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Research article

Fixed point theorems for controlled neutrosophic metric-like spaces

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Abstract: In this paper, we establish the concept of controlled neutrosophic metric-like spaces as a generalization of neutrosophic metric spaces and provide several non-trivial examples to show the spuriousness of the new concept in the existing literature. Furthermore, we prove several fixed point results for contraction mappings and provide the examples with their graphs to show the validity of the results. At the end of the manuscript, we establish an application to integral equations, in which we use the main result to find the solution of the integral equation.

Keywords: fixed point; controlled metric space; metric-like space; controlled neutrosophic metric-like space; integral equations; unique solution
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1. Introduction

The foundation of fixed point theory is the idea of metric spaces and the Banach contraction principle. An enormous number of academics are motivated to the axiomatic interpretation of metric space because of its spaciousness. The metric space has experienced numerous generalizations until now. This demonstrates the attraction, enchantment, and development of the idea of metric spaces.

After being given the notion of fuzzy sets (FSs) by Zadeh [1], researchers provided various generalizations for classical structures [2–5]. In this continuation, Kramosil and Michalek [6] originated the approach of fuzzy metric spaces, while George and Veeramani [7] introduced the concept of fuzzy metric spaces. Garbiec [8] gave the fuzzy interpretation of Banach contraction principle in fuzzy metric spaces.

The idea of fuzzy extended b-metric spaces was first established by Mehmood [9]. Metric-like spaces (MLSs), which is generalization of the idea of metric spaces, were introduced by Harandi [10]. The notions controlled metric type spaces and controlled metric-like spaces were first introduced by Mlaiki [11,12]. Recently, Sezen [13] generalized the concept of controlled type metric spaces and introduced the concept of Controlled fuzzy metric spaces (CFMS). Shukla and Abbas [14] reformulated the definition of MLSs and introduced the concept of fuzzy metric like spaces (FMLSs). Later, Javed et al. [15] obtained fixed point results in the context of fuzzy b-metric-like spaces. The approach of intuitionistic fuzzy metric spaces was tossed by Park [16] that deals with membership and non-membership functions.

Smarandache [17] established the concept of neutrosophic logic and the concept of neutrosophic set in 1998. The concept of neutrosophic sets have three functions, which are membership function, non-membership function and naturalness respectively. Thus, neutrosophic sets are the more general form of fuzzy sets [1] and intuitionistic fuzzy sets [18]. Hence, researchers in [19–22] have made studies on the concept of neutrosophic sets. Recently, Aslan et al. [23] obtained decision making applications for neutrosophic modeling of Talcott Parsons's Action and Kargın et al. [24] introduced decision making applications for law based on generalized set valued neutrosophic quadruple numbers. Şahin et al. [25] studied adequacy of online education using Hausdorff Measures based on neutrosophic quadruple sets. Also, Researchers in [26,27] studied types of metric space based on neutrosophic theory. Recently, Şahin and Kargın [28] obtained neutrosophic triplet metric spaces and neutrosophic triplet normed spaces. Kirişci and Simsek [29] established the concept of neutrosophic metric spaces (NMSs) that deals with membership, non-membership and naturalness functions. Şahin and Kargın [30] studied neutrosophic triplet v-generalized metric spaces and Şahin et al. [31] introduced the concept of neutrosophic triplet bipolar metric spaces. Simsek and Kirişci [32] derived various fixed point theorems for neutrosophic metric space. Şahin and Kargın [33] introduced the concept of neutrosophic triplet b–metric space. Şahin and Kargın [32] established neutrosophic triplet b-metric space and Sowndrarajan et al. [34] studied contradiction mapping results for neutrosophic metric space. Saleem et al. [35–37] proved various fixed point results for contraction mappings. Khater [38] did nice work on diverse solitary and Jacobian solutions in a continually laminated fluid with respect to shear flows through the Ostrovsky equation and Khater [39] worked on numerical simulations of Zakharov's (ZK) non-dimensional equation arising in Langmuir and ion-acoustic waves.

In this manuscript, we introduce the notion of controlled neutrosophic metric-like spaces as a generalization of a NMSs introduced in [29]. We replaced the following conditions of NMS

$$P(\varpi, \nu, \boldsymbol{\tau}) = 1 \text{ for all } \boldsymbol{\tau} > 0, \text{if and only if } \varpi = \nu,$$
$$Q(\varpi, \nu, \boldsymbol{\tau}) = 1 \text{ for all } \boldsymbol{\tau} > 0, \text{if and only if } \varpi = \nu,$$
$$S(\varpi, \nu, \boldsymbol{\tau}) = 1 \text{ for all } \boldsymbol{\tau} > 0, \text{if and only if } \varpi = \nu,$$

with

$$P(\varpi, \nu, \boldsymbol{\tau}) = 1 \text{ implies } \varpi = \nu,$$
$$Q(\varpi, \nu, \boldsymbol{\tau}) = 1 \text{ implies } \varpi = \nu,$$
$$S(\varpi, \nu, \boldsymbol{\tau}) = 1 \text{ implies } \varpi = \nu.$$

Also, we used a controlled function $\phi: \Xi \times \Xi \to [1, \infty)$ in the triangle inequalities of NMS. These both things generalized the defined notions existing in the literature. We also, derived several fixed-point results for contraction mappings in the context of new introduced space with non-trivial examples and graphical structure. At the end, we established an application to integral equation to show the validity of our main result.

In Section 2, we give basic definitions and basic properties for fuzzy metric spaces and neutrosophic metric spaces from [4,10,12–16,29]. In Section 3, we define controlled neutrosophic metric-like spaces and definitions of open ball, G-convergent sequence, G-Cauchy sequence, G-complete space and some examples for controlled neutrosophic metric-like spaces. Also, we give some fixed point (FP) results and illustrative examples. In Section 4, we give conclusions.

2. Preliminaries

The following definitions are useful in the sequel.

Definition 2.1. [15] A binary operation $*: [0, 1] \times [0, 1] \to [0, 1]$ is called a continuous triangle norm (briefly CTN), if it meets the below assertions:

1) $\Upsilon * \varrho = \varrho * \Upsilon, (\forall)\ \Upsilon, \varrho \in [0, 1]$;
2) $*$ is continuous;
3) $\Upsilon * 1 = \Upsilon, (\forall)\ \Upsilon \in [0, 1]$;
4) $(\Upsilon * \varrho) * \varkappa = \Upsilon * (\varrho * \varkappa), (\forall)\ \Upsilon, \varrho, \varkappa \in [0, 1]$;
5) If $\Upsilon \leq \varkappa$ and $\varrho \leq d$, with $\Upsilon, \varrho, \varkappa, d \in [0, 1]$, then $\Upsilon * \varrho \leq \varkappa * d$.

Example 2.1. [4,15] Some fundamental examples of t-norms are: $\Upsilon * \varrho = \Upsilon \cdot \varrho, \Upsilon * \varrho = \min\{\Upsilon, \varrho\}$ and $\Upsilon * \varrho = \max\{\Upsilon + \varrho - 1, 0\}$.

Definition 2.2. [15] A binary operation $\circ: [0, 1] \times [0, 1] \to [0, 1]$ is called a continuous triangle conorm (briefly CTCN) if it meets the below assertions:

1) $\Upsilon \circ \varrho = \varrho \circ \Upsilon$, for all $\Upsilon, \varrho \in [0, 1]$;
2) $\circ$ is continuous;
3) $\Upsilon \circ 0 = 0$, for all $\Upsilon \in [0, 1]$;
4) $(\Upsilon \circ \varrho) \circ \varkappa = \Upsilon \circ (\varrho \circ \varkappa)$, for all $\Upsilon, \varrho, \varkappa \in [0, 1]$;
5) If $\Upsilon \leq \varkappa$ and $\varrho \leq d$, with $\Upsilon, \varrho, \varkappa, d \in [0, 1]$, then $\Upsilon \circ \varrho \leq \varkappa \circ d$.

Example 2.2. [15] $\Upsilon \circ \varrho = \max\{\Upsilon, \varrho\}$ and $\Upsilon \circ \varrho = \min\{\Upsilon + \varrho, 1\}$ are examples of CTCNs.

Definition 2.3. [10] Suppose $\Xi \neq \emptyset$ be a set. A mapping $\Theta: \Xi \times \Xi \to [1, \infty)$ is known as a metric-like, if it satisfying the following conditions:

1) $\Theta(\varpi, \nu) = 0$ implies $\varpi = \nu$;
2) $\Theta(\varpi, \nu) = \Theta(\nu, \varpi)$;
3) $\Theta(\varpi, \nu) \leq \Theta(\varpi, \boldsymbol{\lambda}) + \Theta(\boldsymbol{\lambda}, \nu)$;

for all $\varpi, \nu, \boldsymbol{\lambda} \in \Xi$.

Also, (Ξ, Θ) is called a metric-like space.

Definition 2.4. [12] Let $\Xi \neq \emptyset$, $\psi: \Xi \times \Xi \to [1, \infty)$ be a function and $\Theta: \Xi \times \Xi \to \mathbb{R}^+$. If the following properties are satisfied:

1) $\Theta(\varpi, \nu) = 0$ implies $\varpi = \nu$;

2) $\Theta(\varpi, \nu) = \Theta(\nu, \varpi)$;
3) $\Theta(\varpi, \nu) \le \psi((\varpi, \boldsymbol{\lambda})\Theta(\varpi, \boldsymbol{\lambda}) + \psi(\boldsymbol{\lambda}, \varpi)\Theta(\boldsymbol{\lambda}, \nu)$;

for all $\varpi, \nu, \boldsymbol{\lambda} \in \Xi$, then Θ is said to be a controlled metric-like and (Ξ, Θ) is known as a controlled metric-like space.

Definition 2.5. [13] Suppose $\Xi \neq \emptyset$, $h: \Xi \times \Xi \to [1, \infty)$ be a mapping, $*$ is a CTN and Δ_h is a FS on $\Xi \times \Xi \times (0, \infty)$. Four-tuple $(\Xi, \Delta_h, *, h)$ is called CFMS if it meets the below assertions for all $\varpi, \nu, \boldsymbol{\lambda} \in \Xi$ and $\boldsymbol{\tau}, \boldsymbol{\varsigma} > 0$:

$h1)$ $\Delta_h(\varpi, \nu, 0) = 0$;
$h2)$ $\Delta_h(\varpi, \nu, \boldsymbol{\tau}) = 1 \iff \varpi = \nu$;
$h3)$ $\Delta_h(\varpi, \nu, \boldsymbol{\tau}) = \Delta_h(\nu, \varpi, \boldsymbol{\tau})$;
$h4)$ $\Delta_h(\varpi, \boldsymbol{\lambda}, (\boldsymbol{\tau} + \boldsymbol{\varsigma})) \ge \Delta_h\left(\varpi, \nu, \frac{\tau}{h(\varpi, \nu)}\right) * \Delta_h\left(\nu, \boldsymbol{\lambda}, \frac{\varsigma}{h(\nu, \lambda)}\right)$;
$h5)$ $\Delta_h(\varpi, \nu, \cdot): (0, \infty) \to [0,1]$ is continuous.

Definition 2.6. [16] Let $\Xi \neq \emptyset$, $*$ be a CTN, P be a FSs on $\Xi \times \Xi \times (0, \infty)$. If triplet $(\Xi, \Theta, *)$ verifies the following for all $\varpi, \nu, \boldsymbol{\lambda} \in \Xi$ and $\boldsymbol{\varsigma}, \boldsymbol{\tau} > 0$:

1) $\Theta(\varpi, \nu, \boldsymbol{\tau}) > 0$;
2) $\Theta(\varpi, \nu, \boldsymbol{\tau}) = 1 \iff \varpi = \nu$;
3) $\Theta(\varpi, \nu, \boldsymbol{\tau}) = \Theta(\nu, \varpi, \boldsymbol{\tau})$;
4) $\Theta\big(\varpi, \boldsymbol{\lambda}, b(\boldsymbol{\tau} + \boldsymbol{\varsigma})\big) \ge \Theta(\varpi, \nu, \boldsymbol{\tau}) * \Theta(\nu, \boldsymbol{\lambda}, \boldsymbol{\tau})$;
5) $\Theta(\varpi, \nu, \cdot): (0, \infty) \to [0, 1]$ is a continuous mapping.

then $(\Xi, \Theta, *)$ is called an FMLS.

Definition 2.7. [14] Let Ξ be a universal set. For $\forall \varpi \in E$, $0^- \le T_{\mathcal{A}}(\varpi) + I_{\mathcal{A}}(\varpi) + F_{\mathcal{A}}(\varpi) \le 3^+$, by the help of the functions $T_{\mathcal{A}}: E \to]^-0, 1^+[$, $I_{\mathcal{A}}: E \to]^-0, 1^+[$ and $F_{\mathcal{A}}: E \to]^-0, 1^+[$ a neutrosophic set $\mathcal{A}$ on Ξ is defined by

$$\mathcal{A} = \{\langle \varpi, T_{\mathcal{A}}(\varpi), I_{\mathcal{A}}(\varpi), F_{\mathcal{A}}(\varpi) \rangle : \varpi \in \Xi\}$$

Here, $T_{\mathcal{A}}(\varpi), I_{\mathcal{A}}(\varpi)$ and $F_{\mathcal{A}}(\varpi)$ are the degrees of trueness, indeterminacy and falsity of $\varpi \in \Xi$ respectively.

Definition 2.8. [29] Let $\Xi \neq \emptyset$, $*$ is a CTN, $\circ$ be a CTCN and

$$\mathcal{A} = \{\langle \varpi, \Theta(\varpi), Q(\varpi), S(\varpi) \rangle : \varpi \in \Xi\}$$

be a neutrosophic set such that $\mathcal{A}: \Xi \times \Xi \times (0, \infty) \to [0, 1]$. If for all $\varpi, \nu, \boldsymbol{\lambda} \in \Xi$, the below circumstances are satisfying:

1) $0 \le P(\varpi, \nu, \boldsymbol{\tau}) \le 1$, $0 \le Q(\varpi, \nu, \tau) \le 1$ and $0 \le S(\varpi, \nu, \boldsymbol{\tau}) \le 1$,
2) $P(\varpi, \nu, \boldsymbol{\tau}) + Q(\varpi, \nu, \boldsymbol{\tau}) + S(\varpi, \nu, \boldsymbol{\tau}) \le 3$;
3) $P(\varpi, \nu, \boldsymbol{\tau}) > 0$;
4) $P(\varpi, \nu, \boldsymbol{\tau}) = 1$ for all $\boldsymbol{\tau} > 0$, if and only if $\varpi = \nu$;
5) $P(\varpi, \nu, \boldsymbol{\tau}) = P(\nu, \varpi, \boldsymbol{\tau})$;
6) $P(\varpi, \boldsymbol{\lambda}, \boldsymbol{\tau} + \boldsymbol{\varsigma}) \ge P(\varpi, \nu, \boldsymbol{\tau}) * P(\nu, \lambda, \varsigma)$;
7) $P(\varpi, \nu, \cdot): (0, \infty) \to [0,1]$ is continuous and $\lim_{\boldsymbol{\tau} \to \infty} P(\varpi, \nu, \boldsymbol{\tau}) = 1$;
8) $Q(\varpi, \nu, \boldsymbol{\tau}) < 1$;
9) $Q(\varpi, \nu, \boldsymbol{\tau}) = 0$ for all $\boldsymbol{\tau} > 0$, if and only if $\varpi = \nu$;
10) $Q(\varpi, \nu, \boldsymbol{\tau}) = Q(\nu, \varpi, \boldsymbol{\tau})$;
11) $Q(\varpi, \boldsymbol{\lambda}, \boldsymbol{\tau} + \boldsymbol{\varsigma}) \le Q(\varpi, \nu, \boldsymbol{\tau}) \circ Q(\nu, \boldsymbol{\lambda}, \boldsymbol{\varsigma})$;
12) $Q(\varpi, \nu, \cdot): (0, \infty) \to [0,1]$ is continuous and $\lim_{\boldsymbol{\tau} \to \infty} Q(\varpi, \nu, \boldsymbol{\tau}) = 0$;

13) $S(\varpi, \nu, \boldsymbol{\tau}) < 1$;
14) $S(\varpi, \nu, \boldsymbol{\tau}) = 0$ for all $\boldsymbol{\tau} > 0$, if and only if $\varpi = \nu$;
15) $S(\varpi, \nu, \boldsymbol{\tau}) = S(\nu, \varpi, \boldsymbol{\tau})$;
16) $S(\varpi, \boldsymbol{\lambda}, \boldsymbol{\tau} + \boldsymbol{\varsigma}) \leq S(\varpi, \nu, \boldsymbol{\tau}) \circ S(\nu, \boldsymbol{\lambda}, \boldsymbol{\varsigma})$;
17) $S(\varpi, \nu, \cdot): (0, \infty) \to [0,1]$ is continuous and $\lim_{\boldsymbol{\tau}\to\infty} S(\varpi, \nu, \boldsymbol{\tau}) = 0$;
18) If $\boldsymbol{\tau} \leq 0$, then $P(\varpi, \nu, \boldsymbol{\tau}) = 0, Q(\varpi, \nu, \boldsymbol{\tau}) = 1$ and $S(\varpi, \nu, \boldsymbol{\tau}) = 1$.

then four-tuple $(\Xi, \mathcal{A}, *, \circ)$ is called an NMS.

