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Research article

On the boundedness stepsizes-coefficients of A-BDF methods

Dumitru Baleanu[1,2,3], Mohammad Mehdizadeh Khalsaraei[4,*], Ali Shokri[4] and Kamal Kaveh[4]

[1] Department of Mathematics, Cankaya University, Balgat 06530, Ankara, Turkey

[2] Institute of Space Sciences, R76900 Magurele-Bucharest, Romania

[3] Department of Medical Research, China Medical University, Taichung 40402, Taiwan

[4] Department of Mathematics, Faculty of Basic Science, University of Maragheh, P.O.Box 55181-83111, Maragheh, Iran

* **Correspondence:** Email: mehdizadeh@maragheh.ac.ir; Tel: +989144137854.

Abstract: Physical constraints must be taken into account in solving partial differential equations (PDEs) in modeling physical phenomenon time evolution of chemical or biological species. In other words, numerical schemes ought to be devised in a way that numerical results may have the same qualitative properties as those of the theoretical results. Methods with monotonicity preserving property possess a qualitative feature that renders them practically proper for solving hyperbolic systems. The need for monotonicity signifies the essential boundedness properties necessary for the numerical methods. That said, for many linear multistep methods (LMMs), the monotonicity demands are violated. Therefore, it cannot be concluded that the total variations of those methods are bounded. This paper investigates monotonicity, especially emphasizing the stepsize restrictions for boundedness of A-BDF methods as a subclass of LMMs. A-stable methods can often be effectively used for stiff ODEs, but may prove inefficient in hyperbolic equations with stiff source terms. Numerical experiments show that if we apply the A-BDF method to Sod's problem, the numerical solution for the density is sharp without spurious oscillations. Also, application of the A-BDF method to the discontinuous diffusion problem is free of temporal oscillations and negative values near the discontinuous points while the SSP RK2 method does not have such properties.

Keywords: monotonicity; linear multistep method; total-variation-diminishing; total-variation-bounded; method of lines; A-BDF method
Mathematics Subject Classification: Primary: 58F15, 58F17; Secondary: 53C35

1. Introduction

One of the favorite methods used in the solution of time-dependent hyperbolic PDEs is the method of lines (MOL). Using an appropriate spatial variables discretization for a PDE problem can lead to a system of ordinary differential equations (ODEs). Then, this system is integrated with an appropriate ODE method. One may find these solvers in the related literature; see [1–7, 20, 41–46]. An important issue at stake is the examination of the stability of ODE methods. Hyperbolic PDE solutions are often discontinuous, so one customary linear stability analysis may prove inadequate [38]. Therefore, the ODE solver based on nonlinear stability is a desirable and necessary requirement. These types of methods are called total-variation-diminishing (TVD) time discretization and are attributed to Shu et al. [47, 49].

In semi-discrete approximation of PDEs, LMMs satisfying TVD, strong stability preservation (SSP), and monotonicity properties have been highlighted in the literature. Step length constraints to establish these properties have been studied in many publications [8–19, 24–35, 37–40, 53–55]. The interesting research carried out by Spijker [51] provides conditions which indicate whether nontrivial stepsize restriction exists for a given LMMS to secure boundedness. This paper deals with the step length limitations for boundedness, for a class of LMM, designated as A-BDF methods.

The remainder of this article is structured as follows. In Section 2, some preliminary settings and definitions of monotonicity and boundedness for LMMs are presented, to be used in the rest of the paper. As well, the paper examines the results of stepsize-coefficients for boundedness obtained in Spijker [51]. Section 3 presents the results for A-BDF obtained by boundedness methods. Numerical simulations results are presented in Section 4.

2. Preliminaries

We consider initial value problem

$$\frac{d}{dt}Z(t) = F(Z(t)) \quad t \geq 0, \quad Z(0) = Z_0. \tag{2.1}$$

Here Z_0 is a given vector in a vector space X and F is a given function from X into itself. We have to find $Z(t) \in X$ for $t > 0$. These systems of ODEs frequently occur in such models as they naturally arise while modeling processes that evolve in time, for instance, in time evolution of physical phenomena or chemical, and biological species. Many other examples can be found in Strogatz [52]. In the following section, we will briefly review the developments of monotonicity and boundedness properties.

2.1. Monotonicity for LMMs

The general LMMs, applied to the problem (2.1), showed that $Z_n = Z(t_n)$ at the points $t_n = n\Delta t$, that Δt is time length and $n = k, k+1, \ldots$. The approximation Z_n can be defined in terms of $Z_{n-1}, Z_{n-2}, \ldots, Z_{n-k}$ by the relation

$$Z_n = \sum_{j=1}^{k} \alpha_j Z_{n-j} + \Delta t \sum_{j=0}^{k} \beta_j F(Z_{n-j}). \tag{2.2}$$

with $k \geq 1$ and coefficients α_j,β_j which specify the method. Necessary and sufficient conditions for consistency of method (2.2) are:

$$\sum_{j=1}^{k}\alpha_j = 1, \quad \sum_{j=1}^{k} j\alpha_j = \sum_{j=0}^{k}\beta_j.$$

