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Research article

Regular local hyperrings and hyperdomains

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Abstract: This paper falls in the area of hypercompositional algebra. In particular it focuses on the class of Krasner hyperrings and it studies the regular local hyperrings. These are Krasner hyperrings R with a unique maximal hyperideal M having the dimension equal to the dimension of the vectorial hyperspace $\frac{M}{M^2}$. The aim of the paper is to show that any regular local hyperring is a hyperdomain. For proving this, we make use of the relationship existing between the dimension of the vectorial hyperspaces related to the hyperring R and to the quotient hyperring $\overline{R} = \frac{R}{\langle a \rangle}$, where a is an element in $M \setminus M^2$, and of the regularity of $\overline{R}$.

Keywords: hyperring; hypermodule; vectorial hyperspace; dimension; regular hyperring; regular parameter element; hyperdomain
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1. Introduction

In classical algebra, it is well known that the Krull dimension d of any Noetherian local ring R with maximal ideal M is less than or equal to the dimension of $\frac{M}{M^2}$. If the equality holds, the ring is called regular. One of the crucial results in the theory of regular rings states that any regular local ring is an integral domain. If the dimension of R is zero, then its maximal ideal must be trivial and by consequence R is a field. The same problem has been started to be investigated by Bordbar and Cristea [5] in the theory of Krasner hyperrings. These are ring-like hypercompositional structures, defined by Krasner in 1956, having the additive part a canonical hypergroup (this is a natural generalization of the abelian group) and the multiplicative one a semigroup with a bilaterally absorbing element 0, such that the multiplication distributes from both sides over the addition [11]. Inspired by the properties of the hyperpolynomials (they are defined in the same manner as polynomials, but with coefficients in a Krasner hyperring), Mittas [16] noticed that the multiplication of hyperpolynomials is a hyperoperation and in 1973 he introduced a new class of hyperrings, called superrings. Their additive part remains a canonical hypergroup, while the multiplicative one becomes a semihypergroup with a bilateral absorbing element and the multiplication is distributive on both sides with respect to the addition. Notice that this structure was then called by Ameri et al. [1] a strongly distributive superring, which is in fact an additive-multiplicative hyperring in the sense of Davvaz and Musavi [9]. The definition of the distributivity created some confusions in the terminology, as explained in [10], that we will also clarify in the next section of this paper. If we relax the conditions on the additive structure and consider it as a hypergroup, while the multiplicative part remains a semihypergroup, we obtain the notion of (strong) general hyperring, introduced by Vougiouklis [21]. The last type of hyperring is the multiplicative one, defined by Rota [19], where the addition is a binary operation and the multiplication is a hyperoperation.

Inspired by similar investigations on classical algebra, Bordbar and Cristea have started the study of regular local hyperrings [5], by defining the regular parameter elements in a commutative local hyperring. They have shown that the dimension of a local hyperring R with maximal hyperideal M is equal to the height of M. Since $\frac{R}{M}$ is a hyperfield, the quotient hypermodule $\frac{M}{M^2}$ becomes a vectorial hyperspace, having the dimension greater than or equal to the dimension of the local hyperring R. In the case of equality, the hyperring is called regular. In this paper we continue to investigate on the properties of regular local hyperrings, with the aim to prove that any regular local hyperring is a hyperdomain, i.e., a commutative hyperring without divisors of zero.

The structure of this paper is as follows. In the preliminary section we gather the basic notions related to Krasner hyperrings, dimension of a Krasner hyperring, height of hyperideals, and hypermodules. After shortly recalling the definitions of a regular local hyperring and regular parameter elements, Section 3 focuses on the study of the relationship existing between the dimensions of the vectorial hyperspaces $\frac{M}{M^2}$ and $\frac{\overline{M}}{\overline{M}^2}$ related to a local Krasner hyperring R with maximal hyperideal M, where $\overline{M} = \frac{M}{<a>}$ and $a \in M \setminus M^2$. First, we prove that, if R is a regular local hyperring, then $\overline{R} = \frac{R}{<a>}$ is regular local, too. Then we prove several properties concerning the dimension of R, primary hyperideals, and the dimension of the vectorial hyperspace related to R and $\overline{R}$. These help us to state and prove the main result of this article: every regular local hyperring is a hyperdomain (see Theorem 3.13). The article ends with a non-trivial example of a regular local hyperring. The conclusive section summarizes the findings of this article and their impact on the existing theory, as well as some new lines of research that this study could open.

2. Preliminaries

This section collects the main results about the Krasner hyperrings, that will help the reader to better understand the topic. For a detailed description of the theory of Krasner hyperrings, we suggest to consult the works [8, 12–15, 18].

A *(Krasner) hyperring* is a hypercompositional structure $(R, +, \cdot)$ where

(1) $(R, +)$ is a canonical hypergroup, i.e.,

  (a) $(\forall a, b \in R \Rightarrow a + b \subseteq R)$,
  (b) $(\forall a, b, c \in R)\ (a + (b + c) = (a + b) + c)$,

(c) $(\forall a, b \in R)\ (a + b = b + a)$,
(d) $(\exists 0 \in R)(\forall a \in R)\ (a + 0 = \{a\})$,
(e) $(\forall a \in R)(\exists -a \in R)\ (0 \in a + x \Leftrightarrow x = -a)$,
(f) $(\forall a, b, c \in R)(c \in a + b \Rightarrow a \in c + (-b))$.