Where; $P(\varpi, \nu, \boldsymbol{\tau})$ is degree of nearness, $Q(\varpi, \nu, \boldsymbol{\tau})$ is degree of neutralness and $S(\varpi, \nu, \boldsymbol{\tau})$ is degree of non-nearness.

3. Main results

In this section, we introduce the notion of a CNMLS and prove some related FP results.

Definition 3.1. Suppose $\Xi \neq \emptyset$, assume a six tuple $(\Xi, P_\phi, Q_\phi, R_\phi, *, \circ)$ where * is a CTN, ○ is a CTCN, $\phi: \Xi \times \Xi \to [1, \infty)$ be a function and P_ϕ, Q_ϕ, R_ϕ are neutrosophic sets (NSs) on $\Xi \times \Xi \times (0, \infty)$. If $(\Xi, P_\phi, Q_\phi, , R_\phi, *, \circ)$ meet the below circumstances for all $\varpi, \nu, \boldsymbol{\lambda} \in \Xi$ and $\boldsymbol{\varsigma}, \boldsymbol{\tau} > 0$:

1) $P_\phi(\varpi, \nu, \boldsymbol{\tau}) + Q_\phi(\varpi, \nu, \boldsymbol{\tau}) + R_\phi(\varpi, \nu, \boldsymbol{\tau}) \leq 3$,
2) $P_\phi(\varpi, \nu, \boldsymbol{\tau}) > 0$,
3) $P_\phi(\varpi, \nu, \boldsymbol{\tau}) = 1$ implies $\varpi = \nu$,
4) $P_\phi(\varpi, \nu, \boldsymbol{\tau}) = P_\phi(\nu, \varpi, \boldsymbol{\tau})$,
5) $P_\phi(\varpi, \boldsymbol{\lambda}, (\boldsymbol{\tau} + \boldsymbol{\varsigma})) \geq P_\phi\left(\varpi, \nu, \frac{\boldsymbol{\tau}}{\phi(\varpi, \nu)}\right) * P_\phi\left(\nu, \boldsymbol{\lambda}, \frac{\varsigma}{\phi(\nu, \lambda)}\right)$,
6) $P_\phi(\varpi, \nu, \cdot)$ is ND function of $\mathbb{R}^+$ and $\lim_{\boldsymbol{\tau}\to\infty} P_\phi(\varpi, \nu, \boldsymbol{\tau}) = 1$,
7) $Q_\phi(\varpi, \nu, \boldsymbol{\tau}) < 1$,
8) $Q_\phi(\varpi, \nu, \boldsymbol{\tau}) = 0$ implies $\varpi = \nu$,
9) $Q_\phi(\varpi, \nu, \boldsymbol{\tau}) = Q_\phi(\nu, \varpi, \boldsymbol{\tau})$,
10) $Q_\phi(\varpi, \boldsymbol{\lambda}, (\boldsymbol{\tau} + \boldsymbol{\varsigma})) \leq Q_\phi\left(\varpi, \nu, \frac{\boldsymbol{\tau}}{\phi(\varpi, \nu)}\right) \circ Q_\phi\left(\nu, \boldsymbol{\lambda}, \frac{\varsigma}{\phi(\nu, \lambda)}\right)$,
11) $Q_\phi(\varpi, \nu, \cdot)$ is NI function of $\mathbb{R}^+$ and $\lim_{\boldsymbol{\tau}\to\infty} Q_\phi(\varpi, \nu, \boldsymbol{\tau}) = 0$,
12) $R_\phi(\varpi, \nu, \boldsymbol{\tau}) < 1$,
13) $R_\phi(\varpi, \nu, \boldsymbol{\tau}) = 0$ implies $\varpi = \nu$,
14) $R_\phi(\varpi, \nu, \boldsymbol{\tau}) = R_\phi(\nu, \varpi, \boldsymbol{\tau})$,
15) $R_\phi(\varpi, \boldsymbol{\lambda}, (\boldsymbol{\tau} + \boldsymbol{\varsigma})) \leq R_\phi\left(\varpi, \nu, \frac{\boldsymbol{\tau}}{\phi(\varpi, \nu)}\right) \circ R_\phi\left(\nu, \boldsymbol{\lambda}, \frac{\varsigma}{\phi(\nu, \lambda)}\right)$,
16) $R_\phi(\varpi, \nu, \cdot)$ is NI function of $\mathbb{R}^+$ and $\lim_{\boldsymbol{\tau}\to\infty} R_\phi(\varpi, \nu, \boldsymbol{\tau}) = 0$,
17) If $\boldsymbol{\tau} \leq 0$, then $P_\phi(\varpi, \nu, \boldsymbol{\tau}) = 0, Q_\phi(\varpi, \nu, \boldsymbol{\tau}) = 1$ and $R_\phi(\varpi, \nu, \boldsymbol{\tau}) = 1$.

Then five-tuple $(\Xi, \mathcal{A}_\phi, \phi, *, \circ)$ is called a CNMLS.

Where; $P_\phi(\varpi, \nu, \boldsymbol{\tau})$ is degree of nearness, $Q_\phi(\varpi, \nu, \boldsymbol{\tau})$ is degree of neutralness and $R_\phi(\varpi, \nu, \boldsymbol{\tau})$ is degree of non-nearness.

Example 3.1. Let $\Xi = (0, \infty)$, define $P_\phi, Q_\phi, R_\phi: \Xi \times \Xi \times (0, \infty) \to [0,1]$ by

$$P_\phi(\varpi, \nu, \boldsymbol{\tau}) = \frac{\boldsymbol{\tau}}{\boldsymbol{\tau} + \max\{\varpi, \nu\}^2}, \qquad Q_\phi(\varpi, \nu, \boldsymbol{\tau}) = \frac{\max\{\varpi, \nu\}^2}{\boldsymbol{\tau} + \max\{\varpi, \nu\}^2}, \qquad R_\phi(\varpi, \nu, \boldsymbol{\tau}) = \frac{\max\{\varpi, \nu\}^2}{\boldsymbol{\tau}}$$

for all $\varpi, \nu \in \Xi$ and $\boldsymbol{\tau} > 0$, define CTN "*" by $\Upsilon * \varrho = \Upsilon \cdot \varrho$ and CTCN "○" by $\Upsilon \circ \varrho = \max\{\Upsilon, \varrho\}$ and define "ϕ" by

$$\phi(\varpi,\nu)=\begin{cases}1 & \text{if } \varpi=\nu,\\ \dfrac{1+\max\{\varpi,\nu\}}{\min\{\varpi,\nu\}} & \text{if } \varpi\neq\nu.\end{cases}$$

Then five-tuple $(\Xi,\mathcal{A}_\phi,\phi,*,\circ)$ is a CNMS.

Proof. (i) − (iv), (vi) − (ix), (ix) − (xiv), (xvi) and (xvii) are trivial, here we examine (v), (x) and (xv),

$$\max\{\varpi,\boldsymbol{\lambda}\}^2\leq\phi(\varpi,\nu)\max\{\varpi,\nu\}^2+\phi(\nu,\boldsymbol{\lambda})\max\{\nu,\boldsymbol{\lambda}\}^2$$

Therefore,

$$\boldsymbol{\tau\varsigma}\max\{\varpi,\boldsymbol{\lambda}\}^2\leq\phi(\varpi,\nu)(\boldsymbol{\tau\varsigma}+\boldsymbol{\varsigma}^2)\max\{\varpi,\nu\}^2+\phi(\nu,\boldsymbol{\lambda})(\boldsymbol{\tau\varsigma}+\boldsymbol{\tau}^2)\max\{\nu,\boldsymbol{\lambda}\}^2,$$

$$\Rightarrow\boldsymbol{\tau\varsigma}\max\{\varpi,\boldsymbol{\lambda}\}^2\leq\phi(\varpi,\nu)(\boldsymbol{\tau}+\varsigma)\boldsymbol{\varsigma}\max\{\varpi,\nu\}^2+\phi(\nu,\boldsymbol{\lambda})(\boldsymbol{\tau}+\boldsymbol{\varsigma})\boldsymbol{\tau}\max\{\nu,\boldsymbol{\lambda}\}^2,$$

$$\Rightarrow\boldsymbol{\tau\varsigma}(\boldsymbol{\tau}+\boldsymbol{\varsigma})+\boldsymbol{\tau\varsigma}\max\{\varpi,\boldsymbol{\lambda}\}^2,$$

$$\leq\boldsymbol{\tau\varsigma}(\boldsymbol{\tau}+\boldsymbol{\varsigma})+\phi(\varpi,\nu)(\boldsymbol{\tau}+\boldsymbol{\varsigma})\boldsymbol{\varsigma}\max\{\varpi,\nu\}^2+\phi(\nu,\boldsymbol{\lambda})(\boldsymbol{\tau}+\boldsymbol{\varsigma})\boldsymbol{\tau}\max\{\nu,\boldsymbol{\lambda}\}^2$$

That is,

$$\boldsymbol{\tau\varsigma}[(\boldsymbol{\tau}+\boldsymbol{\varsigma})+\max\{\varpi,\boldsymbol{\lambda}\}^2]\leq(\boldsymbol{\tau}+\boldsymbol{\varsigma})[\boldsymbol{\tau}\varsigma+\phi(\varpi,\nu)\boldsymbol{\varsigma}\max\{\varpi,\nu\}^2+\phi(\nu,\boldsymbol{\lambda})\boldsymbol{\tau}\max\{\nu,\boldsymbol{\lambda}\}^2],$$

$$\Rightarrow\boldsymbol{\tau\varsigma}[(\boldsymbol{\tau}+\boldsymbol{\varsigma})+\max\{\varpi,\lambda\}^2],$$

$$\leq(\boldsymbol{\tau}+\boldsymbol{\varsigma})[\boldsymbol{\tau\varsigma}+\phi(\varpi,\nu)\boldsymbol{\varsigma}\max\{\varpi,\nu\}^2+\phi(\nu,\boldsymbol{\lambda})\boldsymbol{\tau}\max\{\nu,\boldsymbol{\lambda}\}^2+\phi(\varpi,\nu)\phi(\nu,\boldsymbol{\lambda})\max\{\varpi,\nu\}^2\max\{\nu,\boldsymbol{\lambda}\}^2],$$

$$\Rightarrow\boldsymbol{\tau\varsigma}[(\boldsymbol{\tau}+\boldsymbol{\varsigma})+\max\{\varpi,\boldsymbol{\lambda}\}^2]\leq(\boldsymbol{\tau}+\boldsymbol{\varsigma})[\boldsymbol{\tau}+\phi(\varpi,\nu)\max\{\varpi,\nu\}^2][\boldsymbol{\varsigma}+\phi(\nu,\boldsymbol{\lambda})\max\{\nu,\boldsymbol{\lambda}\}^2]$$

Then,

$$\frac{(\boldsymbol{\tau}+\boldsymbol{\varsigma})}{(\boldsymbol{\tau}+\boldsymbol{\varsigma})+\max\{\varpi,\boldsymbol{\lambda}\}^2}\geq\frac{\boldsymbol{\tau\varsigma}}{[\boldsymbol{\tau}+\phi(\varpi,\nu)\max\{\varpi,\nu\}^2][\boldsymbol{\varsigma}+\phi(\nu,\lambda)\max\{\nu,\boldsymbol{\lambda}\}^2]},$$

$$\Rightarrow\frac{(\boldsymbol{\tau}+\boldsymbol{\varsigma})}{(\boldsymbol{\tau}+\boldsymbol{\varsigma})+\max\{\varpi,\boldsymbol{\lambda}\}^2}\geq\frac{\boldsymbol{\tau}}{\boldsymbol{\tau}+\phi(\varpi,\nu)\max\{\varpi,\nu\}^2}\cdot\frac{\boldsymbol{\varsigma}}{\boldsymbol{\varsigma}+\phi(\nu,\boldsymbol{\lambda})\max\{\nu,\boldsymbol{\lambda}\}^2},$$

$$\Rightarrow\frac{(\boldsymbol{\tau}+\boldsymbol{\varsigma})}{(\boldsymbol{\tau}+\boldsymbol{\varsigma})+\max\{\varpi,\boldsymbol{\lambda}\}^2}\geq\frac{\frac{\boldsymbol{\tau}}{\phi(\varpi,\nu)}}{\frac{\boldsymbol{\tau}}{\phi(\varpi,\nu)}+\max\{\varpi,\nu\}^2}\cdot\frac{\frac{\boldsymbol{\varsigma}}{\phi(\nu,\boldsymbol{\lambda})}}{\frac{\boldsymbol{\varsigma}}{\phi(\nu,\boldsymbol{\lambda})}+\max\{\nu,\boldsymbol{\lambda}\}^2}$$

Hence,

$$P_\phi(\varpi,\boldsymbol{\lambda},(\boldsymbol{\tau}+\boldsymbol{\varsigma}))\geq P_\phi\left(\varpi,\nu,\frac{\boldsymbol{\tau}}{\phi(\varpi,\nu)}\right)*P_\phi\left(\nu,\boldsymbol{\lambda},\frac{\boldsymbol{\varsigma}}{\phi(\nu,\boldsymbol{\lambda})}\right)$$

(v) is satisfied.

$$\max\{\varpi,\boldsymbol{\lambda}\}^2=\max\{\varpi,\boldsymbol{\lambda}\}^2\max\{1,1\}$$