Let $\| \,.\, \|$ be a norm or seminorm (i.e, $\| -\nu \| = \| \nu \| \geq 0$, $\| \lambda x \| = | \lambda | \| \nu \|$ *and* $\| \nu + \omega \| \leq \| \nu \| + \| \omega \|$ for all $\nu, \omega \in X$ and real λ) on X, such as the maximum norm $\| \nu \| = \| \nu \|_\infty = \max_i | \nu_i |$ and the total variation seminorm $\| \nu \| = \| \nu \|_{TV} = \sum_i | \nu_i - \nu_{i-1} |$ (ν is a vector). Assume for $\tau_0 > 0$

$$\| \nu + \Delta t F(\nu) \| \leq \| \nu \|, \qquad \text{for all } \nu \in X, \quad \Delta t \in (0, \tau_0], \tag{2.3}$$

as a result, using method $Z_n = Z_{n-1} + \Delta t F(Z_{n-1}), n \geq 1$ with $\Delta t > 0$ to estimate $Z_n \simeq Z(t_n)$ *at* $t_n = n\Delta t$, we have

$$\| Z_n \| \leq \| Z_0 \|, \qquad n \geq 1, \tag{2.4}$$

under condition $\Delta t \leq \tau_0$.

Definition 2.1. (*Monotonicity*). Property (2.4) subject to $\Delta t \leq \gamma\tau_0$, with $\gamma > 0$ is known as monotonicity for general one-step methods.

A method with property $\| Z_n \|_{TV} \leq \| Z_{n-1} \|_{TV}$ has a special importance in the integration of hyperbolic systems and is named TVD; see for example [21–23, 47]. Consequently, total variation of such processes is bounded (it is known as TVB property), i.e. we can find μ in a way that for all $n \geq 1$,

$$\| Z_n \|_{TV} \leq \mu \| Z_0 \|_{TV} .$$

In addition, with monotonicity [25] for multistep methods (2.2), we mean

$$\| Z_n \| \leq \max_{0 \leq j \leq k-1} \| Z_j \|, \qquad (n \geq k). \tag{2.5}$$

Clearly, (2.5) with $\| \,.\, \|_{TV}$ implies

$$\| Z_n \|_{TV} \leq \mu \max_{0 \leq j < k} \| Z_j \|_{TV},$$

which is again a TVB-property.

For (2) coefficient γ has been determined, with the property (2.3) and step size restriction $\Delta t \leq \gamma\tau_0$ guarantees (2.5). For example, if in (2.2) all coefficients $\alpha_j,\beta_j \geq 0$, then monotonicity property holds true provided that $\Delta t \leq \gamma\tau_0$, where $\gamma = \min_j \frac{\alpha_j}{\beta_j}$. For more details see [9, 24, 25, 50].

Based on the recent results in Spijker [51], necessary and sufficient conditions Monotonicity for LMMs is preserved if and only if

$$\begin{aligned} &\beta_0 \geq 0, \quad \alpha_j \geq 0, \quad \beta_j \geq 0, \quad (for\ 1 \leq j \leq k) \\ &and\ \ \alpha_i > 0, \quad for\ all\ i \in \{1, ..., k\}\ with\ \beta_i > 0, \end{aligned} \tag{2.6}$$

for more details and theoretical analysis, see [50, 51].

2.2. Boundedness for LMMs

There are some LMMs in which the existence of $\gamma > 0$ for (2.6) does not hold in practice. It is because of this strict condition that monotonicity is not ensured in Adams or backward differentiation formula (BDF) type methods. Also, for the methods with nonnegative coefficients the stepsize requirement can be very restrictive (see [35]). For example, for an explicit k-step method ($k > 1$) of order p which often $p = k$, $\gamma \leq \frac{(k-p)}{(k-1)}$ so that, for such methods we cannot have a stepsize coefficient $\gamma > 0$ for monotonicity.

Hence, studying properties that are more relaxed than (2.5) is desirable. Therefor, along with monotonicity one can also use the somewhat weaker boundedness property as follows.

Definition 2.2. (*Boundedness*). For linear multistep methods, the property

$$\parallel Z_n \parallel \leq \mu \max_{0\leq j\leq k-1} \parallel Z_j \parallel, \qquad n \geq k, \tag{2.7}$$

with $\Delta t \leq \gamma\tau_0$ is called boundedness property, where the stepsize coefficient $\gamma > 0$ and the factor $\mu \geq 1$ are determined by the multistep method.

Usually, stepsize coefficient γ, (which we call stepsize coefficient bounded) is determined in a way that property (2.7) holds. General boundedness results have been obtained in [51] for the LMMs. In that paper [51], it has been shown that paper shows that stepsize restrictions in (2.7) are necessary and sufficient for boundedness in seminorm. As for many LMMs, nonetheless, one can not conclude that they are TVB. In the following, we review the more recent advancement in the boundedness issue made by Spijker [51], which makes it possible to derive a sufficient condition on boundedness for LMMs. The following concepts about LMMs for formulating results can be found in [7, 20]. We denote the region of stability of LMM by S and its interior by int(S). The method is zero-stable whenever $0 \in S$. For the multistep methods (2.2), we define the characteristic polynomials,

$$\rho(\xi) = \xi^k - \sum_{j=1}^{k} \alpha_j \xi^{k-j}, \qquad \sigma(\xi) = \sum_{j=0}^{k} \beta_j \xi^{k-j}.$$

Definition 2.3. If no root of a polynomial has a modulus greater than one, and if every root with modulus one is simple, it is said the polynomial has the root condition.