(2) $(R, \cdot)$ is a semigroup with a bilaterally absorbing element 0, i.e.,

(a) $(a, b \in R \Rightarrow a \cdot b \in R)$,
(b) $(\forall a, b, c \in R)\ (a \cdot (b \cdot c) = (a \cdot b) \cdot c)$,
(c) $(\forall a \in R)\ (0 \cdot a = a \cdot 0 = 0)$.

(3) The multiplication strongly distributes bilaterally over the addition:

$(\forall a, b, c \in R)\ (a \cdot (b + c) = a \cdot b + a \cdot c \textit{ and } (b + c) \cdot a = b \cdot a + c \cdot a)$.

If moreover $(R, \cdot)$ is commutative, i.e.,

(4) $(\forall a, b \in R)\ (a \cdot b = b \cdot a)$, then the hyperring is called *commutative*.

Finally, if $(R, \cdot)$ is a monoid, i.e.,

(5) $(\exists 1 \in R)(\forall a \in R)\ (a \cdot 1 = a = 1 \cdot a)$, we say that R is *with a unit element*.

(6) A hyperring with a multiplicative identity 1, where every nonzero element is invertible, is called a *hyperfield*.

Throughout this paper, R denotes a *Krasner hyperring*, called here by short, *hyperring*, unless otherwise stated.

A nonempty set I of a hyperring R is called a *right hyperideal* if, for all $a, b \in I$ and $r \in R$, we have $a - b \subseteq I$ and $a \cdot r \in I$. Similarly, a left hyperideal is defined dually and a set I is called a hyperideal if it is a left and right hyperideal of R. A proper hyperideal M of a hyperring R is called *maximal* if M itself and R are the only hyperideals of R containing M. A hyperideal P of a hyperring R is called *prime* if, for every pair of elements a and b of R, the fact that $ab \in P$ implies either $a \in P$ or $b \in P$. Moreover, a nonzero hyperring R having exactly one maximal hyperideal is called a *local hyperring*. A prime hyperideal P of R is called a *minimal prime hyperideal over a hyperideal I of R* if it is minimal (with respect to the inclusion) among all prime hyperideals of R containing I. A prime hyperideal P is called a *minimal prime hyperideal* if it is a minimal prime hyperideal over the zero hyperideal of R.

A hyperring R is called *Noetherian* if it satisfies the *ascending chain condition* on the hyperideals of R, in the sense that, for every ascending chain of hyperideals $I_1 \subseteq I_2 \subseteq I_3 \subseteq \ldots$, there exists $N \in \mathbb{N}$ such that $I_n = I_N$, for every natural number $n \geq N$.

The next result fixes the form of the hyperideals of a quotient hyperring.

Proposition 2.1. *Let R be a commutative hyperring and I be a hyperideal of R. For any hyperideal J of R such that $I \subseteq J$, the set $\frac{J}{I}$ is a hyperideal of $\frac{R}{I}$. Moreover, for any arbitrary element $r \in R$, we have*

$$r + I \subseteq \frac{J}{I} \iff r \in J.$$

Let us recall now some results introduced and studied in [4, 6, 7] related to the dimension of a hyperring. Let P be a prime hyperideal of a commutative hyperring R. The *height* of P, denoted by $ht_R P$, is defined to be the supremum of the lengths of all chains

$$P_0 \subset P_1 \subset \ldots \subset P_n$$

of prime hyperideals of R, for which $P_n = P$, if this supremum exists, and it is ∞, otherwise. We say that the *dimension* of a hyperring R, denoted by $dimR$, is the supremum of the lengths of all chains of prime hyperideals of R. If R is a local hyperring with maximal hyperideal M, then $dimR = ht_R M$.

Proposition 2.2. *Let R be a commutative Noetherian hyperring and I a proper hyperideal of R generated by n elements. Then $ht_R I \leq n$.*

On a canonical hypergroup, endowed with an external operation using elements from a Krasner hyperring, a structure of Krasner hypermodule [2, 3, 13] can be defined as follows.

Let R be a hyperring with unit element 1. An *R-hypermodule M* is a canonical hypergroup $(M, +)$ together with a map $R \times M \longrightarrow M$ defined by

$$(a, m) \mapsto a \cdot m = am \in M \tag{2.1}$$

such that for all $a, b \in R$ and $m_1, m_2 \in M$ we have

(1) $(a + b)m_1 = am_1 + bm_1$.
(2) $a(m_1 + m_2) = am_1 + am_2$.
(3) $(ab)m_1 = a(bm_1)$.
(4) $a0_M = 0_R m_1 = 0_M$.
(5) $1m_1 = m_1$, where 0_M denotes the zero element of the canonical hypergroup $(M, +)$ and 0_R the zero element of the additive part of R, which is again a canonical hypergroup. Besides, if R is a hyperfield, then M is called a *vectorial hyperspace*.

This definition was given by Stratigopoulos [20] and later, Mittas [17] relaxed the first axiom, without changing the name of the structure, asking only the inclusion to be satisfied, i.e., $(a + b)m_1 \subseteq am_1 + bm_1$, in order to state that the product F^n of n-copies of a hyperfield F is an F-vectorial hyperspace with the canonical addition and multiplication by scalars of the n-tuples of elements of F. Recently, this structure has been re-named a *weak hypermodule* [15] (to not be confused with the H_v-modules, where the weak distributivity [22] is involved: $x \cdot (y + z) \cap x \cdot y + x \cdot z \neq \emptyset$ and $(x + y) \cdot z \cap x \cdot z + y \cdot z \neq \emptyset$, for any three elements x, y, z). In our paper we will use the initial definition given by Statigopoulous and call the structure a Krasner hypermodule, or simply a hypermodule.