Therefore,

$$\max\{\varpi,\boldsymbol{\lambda}\}^2 = \max\{\varpi,\boldsymbol{\lambda}\}^2 \max\left\{\frac{\max\{\varpi,\nu\}^2}{\max\{\varpi,\nu\}^2}, \frac{\max\{\nu,\boldsymbol{\lambda}\}^2}{\max\{\nu,\boldsymbol{\lambda}\}^2}\right\}$$

$$\max\{\varpi,\boldsymbol{\lambda}\}^2 \leq [(\boldsymbol{\tau}+\boldsymbol{\varsigma}) + \max\{\varpi,\boldsymbol{\lambda}\}^2] \max\left\{\frac{\max\{\varpi,\nu\}^2}{\max\{\varpi,\nu\}^2}, \frac{\max\{\nu,\boldsymbol{\lambda}\}^2}{\max\{\nu,\boldsymbol{\lambda}\}^2}\right\}$$

$$\max\{\varpi,\boldsymbol{\lambda}\}^2 \leq [(\boldsymbol{\tau}+\boldsymbol{\varsigma}) + \max\{\varpi,\boldsymbol{\lambda}\}^2] \max\left\{\frac{\phi(\varpi,\nu)\max\{\varpi,\nu\}^2}{\phi(\varpi,\nu)\max\{\varpi,\nu\}^2}, \frac{\phi(\nu,\boldsymbol{\lambda})\max\{\nu,\boldsymbol{\lambda}\}^2}{\phi(\nu,\boldsymbol{\lambda})\max\{\nu,\boldsymbol{\lambda}\}^2}\right\}$$

Then,

$$\frac{\max\{\varpi,\boldsymbol{\lambda}\}^2}{(\boldsymbol{\tau}+\boldsymbol{\varsigma}) + \max\{\varpi,\boldsymbol{\lambda}\}^2} \leq \max\left\{\frac{\phi(\varpi,\nu)\max\{\varpi,\nu\}^2}{\boldsymbol{\tau} + \phi(\varpi,\nu)\max\{\varpi,\nu\}^2}, \frac{\phi(\nu,\boldsymbol{\lambda})\max\{\nu,\boldsymbol{\lambda}\}^2}{\boldsymbol{\varsigma} + \phi(\nu,\boldsymbol{\lambda})\max\{\nu,\boldsymbol{\lambda}\}^2}\right\}$$

That is,

$$\frac{\max\{\varpi,\boldsymbol{\lambda}\}^2}{(\boldsymbol{\tau}+\boldsymbol{\varsigma}) + \max\{\varpi,\boldsymbol{\lambda}\}^2} \leq \max\left\{\frac{\max\{\varpi,\nu\}^2}{\frac{\boldsymbol{\tau}}{\phi(\varpi,\nu)} + \max\{\varpi,\nu\}^2}, \frac{\max\{\nu,\boldsymbol{\lambda}\}^2}{\frac{\boldsymbol{\varsigma}}{\phi(\nu,\boldsymbol{\lambda})} + \max\{\nu,\boldsymbol{\lambda}\}^2}\right\}$$

Hence,

$$Q_\phi\big(\varpi,\boldsymbol{\lambda},(\boldsymbol{\tau}+\boldsymbol{\varsigma})\big) \leq Q_\phi\left(\varpi,\nu,\frac{\boldsymbol{\tau}}{\phi(\varpi,\nu)}\right) * Q_\phi\left(\nu,\boldsymbol{\lambda},\frac{\boldsymbol{\varsigma}}{\phi(\nu,\boldsymbol{\lambda})}\right)$$

(x) is satisfied.
It is easy to see that

$$\frac{\max\{\varpi,\boldsymbol{\lambda}\}^2}{\boldsymbol{\tau}+\boldsymbol{\varsigma}} \leq \max\left\{\frac{\phi(\varpi,\nu)\max\{\varpi,\nu\}^2}{\boldsymbol{\tau}}, \frac{\phi(\nu,\boldsymbol{\lambda})\max\{\nu,\boldsymbol{\lambda}\}^2}{\boldsymbol{\varsigma}}\right\}$$

That is,

$$\frac{\max\{\varpi,\boldsymbol{\lambda}\}^2}{(\boldsymbol{\tau}+\boldsymbol{\varsigma})} \leq \max\left\{\frac{\max\{\varpi,\nu\}^2}{\frac{\boldsymbol{\tau}}{\phi(\varpi,\nu)}}, \frac{\max\{\nu,\boldsymbol{\lambda}\}^2}{\frac{\boldsymbol{\varsigma}}{\phi(\nu,\boldsymbol{\lambda})}}\right\}$$

Hence,

$$R_\phi\big(\varpi,\boldsymbol{\lambda},(\boldsymbol{\tau}+\boldsymbol{\varsigma})\big) \leq R_\phi\left(\varpi,\nu,\frac{\boldsymbol{\tau}}{\phi(\varpi,\nu)}\right) * R_\phi\left(\nu,\boldsymbol{\lambda},\frac{\boldsymbol{\varsigma}}{\phi(\nu,\boldsymbol{\lambda})}\right)$$

(xv) is satisfied.
Remark 3.1. If we let, $\Upsilon * \varrho = \min\{\Upsilon,\varrho\}$ and $\Upsilon \circ \varrho = \max\{\Upsilon,\varrho\}$, then above example is also a CNMLS.

Example 3.2. Suppose $\Xi = (0,\infty)$, define $P_\phi, Q_\phi, R_\phi\colon \Xi \times \Xi \times (0,\infty) \to [0,1]$ by

$$P_\phi(\varpi,\nu,\boldsymbol{\tau}) = \frac{\boldsymbol{\tau}}{\boldsymbol{\tau} + \max\{\varpi,\nu\}}$$

$$Q_\phi(\varpi,\nu,\boldsymbol{\tau}) = \frac{\max\{\varpi,\nu\}}{\boldsymbol{\tau} + \max\{\varpi,\nu\}},$$

and

$$R_\phi(\varpi,\nu,\tau)=\frac{\max\{\varpi,\nu\}}{\boldsymbol{\tau}}$$

for all $\varpi,\nu\in\varXi$ and $\boldsymbol{\tau}>0$, define CTN "*" by $\varUpsilon*\varrho=\varUpsilon\cdot\varrho$ and CTCN "$\circ$" by $\varUpsilon\circ\varrho=\max\{\varUpsilon,\varrho\}$ and define "ϕ" by

$$\phi(\varpi,\nu)=1+\varpi+\nu$$

Then $(\varXi,P_\phi,Q_\phi,R_\phi,*,\circ)$ be a CNMLS.

Remark 3.2. The above Examples 3.1 and 3.2 are not neutrosophic metric spaces.

Definition 3.2. Let $(\varXi,P_\phi,Q_\phi,R_\phi,*,\circ)$ is a CNMLS, then we define an open ball $B(\varpi,r,\boldsymbol{\tau})$ with centre ϖ, radius $r, 0<r<1$ and $\boldsymbol{\tau}>0$ as follows:

$$B(\varpi,r,\boldsymbol{\tau})=\{\nu\in\varXi: P(\varpi,\nu,\boldsymbol{\tau})>1-r, Q(\varpi,\nu,\boldsymbol{\tau})<r, R(\varpi,\nu,\boldsymbol{\tau})<r\}.$$

Definition 3.3. Let $(\varXi,P_\phi,Q_\phi,R_\phi,*,\circ)$ be a CNMLS. Then

1) a sequence $\{\varpi_n\}$ in $\varXi$ is named to be G-Cauchy sequence (GCS) if and only if for all $q>0$ and $\boldsymbol{\tau}>0$,

$$\lim_{n\to\infty}P_\phi(\varpi_n,\varpi_{n+q},\boldsymbol{\tau}),\ \lim_{n\to\infty}Q_\phi(\varpi_n,\varpi_{n+q},\boldsymbol{\tau}) \text{ and } \lim_{n\to\infty}R_\phi(\varpi_n,\varpi_{n+q},\boldsymbol{\tau}) \text{ exists and finite}$$

2) a sequence $\{\varpi_n\}$ in $\varXi$ is named to be G-convergent (GC) to ϖ in $\varXi$, if and only if for all $\boldsymbol{\tau}>0$,

$$\lim_{n\to\infty}P_\phi(\varpi_n,\varpi,\boldsymbol{\tau})=P_\phi(\varpi,\varpi,\boldsymbol{\tau}),\ \lim_{n\to\infty}Q_\phi(\varpi_n,\varpi,\boldsymbol{\tau})=Q_\phi(\varpi,\varpi,\boldsymbol{\tau})$$
$$\text{and }\lim_{n\to\infty}R_\phi(\varpi_n,\varpi,\boldsymbol{\tau})=R_\phi(\varpi,\varpi,\boldsymbol{\tau}).$$

3) a CNMLS is named to be complete if each GCS is convergent i.e.,

$$\lim_{n\to\infty}P_\phi(\varpi_n,\varpi_{n+q},\boldsymbol{\tau})=\lim_{n\to\infty}P_\phi(\varpi_n,\varpi,\boldsymbol{\tau})=P_\phi(\varpi,\varpi,\boldsymbol{\tau}),$$
$$\lim_{n\to\infty}Q_\phi(\varpi_n,\varpi_{n+q},\boldsymbol{\tau})=\lim_{n\to\infty}Q_\phi(\varpi_n,\varpi,\boldsymbol{\tau})=Q_\phi(\varpi,\varpi,\boldsymbol{\tau}),$$
$$\lim_{n\to\infty}R_\phi(\varpi_n,\varpi_{n+q},\boldsymbol{\tau})=\lim_{n\to\infty}R_\phi(\varpi_n,\varpi,\boldsymbol{\tau})=R_\phi(\varpi,\varpi,\boldsymbol{\tau})$$

Theorem 3.1. Suppose $(\varXi,P_\phi,Q_\phi,R_\phi,*,\circ)$ be a G-complete CNMLS with $\phi:\varXi\times\varXi\to[1,\infty)$ and assume that

$$\lim_{\boldsymbol{\tau}\to\infty}P_\phi(\varpi,\nu,\boldsymbol{\tau})=1,\ \lim_{\boldsymbol{\tau}\to\infty}Q_\phi(\varpi,\nu,\boldsymbol{\tau})=0 \text{ and } \lim_{\boldsymbol{\tau}\to\infty}R_\phi(\varpi,\nu,\boldsymbol{\tau})=0 \qquad (1)$$

for all $\varpi,\nu\in\varXi$ and $\boldsymbol{\tau}>0$. Suppose $\xi:\varXi\to\varXi$ be a mapping verifying

$$P_\phi(\xi\varpi,\xi\nu,£\boldsymbol{\tau})\geq P_\phi(\varpi,\nu,\boldsymbol{\tau}),$$
$$Q_\phi(\xi\varpi,\xi\nu,£\boldsymbol{\tau})\leq Q_\phi(\varpi,\nu,\boldsymbol{\tau}) \text{ and } R_\phi(\xi\varpi,\xi\nu,£\boldsymbol{\tau})\leq R_\phi(\varpi,\nu,\boldsymbol{\tau}) \qquad (2)$$

for all $\varpi,\nu\in\varXi$, $0<£<1$ and $\boldsymbol{\tau}>0$. Also assume that for every $\varpi\in Z$,

$$\lim_{n\to\infty}\phi(\varpi_n,\nu) \text{ and } \lim_{n\to\infty}\phi(\nu,\varpi_n) \tag{3}$$

exists and finite. Then ζ has a unique fixed point in Z. Then ξ has a unique FP.

Proof. Let ϖ_0 be an arbitrary point of $\boldsymbol{\Xi}$ and define a sequence ϖ_n by $\varpi_n=\xi^n\varpi_0=\xi\varpi_{n-1}$, $n\in\mathbb{N}$. By utilizing (2) for all $\boldsymbol{\tau}>0$, we get

$$P_\phi(\varpi_n,\varpi_{n+1},£\boldsymbol{\tau})=P_\phi(\xi\varpi_{n-1},\xi\varpi_n,£\boldsymbol{\tau})\geq P_\phi(\varpi_{n-1},\varpi_n,\boldsymbol{\tau})\geq P_\phi\left(\varpi_{n-2},\varpi_{n-1},\frac{\boldsymbol{\tau}}{£}\right)$$
$$\geq P_\phi\left(\varpi_{n-3},\varpi_{n-2},\frac{\boldsymbol{\tau}}{£^2}\right)\geq\cdots\geq P_\phi\left(\varpi_0,\varpi_1,\frac{\boldsymbol{\tau}}{£^{n-1}}\right),$$
$$Q_\phi(\varpi_n,\varpi_{n+1},£\boldsymbol{\tau})=Q_\phi(\xi\varpi_{n-1},\xi\varpi_n,£\boldsymbol{\tau})\leq Q_\phi(\varpi_{n-1},\varpi_n,\boldsymbol{\tau})\leq Q_\phi\left(\varpi_{n-2},\varpi_{n-1},\frac{\boldsymbol{\tau}}{£}\right)$$
$$\leq Q_\phi\left(\varpi_{n-3},\varpi_{n-2},\frac{\boldsymbol{\tau}}{£^2}\right)\leq\cdots\leq Q_\phi\left(\varpi_0,\varpi_1,\frac{\boldsymbol{\tau}}{£^{n-1}}\right)$$

and

$$R_\phi(\varpi_n,\varpi_{n+1},£\boldsymbol{\tau})=R_\phi(\xi\varpi_{n-1},\xi\varpi_n,£\boldsymbol{\tau})\leq R_\phi(\varpi_{n-1},\varpi_n,\boldsymbol{\tau})\leq R_\phi\left(\varpi_{n-2},\varpi_{n-1},\frac{\boldsymbol{\tau}}{£}\right)$$
$$\leq R_\phi\left(\varpi_{n-3},\varpi_{n-2},\frac{\boldsymbol{\tau}}{£^2}\right)\leq\cdots\leq R_\phi\left(\varpi_0,\varpi_1,\frac{\boldsymbol{\tau}}{£^{n-1}}\right)$$