In the following, we suppose that the root condition holds for polynomial $\rho(\xi)$ and the number of common zeros of the $\rho(\xi)$ and $\sigma(\xi)$ is 0. Obtaining boundedness coefficient for LMMs on the basis of the classical theory on boundedness provided in [51] by Spijker et al. is complicated. One way to avoid this complexity is to concentrate directly on stepsize restrictions for boundedness, which is again taken from Spijker [51] given as follows. By adjoining to method (2.2) the following values τ_n $(for\ n \in \mathbb{Z})$,

$$\tau_n = 0 \quad (n < 0), \qquad \tau_n = \sum_{j=1}^{k} \alpha_j \tau_{n-j} + \beta_n \quad (0 \leq n \leq k),$$

$$\tau_n = \sum_{j=1}^{k} \alpha_j \tau_{n-j}, \qquad (n > k), \tag{2.8}$$

we recall a criterion, using the following theorem given in [51] for the existence of $\gamma > 0$ satisfying (2.7). The roots of modulus equal to one is called the *essential roots* of $\rho(\xi)$.

Theorem 2.4. *If $\xi = 1$ is the only essential root of polynomial $\rho(\xi)$ and $\tau_n > 0$ ($\forall\ n \geq n_0$), then the existence of stepsize coefficient for boundedness is guaranteed. In other words, if condition $\tau_n \leq 0$ holds, then no such coefficient exists, where*

$$n_0 = \min\{n :\ 1 \leq n \leq k,\ \tau_n \neq 0\}. \tag{2.9}$$

We briefly review the boundedness for BDFs. The k-step BDF is implicit and can be written as

$$Z_n = \alpha_1 Z_{n-1} + ... + \alpha_k Z_{n-k} + \Delta t \beta_0 F(Z_n), \tag{2.10}$$

where the coefficients $\alpha_j = \alpha_{k,j}, \beta_0 = \beta_{k,0}$ are chosen so that BDF has order k [13]. These methods have special properties, for example, A-stability up to order 2 and $A(\alpha)$-stability up to order 6. For $k = 1$, since the coefficients α_1, β_0 are positive, so that the BDF satisfies in the monotonicity property, see (2.6). When $k \geq 2$, the coefficients $\alpha_{k,2} < 0$ then monotonicity is not preserved. These methods with $2 \leq k \leq 6$ are zero stable and $\xi = 1$ is the only essential root of polynomial $\rho(\xi)$. It can be shown that the values τ_n satisfy

$$\tau_n = 0 \quad (for\ n < 0), \qquad \tau_0 = \beta_0, \qquad \tau_n = \alpha_1 \tau_{n-1} + ... + \alpha_k \tau_{n-k} \quad (for\ n \geq 1).$$

We have $\tau_1 = \alpha_1 \beta_0 \neq 0$ (for $2 \leq k \leq 6$), so that $n_0 = 1$. Furthermore, it can be shown that $\tau_n > 0 \quad (for\ n \geq n_0$, k=2). For $k = 3, 4, 5, 6$ by using MATLAB this condition is fulfilled. Theorem 1 thus leads to the following conclusion:

Conclusion 1.

With $k = 1$ monotonicity is guaranteed.

With $2 \leq k \leq 6$, the above coefficient does not exist for monotonicity, but one can find the above coefficient to boundedness. For a review of more details, see [51].

The results on the boundedness for Adams-Moulton, Adams-Bashforth, and E-BDF methods can be found in [51].

3. Main results

In this section, given the A-BDF algorithm and its properties, we attain some results to boundedness. The method proposed by Fredebule [13] has a larger stability region than those of BDFs and is defined as follows:

$$\begin{aligned} BDF_k^{(i)} &:\ Z_n = a_1 Z_{n-1} + ... + a_k Z_{n-k} + \Delta t b_0 F(Z_n), \\ BDF_k^{(e)} &:\ Z_n = \bar{a}_1 Z_{n-1} + \bar{a}_2 Z_{n-2} + ... + \bar{a}_k Z_{n-k} + \Delta t \bar{b}_1 F(Z_{n-1}). \end{aligned} \tag{3.1}$$

Where the coefficients are chosen in a way that the order of (3.1) is k. We show the corresponding modified BDF of degree k with A-BDF$_k$ and is given as follows:

$$BDF_k^{(i)} - sBDF_k^{(e)} = 0,$$

or

$$(1-s)Z_n = (a_1 - s\bar{a}_1)Z_{n-1} + ... + (a_k - s\bar{a}_k)Z_{n-k} + \Delta t\big(b_0 F(Z_n) - s\,\bar{b}_1 F(Z_{n-1})\big). \tag{3.2}$$

After dividing on $(1-s)$, (3.2) gives

$$Z_n = \alpha_1 Z_{n-1} + ... + \alpha_k Z_{n-k} + \Delta t\big(\beta_0 F(Z_n) + \beta_1 F(Z_{n-1})\big), \tag{3.3}$$

where $\alpha_i = \frac{(a_i - s\bar{a}_i)}{1-s}$ $(for\ i = 1,..,k)\ and\ \beta_0 = \frac{b_0}{1-s},\ \beta_1 = \frac{-s\bar{b}_1}{1-s}$.

It has to be mentioned that A-BDF is of order k, for all $s \in \mathbb{R} - \{1\}$. Also, zero-stability is established with some value of s $(-1 \leq s < 1)$ for $k = 1, 2, ..., 7$, and the corresponding regions of A(α)-stability are given in Table 1. Moreover, a considerably larger set of stiff problems with oscillatory modes can be solved efficiently and accurately at the expense of minor disadvantages in the solution of ordinary stiff problems.