Let V be a vectorial hyperspace over a hyperfield F. A set A of elements of V is called:

(1) Linearly independent if for every finite set of vectors $\{v_1, v_2, \ldots, v_n\} \subseteq A$, $0 \in \sum_{i=1}^{n} r_i v_i$ implies $r_i = 0$ for all i such that $1 \leq i \leq n$. If A is not linearly independent, then A is called linearly dependent.
(2) Spanning set if each vector of V is contained in a linear combination from A. We say that V is the span of A.

A basis for a vectorial hyperspace V over a hyperfield F is a set $B \subseteq V$ which is both spanning set and linearly independent. A vectorial hyperspace is finite-dimensional if it has a finite basis. The number of elements in an arbitrary basis of a vectorial hyperspace is called the *dimension* of the vectorial hyperspace and we denote it by $vdim_F V$, where "v" stays for "vectorial hyperspace". This is to differentiate it from the dimension of a hyperring R, denoted simply by $dimR$.

We conclude this section by recalling one important result in hypermodule theory. It comes from the original Nakayama's lemma in classical algebra, that establishes the interactions between the Jacobson radical ideal of a commutative ring and its finitely generated module. In a commutative hyperring R, the intersection between all maximal hyperideals is called the Jacobson radical of the hyperring, denoted by $Jac(R)$.

Lemma 2.3. [4] (Nakayama's lemma) *Let M be a finitely generated R-hypermodule and I a hyperideal of R contained by the Jacobson radical $Jac(R)$ of R. Then $M = IM$ implies that $M = \{0\}$.*

3. Regular local hyperrings and hyperdomains

The aim of this section is to show that every local regular hyperring is a hyperdomain. For doing this, we use elements from hypermodule theory, where by a hypermodule we mean the structure defined by Stratigopoulos, where the distributivity property is defined using the equality.

We start with the result about the relationship existing between the dimension of a local hyperring with maximal hyperideal M and the dimension of the associated vectorial hyperspace $\frac{M}{M^2}$.

Theorem 3.1. [5] *Let R be a local hyperring with maximal hyperideal M. Then*

$$dimR \leq vdim_{\frac{R}{M}} \frac{M}{M^2}.$$

This naturally leads to a new definition, the one of a regular hyperring, as follows.

Definition 3.2. [5] Let R be a local Noetherian hyperring with maximal hyperideal M. We say that R is a regular hyperring when $dimR = vdim_{\frac{R}{M}} \frac{M}{M^2}$. Consider $d = dimR$. By regular parameter elements of R we mean a set of d elements of R that generate an M-primary hyperideal of R.

Some properties of a finitely generated R-hypermodule over a local hyperring R are covered in the next results.

Proposition 3.3. [5] *Let R be a local hyperring with maximal hyperideal M and consider the hyperfield $F = \frac{R}{M}$. Let N be a finitely generated R-hypermodule. Then the R-hypermodule $\frac{N}{MN}$ has a natural structure as a hypermodule over $\frac{R}{M}$ as a F-vectorial hyperspace.*

Moreover, let $n_1, n_2, \ldots, n_t \in N$. Then the following statements are equivalent.

(*i*) *N is generated by $n_1, n_2, \ldots, n_t$.*
(*ii*) *The R-hypermodule $\frac{N}{MN}$ is generated by the elements $n_1 + MN$, $n_2 + MN, \ldots, n_t + MN$.*
(*iii*) *The F-vectorial hyperspace $\frac{N}{MN}$ is generated by the elements $n_1 + MN$, $n_2 + MN, \ldots, n_t + MN$.*

Corollary 3.4. *For a local hyperring R with maximal hyperideal M and $dimR = d$, the following properties are fulfilled.*

(*i*) *The dimension of the $\frac{R}{M}$-vectorial hyperspace $\frac{M}{M^2}$ is the number of elements in each minimal generating set for the hyperideal M. Moreover, at least d elements are needed to generate M, and R is a regular hyperring when the hyperideal M can be generated by exactly d elements.*
(*ii*) *Suppose that R is a regular hyperring and $a_1, a_2, \ldots, a_d \in M$. Using Proposition 3.3, the elements $a_1, a_2, \ldots, a_d$ generate M if and only if the elements $a_1 + M^2, a_2 + M^2, \ldots, a_d + M^2$ in $\frac{M}{M^2}$ form a basis for this $\frac{R}{M}$-vectorial hyperspace. Equivalently, the elements $a_1 + M^2, a_2 + M^2, \ldots, a_d + M^2$ form a linearly independent set.*

In the next theorem, we study the relationship between the dimensions of the two vectorial hyperspaces related to a local hyperring.

Theorem 3.5. *Suppose that R is a local hyperring with maximal hyperideal M. For an arbitrary element $a \in M \setminus M^2$, take $\overline{R} = \frac{R}{\langle a \rangle}$, $\overline{M} = \frac{M}{\langle a \rangle}$, and $\overline{M^2} = \frac{M^2}{\langle a \rangle}$. Moreover, suppose that the application $nat : R \to \overline{R}$ is the natural surjective hyperring homomorphism defined, for each $r \in R$, as follows:*

$$nat(r) = r + \langle a \rangle. \tag{3.1}$$

Then, we have

$$vdim_{\frac{R}{M}} \frac{M}{M^2} = vdim_{\frac{\overline{R}}{\overline{M}}} \frac{\overline{M}}{\overline{M^2}} + 1.$$