We obtain

$$P_\phi(\varpi_n,\varpi_{n+1},£\boldsymbol{\tau})\geq P_\phi\left(\varpi_0,\varpi_1,\frac{\boldsymbol{\tau}}{£^{n-1}}\right),$$
$$Q_\phi(\varpi_n,\varpi_{n+1},£\boldsymbol{\tau})\leq Q_\phi\left(\varpi_0,\varpi_1,\frac{\tau}{£^{n-1}}\right) \text{ and } R_\phi(\varpi_n,\varpi_{n+1},£\boldsymbol{\tau})\leq R_\phi\left(\varpi_0,\varpi_1,\frac{\tau}{£^{n-1}}\right) \tag{4}$$

for any $q\in\mathbb{N}$, using (v), (x) and (xv) , we deduce

$$P_\phi(\varpi_n,\varpi_{n+q},\boldsymbol{\tau})\geq P_\phi\left(\varpi_n,\varpi_{n+1},\frac{\boldsymbol{\tau}}{2(\phi(\varpi_n,\varpi_{n+1}))}\right)*P_\phi\left(\varpi_{n+1},\varpi_{n+q},\frac{\boldsymbol{\tau}}{2\left(\phi(\varpi_{n+1},\varpi_{n+q})\right)}\right)$$
$$\geq P_\phi\left(\varpi_n,\varpi_{n+1},\frac{\boldsymbol{\tau}}{2(\phi(\varpi_n,\varpi_{n+1}))}\right)*P_\phi\left(\varpi_{n+1},\varpi_{n+2},\frac{\boldsymbol{\tau}}{(2)^2\left(\phi(\varpi_{n+1},\varpi_{n+q})\phi(\varpi_{n+1},\varpi_{n+2})\right)}\right)$$
$$*P_\phi\left(\varpi_{n+2},\varpi_{n+q},\frac{\boldsymbol{\tau}}{(2)^2\left(\phi(\varpi_{n+1},\varpi_{n+q})\phi(\varpi_{n+2},\varpi_{n+q})\right)}\right)$$
$$\geq P_\phi\left(\varpi_n,\varpi_{n+1},\frac{\boldsymbol{\tau}}{2(\phi(\varpi_n,\varpi_{n+1}))}\right)*P_\phi\left(\varpi_{n+1},\varpi_{n+2},\frac{\boldsymbol{\tau}}{(2)^2\left(\phi(\varpi_{n+1},\varpi_{n+q})\phi(\varpi_{n+1},\varpi_{n+2})\right)}\right)$$
$$*P_\phi\left(\varpi_{n+2},\varpi_{n+3},\frac{\boldsymbol{\tau}}{(2)^3\left(\phi(\varpi_{n+1},\varpi_{n+q})\phi(\varpi_{n+2},\varpi_{n+q})\phi(\varpi_{n+2},\varpi_{n+3})\right)}\right)$$

$$* P_\phi\left(\varpi_{n+3},\varpi_{n+q},\frac{\boldsymbol{\tau}}{(2)^3\left(\phi\left(\varpi_{n+1},\varpi_{n+q}\right)\phi\left(\varpi_{n+2},\varpi_{n+q}\right)\phi\left(\varpi_{n+3},\varpi_{n+q}\right)\right)}\right)$$

$$\geq P_\phi\left(\varpi_n,\varpi_{n+1},\frac{\boldsymbol{\tau}}{2\left(\phi(\varpi_n,\varpi_{n+1})\right)}\right) * P_\phi\left(\varpi_{n+1},\varpi_{n+2},\frac{\boldsymbol{\tau}}{(2)^2\left(\phi\left(\varpi_{n+1},\varpi_{n+q}\right)\phi(\varpi_{n+1},\varpi_{n+2})\right)}\right)$$

$$* P_\phi\left(\varpi_{n+2},\varpi_{n+3},\frac{\boldsymbol{\tau}}{(2)^3\left(\phi\left(\varpi_{n+1},\varpi_{n+q}\right)\phi\left(\varpi_{n+2},\varpi_{n+q}\right)\phi(\varpi_{n+2},\varpi_{n+3})\right)}\right)$$

$$* P_\phi\left(\varpi_{n+3},\varpi_{n+4},\frac{\boldsymbol{\tau}}{(2)^4\left(\phi\left(\varpi_{n+1},\varpi_{n+q}\right)\phi\left(\varpi_{n+2},\varpi_{n+q}\right)\phi\left(\varpi_{n+3},\varpi_{n+q}\right)\phi(\varpi_{n+3},\varpi_{n+4})\right)}\right) * \cdots *$$

$$P_\phi\left(\varpi_{n+q-2},\varpi_{n+q-1},\frac{\boldsymbol{\tau}}{(2)^{q-1}\left(\phi\left(\varpi_{n+1},\varpi_{n+q}\right)\phi\left(\varpi_{n+2},\varpi_{n+q}\right)\phi\left(\varpi_{n+3},\varpi_{n+q}\right)\cdots\phi\left(\varpi_{n+q-2},\varpi_{n+q-1}\right)\right)}\right)$$

$$* P_\phi\left(\varpi_{n+q-1},\varpi_{n+q},\frac{\boldsymbol{\tau}}{(2)^{q-1}\left(\phi\left(\varpi_{n+1},\varpi_{n+q}\right)\phi\left(\varpi_{n+2},\varpi_{n+q}\right)\phi\left(\varpi_{n+3},\varpi_{n+q}\right)\cdots\phi\left(\varpi_{n+q-1},\varpi_{n+q}\right)\right)}\right),$$

$$Q_\phi\left(\varpi_n,\varpi_{n+q},\boldsymbol{\tau}\right) \leq Q_\phi\left(\varpi_n,\varpi_{n+1},\frac{\boldsymbol{\tau}}{2\left(\phi(\varpi_n,\varpi_{n+1})\right)}\right) \circ Q_\phi\left(\varpi_{n+1},\varpi_{n+q},\frac{\boldsymbol{\tau}}{2\left(\phi\left(\varpi_{n+1},\varpi_{n+q}\right)\right)}\right)$$

$$\leq Q_\phi\left(\varpi_n,\varpi_{n+1},\frac{\boldsymbol{\tau}}{2\left(\phi(\varpi_n,\varpi_{n+1})\right)}\right) \circ Q_\phi\left(\varpi_{n+1},\varpi_{n+2},\frac{\boldsymbol{\tau}}{(2)^2\left(\phi\left(\varpi_{n+1},\varpi_{n+q}\right)\phi(\varpi_{n+1},\varpi_{n+2})\right)}\right)$$

$$\circ Q_\phi\left(\varpi_{n+2},\varpi_{n+q},\frac{\boldsymbol{\tau}}{(2)^2\left(\phi\left(\varpi_{n+1},\varpi_{n+q}\right)\phi\left(\varpi_{n+2},\varpi_{n+q}\right)\right)}\right)$$

$$\leq Q_\phi\left(\varpi_n,\varpi_{n+1},\frac{\boldsymbol{\tau}}{2\left(\phi(\varpi_n,\varpi_{n+1})\right)}\right) \circ Q_\phi\left(\varpi_{n+1},\varpi_{n+2},\frac{\boldsymbol{\tau}}{(2)^2\left(\phi\left(\varpi_{n+1},\varpi_{n+q}\right)\phi(\varpi_{n+1},\varpi_{n+2})\right)}\right)$$

$$\circ Q_\phi\left(\varpi_{n+2},\varpi_{n+3},\frac{\boldsymbol{\tau}}{(2)^3\left(\phi\left(\varpi_{n+1},\varpi_{n+q}\right)\phi\left(\varpi_{n+2},\varpi_{n+q}\right)\phi(\varpi_{n+2},\varpi_{n+3})\right)}\right)$$

$$\circ Q_\phi\left(\varpi_{n+3},\varpi_{n+q},\frac{\boldsymbol{\tau}}{(2)^3\left(\phi\left(\varpi_{n+1},\varpi_{n+q}\right)\phi\left(\varpi_{n+2},\varpi_{n+q}\right)\phi\left(\varpi_{n+3},\varpi_{n+q}\right)\right)}\right)$$

$$\leq Q_\phi\left(\varpi_n,\varpi_{n+1},\frac{\boldsymbol{\tau}}{2\left(\phi(\varpi_n,\varpi_{n+1})\right)}\right) \circ Q_\phi\left(\varpi_{n+1},\varpi_{n+2},\frac{\boldsymbol{\tau}}{(2)^2\left(\phi\left(\varpi_{n+1},\varpi_{n+q}\right)\phi(\varpi_{n+1},\varpi_{n+2})\right)}\right)$$

$$\circ\, Q_\phi\left(\varpi_{n+2},\varpi_{n+3},\frac{\boldsymbol{\tau}}{(2)^3\left(\phi\left(\varpi_{n+1},\varpi_{n+q}\right)\phi\left(\varpi_{n+2},\varpi_{n+q}\right)\phi(\varpi_{n+2},\varpi_{n+3})\right)}\right)$$

$$\circ\, Q_\phi\left(\varpi_{n+3},\varpi_{n+4},\frac{\boldsymbol{\tau}}{(2)^4\left(\phi\left(\varpi_{n+1},\varpi_{n+q}\right)\phi\left(\varpi_{n+2},\varpi_{n+q}\right)\phi\left(\varpi_{n+3},\varpi_{n+q}\right)\phi(\varpi_{n+3},\varpi_{n+4})\right)}\right)\circ\cdots\circ$$

$$Q_\phi\left(\varpi_{n+q-2},\varpi_{n+q-1},\frac{\boldsymbol{\tau}}{(2)^{q-1}\left(\phi\left(\varpi_{n+1},\varpi_{n+q}\right)\phi\left(\varpi_{n+2},\varpi_{n+q}\right)\phi\left(\varpi_{n+3},\varpi_{n+q}\right)\cdots\phi\left(\varpi_{n+q-2},\varpi_{n+q-1}\right)\right)}\right)$$

$$\circ\, Q_\phi\left(\varpi_{n+q-1},\varpi_{n+q},\frac{\boldsymbol{\tau}}{(2)^{q-1}\left(\phi\left(\varpi_{n+1},\varpi_{n+q}\right)\phi\left(\varpi_{n+2},\varpi_{n+q}\right)\phi\left(\varpi_{n+3},\varpi_{n+q}\right)\cdots\phi\left(\varpi_{n+q-1},\varpi_{n+q}\right)\right)}\right)$$

and

$$R_\phi\left(\varpi_n,\varpi_{n+q},\boldsymbol{\tau}\right)\le R_\phi\left(\varpi_n,\varpi_{n+1},\frac{\boldsymbol{\tau}}{2\left(\phi(\varpi_n,\varpi_{n+1})\right)}\right)\circ R_\phi\left(\varpi_{n+1},\varpi_{n+q},\frac{\boldsymbol{\tau}}{2\left(\phi\left(\varpi_{n+1},\varpi_{n+q}\right)\right)}\right)$$

$$\le R_\phi\left(\varpi_n,\varpi_{n+1},\frac{\boldsymbol{\tau}}{2\left(\phi(\varpi_n,\varpi_{n+1})\right)}\right)\circ R_\phi\left(\varpi_{n+1},\varpi_{n+2},\frac{\boldsymbol{\tau}}{(2)^2\left(\phi\left(\varpi_{n+1},\varpi_{n+q}\right)\phi(\varpi_{n+1},\varpi_{n+2})\right)}\right)$$

$$\circ\, R_\phi\left(\varpi_{n+2},\varpi_{n+q},\frac{\boldsymbol{\tau}}{(2)^2\left(\phi\left(\varpi_{n+1},\varpi_{n+q}\right)\phi\left(\varpi_{n+2},\varpi_{n+q}\right)\right)}\right)$$

$$\le R_\phi\left(\varpi_n,\varpi_{n+1},\frac{\boldsymbol{\tau}}{2\left(\phi(\varpi_n,\varpi_{n+1})\right)}\right)\circ R_\phi\left(\varpi_{n+1},\varpi_{n+2},\frac{\boldsymbol{\tau}}{(2)^2\left(\phi\left(\varpi_{n+1},\varpi_{n+q}\right)\phi(\varpi_{n+1},\varpi_{n+2})\right)}\right)$$

$$\circ\, R_\phi\left(\varpi_{n+2},\varpi_{n+3},\frac{\boldsymbol{\tau}}{(2)^3\left(\phi\left(\varpi_{n+1},\varpi_{n+q}\right)\phi\left(\varpi_{n+2},\varpi_{n+q}\right)\phi(\varpi_{n+2},\varpi_{n+3})\right)}\right)$$

$$\circ\, R_\phi\left(\varpi_{n+3},\varpi_{n+q},\frac{\boldsymbol{\tau}}{(2)^3\left(\phi\left(\varpi_{n+1},\varpi_{n+q}\right)\phi\left(\varpi_{n+2},\varpi_{n+q}\right)\phi\left(\varpi_{n+3},\varpi_{n+q}\right)\right)}\right)$$

$$\le R_\phi\left(\varpi_n,\varpi_{n+1},\frac{\boldsymbol{\tau}}{2\left(\phi(\varpi_n,\varpi_{n+1})\right)}\right)\circ R_\phi\left(\varpi_{n+1},\varpi_{n+2},\frac{\boldsymbol{\tau}}{(2)^2\left(\phi\left(\varpi_{n+1},\varpi_{n+q}\right)\phi(\varpi_{n+1},\varpi_{n+2})\right)}\right)$$

$$\circ\, R_\phi\left(\varpi_{n+2},\varpi_{n+3},\frac{\boldsymbol{\tau}}{(2)^3\left(\phi\left(\varpi_{n+1},\varpi_{n+q}\right)\phi\left(\varpi_{n+2},\varpi_{n+q}\right)\phi(\varpi_{n+2},\varpi_{n+3})\right)}\right)$$

$$\circ R_\phi\left(\varpi_{n+3},\varpi_{n+4},\frac{\boldsymbol{\tau}}{(2)^4\left(\phi\left(\varpi_{n+1},\varpi_{n+q}\right)\phi\left(\varpi_{n+2},\varpi_{n+q}\right)\phi\left(\varpi_{n+3},\varpi_{n+q}\right)\phi(\varpi_{n+3},\varpi_{n+4})\right)}\right)\circ\cdots\circ$$

$$R_\phi\left(\varpi_{n+q-2},\varpi_{n+q-1},\frac{\boldsymbol{\tau}}{(2)^{q-1}\left(\phi\left(\varpi_{n+1},\varpi_{n+q}\right)\phi\left(\varpi_{n+2},\varpi_{n+q}\right)\phi\left(\varpi_{n+3},\varpi_{n+q}\right)\cdots\phi\left(\varpi_{n+q-2},\varpi_{n+q-1}\right)\right)}\right)$$

$$\circ R_\phi\left(\varpi_{n+q-1},\varpi_{n+q},\frac{\boldsymbol{\tau}}{(2)^{q-1}\left(\phi\left(\varpi_{n+1},\varpi_{n+q}\right)\phi\left(\varpi_{n+2},\varpi_{n+q}\right)\phi\left(\varpi_{n+3},\varpi_{n+q}\right)\cdots\phi\left(\varpi_{n+q-1},\varpi_{n+q}\right)\right)}\right)$$

Using (4) in the above inequalities, we deduce

$$\geq P_\phi\left(\varpi_0,\varpi_1,\frac{\boldsymbol{\tau}}{2(£)^{n-1}\left(\phi(\varpi_n,\varpi_{n+1})\right)}\right)*P_\phi\left(\varpi_0,\varpi_1,\frac{\boldsymbol{\tau}}{(2)^2(£)^n\left(\phi\left(\varpi_{n+1},\varpi_{n+q}\right)\phi(\varpi_{n+1},\varpi_{n+2})\right)}\right)$$

$$*P_\phi\left(\varpi_0,\varpi_1,\frac{\boldsymbol{\tau}}{(2)^3(£)^{n+1}\left(\phi\left(\varpi_{n+1},\varpi_{n+q}\right)\phi\left(\varpi_{n+2},\varpi_{n+q}\right)\phi(\varpi_{n+2},\varpi_{n+3})\right)}\right)$$

$$*P_\phi\left(\varpi_0,\varpi_1,\frac{\boldsymbol{\tau}}{(2)^4(£)^{n+2}\left(\phi\left(\varpi_{n+1},\varpi_{n+q}\right)\phi\left(\varpi_{n+2},\varpi_{n+q}\right)\phi\left(\varpi_{n+3},\varpi_{n+q}\right)\phi(\varpi_{n+3},\varpi_{n+4})\right)}\right)$$

$$*\cdots*$$

$$P_\phi\left(\varpi_0,\varpi_1,\frac{\boldsymbol{\tau}}{(2)^{q-1}(£)^{n+q-2}\left(\phi\left(\varpi_{n+1},\varpi_{n+q}\right)\phi\left(\varpi_{n+2},\varpi_{n+q}\right)\phi\left(\varpi_{n+3},\varpi_{n+q}\right)\cdots\phi\left(\varpi_{n+q-2},\varpi_{n+q-1}\right)\right)}\right)$$

$$*P_\phi\left(\varpi_0,\varpi_1,\frac{\boldsymbol{\tau}}{(2)^{q-1}(£)^{n+q-1}\left(\phi\left(\varpi_{n+1},\varpi_{n+q}\right)\phi\left(\varpi_{n+2},\varpi_{n+q}\right)\phi\left(\varpi_{n+3},\varpi_{n+q}\right)\cdots\phi\left(\varpi_{n+q-1},\varpi_{n+q}\right)\right)}\right),$$

$$\leq Q_\phi\left(\varpi_0,\varpi_1,\frac{\boldsymbol{\tau}}{2(£)^{n-1}\left(\phi(\varpi_n,\varpi_{n+1})\right)}\right)$$