Table 1. A(α)-stability of A-BDF methods.

k	1	2	3	4	5	6	7	8
p	1	2	3	4	5	6	7	8
(α_{max})	90^0	90^0	90^0	88^0	73^0	51^0	18^0	-

In addition to the above description the central results are the following [13]:

- The A-BDF$_k$ are consistent (at least) of order k for any $k \in \mathbb{N}$ and $s \in \mathbb{R}$.
- They are ∞-stable for any $k \in \mathbb{N}$ and $s \in [-1, 1]$; in particular, for $s = 1$, they are reducible and *not* zero-stable.
- For $k = 1, \ldots, 7$, there exist intervals I_k and J_k, such that the A-BDF$_k$ are zero-stable for $s \in I_k$ and A$_0$-stable for $s \in J_k$. Here, $J_k \subset I_k \cap [-1, 1)$.
- The A$_0$-stable A-BDF$_k$ are A-stable for $k = 1, 2$ and A(α)-stable for $k = 3, \ldots, 7$.
- There are no A$_0$-stable A-BDF$_k$ for any $k \geq 8$.

Note that these results include classic cases BDF of order 1 through 6 as a special case setting $s = 0$.

Theorem 3.1. *The existence of a stepsize coefficient for A-BDF$_k$, $k = 1, 2, 3$ is guaranteed with respect to boundedness.*

Proof. Case 1. $k = 1$

When $k = 1$ we have $\alpha_1 = 1, \beta_0 = \frac{1}{1-s}$, $\beta_1 = \frac{-s}{1-s}$. These coefficients for $-1 \leq s \leq 0$ are non-negative, so one can find γ for monotonicity; see (2.6). If for $0 < s < 1$, we have $\beta_1 < 0$, monotonicity is not guaranteed, but boundedness holds in this case. In fact, under this assumption, we have

$$\tau_0 = \beta_0 = \frac{1}{1-s} > 0, \qquad \tau_1 = \alpha_1\tau_0 + \beta_1 = \frac{1-s}{1-s} = 1 > 0,$$

therefore, we have $n_0 = 1$ and $\tau_n = \alpha_1\tau_{n-1} = 1$, $(\forall\ n \geq 2)$. It follows that the A-BDF is bounded with respect to Theorem 1.

Hence, for $k = 1$ and $0 \leq s \leq 1$ there is a stepsize coefficient for boundedness.
$n_0 = 1,\ \tau_n > 0\ \ (\forall\ n \geq n_0)$.

Case 2. $k = 2$

In this case we have

$$\alpha_1 = \frac{4}{3(1-s)} > 0 \ \text{ for } \ s \in [-1, 1),$$
$$\alpha_2 = \frac{-1-3s}{3(1-s)} > 0 \ \text{ for } \ s \in [-1, -\frac{1}{3}),$$
$$\beta_0 = \frac{2}{3(1-s)} > 0 \ \text{ for } \ s \in [-1, 1),$$
$$\beta_1 = \frac{-2s}{1-s} > 0 \ \text{ for } \ s \in [-1, 0).$$

We see that for $s \in [-1, \frac{-1}{3})$, all of the coefficients$\alpha_1, \alpha_2, \beta_1$ and β_1 are positive and then there is a stepsize coefficient for monotonicity, (2.2). For $s \in [-1, -\frac{1}{3})$, $\xi = 1$ is the only essential root of polynomial $\rho(\xi)$. Also, for this interval, the number of common zeros of the $\rho(\xi)$ and $\sigma(\xi)$ is 0. Considering

$$\tau_0 = \beta_0 = \frac{2}{3(1-s)} > 0 \ \text{ for } \ s \in [-1, 1),$$
$$\tau_1 = \frac{18s^2 - 18s + 8}{9(1-s)^2} > 0 \ \text{ for } \ s \in [-1, 1),$$

when $s \in [-1, -\frac{1}{3})$ from (2.8), we derive $n_0 = 1$ and also τ_0 and τ_1 are positive. Since $\alpha_1, \alpha_2 > 0$ for $s \in [-1, -\frac{1}{3})$ therefore, it can be easily seen that $\tau_2 = \alpha_1\tau_1 + \alpha_2\tau_0 > 0$ and by induction,we have

$$\tau_n = \alpha_1\tau_{n-1} + \alpha_2\tau_{n-2} > 0 \ \ (\forall n \geq 3). \tag{3.4}$$

On the other hand, boundedness is guaranteed for A-BDF. The behaviors of the coefficients α_1, α_2 and the values of τ_0, τ_1 are given in Figures 1 and 2.

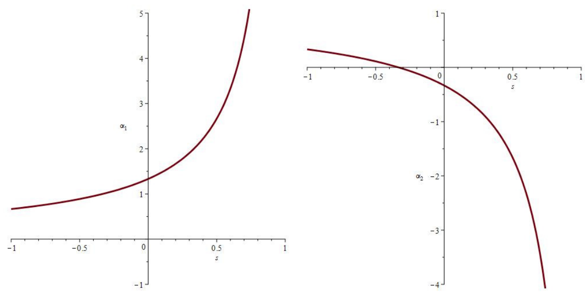

Figure 1. Behavior of the coefficients α_1 (left) and α_2 (right) in case $k = 2$.

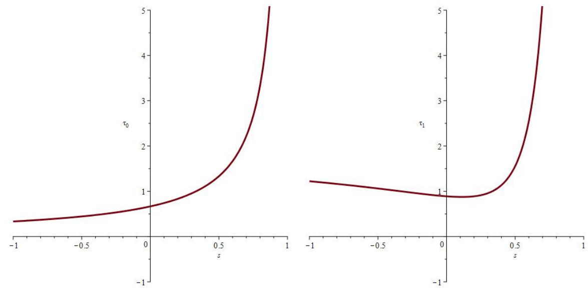

Figure 2. Behavior of the τ_0 (left), τ_1 (right) in case $k = 2$.