Proof. Since M is a unique maximal hyperideal of R, it follows that $\overline{R}$ is a local hyperring with $\bar{M}$ its maximal hyperideal. Hence, $\frac{\overline{R}}{\overline{M}}$ is a hyperfield and thus $\frac{\overline{M}}{\overline{M^2}}$ is a $\frac{\overline{R}}{\overline{M}}$-vectorial hyperspace. Suppose that $d = vdim_{\frac{\overline{R}}{\overline{M}}} \frac{\overline{M}}{\overline{M^2}}$. Then, using Proposition 3.3, there exist the elements $m_1, m_2, \ldots, m_d \in M$ such that their images under the composition of the following two homomorphisms, i.e., the homomorphism $g \circ f$, is a basis for the $\frac{\overline{R}}{\overline{M}}$-vectorial hyperspace $\frac{\overline{M}}{\overline{M^2}}$: For any $i = 1, 2, \ldots, d$,

$$M \xrightarrow{f} \overline{M} \xrightarrow{g} \frac{\overline{M}}{\overline{M^2}}, \quad f(m_i) = \overline{m_i} = m_i + \langle a \rangle, \quad g(\overline{m_i}) = \overline{m_i} + \overline{M^2}. \tag{3.2}$$

Moreover, Proposition 3.3 shows that the hyperideal $\overline{M}$ of $\overline{R}$ is generated by d elements $\overline{m_1}, \overline{m_2}, \ldots, \overline{m_d}$. This means that,

$$\overline{M} = \sum_{i=1}^{d} \langle \overline{m_i} \rangle = \sum_{i=1}^{d} \overline{R}\,\overline{m_i} = \sum_{i=1}^{d} \frac{R}{\langle a \rangle}(m_i + \langle a \rangle) = \frac{(\sum_{i=1}^{d} \langle m_i \rangle + \langle a \rangle)}{\langle a \rangle}.$$

Then, based on Proposition 2.1, we can write

$$M = \sum_{i=1}^{d} \langle m_i \rangle + \langle a \rangle.$$

In other words, the set $\{m_1, m_2, \ldots, m_d, a\}$ is a generating set for M. Corollary 3.4 part (*ii*) shows that the $\frac{R}{M}$-vectorial hyperspace $\frac{M}{M^2}$ is generated by the elements $m_1 + M^2, m_2 + M^2, \ldots, m_d + M^2, a + M^2$. In order to complete the proof, we need to prove that $m_1 + M^2, m_2 + M^2, \ldots, m_d + M^2, a + M^2$ are linearly independent elements of $\frac{M}{M^2}$ over the hyperfield $\frac{R}{M}$. For doing this, let $r_1 + M, r_2 + M, \ldots, r_d + M, r_{d+1} + M$ be elements in $\frac{R}{M}$ such that

$$(r_1 + M)(m_1 + M^2) + \cdots + (r_d + M)(m_d + M^2) + (r_{d+1} + M)(a + M^2) = 0_{\frac{M}{M^2}}. \tag{3.3}$$

Since M is a hyperideal of R and $M^2 \subseteq M$, we have

$$((r_1 \cdot m_1) + M^2) + \cdots + ((r_d \cdot m_d) + M^2) + ((r_{d+1} \cdot a) + M^2) = 0_{\frac{M}{M^2}} = M^2,$$

which means that

$$r_1 \cdot m_1 + r_2 \cdot m_2 + \cdots + r_d \cdot m_d + r_{d+1} \cdot a \in M^2. \tag{3.4}$$

Therefore,

$$\Big(r_1 \cdot m_1 + r_2 \cdot m_2 + \cdots + r_d \cdot m_d + r_{d+1} \cdot a\Big) + \langle a\rangle \in \overline{M^2} = \frac{M^2}{\langle a\rangle}.$$

Since $r_{d+1} \cdot a \in \langle a\rangle$, we have

$$\Big(r_1 \cdot m_1 + r_2 \cdot m_2 + \cdots + r_d \cdot m_d\Big) + \langle a\rangle \in \overline{M^2},$$

which means that

$$((r_1 \cdot m_1) + \langle a\rangle) + ((r_2 \cdot m_2) + \langle a\rangle) + \cdots + ((r_d \cdot m_d) + \langle a\rangle) \in \overline{M^2}. \tag{3.5}$$

Using (3.1), we know that $(r_i \cdot m_i) + \langle a\rangle = nat(r_i \cdot m_i)$ and clearly $nat(r_i \cdot m_i) = nat(r_i) \cdot nat(m_i)$. Thus, we have the following equation in $\frac{\overline{M}}{\overline{M^2}}$:

$$\Big(nat(r_1) \cdot nat(m_1) + nat(r_2) \cdot nat(m_2) + \cdots + nat(r_d) \cdot nat(m_d)\Big) + \overline{M^2} = \overline{M^2},$$

equivalently with

$$(nat(r_1) + \overline{M}) \cdot (nat(m_1) + \overline{M^2}) + \cdots + (nat(r_d) + \overline{M}) \cdot (nat(m_d) + \overline{M^2}) = 0_{\frac{\overline{M}}{\overline{M^2}}}. \tag{3.6}$$