$$\circ Q_\phi\left(\varpi_0,\varpi_1,\frac{\boldsymbol{\tau}}{(2)^2(£)^n\left(\phi\left(\varpi_{n+1},\varpi_{n+q}\right)\phi(\varpi_{n+1},\varpi_{n+2})\right)}\right)$$

$$\circ Q_\phi\left(\varpi_0,\varpi_1,\frac{\boldsymbol{\tau}}{(2)^3(£)^{n+1}\left(\phi\left(\varpi_{n+1},\varpi_{n+q}\right)\phi\left(\varpi_{n+2},\varpi_{n+q}\right)\phi(\varpi_{n+2},\varpi_{n+3})\right)}\right)$$

$$\circ Q_\phi\left(\varpi_0,\varpi_1,\frac{\boldsymbol{\tau}}{(2)^4(£)^{n+2}\left(\phi\left(\varpi_{n+1},\varpi_{n+q}\right)\phi\left(\varpi_{n+2},\varpi_{n+q}\right)\phi\left(\varpi_{n+3},\varpi_{n+q}\right)\phi(\varpi_{n+3},\varpi_{n+4})\right)}\right)$$

$$\circ\cdots\circ$$

$$Q_\phi\left(\varpi_0,\varpi_1,\frac{\boldsymbol{\tau}}{(2)^{q-1}(£)^{n+q-2}\left(\phi(\varpi_{n+1},\varpi_{n+q})\phi(\varpi_{n+2},\varpi_{n+q})\phi(\varpi_{n+3},\varpi_{n+q})\cdots\phi(\varpi_{n+q-2},\varpi_{n+q-1})\right)}\right)$$

$$\circ\, Q_\phi\left(\varpi_0,\varpi_1,\frac{\boldsymbol{\tau}}{(2)^{q-1}(£)^{n+q-1}\left(\phi(\varpi_{n+1},\varpi_{n+q})\phi(\varpi_{n+2},\varpi_{n+q})\phi(\varpi_{n+3},\varpi_{n+q})\cdots\phi(\varpi_{n+q-1},\varpi_{n+q})\right)}\right)$$

and

$$\leq R_\phi\left(\varpi_0,\varpi_1,\frac{\boldsymbol{\tau}}{2(£)^{n-1}(\phi(\varpi_n,\varpi_{n+1}))}\right)$$

$$\circ\, R_\phi\left(\varpi_0,\varpi_1,\frac{\boldsymbol{\tau}}{(2)^2(£)^n\left(\phi(\varpi_{n+1},\varpi_{n+q})\phi(\varpi_{n+1},\varpi_{n+2})\right)}\right)$$

$$\circ\, R_\phi\left(\varpi_0,\varpi_1,\frac{\boldsymbol{\tau}}{(2)^3(£)^{n+1}\left(\phi(\varpi_{n+1},\varpi_{n+q})\phi(\varpi_{n+2},\varpi_{n+q})\phi(\varpi_{n+2},\varpi_{n+3})\right)}\right)$$

$$\circ\, R_\phi\left(\varpi_0,\varpi_1,\frac{\boldsymbol{\tau}}{(2)^4(£)^{n+2}\left(\phi(\varpi_{n+1},\varpi_{n+q})\phi(\varpi_{n+2},\varpi_{n+q})\phi(\varpi_{n+3},\varpi_{n+q})\phi(\varpi_{n+3},\varpi_{n+4})\right)}\right)$$

$$\circ\cdots\circ$$

$$R_\phi\left(\varpi_0,\varpi_1,\frac{\boldsymbol{\tau}}{(2)^{q-1}(£)^{n+q-2}\left(\phi(\varpi_{n+1},\varpi_{n+q})\phi(\varpi_{n+2},\varpi_{n+q})\phi(\varpi_{n+3},\varpi_{n+q})\cdots\phi(\varpi_{n+q-2},\varpi_{n+q-1})\right)}\right)$$

$$\circ\, R_\phi\left(\varpi_0,\varpi_1,\frac{\boldsymbol{\tau}}{(2)^{q-1}(£)^{n+q-1}\left(\phi(\varpi_{n+1},\varpi_{n+q})\phi(\varpi_{n+2},\varpi_{n+q})\phi(\varpi_{n+3},\varpi_{n+q})\cdots\phi(\varpi_{n+q-1},\varpi_{n+q})\right)}\right)$$

Using (1), for $n\to\infty$, we deduce

$$\lim_{n\to\infty} P_\phi(\varpi_n,\varpi_{n+q},\boldsymbol{\tau}) = 1*1*\cdots*1 = 1,$$

$$\lim_{n\to\infty} Q_\phi(\varpi_n,\varpi_{n+q},\boldsymbol{\tau}) = 0\circ 0\circ\cdots\circ 0 = 0,$$

and

$$\lim_{n\to\infty} R_\phi(\varpi_n,\varpi_{n+q},\boldsymbol{\tau}) = 0\circ 0\circ\cdots\circ 0 = 0$$

i.e., $\{\varpi_n\}$ is a GCS. Therefore, $(\boldsymbol{\Xi},P_\phi,Q_\phi,R_\phi,*,\circ)$ be a G-complete CNMS, there exists $\varpi\in\boldsymbol{\Xi}$.

Now investigate that ϖ is a FP of ξ, using (v), (x), (xv) and (1), we obtain

$$P_\phi(\varpi,\xi\varpi,\boldsymbol{\tau}) \geq P_\phi\left(\varpi,\varpi_{n+1},\frac{\boldsymbol{\tau}}{2(\phi(\varpi,\varpi_{n+1}))}\right) * P_\phi\left(\varpi_{n+1},\xi\varpi,\frac{\boldsymbol{\tau}}{2(\phi(\varpi_{n+1},\xi\varpi))}\right)$$

$$P_\phi(\varpi,\xi\varpi,\boldsymbol{\tau}) \geq P_\phi\left(\varpi,\varpi_{n+1},\frac{\boldsymbol{\tau}}{2\big(\phi(\varpi,\varpi_{n+1})\big)}\right) * P_\phi\left(\xi\varpi_n,\xi\varpi,\frac{\boldsymbol{\tau}}{2\big(\phi(\varpi_{n+1},\xi\varpi)\big)}\right)$$

$$P_\phi(\varpi,\xi\varpi,\boldsymbol{\tau}) \geq P_\phi\left(\varpi,\varpi_{n+1},\frac{\boldsymbol{\tau}}{2\big(\phi(\varpi,\varpi_{n+1})\big)}\right) * P_\phi\left(\varpi_n,\varpi,\frac{\boldsymbol{\tau}}{2£\big(\phi(\varpi_{n+1},\xi\varpi)\big)}\right) \to 1 * 1 = 1$$

as $n \to \infty$,

$$Q_\phi(\varpi,\xi\varpi,\boldsymbol{\tau}) \leq Q_\phi\left(\varpi,\varpi_{n+1},\frac{\boldsymbol{\tau}}{2\big(\phi(\varpi,\varpi_{n+1})\big)}\right) \circ Q_\phi\left(\varpi_{n+1},\xi\varpi,\frac{\boldsymbol{\tau}}{2\big(\phi(\varpi_{n+1},\xi\varpi)\big)}\right)$$

$$Q_\phi(\varpi,\xi\varpi,\boldsymbol{\tau}) \leq Q_\phi\left(\varpi,\varpi_{n+1},\frac{\boldsymbol{\tau}}{2\big(\phi(\varpi,\varpi_{n+1})\big)}\right) \circ Q_\phi\left(\xi\varpi_n,\xi\varpi,\frac{\boldsymbol{\tau}}{2\big(\phi(\varpi_{n+1},\xi\varpi)\big)}\right)$$

$$Q_\phi(\varpi,\xi\varpi,\boldsymbol{\tau}) \leq Q_\phi\left(\varpi,\varpi_{n+1},\frac{\boldsymbol{\tau}}{2\big(\phi(\varpi,\varpi_{n+1})\big)}\right) \circ Q_\phi\left(\varpi_n,\varpi,\frac{\boldsymbol{\tau}}{2£\big(\phi(\varpi_{n+1},\xi\varpi)\big)}\right) \to 0 \circ 0 = 0$$

as $n \to \infty$, and

$$R_\phi(\varpi,\xi\varpi,\boldsymbol{\tau}) \leq R_\phi\left(\varpi,\varpi_{n+1},\frac{\boldsymbol{\tau}}{2\big(\phi(\varpi,\varpi_{n+1})\big)}\right) \circ R_\phi\left(\varpi_{n+1},\xi\varpi,\frac{\boldsymbol{\tau}}{2\big(\phi(\varpi_{n+1},\xi\varpi)\big)}\right)$$

$$R_\phi(\varpi,\xi\varpi,\boldsymbol{\tau}) \leq R_\phi\left(\varpi,\varpi_{n+1},\frac{\boldsymbol{\tau}}{2\big(\phi(\varpi,\varpi_{n+1})\big)}\right) \circ R_\phi\left(\xi\varpi_n,\xi\varpi,\frac{\boldsymbol{\tau}}{2\big(\phi(\varpi_{n+1},\xi\varpi)\big)}\right)$$

$$R_\phi(\varpi,\xi\varpi,\boldsymbol{\tau}) \leq R_\phi\left(\varpi,\varpi_{n+1},\frac{\boldsymbol{\tau}}{2\big(\phi(\varpi,\varpi_{n+1})\big)}\right) \circ R_\phi\left(\varpi_n,\varpi,\frac{\boldsymbol{\tau}}{2£\big(\phi(\varpi_{n+1},\xi\varpi)\big)}\right) \to 0 \circ 0 = 0$$

as $n \to \infty$. This implies that $\xi\varpi = \varpi$, a FP. Now we show the uniqueness, suppose $\xi c = c$ for some $c \in \boldsymbol{\Xi}$, then

$$1 \geq P_\phi(c,\varpi,\boldsymbol{\tau}) = P_\phi(\xi c,\xi\varpi,\boldsymbol{\tau}) \geq P_\phi\left(c,\varpi,\frac{\boldsymbol{\tau}}{£}\right) = P_\phi\left(\xi c,\xi\varpi,\frac{\boldsymbol{\tau}}{£}\right)$$

$$\geq P_\phi\left(c,\varpi,\frac{\boldsymbol{\tau}}{£^2}\right) \geq \cdots \geq P_\phi\left(c,\varpi,\frac{\boldsymbol{\tau}}{£^n}\right) \to 1 \text{ as } n \to \infty,$$

$$0 \leq Q_\phi(c,\varpi,\boldsymbol{\tau}) = Q_\phi(\xi c,\xi\varpi,\boldsymbol{\tau}) \leq Q_\phi\left(c,\varpi,\frac{\boldsymbol{\tau}}{£}\right) = Q_\phi\left(\xi c,\xi\varpi,\frac{\boldsymbol{\tau}}{£}\right)$$

$$\leq Q_\phi\left(c,\varpi,\frac{\boldsymbol{\tau}}{£^2}\right) \leq \cdots \leq Q_\phi\left(c,\varpi,\frac{\boldsymbol{\tau}}{£^n}\right) \to 0 \text{ as } n \to \infty,$$

and

$$0 \leq R_\phi(c,\varpi,\boldsymbol{\tau}) = R_\phi(\xi c,\xi\varpi,\boldsymbol{\tau}) \leq R_\phi\left(c,\varpi,\frac{\boldsymbol{\tau}}{£}\right) = R_\phi\left(\xi c,\xi\varpi,\frac{\boldsymbol{\tau}}{£}\right)$$

$$\leq R_\phi\left(c,\varpi,\frac{\boldsymbol{\tau}}{£^2}\right) \leq \cdots \leq R_\phi\left(c,\varpi,\frac{\boldsymbol{\tau}}{£^n}\right) \to 0 \text{ as } n \to \infty,$$

by using (iii), (viii) and (xii), $\varpi = c$.

Definition 3.4. Let $\left(\boldsymbol{\Xi},P_\phi,Q_\phi,R_\phi,*,\circ\right)$ be a CNMLS. A map $\xi:\boldsymbol{\Xi} \to \boldsymbol{\Xi}$ is CNL-contraction if there exists $0 < £ < 1$, such that

$$\frac{1}{P_\phi(\xi\varpi,\xi\nu,\boldsymbol{\tau})}-1 \leq £\left[\frac{1}{P_\phi(\varpi,\nu,\boldsymbol{\tau})}-1\right] \tag{5}$$

and

$$Q_\phi(\xi\varpi,\xi\nu,\boldsymbol{\tau}) \leq £Q_\phi(\varpi,\nu,\boldsymbol{\tau}), R_\phi(\xi\varpi,\xi\nu,\boldsymbol{\tau}) \leq £R_\phi(\varpi,\nu,\boldsymbol{\tau}) \tag{6}$$

for all $\varpi,\nu \in \Xi$ and $\boldsymbol{\tau} > 0$.

Theorem 3.2. Let $(\Xi, P_\phi, Q_\phi, R_\phi, *, \circ)$ be a G-complete CNMLS with $\phi: \Xi \times \Xi \to [1,\infty)$ and suppose that

$$\lim_{\boldsymbol{\tau}\to\infty} P_\phi(\varpi,\nu,\boldsymbol{\tau}) = 1, \lim_{\boldsymbol{\tau}\to\infty} Q_\phi(\varpi,\nu,\boldsymbol{\tau}) = 0 \text{ and } \lim_{\boldsymbol{\tau}\to\infty} R_\phi(\varpi,\nu,\boldsymbol{\tau}) = 0 \tag{7}$$

for all $\varpi,\nu \in \Xi$ and $\boldsymbol{\tau} > 0$. Let $\xi: \Xi \to \Xi$ be a CN-contraction. Further, assume that for an arbitrary $\varpi_0 \in \Xi$, and $n, q \in \mathbb{N}$, where $\varpi_n = \xi^n\varpi_0 = \xi\varpi_{n-1}$ also $\lim_{n\to\infty}\phi(\varpi_n,\nu)$ and $\lim_{n\to\infty}\phi(\nu,\varpi_n)$ exists and finite. Then ξ has a unique FP.