Case 3. k=3

In this case the coefficients are

$$\alpha_1 = \frac{36+33s}{22(1-s)} > 0 \;\text{ for }\; s \in [-1,1),$$
$$\alpha_2 = \frac{-9-33s}{11(1-s)} > 0 \;\text{ for }\; s \in [-1,-\frac{3}{11}),$$
$$\alpha_3 = \frac{4+11s}{22(1-s)} > 0 \;\text{ for }\; s \in (-\frac{4}{11},1),$$
$$\beta_0 = \frac{6}{11(1-s)} > 0 \;\text{ for }\; s \in [-1,1),$$
$$\beta_1 = \frac{-3s}{1-s} > 0 \;\text{ for }\; s \in [-1,0).$$

We see that for $s \in (-0.36,-0.27)$, all of the coefficients are positive and $\xi = 1$ is the only essential root of polynomial $\rho(\xi)$. Moreover, the number of common zeros of the $\rho(\xi)$ and $\sigma(\xi)$ is 0. We now have the following expression for τ_0, τ_1 , and τ_2

$$\tau_0 = \beta_0 = \frac{6}{11(1-s)} > 0 \;\text{ for }\; s \in [-1,1),$$
$$\tau_1 = \alpha_1\tau_0 + \beta_1 = \frac{3\left(121s^2-88s+36\right)}{121(s-1)^2} > 0 \;\text{ for }\; s \in [-1,1),$$
$$\tau_2 = \alpha_1\tau_1 + \alpha_2\tau_0 = -\frac{9\left(1331s^3+968s^2-1012s+300\right)}{2662(s-1)^3} > 0 \;\text{ for }\; s \in [-1,1)$$

which give $\tau_0, \tau_1, \tau_2 > 0$ for $s \in (-0.36,-0.27)$. Since coefficients α_0, α_1 and α_2 are positive in the interval $s \in (-0.36,-0.27)$, it can be easily seen that $\tau_3 = \alpha_2\tau_0 + \alpha_1\tau_1 + \alpha_2\tau_0 > 0$ and by induction, remaining elements of τ_n, $n \geq 4$ which is determined uniquely by the underlying recurrence

$$\tau_n = \alpha_1\tau_{n-1} + \alpha_2\tau_{n-2} + \alpha_3\tau_{n-3}, \tag{3.5}$$

are positive. On the other hand, boundedness is guaranteed for A-BDF. This completes the proof. □

The behaviors of the coefficients $\alpha_1, \alpha_2, \alpha_2$ and the values of τ_0, τ_1, τ_2 are given in Figures 3 and 4.

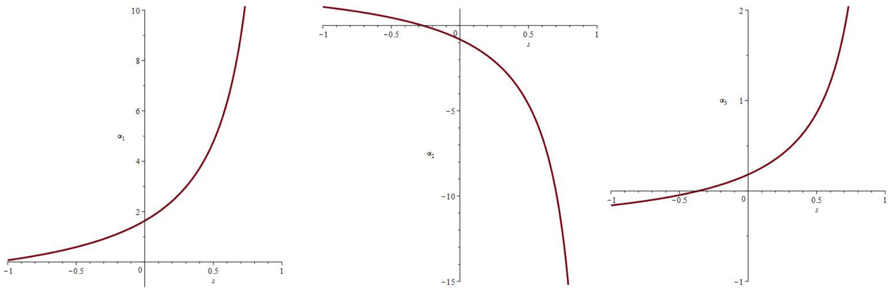

Figure 3. Behavior of the coefficients α_1 (left), α_2 (center) and α_3 (right) for $k = 3$.

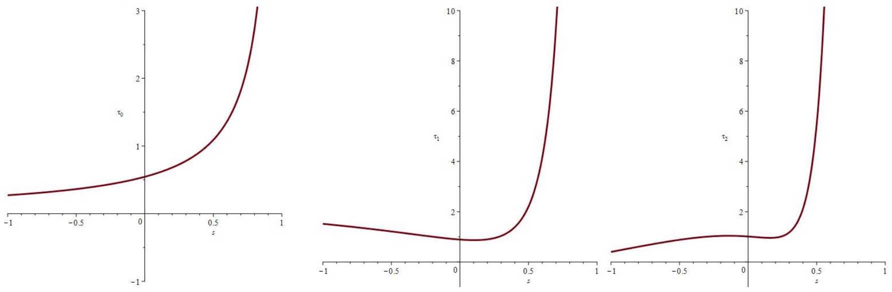

Figure 4. Behavior τ_0 (left), τ_1 (center) and τ_2 (right) for $k = 3$.