Since $\frac{\overline{M}}{\overline{M^2}}$ is a vectorial hyperspace over the hyperfield $\frac{\overline{R}}{\overline{M}}$ and using (3.2), it follows that $\{\overline{m_1} + \overline{M}^2, \overline{m_2} + \overline{M}^2, \ldots, \overline{m_d} + \overline{M}^2\}$ is a generating set for the vectorial hyperspace $\frac{\overline{M}}{\overline{M^2}}$. Threfore the elements $\overline{m_1} + \overline{M}^2, \overline{m_2} + \overline{M}^2, \ldots, \overline{m_d} + \overline{M}^2$ are linearly independent in $\frac{\overline{R}}{\overline{M}}$. Based on (3.6) we conclude that $\overline{r_i} + \overline{M} = 0_{\frac{\overline{R}}{\overline{M}}} = \overline{M}$ for $1 \le i \le d$. Hence for each i, $1 \le i \le d$, $\overline{r_i} \in \overline{M}$ and by using Proposition 2.1, we have

$$r_1, r_2, \ldots, r_d \in M = 0_{\frac{R}{M}}. \tag{3.7}$$

Therefore in (3.3), for each i, $1 \le i \le d$, $r_i + M = M = 0_{\frac{R}{M}}$. It is enough to show that $r_{d+1} \in M$ because in this case $r_{d+1} + M = M = 0_{\frac{R}{M}}$ and therefore in (3.3) the elements $m_1 + M^2, m_2 + M^2, \ldots, m_d + M^2, a + M^2$ are all linearly independent.

For this goal, suppose that $r_{d+1} \notin M$. In this case, r_{d+1} should be a unit element of the hyperring R. From (3.4) and (3.7), we can conclude that $r_{d+1}a \in M^2$. Thus, $a = a\, r_{d+1}^{-1}\, r_{d+1} \in M^2$ which contradicts the hypothesis $a \in M \setminus M^2$. Therefore, $r_{d+1} \in M$ and the set $\{m_1 + M^2, m_2 + M^2, \ldots, m_d + M^2, a + M^2\}$ is a basis for $\frac{\overline{M}}{\overline{M^2}}$, equivalently with

$$vdim_{\frac{\overline{R}}{\overline{M}}} \frac{\overline{M}}{\overline{M}^2} = d + 1.$$

□

Theorem 3.6. [6] Let R be a commutative Noetherian hyperring, P a prime hyperideal of R and I a proper hyperideal of R generated by n elements, such that $I \subseteq P$. Then:

$$ht_{\frac{R}{I}}\frac{P}{I} \leq ht_R P \leq ht_{\frac{R}{I}}\frac{P}{I} + n.$$

As a consequence of this theorem, we can state the next result.

Theorem 3.7. *Suppose that R is a local regular hyperring with maximal hyperideal M and $a \in M \setminus M^2$ is an arbitrary element. Let $\overline{R} = \frac{R}{\langle a \rangle}$ and $\overline{M} = \frac{M}{\langle a \rangle}$. Then $\overline{R} = \frac{R}{\langle a \rangle}$ is a regular hyperring.*

Proof. Since R is a regular local hyperring with maximal hyperideal M, using Definition 3.2, we state that

$$dimR = vdim_{\frac{R}{M}}\frac{M}{M^2}.$$

Moreover, the hyperring $\overline{R}$ is a local hyperring with maximal hyperideal $\overline{M}$. Using Theorem 3.6, we conclude that

$$ht_R M - 1 \leq ht_{\overline{R}}\overline{M}.$$

Besides, R is a local hyperring, hence $dimR = ht_R M$. Applying Theorem 3.1 to the hyperring $\overline{R}$, we get

$$dimR - 1 = ht_R M - 1 \leq ht_{\overline{R}}\overline{M} = dim\overline{R} \leq vdim_{\frac{\overline{R}}{\overline{M}}}\frac{\overline{M}}{\overline{M}^2}.$$

Using now Definition 3.2 and Theorem 3.5, we conclude that

$$vdim_{\frac{\overline{R}}{\overline{M}}}\frac{\overline{M}}{\overline{M}^2} = vdim_{\frac{R}{M}}\frac{M}{M^2} - 1 = dimR - 1.$$

Thus,

$$dim\overline{R} = vdim_{\frac{\overline{R}}{\overline{M}}}\frac{\overline{M}}{\overline{M}^2}.$$

Therefore, $\overline{R}$ is a regular hyperring. □

In the following, we present two results on Noetherian hyperrings, needed in the proof of the Theorem 3.10.

Proposition 3.8. *Let R be a Noetherian hyperring, A be a hyperideal of R and $B = \cap_{n=1}^{\infty} A^n$. Then, $B = AB$.*

Proof. If $A = R$, then it is clear that $B = AB$. So, assume that A is a proper hyperideal of R. A routine verification shows that $AB \subseteq B$. Since $AB \subseteq B \subseteq A \neq R$, we conclude that AB is a proper hyperideal of R.

R is a Noetherian hyperring, thus AB has a decomposition. Suppose that $AB = P_1 \cap P_2 \cap \cdots \cap P_n$ and $r(P_i) = Q_i$ for $1 \leq i \leq n$, is a minimal primary decomposition of AB. If there exists i, where $1 \leq i \leq n$, such that $B \nsubseteq P_i$, then we can choose an element $b \in B \setminus P_i$. Besides, $bA \subseteq AB = P_1 \cap P_2 \cap \cdots \cap P_n \subseteq P_i$. Since P_i is a primary hyperideal and $b \notin P_i$, there exists $m \in \mathbb{N}$ such that $A \subseteq P_i^m$. Thus,

$$B = \cap_{n=1}^{\infty} A^n \subseteq A \subseteq P_i^m \subseteq P_i.$$

This is a contradiction. Hence, for every i where $1 \leq i \leq n$, we have $B \subseteq P_i$. This means that $B \subseteq AB$ and therefore $AB = B$. □

Lemma 3.9. *Let R be a Noetherian hyperring R and A be a hyperideal such that $A \subseteq Jac(R)$. Then $\cap_{n=1}^{\infty} A^n = 0$.*

