.3		0.99990	0.999300	6.06200×10^{-04}	0.99990	0.99990	2.5230×10^{-09}
0.1	10	1.00000	0.99990	7.3200×10^{-05}	1.00000	1.00000	1.1700×10^{-11}
0.2		1.00000	0.99980	1.4760×10^{-04}	1.00000	1.00000	1.8530×10^{-10}
0.3		1.00000	0.99980	2.2310×10^{-04}	1.00000	1.00000	9.2900×10^{-09}

Table 4. Comparison between $\psi(x,t)$ solution with q-HATM [20] $S^3(x,t)$ for Example 1.

t	x	$Exact$[20]	$q-HATM$[20]	|Exact−q-HATM|	Exactϕ	MDLDM	|Exact−ϕ|
0.1	5.0	1.99360	1.98290	1.0700×10^{-03}	1.99360	1.99730	3.7000×10^{-03}
0.2		0.99390	1.97230	2.1600×10^{-02}	1.99390	2.00140	7.4000×10^{-03}
0.3		1.99420	1.96160	3.2700×10^{-02}	1.99420	2.00540	1.1100×10^{-02}
0.1	6.0	1.99760	1.99370	4.0000×10^{-03}	1.99760	1.99900	1.4000×10^{-03}
0.2		1.99780	1.98970	8.0000×10^{-02}	1.99780	2.00050	2.8000×10^{-03}
0.3		1.99790	1.98580	1.2100×10^{-02}	1.99790	2.00200	4.1000×10^{-03}
0.1	7.0	1.99910	1.99770	1.5000×10^{-03}	1.99910	1.99960	5.0000×10^{-04}
0.2		1.99920	1.99620	3.0000×10^{-03}	1.99920	2.00020	1.0000×10^{-03}
0.3		1.99920	1.99470	4.5000×10^{-03}	1.99920	2.00070	1.5000×10^{-03}
0.1	8.0	1.99970	1.99910	5.0000×10^{-04}	1.99970	1.99990	1.8720×10^{-04}
0.2		1.99970	1.99860	1.1000×10^{-03}	1.99970	2.00010	3.7450×10^{-04}
0.3		1.99970	1.99810	1.6000×10^{-03}	1.99990	2.00030	5.6170×10^{-04}
0.1	9.0	1.99990	9.99700	1.9900×10^{-04}	1.99990	2.00000	6.8900×10^{-05}
0.2		1.99990	9.99500	4.0100×10^{-04}	1.99990	2.00000	1.3780×10^{-04}
0.3		1.99990	1.99930	6.0620×10^{-04}	1.99990	2.00000	2.0670×10^{-04}
0.1	10	2.00000	1.99990	7.3200×10^{-05}	2.00000	2.00000	2.5350×10^{-05}
0.2		2.00000	1.99980	1.4760×10^{-04}	2.00000	2.00000	5.0710×10^{-05}
0.3		2.00000	1.99980	2.2310×10^{-04}	2.00000	2.00000	7.6060×10^{-05}

Table 5. Comparison between $\phi(x,t)$ solution with q-HATM [20] $Q^3(x,t)$ for Example 2.