4. Numerical results

Here we will first consider the one-dimensional linear advection equation (AE)

$$\frac{\partial}{\partial t}Z(x,t) + \frac{\partial}{\partial x}Z(x,t) = 0, \qquad (t > 0 \textit{ and } 0 < x < 1),$$

with periodic boundary conditions. The computational domain has been discretized by a uniform mesh with grid points $x_i = i\Delta x$, and $\Delta x = \frac{1}{220}$ by using the third-order upwind biased in flux form

$$Z_i^{'}(t) = \frac{1}{\Delta x}(F_{i-\frac{1}{2}}(t, Z(t)) - F_{i+\frac{1}{2}}(t, Z(t))), \quad F_{i\pm\frac{1}{2}}(t, Z) = Z_{i\pm\frac{1}{2}} \quad i \in \{1, 2, ..., 220\}, \tag{4.1}$$

then we get

$$F_{i\pm\frac{1}{2}}(t, Z) = \frac{1}{6}(-Z_{i-1} + 5Z_i + 2Z_{i+1}) = Z_i + (\frac{1}{3} + \frac{1}{6}\vartheta_i)(Z_{i+1} - Z_i),$$

where ϑ_i is the ratio

$$\vartheta_i = \frac{Z_i - Z_{i-1}}{Z_{i+1} - Z_i} \quad i \in \{1, 2, ..., 220\}.$$

Then (4.1) gives

$$Z_i^{'} = \frac{1}{\Delta x}\Big(1 - \phi(\vartheta_{i-1}) + \frac{1}{\vartheta_i}\phi(\vartheta_i)\Big)(Z_{i-1} - Z_i) \quad i \in \{1, 2, ..., 220\},$$

with

$$\phi(\vartheta) = \max\Big(0, \min(1, \frac{1}{3} + \frac{1}{6}\vartheta, \vartheta)\Big),$$

as the limiter function [32]. Subsequently, we solve the obtained nonlinear system in time with $s = -0.35, 0.7$, and $s = 1.2$. Figures 5-7 shows the numerical solution for three schemes (A-BDFs for $k = 1, 2, 3$) with block (nonsmooth) initial profile: $Z_0(x,t) = 1$ for $0.3 < x < 0.7$ and 0 elsewhere, at output time T. The starting approximations are computed by RK2 method. As we can see in Figures 5–7 the qualitative behavior of the three schemes coincides with the results obtained in section 3. On the other hand, for the s in the boundedness interval, we get nice boundedness property with A-BDFs. Whereas these schemes produce large oscillations with $s = 0.7$ (for $k = 2, 3$) and $s = 1.2$ (for $k = 1$), they become increasingly pronounced for increasing time.

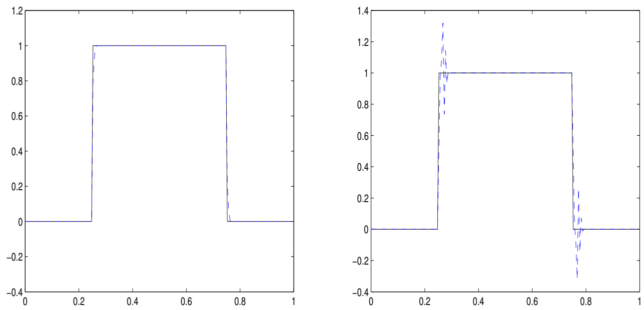

Figure 5. Numerical solutions for the one-step A-BDF method with parameter $s = -0.35$, $T = 1$ (left), and $s = 1.2$, $T = 0.007$ (right). solid line for exact solution and dashed line for numerical solutions.

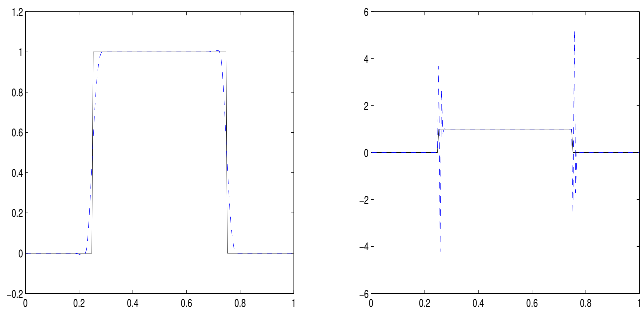

Figure 6. Numerical solutions for the two-step A-BDF method with parameter $s = -0.35$, $T = 1$ (left), and $s = 0.7$, $T = 0.004$ (right). solid line for exact solution and dashed line for numerical solutions.

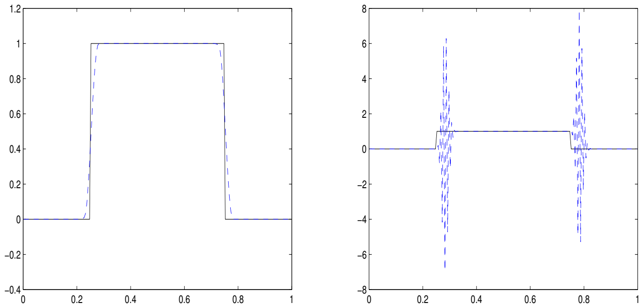

Figure 7. Numerical solutions for the three-step A-BDF method with parameter $s = -0.35$, $T = 1$ (left), and $s = 0.7$, $T = 0.05$ (right). solid line for exact solution and dashed line for numerical solutions.

In addition, in Tables 2 and 3 one can find more details of numerical approximations for these schemes with the number step $N = 440, 700, 1000$, and two initial curves, namely, the smooth profile $Z_0(x,t) = sin^{100}(\pi x)$ and the above-mentioned block function. These lead to value of $\Delta t = 0.0023, 0.0014, 0.001$ and $\nu = \frac{\Delta t}{\Delta x} = \frac{220}{N} \simeq 0.5, 0.31, 0.22$ (the Courant numbers). Moreover, to characterize boundedness, the biggest component of the solution is presented. We observe different behavior with regards to boundedness, with peaked function. Moreover, A-BDFs perform well and allow much larger stepsize than the block initial profile.

Table 2. Results for the scalar linear advection with smooth profile. N denotes the number of time steps.