Proof. Suppose that $B = \cap_{n=1}^{\infty} A^n$. Using Proposition 3.8, we have $B = AB$. Since R is a Noetherian hyperring, B has a finitely generated R-hypermodule structure. Thus, using Lemma 2.3, we conclude that $B = 0$. □

Theorem 3.10. *Let R be a local hyperring with maximal hyperideal M. Suppose that R is not a hyperdomain and P is a principal prime hyperideal of R. Then $ht_R P = 0$.*

Proof. Suppose that P is a principal hyperideal of R generated by the element a and $ht_R P \neq 0$. Then, there exists at least a prime hyperideal Q of R such that $Q \subset P$. If we show that, for all $n \in \mathbb{N}$, $Q \subseteq P^n$ then, using Lemma 3.9, we conclude that $Q = 0$. We will prove this statement by induction on n. If $n = 1$, take an arbitrary $q \in Q$. It is clear that $q \in P$ and $Q \subseteq P^1$. Suppose we have shown that $q \in P^n$. Since $P =< a >$, there exists $r \in R$ such that $q = r \cdot a^n \in Q$. Since $a \notin Q$ and Q is a prime hyperideal, it follows that $r \in Q$. Hence, $r \in P$ and $q = ra^n \in P^{n+1}$. Therefore, by induction on n, we conclude that

$$Q \subseteq \cap_{n=1}^{\infty} P^n.$$

Lemma 3.9 shows that $\cap_{n=1}^{\infty} P^n = 0$ and therefore $Q = 0$. In this case R should be a hyperdomain and this is a contradiction. So $ht_R P = 0$. □

Corollary 3.11. *Let R be a local hyperring with maximal hyperideal M. Suppose that R is not a hyperdomain and P is a principal prime hyperideal of R. Then P is a minimal prime hyperideal.*

Proposition 3.12. *Suppose that $P_1, P_2, \ldots, P_n$ are hyperideals of a hyperring R such that at least $n-2$ of them are prime hyperideals, where $n \geq 2$. Let A be a hyperideal of R with the property*

$$A \subseteq P_1 \cup P_2 \cup \cdots \cup P_n.$$

Then, $A \subseteq P_i$, for some i, $1 \leq i \leq n$.

Proof. In order to prove the statement, we will use the mathematical induction method on n. First, for $n = 2$ suppose that $A \subseteq P_1 \cup P_2$. If $A \nsubseteq P_1$ and $A \nsubseteq P_2$, then there exists $x \in A \setminus P_1$ and $y \in A \setminus P_2$. Since $x \in A \setminus P_1$ and $A \subseteq P_1 \cup P_2$, we conclude that $x \in P_2$ and similarly, $y \in P_1$. Besides, A is a hyperideal and $x, y \in A$, hence, $x - y \subseteq A \subseteq P_1 \cup P_2$. For an arbitrary element $z \in x - y$, we have $z \in P_1 \cup P_2$. Thus, $z \in P_1$ or $z \in P_2$. Now suppose that $z \in P_1$. Since $z \in x - y$, it follows that $x \in z + y$. Moreover, z and y are elements of P_1 and P_1 is a hyperideal. Hence, $x \in z + y \subseteq P_1$ which is a contradiction with $x \in A \setminus P_1$.

Similarly, the other possibility ($z \in P_2$) leads us to another contradiction ($y \in P_2$). Therefore, $A \subseteq P_1$ or $A \subseteq P_2$.

Assume that $k \geq 2$ and that for $n = k$ the result has been proved. Now we prove it for $n = k + 1$. Then,

$$A \subseteq \cup_{i=1}^{k+1} P_i,$$

where $k - 1$ hyperideals of P_i, for $1 \leq i \leq k + 1$, are prime. Suppose that P_1 is one of the prime hyperideals. If for each j, where $1 \leq j \leq k + 1$, $A \nsubseteq \cup_{i=1, i \neq j}^{k+1} P_i$, then there exists $a_j \in A$ such that

$$a_j \in A \setminus \cup_{i=1, i \neq j}^{k+1} P_i.$$

Clearly, $a_j \in P_j$, for each $1 \leq j \leq k+1$. Moreover, since P_1 is a prime hyperideal and $a_2, a_3, \dots, a_{k+1} \notin P_1$ it follows that $a_2a_3 \cdots a_{k+1} \notin P_1$. Let define the set B as $B = a_1 + a_2a_3 \cdots a_{k+1}$. Then B can not be a subset of P_1, because if $B \subseteq P_1$, then

$$a_2a_3 \cdots a_{k+1} \in B - a_1 \subseteq P_1,$$

that is a contradiction. Similarly, B can not be a subset of any P_t, where $2 \leq t \leq k+1$.

Since A is a hyperideal and $a_1, a_2, \dots, a_{k+1} \in A$, a routine verification shows that

$$B = a_1 + a_2a_3 \cdots a_{k+1} \subseteq A.$$

Let $b \in B$ be an arbitrary element. Then $b \in A$ and using the hypothesis $A \subseteq \cup_{i=1}^{k+1} P_i$, we conclude that $b \in P_1 \cup P_2 \cup \cdots \cup P_{k+1}$. Thus, $b \in P_1$ or $b \in P_t$ for some t, where $2 \leq t \leq k+1$. In both case this is a contradiction, because B can not be a subset of P_1 or P_t for some t, where $2 \leq t \leq k+1$.

Therefore, at least for one j, with $1 \leq j \leq k+1$, we have $A \subseteq \cup_{i=1, i \neq j}^{k+1} P_i$ and using the induction we conclude that $A \subseteq P_i$ for some i, with $1 \leq i \leq k+1$. □

Now we have all the elements to prove the main result of this article.