Proof. Suppose ϖ_0 be an arbitrary point of Ξ and define a sequence ϖ_n by $\varpi_n = \xi^n\varpi_0 = \xi\varpi_{n-1}$, $n \in \mathbb{N}$. By utilizing (5) and (6) for all $\boldsymbol{\tau} > 0, n > q$, we get

$$\frac{1}{P_\phi(\varpi_n,\varpi_{n+1},\boldsymbol{\tau})}-1 = \frac{1}{P_\phi(\xi\varpi_{n-1},\varpi_n,\boldsymbol{\tau})}-1$$

$$\leq £\left[\frac{1}{P_\phi(\varpi_{n-1},\varpi_n,\boldsymbol{\tau})}-1\right] = \frac{£}{P_\phi(\varpi_{n-1},\varpi_n,\boldsymbol{\tau})}-£$$

$$\Rightarrow \frac{1}{P_\phi(\varpi_n,\varpi_{n+1},\boldsymbol{\tau})} \leq \frac{£}{P_\phi(\varpi_{n-1},\varpi_n,\boldsymbol{\tau})}+(1-£)$$

$$\leq \frac{£^2}{P_\phi(\varpi_{n-2},\varpi_{n-1},\boldsymbol{\tau})}+£(1-£)+(1-£)$$

Continuing in this way, we get

$$\frac{1}{P_\phi(\varpi_n,\varpi_{n+1},\boldsymbol{\tau})} \leq \frac{£^n}{P_\phi(\varpi_0,\varpi_1,\boldsymbol{\tau})}+£^{n-1}(1-£)+£^{n-2}(1-£)+\cdots+£(1-£)+(1-£)$$

$$\leq \frac{£^n}{P_\phi(\varpi_0,\varpi_1,\boldsymbol{\tau})}+(£^{n-1}+£^{n-2}+\cdots+1)(1-£) \leq \frac{£^n}{P_\phi(\varpi_0,\varpi_1,\boldsymbol{\tau})}+(1-£^n)$$

We obtain

$$\frac{1}{\frac{£^n}{P_\phi(\varpi_0,\varpi_1,\boldsymbol{\tau})}+(1-£^n)} \leq P_\phi(\varpi_n,\varpi_{n+1},\boldsymbol{\tau}) \tag{8}$$

and

$$Q_\phi(\varpi_n,\varpi_{n+1},\boldsymbol{\tau}) = Q_\phi(\xi\varpi_{n-1},\varpi_n,\boldsymbol{\tau}) \leq £Q_\phi(\varpi_{n-1},\varpi_n,\boldsymbol{\tau}) = Q_\phi(\xi\varpi_{n-2},\varpi_{n-1},\boldsymbol{\tau})$$

$$\leq £^2Q_\phi(\varpi_{n-2},\varpi_{n-1},\boldsymbol{\tau}) \leq \cdots \leq £^nQ_\phi(\varpi_0,\varpi_1,\boldsymbol{\tau}) \tag{9}$$

$$R_\phi(\varpi_n,\varpi_{n+1},\boldsymbol{\tau}) = R_\phi(\xi\varpi_{n-1},\varpi_n,\boldsymbol{\tau}) \le \pounds R_\phi(\varpi_{n-1},\varpi_n,\boldsymbol{\tau}) = R_\phi(\xi\varpi_{n-2},\varpi_{n-1},\boldsymbol{\tau})$$

$$\le \pounds^2 R_\phi(\varpi_{n-2},\varpi_{n-1},\boldsymbol{\tau}) \le \cdots \le \pounds^n R_\phi(\varpi_0,\varpi_1,\boldsymbol{\tau}) \tag{10}$$

for any $q \in \mathbb{N}$, using (v), (x) and (xv) , we deduce

$$P_\phi(\varpi_n,\varpi_{n+q},\boldsymbol{\tau}) \ge P_\phi\left(\varpi_n,\varpi_{n+1},\frac{\boldsymbol{\tau}}{2\big(\phi(\varpi_n,\varpi_{n+1})\big)}\right) * P_\phi\left(\varpi_{n+1},\varpi_{n+q},\frac{\boldsymbol{\tau}}{2\left(\phi\big(\varpi_{n+1},\varpi_{n+q}\big)\right)}\right)$$

$$\ge P_\phi\left(\varpi_n,\varpi_{n+1},\frac{\boldsymbol{\tau}}{2\big(\phi(\varpi_n,\varpi_{n+1})\big)}\right) * P_\phi\left(\varpi_{n+1},\varpi_{n+2},\frac{\boldsymbol{\tau}}{(2)^2\left(\phi\big(\varpi_{n+1},\varpi_{n+q}\big)\phi(\varpi_{n+1},\varpi_{n+2})\right)}\right)$$

$$* P_\phi\left(\varpi_{n+2},\varpi_{n+q},\frac{\boldsymbol{\tau}}{(2)^2\left(\phi\big(\varpi_{n+1},\varpi_{n+q}\big)\phi\big(\varpi_{n+2},\varpi_{n+q}\big)\right)}\right)$$

$$\ge P_\phi\left(\varpi_n,\varpi_{n+1},\frac{\boldsymbol{\tau}}{2\big(\phi(\varpi_n,\varpi_{n+1})\big)}\right) * P_\phi\left(\varpi_{n+1},\varpi_{n+2},\frac{\boldsymbol{\tau}}{(2)^2\left(\phi\big(\varpi_{n+1},\varpi_{n+q}\big)\phi(\varpi_{n+1},\varpi_{n+2})\right)}\right)$$

$$* P_\phi\left(\varpi_{n+2},\varpi_{n+3},\frac{\boldsymbol{\tau}}{(2)^3\left(\phi\big(\varpi_{n+1},\varpi_{n+q}\big)\phi\big(\varpi_{n+2},\varpi_{n+q}\big)\phi(\varpi_{n+2},\varpi_{n+3})\right)}\right)$$

$$* P_\phi\left(\varpi_{n+3},\varpi_{n+q},\frac{\boldsymbol{\tau}}{(2)^3\left(\phi\big(\varpi_{n+1},\varpi_{n+q}\big)\phi\big(\varpi_{n+2},\varpi_{n+q}\big)\phi\big(\varpi_{n+3},\varpi_{n+q}\big)\right)}\right)$$

$$\ge P_\phi\left(\varpi_n,\varpi_{n+1},\frac{\boldsymbol{\tau}}{2\big(\phi(\varpi_n,\varpi_{n+1})\big)}\right) * P_\phi\left(\varpi_{n+1},\varpi_{n+2},\frac{\boldsymbol{\tau}}{(2)^2\left(\phi\big(\varpi_{n+1},\varpi_{n+q}\big)\phi(\varpi_{n+1},\varpi_{n+2})\right)}\right)$$

$$* P_\phi\left(\varpi_{n+2},\varpi_{n+3},\frac{\boldsymbol{\tau}}{(2)^3\left(\phi\big(\varpi_{n+1},\varpi_{n+q}\big)\phi\big(\varpi_{n+2},\varpi_{n+q}\big)\phi(\varpi_{n+2},\varpi_{n+3})\right)}\right)$$

$$* P_\phi\left(\varpi_{n+3},\varpi_{n+4},\frac{\boldsymbol{\tau}}{(2)^4\left(\phi\big(\varpi_{n+1},\varpi_{n+q}\big)\phi\big(\varpi_{n+2},\varpi_{n+q}\big)\phi\big(\varpi_{n+3},\varpi_{n+q}\big)\phi(\varpi_{n+3},\varpi_{n+4})\right)}\right) * \cdots *$$

$$P_\phi\left(\varpi_{n+q-2},\varpi_{n+q-1},\frac{\boldsymbol{\tau}}{(2)^{q-1}\left(\phi\big(\varpi_{n+1},\varpi_{n+q}\big)\phi\big(\varpi_{n+2},\varpi_{n+q}\big)\phi\big(\varpi_{n+3},\varpi_{n+q}\big)\cdots\phi\big(\varpi_{n+q-2},\varpi_{n+q-1}\big)\right)}\right)$$

$$* P_\phi\left(\varpi_{n+q-1},\varpi_{n+q},\frac{\boldsymbol{\tau}}{(2)^{q-1}\left(\phi\big(\varpi_{n+1},\varpi_{n+q}\big)\phi\big(\varpi_{n+2},\varpi_{n+q}\big)\phi\big(\varpi_{n+3},\varpi_{n+q}\big)\cdots\phi\big(\varpi_{n+q-1},\varpi_{n+q}\big)\right)}\right)$$

and

$$Q_\phi(\varpi_n,\varpi_{n+q},\boldsymbol{\tau}) \leq Q_\phi\left(\varpi_n,\varpi_{n+1},\frac{\boldsymbol{\tau}}{2\big(\phi(\varpi_n,\varpi_{n+1})\big)}\right) \circ Q_\phi\left(\varpi_{n+1},\varpi_{n+q},\frac{\boldsymbol{\tau}}{2\left(\phi\big(\varpi_{n+1},\varpi_{n+q}\big)\right)}\right)$$

$$\leq Q_\phi\left(\varpi_n,\varpi_{n+1},\frac{\boldsymbol{\tau}}{2\big(\phi(\varpi_n,\varpi_{n+1})\big)}\right) \circ Q_\phi\left(\varpi_{n+1},\varpi_{n+2},\frac{\boldsymbol{\tau}}{(2)^2\left(\phi\big(\varpi_{n+1},\varpi_{n+q}\big)\phi(\varpi_{n+1},\varpi_{n+2})\right)}\right)$$

$$\circ\, Q_\phi\left(\varpi_{n+2},\varpi_{n+q},\frac{\boldsymbol{\tau}}{(2)^2\left(\phi\big(\varpi_{n+1},\varpi_{n+q}\big)\phi\big(\varpi_{n+2},\varpi_{n+q}\big)\right)}\right)$$

$$\leq Q_\phi\left(\varpi_n,\varpi_{n+1},\frac{\boldsymbol{\tau}}{2\big(\phi(\varpi_n,\varpi_{n+1})\big)}\right) \circ Q_\phi\left(\varpi_{n+1},\varpi_{n+2},\frac{\boldsymbol{\tau}}{(2)^2\left(\phi\big(\varpi_{n+1},\varpi_{n+q}\big)\phi(\varpi_{n+1},\varpi_{n+2})\right)}\right)$$

$$\circ\, Q_\phi\left(\varpi_{n+2},\varpi_{n+3},\frac{\boldsymbol{\tau}}{(2)^3\left(\phi\big(\varpi_{n+1},\varpi_{n+q}\big)\phi\big(\varpi_{n+2},\varpi_{n+q}\big)\phi(\varpi_{n+2},\varpi_{n+3})\right)}\right)$$

$$\circ\, Q_\phi\left(\varpi_{n+3},\varpi_{n+q},\frac{\boldsymbol{\tau}}{(2)^3\left(\phi\big(\varpi_{n+1},\varpi_{n+q}\big)\phi\big(\varpi_{n+2},\varpi_{n+q}\big)\phi\big(\varpi_{n+3},\varpi_{n+q}\big)\right)}\right)$$

$$\leq Q_\phi\left(\varpi_n,\varpi_{n+1},\frac{\boldsymbol{\tau}}{2\big(\phi(\varpi_n,\varpi_{n+1})\big)}\right) \circ Q_\phi\left(\varpi_{n+1},\varpi_{n+2},\frac{\boldsymbol{\tau}}{(2)^2\left(\phi\big(\varpi_{n+1},\varpi_{n+q}\big)\phi(\varpi_{n+1},\varpi_{n+2})\right)}\right)$$

$$\circ\, Q_\phi\left(\varpi_{n+2},\varpi_{n+3},\frac{\boldsymbol{\tau}}{(2)^3\left(\phi\big(\varpi_{n+1},\varpi_{n+q}\big)\phi\big(\varpi_{n+2},\varpi_{n+q}\big)\phi(\varpi_{n+2},\varpi_{n+3})\right)}\right)$$

$$\circ\, Q_\phi\left(\varpi_{n+3},\varpi_{n+4},\frac{\boldsymbol{\tau}}{(2)^4\left(\phi\big(\varpi_{n+1},\varpi_{n+q}\big)\phi\big(\varpi_{n+2},\varpi_{n+q}\big)\phi\big(\varpi_{n+3},\varpi_{n+q}\big)\phi(\varpi_{n+3},\varpi_{n+4})\right)}\right) \circ \cdots \circ$$

$$Q_\phi\left(\varpi_{n+q-2},\varpi_{n+q-1},\frac{\boldsymbol{\tau}}{(2)^{q-1}\left(\phi\big(\varpi_{n+1},\varpi_{n+q}\big)\phi\big(\varpi_{n+2},\varpi_{n+q}\big)\phi\big(\varpi_{n+3},\varpi_{n+q}\big)\cdots\phi\big(\varpi_{n+q-2},\varpi_{n+q-1}\big)\right)}\right)$$

$$\circ\, Q_\phi\left(\varpi_{n+q-1},\varpi_{n+q},\frac{\boldsymbol{\tau}}{(2)^{q-1}\left(\phi\big(\varpi_{n+1},\varpi_{n+q}\big)\phi\big(\varpi_{n+2},\varpi_{n+q}\big)\phi\big(\varpi_{n+3},\varpi_{n+q}\big)\cdots\phi\big(\varpi_{n+q-1},\varpi_{n+q}\big)\right)}\right),$$

$$R_\phi(\varpi_n,\varpi_{n+q},\boldsymbol{\tau}) \leq R_\phi\left(\varpi_n,\varpi_{n+1},\frac{\boldsymbol{\tau}}{2\big(\phi(\varpi_n,\varpi_{n+1})\big)}\right) \circ R_\phi\left(\varpi_{n+1},\varpi_{n+q},\frac{\boldsymbol{\tau}}{2\left(\phi\big(\varpi_{n+1},\varpi_{n+q}\big)\right)}\right)$$

$$\leq R_\phi\left(\varpi_n,\varpi_{n+1},\frac{\tau}{2\big(\phi(\varpi_n,\varpi_{n+1})\big)}\right)\circ R_\phi\left(\varpi_{n+1},\varpi_{n+2},\frac{\tau}{(2)^2\left(\phi\big(\varpi_{n+1},\varpi_{n+q}\big)\phi(\varpi_{n+1},\varpi_{n+2})\right)}\right)$$

$$\circ R_\phi\left(\varpi_{n+2},\varpi_{n+q},\frac{\tau}{(2)^2\left(\phi\big(\varpi_{n+1},\varpi_{n+q}\big)\phi\big(\varpi_{n+2},\varpi_{n+q}\big)\right)}\right)$$

$$\leq R_\phi\left(\varpi_n,\varpi_{n+1},\frac{\tau}{2\big(\phi(\varpi_n,\varpi_{n+1})\big)}\right)\circ R_\phi\left(\varpi_{n+1},\varpi_{n+2},\frac{\tau}{(2)^2\left(\phi\big(\varpi_{n+1},\varpi_{n+q}\big)\phi(\varpi_{n+1},\varpi_{n+2})\right)}\right)$$

$$\circ R_\phi\left(\varpi_{n+2},\varpi_{n+3},\frac{\tau}{(2)^3\left(\phi\big(\varpi_{n+1},\varpi_{n+q}\big)\phi\big(\varpi_{n+2},\varpi_{n+q}\big)\phi(\varpi_{n+2},\varpi_{n+3})\right)}\right)$$

$$\circ R_\phi\left(\varpi_{n+3},\varpi_{n+q},\frac{\tau}{(2)^3\left(\phi\big(\varpi_{n+1},\varpi_{n+q}\big)\phi\big(\varpi_{n+2},\varpi_{n+q}\big)\phi\big(\varpi_{n+3},\varpi_{n+q}\big)\right)}\right)$$

$$\leq R_\phi\left(\varpi_n,\varpi_{n+1},\frac{\tau}{2\big(\phi(\varpi_n,\varpi_{n+1})\big)}\right)\circ R_\phi\left(\varpi_{n+1},\varpi_{n+2},\frac{\tau}{(2)^2\left(\phi\big(\varpi_{n+1},\varpi_{n+q}\big)\phi(\varpi_{n+1},\varpi_{n+2})\right)}\right)$$

$$\circ R_\phi\left(\varpi_{n+2},\varpi_{n+3},\frac{\tau}{(2)^3\left(\phi\big(\varpi_{n+1},\varpi_{n+q}\big)\phi\big(\varpi_{n+2},\varpi_{n+q}\big)\phi(\varpi_{n+2},\varpi_{n+3})\right)}\right)$$

$$\circ R_\phi\left(\varpi_{n+3},\varpi_{n+4},\frac{\tau}{(2)^4\left(\phi\big(\varpi_{n+1},\varpi_{n+q}\big)\phi\big(\varpi_{n+2},\varpi_{n+q}\big)\phi\big(\varpi_{n+3},\varpi_{n+q}\big)\phi(\varpi_{n+3},\varpi_{n+4})\right)}\right)\circ\cdots\circ$$