		One step A-BDF		Two step A-BDF		Three step A-BDF	
s	N	$\min_{i,n}(Z_i^n)$	$\max_{i,n}(Z_i^n)$	$\min_{i,n}(Z_i^n)$	$\max_{i,n}(Z_i^n)$	$\min_{i,n}(Z_i^n)$	$\max_{i,n}(Z_i^n)$
	440	$-1.23e-0-14$	$9.95e-01$	$-5.76e-01$	$1.06e+0$	$-3.71e-009$	$9.95e-01$
$\frac{-35}{100}$	700	$1.71e-122$	$9.98e-01$	$-4.13e-04$	$9.96e-01$	$+8.19e-122$	$9.97e-01$
	1000	$1.74e-123$	$9.98e-01$	$-8.31e-05$	$9.98e-01$	$+4.72e-123$	$9.98e-01$
	440	$-1.12e+24$	$9.64e+23$	$-9.44e+154$	$9.63e+154$	$-1.98e+73$	$1.98e+73$
$\frac{7}{10}$	700	$-1.56e-19$	$9.97e-01$	$-6.11e+154$	$6.04e+154$	$-1.29e+47$	$9.29e+47$
	1000	$-6.56e-48$	$9.97e-01$	$-4.27e+154$	$4.32e+154$	$-9.30e+12$	$9.29e+12$

Table 3. Results for the scalar linear advection with nonsmooth profile. N denotes the number of time steps.

		One step A-BDF		Two step A-BDF		Three step A-BDF	
s	N	$\min_{i,n}(Z_i^n)$	$\max_{i,n}(Z_i^n)$	$\min_{i,n}(Z_i^n)$	$\max_{i,n}(Z_i^n)$	$\min_{i,n}(Z_i^n)$	$\max_{i,n}(Z_i^n)$
	440	$-2.78e-3$	$1+2e-03$	$-2.83e+0$	$2.83e+0$	$-8.00e-2$	$1.04e+0$
$\frac{-35}{100}$	700	$-7.55e-10$	$1+7e-03$	$-7.01e-2$	$1.07e+0$	$0.00e+0$	$1.00e+0$
	1000	$-3.82e-11$	$1+3e-03$	$-4.33e-02$	$1.00e+0$	$0.00e+0$	$1.00e+0$
	440	$-2.21e+22$	$2.33e+22$	$-5.77e+154$	$5.75e+154$	$-1.10e+74$	$1.96e+74$
$\frac{7}{10}$	700	$-2.53e-02$	$1+3e-02$	$-3.83e+154$	$3.83e+154$	$-4.55e+47$	$4.55e+47$
	1000	$-1.53e-02$	$1+4e-02$	$-5.20e+154$	$5.20e+154$	$-6.44e+12$	$6.44e+12$

As second example consider two-dimentional AE, defined by

$$Z_t + Z_x + Z_y = 0, \tag{4.2}$$

on the unit square. After discretization, we get

$$Z'_{ij}(t) = \alpha_{ij}(Z(t))(Z_{i-1,j}(t) - Z_{ij}(t)) + \beta_{ij}(Z(t))(Z_{i,j-1}(t) - Z_{ij}(t)), \tag{4.3}$$

where α_{ij}, β_{ij} satisfying

$$0 \leq \alpha_{ij}(Z) \leq \frac{2}{\Delta x}, \quad 0 \leq \beta_{ij}(Z) \leq \frac{2}{\Delta y},$$

where Δx and Δy are the step lengths. Results for the three-step A-BDF methods have been shown in Figure 8. Figure 9 indicates that when the RK4 method is applied with CFL numbers ≥ 1, the numerical results are quickly corrupted.

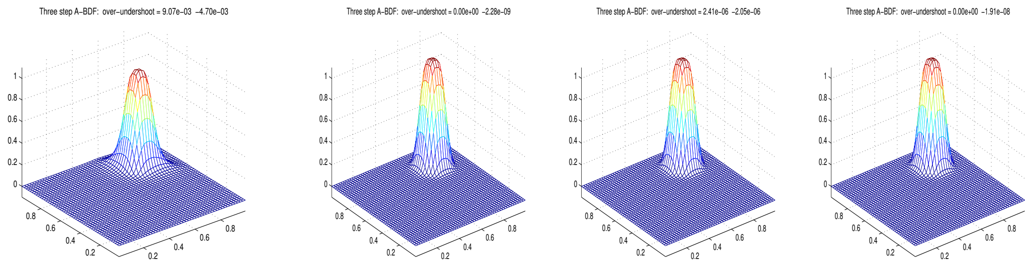

Figure 8. From left, solutions for the three A-BDF method with $N = 220, 110, 55, 28$ time steps, respectively. Corresponding Courant numbers are 2, 1, 0.5, 0.25.

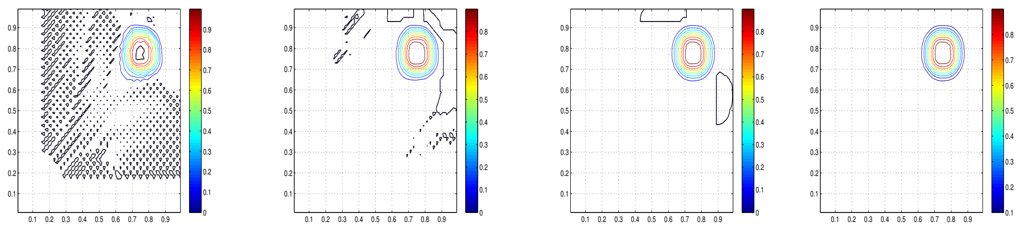

Figure 9. Advection for the cylinder profile on a 50×50 grid. Contour lines at different levels 0.1, 0.2, . . ., 0.9.