Theorem 3.13. *Every regular hyperring is a hyperdomain.*

Proof. Let R be a regular hyperring of dimension d with its maximal hyperideal M. We will use the mathematical induction on d to prove this theorem. If $d = 0$, then using Definition 3.2 and Corollary 3.4, it follows that $vdim_{\frac{R}{M}} \frac{M}{M^2} = 0$ and the number of the elements in each generating set for M is 0. Hence $M = 0$ and R should be a hyperfield. Since every hyperfield is a hyperdomain, the theorem is proved. Now suppose that $d > 0$ and every regular hyperring of dimension less than d is a hyperdomain.

Suppose by absurd that R is not a hyperdomain. Then $M^2 \subsetneq M$ because

$$0 < d = dimR = vdim_{\frac{R}{M}} \frac{M}{M^2}.$$

So there exists an element $a \in M \setminus M^2$. Moreover, using Theorem 3.7, we conclude that the hyperring $\frac{R}{<a>} = \bar{R}$ is regular having the dimension $d-1$. Thus, based on the induction assumption, the hyperring $\bar{R}$ is a hyperdomain. Therefore, the hyperideal $< a >$ is prime. Theorem 3.10 indicates that $ht < a >= 0$ and so $< a >$ is a minimal prime hyperideal.

Since R is a Noetherian hyperring it follows that every hyperideal of R has a finite set of minimal prime hyperideals. Suppose that the set $\{P_1, P_2, \dots, P_n\}$ is the set of minimal prime hyperideals of 0. Then, for an arbitrary element $a \in M \setminus M^2$, there exists i, $1 \leq i \leq n$, such that $< a >= P_i$. Thus, we conclude that

$$M \setminus M^2 \subseteq P_1 \cup P_2 \cup \cdots \cup P_n.$$

In other words

$$M \subseteq M^2 \cup P_1 \cup P_2 \cup \cdots \cup P_n.$$

Using Proposition 3.12, we conclude that $M \subseteq M^2$ or $M \subseteq P_i$, for some i, $1 \leq i \leq n$. In both cases we get a contradiction since we have shown that $M^2 \subset M$ and besides if $M \subseteq P_i$ for some i, where $1 \leq i \leq n$, then

$$dimR = ht_R M \leq htP_i = 0,$$

which contradicts our hypothesis. Therefore, R is a hyperdomin and the proof is complete. □

In the next example, we construct a local hyperring which is also a principle hyperideal hyperdomain, containing non-trivial regular parameter elements.

Example 3.14. Let $R = \{0, 1\} \cup \{\frac{1}{p^n} \mid \ p \ is \ a \ prime \ number \ and \ n \in \mathbb{N}\}$. First we endow the carrier set R with a canonical hypergroup structure, by defining on R the following hyperoperation:

$$x + y = \begin{cases} \max\{x, y\}, & \text{if } x \neq y, \\ \{r \in R \mid r \leq x\} = [0, x]_R, & \text{if } x = y. \end{cases} \tag{3.8}$$

It is clear that this is a commutative hyperoperation. Besides, also the associativity property holds. Indeed, suppose that x, y and z are arbitrary elements of R. If $x = y = z$, then obviously,

$$x + (y + z) = (x + y) + z = [0, x]_R.$$

If all three elements are different, then $(x + y) + z = \max\{x, y\} + z = \max\{x, y, z\}$. Similarly, we have $x + (y + z) = x + \max\{y, z\} + z = \ max\{x, y, z\}$. Thus, in this case, $x + (y + z) = (x + y) + z$.

Now suppose that $x = y \neq z$. Then we distinguish two possibilities.

Case I: $z < x = y$. Then

$$(x + y) + z = [0, x]_R + z = \cup_{w \in [0,x]_R}(w + z). \tag{3.9}$$

Since $z, w \in [0, x]_R$, for the last equation in (3.9), we have

$$w + z = \begin{cases} [0, w]_R, \ if \ w = z, \\ z, \ if \ w < z, \\ w, \ if \ z < w. \end{cases}$$

Thus,

$$\cup_{w \in [0,x]_R}(w + z) = [0, w]_R \cup \{w \in R \mid z < w \leq x\} \cup \{z\} = [0, x]_R.$$

In a similar way one proves that $x + (y + z) = [0, x]_R$, meaning that the associativity holds.

Case II: $x = y < z$. Then,

$$(x + y) + z = [0, x]_R + z = \cup_{w \in [0,x]_R}(w + z) = z.$$

Besides, $x + (y + z) = x + z = z$, thus again the hyperoperation is associative.

It is a routine to verify that, for any $x \in R$, $0 + x = x + 0 = \{x\}$. Moreover, since for each $x \in R$, $0 \in x + x = [0, x]_R$, we conclude that for every element $x \in R$, there exist an element $y \in R$ such that $0 \in x + y$.

Finally, we prove that for all $x, y, z \in R$,

$$z \in x + y \Longrightarrow x \in z + y.$$

Suppose that $z \in x + y$ and $x \neq y$. Thus $z = \max\{x, y\}$ and $z + y = \max\{x, y\} + y$. Hence,

$$z + y = \max\{x, y\} + y = \begin{cases} x + y, \ if \ x > y, \\ y + y, \ if \ x < y \end{cases} = \begin{cases} x, \ if \ x > y, \\ [0, y]_R, \ if \ x < y. \end{cases}$$

In both cases, it results $x \in z + y$.