$$R_\phi\left(\varpi_{n+q-2},\varpi_{n+q-1},\frac{\tau}{(2)^{q-1}\left(\phi\big(\varpi_{n+1},\varpi_{n+q}\big)\phi\big(\varpi_{n+2},\varpi_{n+q}\big)\phi\big(\varpi_{n+3},\varpi_{n+q}\big)\cdots\phi\big(\varpi_{n+q-2},\varpi_{n+q-1}\big)\right)}\right)$$

$$\circ R_\phi\left(\varpi_{n+q-1},\varpi_{n+q},\frac{\tau}{(2)^{q-1}\left(\phi\big(\varpi_{n+1},\varpi_{n+q}\big)\phi\big(\varpi_{n+2},\varpi_{n+q}\big)\phi\big(\varpi_{n+3},\varpi_{n+q}\big)\cdots\phi\big(\varpi_{n+q-1},\varpi_{n+q}\big)\right)}\right)$$

$$P_\phi\big(\varpi_n,\varpi_{n+q},\tau\big)$$

$$\geq\frac{1}{\dfrac{£^n}{P_\phi\left(\varpi_0,\varpi_1,\dfrac{\tau}{2\big(\phi(\varpi_n,\varpi_{n+1})\big)}\right)}+(1-£^n)}$$

$$*\frac{1}{\frac{\frac{1}{\text{£}^{n+1}}}{P_\phi\left(\varpi_0,\varpi_1,\frac{\boldsymbol{\tau}}{(2)^2\left(\phi(\varpi_{n+1},\varpi_{n+q})\phi(\varpi_{n+1},\varpi_{n+2})\right)}\right)}+(1-\text{£}^{n+1})}$$

$$*\frac{1}{\frac{\frac{1}{\text{£}^{n+2}}}{P_\phi\left(\varpi_0,\varpi_1,\frac{\boldsymbol{\tau}}{(2)^3\left(\phi(\varpi_{n+1},\varpi_{n+q})\phi(\varpi_{n+2},\varpi_{n+q})\phi(\varpi_{n+2},\varpi_{n+3})\right)}\right)}+(1-\text{£}^{n+2})}*\cdots*$$

$$\frac{1}{\frac{\frac{1}{\text{£}^{n+q-2}}}{P_\phi\left(\varpi_0,\varpi_1,\frac{\boldsymbol{\tau}}{(2)^{q-1}\left(\phi(\varpi_{n+1},\varpi_{n+q})\phi(\varpi_{n+2},\varpi_{n+q})\phi(\varpi_{n+3},\varpi_{n+q})\cdots\phi(\varpi_{n+q-2},\varpi_{n+q-1})\right)}\right)}+(1-\text{£}^{n+q-2})}$$

$$*\frac{1}{\frac{\frac{1}{\text{£}^{n+q-1}}}{P_\phi\left(\varpi_0,\varpi_1,\frac{\boldsymbol{\tau}}{(2)^{q-1}\left(\phi(\varpi_{n+1},\varpi_{n+q})\phi(\varpi_{n+2},\varpi_{n+q})\phi(\varpi_{n+3},\varpi_{n+q})\cdots\phi(\varpi_{n+q-1},\varpi_{n+q})\right)}\right)}+(1-\text{£}^{n+q-1})}$$

and

$$Q_\phi(\varpi_n,\varpi_{n+q},\boldsymbol{\tau})$$

$$\leq \text{£}^n Q_\phi\left(\varpi_0,\varpi_1,\frac{\boldsymbol{\tau}}{2(\phi(\varpi_n,\varpi_{n+1}))}\right)\circ \text{£}^{n+1}Q_\phi\left(\varpi_0,\varpi_1,\frac{\boldsymbol{\tau}}{(2)^2\left(\phi(\varpi_{n+1},\varpi_{n+q})\phi(\varpi_{n+1},\varpi_{n+2})\right)}\right)$$

$$\circ\, \text{£}^{n+2}Q_\phi\left(\varpi_0,\varpi_1,\frac{\boldsymbol{\tau}}{(2)^3\left(\phi(\varpi_{n+1},\varpi_{n+q})\phi(\varpi_{n+2},\varpi_{n+q})\phi(\varpi_{n+2},\varpi_{n+3})\right)}\right)\circ\cdots\circ$$

$$\text{£}^{n+q-2}Q_\phi\left(\varpi_0,\varpi_1,\frac{\boldsymbol{\tau}}{(2)^{q-1}\left(\phi(\varpi_{n+1},\varpi_{n+q})\phi(\varpi_{n+2},\varpi_{n+q})\phi(\varpi_{n+3},\varpi_{n+q})\cdots\phi(\varpi_{n+q-2},\varpi_{n+q-1})\right)}\right)$$

$$\circ\, \text{£}^{n+q-1}Q_\phi\left(\varpi_0,\varpi_1,\frac{\boldsymbol{\tau}}{(2)^{q-1}\left(\phi(\varpi_{n+1},\varpi_{n+q})\phi(\varpi_{n+2},\varpi_{n+q})\phi(\varpi_{n+3},\varpi_{n+q})\cdots\phi(\varpi_{n+q-1},\varpi_{n+q})\right)}\right),$$

$$R_\phi(\varpi_n,\varpi_{n+q},\boldsymbol{\tau})$$

$$\leq \text{£}^n R_\phi\left(\varpi_0,\varpi_1,\frac{\boldsymbol{\tau}}{2(\phi(\varpi_n,\varpi_{n+1}))}\right)\circ \text{£}^{n+1}R_\phi\left(\varpi_0,\varpi_1,\frac{\boldsymbol{\tau}}{(2)^2\left(\phi(\varpi_{n+1},\varpi_{n+q})\phi(\varpi_{n+1},\varpi_{n+2})\right)}\right)$$

$$\circ\ £^{n+2}R_\phi\left(\varpi_0,\varpi_1,\frac{\boldsymbol{\tau}}{(2)^3\left(\phi(\varpi_{n+1},\varpi_{n+q})\phi(\varpi_{n+2},\varpi_{n+q})\phi(\varpi_{n+2},\varpi_{n+3})\right)}\right)\circ\cdots\circ$$

$$£^{n+q-2}R_\phi\left(\varpi_0,\varpi_1,\frac{\boldsymbol{\tau}}{(2)^{q-1}\left(\phi(\varpi_{n+1},\varpi_{n+q})\phi(\varpi_{n+2},\varpi_{n+q})\phi(\varpi_{n+3},\varpi_{n+q})\cdots\phi(\varpi_{n+q-2},\varpi_{n+q-1})\right)}\right)$$

$$\circ\ £^{n+q-1}R_\phi\left(\varpi_0,\varpi_1,\frac{\boldsymbol{\tau}}{(2)^{q-1}\left(\phi(\varpi_{n+1},\varpi_{n+q})\phi(\varpi_{n+2},\varpi_{n+q})\phi(\varpi_{n+3},\varpi_{n+q})\cdots\phi(\varpi_{n+q-1},\varpi_{n+q})\right)}\right)$$

Therefore,

$$\lim_{n\to\infty}P_\phi(\varpi_n,\varpi_{n+q},\boldsymbol{\tau})=1*1*\cdots*=1,$$

and

$$\lim_{n\to\infty}Q_\phi(\varpi_n,\varpi_{n+q},\boldsymbol{\tau})=0\circ 0\circ\cdots\circ 0=0,$$

$$\lim_{n\to\infty}R_\phi(\varpi_n,\varpi_{n+q},\boldsymbol{\tau})=0\circ 0\circ\cdots\circ 0=0,$$

i.e., $\{\varpi_n\}$ is a GCS. Therefore, $(\boldsymbol{\Xi},P_\phi,Q_\phi,R_\phi,*,\circ)$ be a G-complete CNMS, there exists $\varpi\in\boldsymbol{\Xi}$.

Now, we show that ϖ is a FP of ξ, utilizing (v), (x) and (xv), we get

$$\frac{1}{P_\phi(\xi\varpi_n,\xi\varpi,\boldsymbol{\tau})}-1\le £\left[\frac{1}{P_\phi(\varpi_n,\varpi,\boldsymbol{\tau})}-1\right]=\frac{£}{P_\phi(\varpi_n,\varpi,\boldsymbol{\tau})}-£$$

$$\Rightarrow\frac{1}{\frac{£}{P_\phi(\varpi_n,\varpi,\boldsymbol{\tau})}+(1-£)}\le P_\phi(\xi\varpi_n,\xi\varpi,\boldsymbol{\tau})$$

Using above inequality, we obtain

$$P_\phi(\varpi,\xi\varpi,\boldsymbol{\tau})\ge P_\phi\left(\varpi,\varpi_{n+1},\frac{\boldsymbol{\tau}}{2\phi(\varpi,\varpi_{n+1})}\right)*P_\phi\left(\varpi_{n+1},\xi\varpi,\frac{\boldsymbol{\tau}}{2\phi(\varpi_{n+1},\xi\varpi)}\right)$$

$$\ge P_\phi\left(\varpi,\varpi_{n+1},\frac{\boldsymbol{\tau}}{2\phi(\varpi,\varpi_{n+1})}\right)*P_\phi\left(\xi\varpi_n,\xi\varpi,\frac{\boldsymbol{\tau}}{2\phi(\varpi_{n+1},\xi\varpi)}\right)$$

$$\ge P_\phi\left(\varpi_n,\varpi_{n+1},\frac{\boldsymbol{\tau}}{2\phi(2\phi(\varpi,\varpi_{n+1}))}\right)*\frac{1}{\frac{£}{P_\phi\left(\varpi_n,\varpi,\frac{\boldsymbol{\tau}}{2\phi(\varpi_{n+1},\xi\varpi)}\right)}+(1-£)}\to 1*1=1$$

as $n\to\infty$, and

$$Q_\phi(\varpi,\xi\varpi,\boldsymbol{\tau})\le P_\phi\left(\varpi,\varpi_{n+1},\frac{\tau}{2\phi(\varpi,\varpi_{n+1})}\right)\circ Q_\phi\left(\varpi_{n+1},\xi\varpi,\frac{\boldsymbol{\tau}}{2\phi(\varpi_{n+1},\xi\varpi)}\right)$$

$$\leq Q_{\phi}\left(\varpi, \varpi_{n+1}, \frac{\boldsymbol{\tau}}{2\phi(\varpi, \varpi_{n+1})}\right) \circ Q_{\phi}\left(\xi\varpi_{n}, \xi\varpi, \frac{\boldsymbol{\tau}}{2\phi(\varpi_{n+1}, \xi\varpi)}\right)$$

$$\leq Q_{\phi}\left(\varpi_{n}, \varpi_{n+1}, \frac{\boldsymbol{\tau}}{2\phi(\varpi, \varpi_{n+1})}\right) \circ £Q_{\phi}\left(\varpi_{n}, \varpi, \frac{\boldsymbol{\tau}}{2\phi(\varpi_{n+1}, \xi\varpi)}\right) \to 0 \circ 0 = 0 \text{ as } n \to \infty,$$

$$R_{\phi}(\varpi, \xi\varpi, \boldsymbol{\tau}) \leq R_{\phi}\left(\varpi, \varpi_{n+1}, \frac{\boldsymbol{\tau}}{2\phi(\varpi, \varpi_{n+1})}\right) \circ R_{\phi}\left(\varpi_{n+1}, \xi\varpi, \frac{\boldsymbol{\tau}}{2\phi(\varpi_{n+1}, \xi\varpi)}\right)$$

$$\leq R_{\phi}\left(\varpi, \varpi_{n+1}, \frac{\boldsymbol{\tau}}{2\phi(\varpi, \varpi_{n+1})}\right) \circ R_{\phi}\left(\xi\varpi_{n}, \xi\varpi, \frac{\boldsymbol{\tau}}{2\phi(\varpi_{n+1}, \xi\varpi)}\right)$$

$$\leq R_{\phi}\left(\varpi_{n}, \varpi_{n+1}, \frac{\boldsymbol{\tau}}{2\phi(\varpi, \varpi_{n+1})}\right) \circ £R_{\phi}\left(\varpi_{n}, \varpi, \frac{\boldsymbol{\tau}}{2\phi(\varpi_{n+1}, \xi\varpi)}\right) \to 0 \circ 0 = 0 \text{ as } n \to \infty.$$

Hence, $\xi\varpi = \varpi$, a FP.
Uniqueness: Assume $\xi c = c$ for some $c \in \Xi$, then

$$\frac{1}{P_{\phi}(\varpi, c, \boldsymbol{\tau})} - 1 = \frac{1}{P_{\phi}(\xi\varpi, \xi c, \boldsymbol{\tau})} - 1$$

$$\leq £\left[\frac{1}{P_{\phi}(\varpi, c, \boldsymbol{\tau})} - 1\right] < \frac{1}{P_{\phi}(\varpi, c, \boldsymbol{\tau})} - 1$$

a contradiction, and

$$Q_{\phi}(\varpi, c, \boldsymbol{\tau}) = Q_{\phi}(\xi\varpi, \xi c, \boldsymbol{\tau}) \leq £Q_{\phi}(\varpi, c, \boldsymbol{\tau}) < Q_{\phi}(\varpi, c, \boldsymbol{\tau}),$$

$$R_{\phi}(\varpi, c, \boldsymbol{\tau}) = R_{\phi}(\xi\varpi, \xi c, \boldsymbol{\tau}) \leq £R_{\phi}(\varpi, c, \boldsymbol{\tau}) < R_{\phi}(\varpi, c, \boldsymbol{\tau}),$$

are contradictions.
Therefore, we must have $P_{\phi}(\varpi, c, \boldsymbol{\tau}) = 1, Q_{\phi}(\varpi, c, \boldsymbol{\tau}) = 0$ and $R_{\phi}(\varpi, c, \boldsymbol{\tau}) = 0$, that is $\varpi = c$.

Example 3.3. Suppose $\boldsymbol{\Xi} = [0,1]$. Define ϕ by

$$\phi(\varpi, \nu) = \begin{cases} 1 & \text{if } \varpi = \nu, \\ \dfrac{1 + \max\{\varpi, \nu\}}{\min\{\varpi, \nu\}} & \text{if } \varpi \neq \nu \neq 0. \end{cases}$$

Also, define

$$P_{\phi}(\varpi, \nu, \boldsymbol{\tau}) = \frac{\boldsymbol{\tau}}{\boldsymbol{\tau} + \max\{\varpi, \nu\}}$$

$$Q_{\phi}(\varpi, \nu, \boldsymbol{\tau}) = \frac{\max\{\varpi, \nu\}}{\boldsymbol{\tau} + \max\{\varpi, \nu\}},$$

and

$$R_{\phi}(\varpi, \nu, \boldsymbol{\tau}) = \frac{\max\{\varpi, \nu\}}{\boldsymbol{\tau}},$$

with $\Upsilon * \varrho = \Upsilon . \varrho$ and $\Upsilon \circ \varrho = \max\{\Upsilon, \varrho\}$. Then $\left(\boldsymbol{\Xi}, P_{\phi}, Q_{\phi}, R_{\phi}, *, \circ\right)$ is a G-complete CNMLS.