As our third example, we consider the one-dimension Euler equations in the conservation form [48]:

$$Z_t + F(Z)_x = 0,$$

with

$$Z^T = (\rho, \rho v, E)^T,$$
$$F(Z) = \left(\rho v, \rho v^2 + p, v(E + p)\right)^T,$$

where ρ and E are the density and the total energy, respectively. v is the velocity, p is the pressure, associated to the total energy in the following form

$$E = \frac{p}{\gamma - 1} + \frac{1}{2}\rho v^2.$$

As for an ideal gas $\gamma = 1.4$ is considered. With initial condition as follows:

$$(\rho, v, p) = \begin{cases} (1, 0.75, 1), & x \leq 0, \\ (0.125, 0, 0.1), & x > 0, \end{cases}$$

this problem is called Sod's problem. To discretize Sod's problem we use the WENO5 [36] scheme in space and A-BDF in time. We set CFL $= 0.1$ and $N = 200$ points in the interval $[0, 1]$ at final time $t_f = 0.17$, where shocks appear in the solution. Figure 10 shows that the numerical solution for the density is sharp and without spurious oscillations.

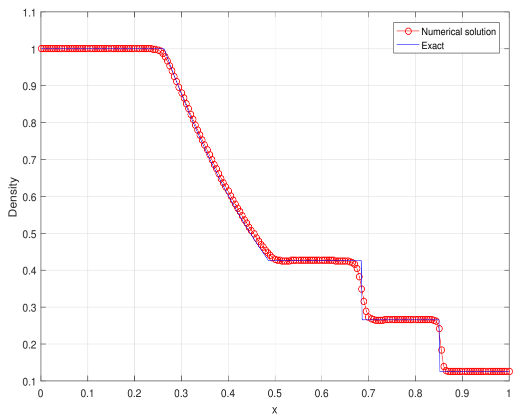

Figure 10. Numerical solution of the Sod's problem obtained by the A-BDF with $N = 200$ and CFL $= 0.1$ in the interval $[0, 1]$ at final time $t_f = 0.17$.

As our final example

$$Z_t = Z_{xx}, \quad t > 0, \ 0 < x < 1,$$

with boundary condition $U(0,2) = U(1,t) = 0$ for $t > 0$ and initial function

$$Z(x,0) = \begin{cases} 0 & 0 < x < \frac{1}{2} \\ 1 & \frac{1}{2} \leq x < 1 \end{cases},$$

giving discontinuities at $x = \frac{1}{2}, 1$ for $t = 0$. Application of the A-BDF method and the SSP RK2 method give the approximate solutions shown in Figure 11. It can be seen that RK2 method gives temporal oscillations and negative values near the points $\frac{1}{2}$ and $x = 1$.

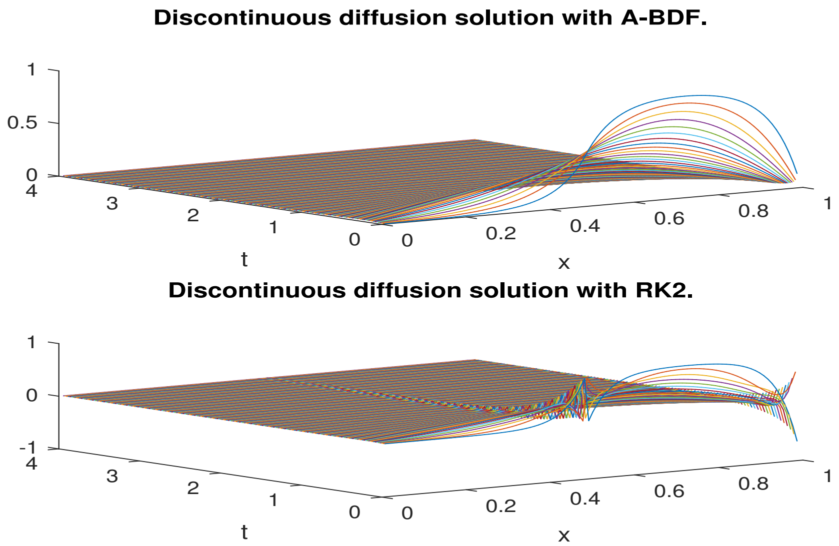

Figure 11. Discontinuous diffusion solution with A-BDF and the SSP RK2 mthod.

5. Conclusions

Schemes preserving monotonicity or boundedness are important in practice because they display high accuracy near discontinuities and shocks. In this article, we illustrated the boundedness property for the A-BDF methods based on the interesting content of [51]. Our numerical experiments show that the RK methods may fail if applied to the hyperbolic equations with stiff source terms. In some cases, employing the second-order A-BDF method, we obtained good accuracy of source terms in regions where they are smooth (or where transients are well-resolved), avoiding oscillations in regions of stiffness. Our numerical evidence shows that if we apply the A-BDF method to the stiff source term in the Burgers equation, oscillations and unphysical states are eliminated. However, it still produces an incorrect solution in the stiff case. We think that the step size control is capable of maintaining accuracy in this case (Our interest for future work). Furthermore, future studies should establish the positivity property of A-BDFs because we have numerical evidence in solving special positive systems–a fact that indicates A-BDF methods keep the positivity of the solutions.

Conflict of interest

The authors declare no conflicts of interest regarding the publication of this paper.

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