Now suppose that $z \in x + y$ and $x = y$. Then from $z \in x + y$, it follows that $z \in [0, x]_R$, meaning that $z \leq x$. Therefore,

$$z + y = z + x = \begin{cases} \max\{z, x\}, \ if \ x > z, \\ [0, x]_R, \ if \ x = z \end{cases} = \begin{cases} x, \ if \ x > z, \\ [0, x]_R, \ if \ x = z. \end{cases}$$

And again it results $x \in z + y$.

Therefore, $(R, +)$ is a canonical hypergroup. We endow it now with the classical multiplication of real numbers, which is associative and has the property that, for all $x \in R$, $0 \cdot x = x \cdot 0 = 0$. In order to show that $(R, +, \cdot)$ is a Krasner hyperring, it is enough to prove that the multiplication distributes over the addition, i.e., for all $x, y, z \in R$,

$$(x + y) \cdot z = x \cdot z + y \cdot z. \tag{3.10}$$

If $z = 0$, then Eq (3.10) is clear. So we suppose that $z \neq 0$. Thus $z = \frac{1}{p^k}$, for some $k \in \mathbb{N}$. Now we have two possibilities, the first one is $x = y$ and the second one is $x \neq y$.

If $x \neq y$, then

$$(x + y) \cdot z = \max\{x, y\} \cdot z = \begin{cases} x \cdot z, \ if \ x > y, \\ y \cdot z, \ if \ y > x, \end{cases}$$

while,

$$x \cdot z + y \cdot z = \max\{x \cdot z, y \cdot z\} = \begin{cases} x \cdot z, \ if \ x > y, \\ y \cdot z, \ if \ y > x. \end{cases}$$

This means that equality (3.10) holds.

Now suppose that $x = y \neq 0$ (if $x = y = 0$, then the Eq (3.10) is clear). Thus, there exists $k' \in \mathbb{N}$ such that $x = y = \frac{1}{p^{k'}}$ and

$$(x + y) \cdot z = [0, x]_R \cdot z = [0, \frac{1}{p^{k'}}]_R \cdot \frac{1}{p^k} = [0, \frac{1}{p^{k+k'}}]_R.$$

Besides,

$$x \cdot z + y \cdot z = \frac{1}{p^{k+k'}} + \frac{1}{p^{k+k'}} = [0, \frac{1}{p^{k+k'}}]_R.$$

Therefore, the equality (3.10) holds and we conclude that $(R, +, \cdot)$ is a Krasner hyperring.

Let search now for the structure of an arbitrary hyperideal of R. Suppose that I is a nonzero hyperideal of R and take an arbitrary element $a \in I$. Using the hyperoperation defined by (3.8), we conclude that $a + a = [0, a]_R \subseteq I$. Thus, $\cup_{a \in I}[0, a]_R \subseteq I$, while the reverse inclusion is clear. Therefore,

$$I = \cup_{a \in I}[0, a]_R \tag{3.11}$$

and each hyperideal I of R has the form $\cup_{a \in I}[0, a]_R$.

Moreover, if $1 \in I$, then using (3.11), we get $I = \cup_{a \in I}[0, a]_R = [0, 1]_R = \langle 1 \rangle$. If $1 \notin I$, denote $A = \{n \in \mathbb{N} \mid \frac{1}{p^n} \in I\}$. In this case, $A \neq \emptyset$ and $A \subseteq \mathbb{N}$. Thus, using the Least Principle Theorem, we know that A has a least element m_0. Then we can check that $I = \langle \frac{1}{p^{m_0}} \rangle$. Therefore, every hyperideal of R is a principle hyperideal.

Let

$$M = \langle \frac{1}{p} \rangle = \{0, \frac{1}{p}, \frac{1}{p^2}, \frac{1}{p^3}, \cdots\}.$$

Then M is a maximal hyperideal because if N is another hyperideal of R such that $M \subseteq N$ and $N \neq R$, then there exists $n_0 \in \mathbb{N}$ such that

$$N = \langle \frac{1}{p^{n_0}} \rangle = \{0, \frac{1}{p^{n_0}}, \frac{1}{p^{n_0+1}}, \frac{1}{p^{n_0+2}}, \cdots\}.$$

It is clear that $N \subseteq M$. Thus, we conclude that $M = N$ and M is a maximal hyperideal. Besides, M is the only maximal hyperideal of R. Therefore, R is a local hyperring, which is also a principle hyperideal hyperdomain, with $dimR = ht_R M = 1$ since $0 \subset M$ is the only chain of prime hyperideals of R.

Finally, using Definition 3.2, we conclude that the element $\frac{1}{p} \in R$ is a regular parameter element of R since $M =< \frac{1}{p} >$ and M is an M-primary hyperideal of R.

4. Conclusions

The main findings of this paper give us the possibility to raise some open questions. Theorem 3.13 states that each regular local hyperring is a hyperdomain. The first question that follows immediately from this is whether a regular local hyperring is a hyperfield or not, or if it could contain a hyperfield or not. It's worth investigating on additional conditions under which the previous statement is true. Moreover, in Theorem 3.7 we use an element $a \in M \setminus M^2$, to show that the quotient $\overline{R} = \frac{R}{\langle a \rangle}$ is a regular local hyperring when R is a regular local hyperring. And now we raise the following question. Can we get the same result if we use an element a in $M \setminus M^3$ or generally $M \setminus M^n$, when n is a natural number and $M^n \neq 0$? Besides, what can we say about the quotient $\overline{R}$ for an arbitrary element $a \in M^2$?

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Conflict of interest

The authors declare no conflicts of interest.

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