t	x	$Exact$[18]	q-HATM[20]	|Exact−q-HATM|	Exactϕ	MDLDM	|Exact−ϕ|
0.1	5.0	0.99363	0.995590	1.9616×10^{-03}	0.99363	0.99363	2.6743×10^{-07}
0.2		0.99394	0.99780	3.8576×10^{-03}	0.99394	0.99394	2.1272×10^{-06}
0.3		0.99423	0.99992	5.6881×10^{-03}	1.00000	0.99423	7.1386×10^{-06}
0.1	6.0	0.99765	0.99838	7.2768×10^{-04}	0.99765	0.99765	1.0070×10^{-07}
0.2		0.99776	0.99919	4.3080×10^{-03}	0.99776	0.99776	8.0150×10^{-07}
0.3		0.99787	0.99998	2.1094×10^{-04}	0.99787	0.99787	2.6891×10^{-07}
0.1	7.0	0.99913	0.99940	2.6852×10^{-04}	0.99913	0.99913	3.7495×10^{-08}
0.2		0.99918	0.9997	5.2796×10^{-04}	0.99910	0.99910	2.9814×10^{-07}
0.3		0.99922	0.99999	7.7833×10^{-04}	0.99922	0.99922	1.0002×10^{-06}
0.1	8.0	0.99980	0.99940	5.0000×10^{-04}	0.99970	0.99970	8.6000×10^{-08}
0.2		0.99980	0.99864	1.1000×10^{-03}	0.99970	0.99970	1.3660×10^{-07}
0.3		0.99980	0.99840	1.6000×10^{-04}	0.99990	0.99970	6.8440×10^{-06}
0.1	9.0	0.99988	0.99992	3.6397×10^{-05}	0.99988	0.99988	5.1053×10^{-08}
0.2		0.99989	0.99996	7.1561×10^{-05}	0.99989	0.99989	4.0593×10^{-08}
0.3		0.99989	0.999900	1.0549×10^{-05}	0.99989	00.99989	1.3618×10^{-07}
0.1	10	0.99996	0.99996	1.3392×10^{-05}	0.99996	0.99996	1.8793×10^{-09}
0.2		0.99997	0.99996	2.633×10^{-05}	0.99996	0.99996	1.4943×10^{-08}
0.3		0.99996	1.00000	3.8815×10^{-05}	0.99996	0.99996	5.0127×10^{-08}

Table 6. Comparison of the MDLDM $\psi(x,t)$ solution with q-HATM [20] $S^3(x,t)$ for Example 2.

t	x	$ExactH$	$q-HATM$[18]	|Exact−q-HATM|	Exactϕ	MDLDM	|Exact−ϕ|
0.1	5.0	1.99360	1.99560	1.9616×10^{-03}	1.99360	1.99730	3.7125×10^{-03}
0.2		0.99390	1.99780	3.8577×10^{-03}	1.99390	2.00140	7.4231×10^{-03}
0.3		1.99420	1.99990	5.6882×10^{-03}	1.99420	2.00540	1.1213×10^{-02}
0.1	6.0	1.99760	1.99840	7.2768×10^{-04}	1.99760	1.99900	1.3774×10^{-03}
0.2		1.99780	1.99920	1.4308×10^{-04}	1.99780	2.00050	2.7542×10^{-03}
0.3		1.99790	2.00000	2.1095×10^{-04}	1.99790	2.00200	41.2970×10^{-03}
0.1	7.0	1.99910	1.99940	2.6852×10^{-04}	1.99910	1.99960	5.0831×10^{-04}
0.2		1.99920	1.99970	5.2797×10^{-04}	1.99920	2.00020	1.0164×10^{-03}
0.3		1.99920	2.00000	7.7833×10^{-04}	1.99920	2.00070	1.5240×10^{-03}
0.1	8.0	1.99970	1.99980	9.8896×10^{-05}	1.99970	1.99990	1.8721×10^{-04}
0.2		1.99970	1.99990	1.9444×10^{-04}	1.99970	2.00010	3.7435×10^{-04}
0.3		1.99970	2.00000	2.8665×10^{-03}	1.99990	2.00030	5.6130×10^{-04}
0.1	9.0	1.99990	1.99990	3.6397×10^{-05}	1.99990	2.00000	6.8902×10^{-05}
0.2		1.99990	2.00000	7.1561×10^{-05}	1.99990	2.00000	1.3777×10^{-04}
0.3		1.99990	2.00000	1.0549×10^{-04}	1.99990	2.00001	2.0658×10^{-04}
0.1	10	2.00000	2.00000	1.3392×10^{-05}	2.00000	2.00000	2.5350×10^{-05}
0.2		2.00000	2.00000	2.6330×10^{-04}	2.00000	2.00000	5.0692×10^{-05}
0.3		2.00000	2.00000	3.8815×10^{-05}	2.00000	2.00000	$7.7.6010\times10^{-05}$

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