Observe that $\lim_{\boldsymbol{\tau}\to\infty} P_\phi(\varpi,\nu,\boldsymbol{\tau}) = 1$, $\lim_{\boldsymbol{\tau}\to\infty} Q_\phi(\varpi,\nu,\boldsymbol{\tau}) = 0$ and $\lim_{\boldsymbol{\tau}\to\infty} R_\phi(\varpi,\nu,\boldsymbol{\tau}) = 0$, satisfied. Define $\xi: \boldsymbol{\Xi} \to \boldsymbol{\Xi}$ by

$$\xi(\varpi) = \frac{\varpi}{9}$$

Then,

$$P_\phi(\xi\varpi,\xi\nu,£\boldsymbol{\tau}) \geq P_\phi(\varpi,\nu,\boldsymbol{\tau}),$$

$$Q_\phi(\xi\varpi,\xi\nu,£\boldsymbol{\tau}) \leq Q_\phi(\varpi,\nu,\boldsymbol{\tau}) \text{ and } R_\phi(\xi\varpi,\xi\nu,£\boldsymbol{\tau}) \leq R_\phi(\varpi,\nu,\boldsymbol{\tau})$$

are satisfied for $£ \in \left[\frac{1}{2}, 1\right)$, as we can see that Figure 1 shows that $P_\phi(\xi\varpi,\xi\nu,£\boldsymbol{\tau}) \geq P_\phi(\varpi,\nu,\boldsymbol{\tau})$, Figure 2 shows that $Q_\phi(\xi\varpi,\xi\nu,£\boldsymbol{\tau}) \leq Q_\phi(\varpi,\nu,\boldsymbol{\tau})$ and Figure 3 shows that $R_\phi(\xi\varpi,\xi\nu,£\boldsymbol{\tau}) \leq R_\phi(\varpi,\nu,\boldsymbol{\tau})$.

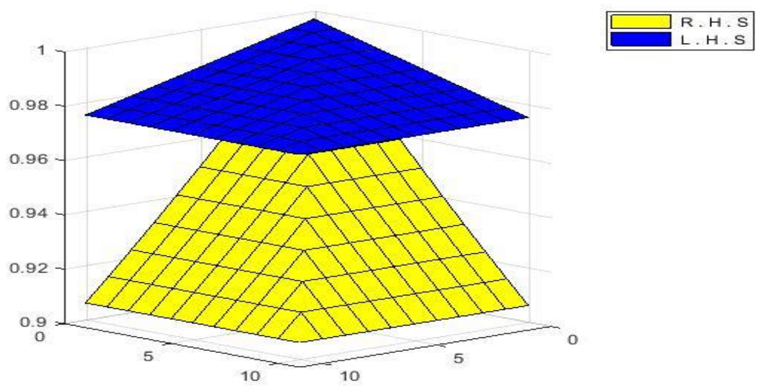

Figure 1. Shows the graphical behavior of $P_\phi(\xi\varpi,\xi\nu,£\boldsymbol{\tau}) \geq P_\phi(\varpi,\nu,\boldsymbol{\tau})$, when $\boldsymbol{\tau} = 10$ and $£ = 0.5$.

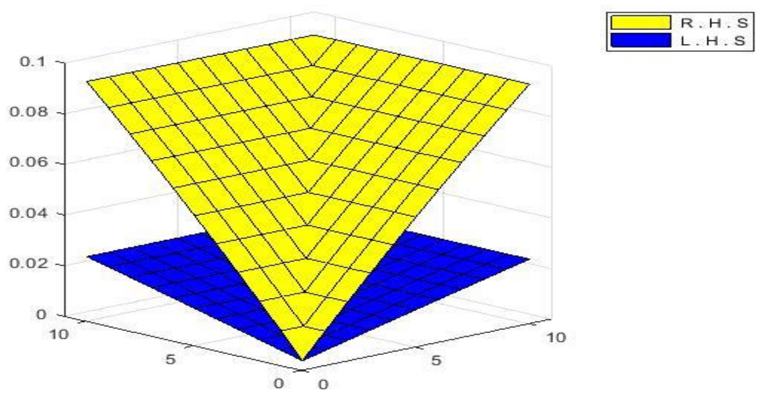

Figure 2. Shows the graphical behavior of $Q_\phi(\xi\varpi,\xi\nu,£\boldsymbol{\tau}) \leq Q_\phi(\varpi,\nu,\boldsymbol{\tau})$, when $\boldsymbol{\tau} = 10$ and $£ = 0.5$.

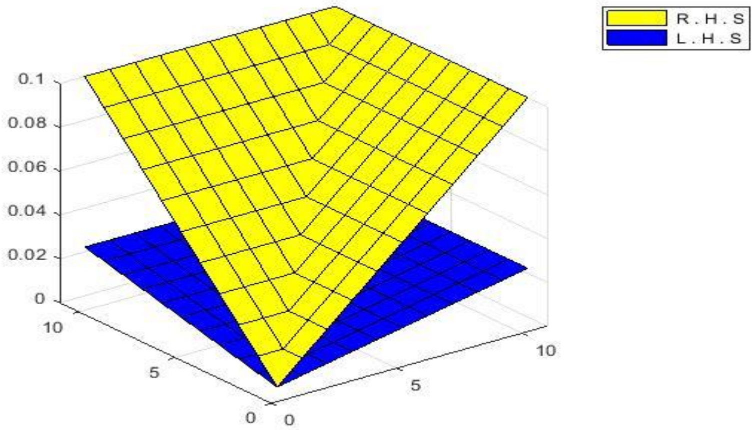

Figure 3. Shows the graphical behavior of $R_\phi(\xi\varpi,\xi\nu,£\boldsymbol{\tau}) \leq R_\phi(\varpi,\nu,\boldsymbol{\tau})$, when $\boldsymbol{\tau}=10$ and $£=0.5$.

Also,

$$\frac{1}{P_\phi(\xi\varpi,\xi\nu,\boldsymbol{\tau})}-1 \leq £\left[\frac{1}{P_\phi(\varpi,\nu,\boldsymbol{\tau})}-1\right] \text{ and}$$

$$Q_\phi(\xi\varpi,\xi\nu,\boldsymbol{\tau}) \leq £Q_\phi(\varpi,\nu,\boldsymbol{\tau}), \text{ and } R_\phi(\xi\varpi,\xi\nu,\boldsymbol{\tau}) \leq £R_\phi(\varpi,\nu,\boldsymbol{\tau}),$$

are satisfied for $£ \in \left[\frac{1}{2},1\right)$, as we can see that Figure 4 shows that $\frac{1}{P_\phi(\xi\varpi,\xi\nu,\boldsymbol{\tau})}-1 \leq £\left[\frac{1}{P_\phi(\varpi,\nu,\boldsymbol{\tau})}-1\right]$, Figure 5 shows that $Q_\phi(\xi\varpi,\xi\nu,\boldsymbol{\tau}) \leq £Q_\phi(\varpi,\nu,\boldsymbol{\tau})$ and Figure 6 shows that $R_\phi(\xi\varpi,\xi\nu,\boldsymbol{\tau}) \leq £R_\phi(\varpi,\nu,\boldsymbol{\tau})$.

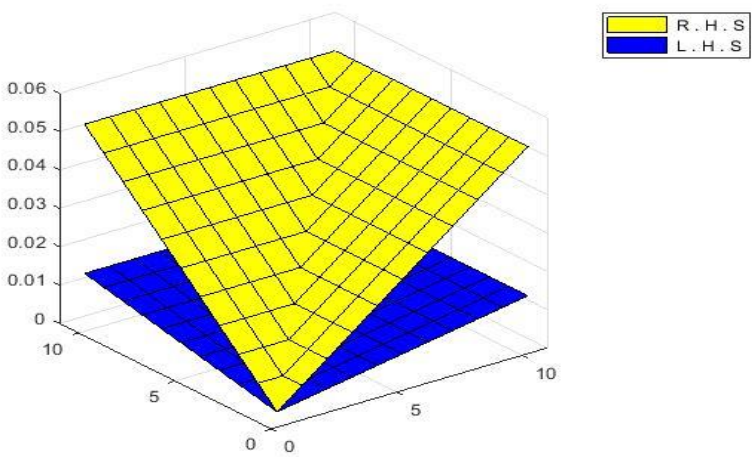

Figure 4. Shows the graphical behavior of $\frac{1}{P_\phi(\xi\varpi,\xi\nu,\boldsymbol{\tau})}-1 \leq £\left[\frac{1}{P_\phi(\varpi,\nu,\boldsymbol{\tau})}-1\right]$, when $\boldsymbol{\tau}=10$ and $£=0.5$.

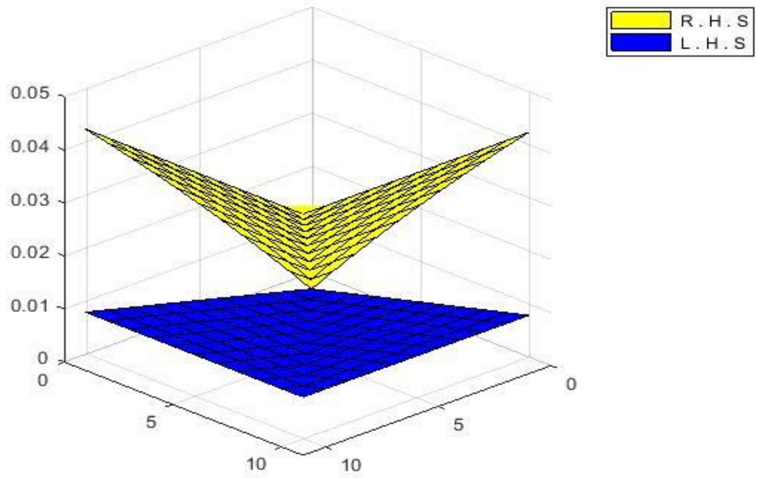

Figure 5. Shows the graphical behavior of $Q_{\phi}(\xi\varpi, \xi\nu, \boldsymbol{\tau}) \leq £Q_{\phi}(\varpi, \nu, \boldsymbol{\tau})$, when $\boldsymbol{\tau} = 10$ and $£ = 0.5$.

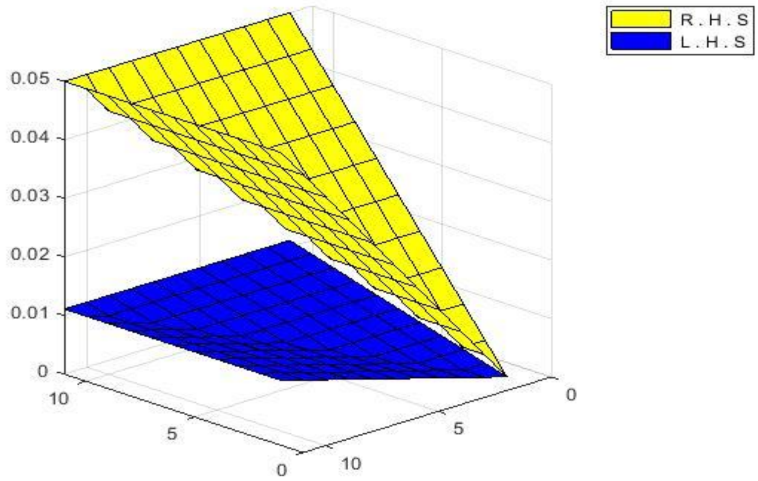

Figure 6 . Shows the graphical behavior of $R_{\phi}(\xi\varpi, \xi\nu, \boldsymbol{\tau}) \leq £R_{\phi}(\varpi, \nu, \boldsymbol{\tau})$, when $\boldsymbol{\tau} = 10$ and $£ = 0.5$.

We can easily see that $\lim_{n\to\infty} \phi(\varpi_n, \nu)$ and $\lim_{n\to\infty} \phi(\nu, \varpi_n)$ exists and finite. Observe that all circumstances of Theorems 3.1 and 3.2 are fulfilled, and 0 is a unique FP of ξ as we can see in the Figure 7.

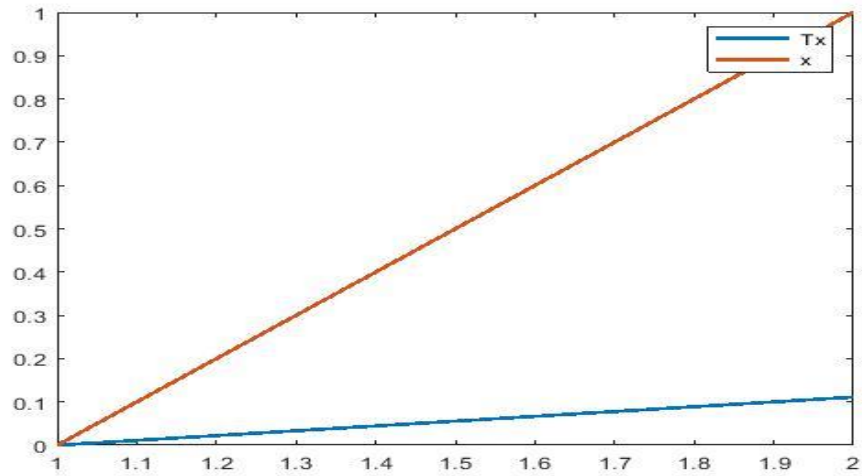

Figure 7. Shows that the fixed point of ξ is 0 and is unique.

4. Application

Suppose $\Xi = C([\mathrm{c},\mathrm{a}],\mathbb{R})$ be the set of real valued continuous functions defined on $[\mathrm{c},\mathrm{a}]$. Suppose the integral equation:

$$\varpi(\tau) = \Lambda(\tau) + \delta\int_{\mathrm{c}}^{\mathrm{a}} Л(\tau,\upsilon)\varpi(\tau)d\upsilon \text{ for } \tau,\upsilon \in [\mathrm{c},\mathrm{a}] \tag{11}$$

where $\delta > 0, \Lambda(\upsilon)$ is a function of $\upsilon: \upsilon \in [\mathrm{c},\mathrm{a}]$ and $Л: C([\mathrm{c},\mathrm{a}] \times \mathbb{R}) \to \mathbb{R}^+$. Define P and Q by

$$P(\varpi(\tau),\nu(\tau),\hat{\mathrm{r}}) = \sup_{\tau\in[\mathrm{c},\mathrm{a}]} \frac{\hat{\mathrm{r}}}{\hat{\mathrm{r}}+|\varpi(\tau)-\nu(\tau)|^2} \text{ for all } \varpi,\nu \in \mathfrak{C} \text{ and } \hat{\mathrm{r}} > 0,$$

$$Q(\varpi(\tau),\nu(\tau),\hat{\mathrm{r}}) = 1 - \sup_{\tau\in[\mathrm{c},\mathrm{a}]} \frac{\hat{\mathrm{r}}}{\hat{\mathrm{r}}+|\varpi(\tau)-\nu(\tau)|^2} \text{ for all } \varpi,\nu \in \mathfrak{C} \text{ and } \hat{\mathrm{r}} > 0,$$

and

$$R(\varpi(\tau),\nu(\tau),\hat{\mathrm{r}}) = \sup_{\tau\in[\mathrm{c},\mathrm{a}]} \frac{|\varpi(\tau)-\nu(\tau)|^2}{\hat{\mathrm{r}}} \text{ for all } \varpi,\nu \in \mathfrak{C} \text{ and } \hat{\mathrm{r}} > 0,$$

with continuous t-norm and continuous t-conorm define by $\hat{\mathrm{e}} * \bar{\mathrm{a}} = \hat{\mathrm{e}}.\bar{\mathrm{a}}$ and $\hat{\mathrm{e}} \circ \bar{\mathrm{a}} = \max\{\hat{\mathrm{e}},\bar{\mathrm{a}}\}$. Define $\xi,\Gamma: \mathfrak{C} \times \mathfrak{C} \to [1,\infty)$ as

$$\xi(\varpi,\nu) = \begin{cases} 1 & \text{if } \varpi = \nu \\ \dfrac{1+\max\{\varpi,\nu\}}{\min\{\varpi,\nu\}} & \text{if } \varpi \neq \nu \neq 0 \end{cases};$$

Then $(\boldsymbol{\Xi},\boldsymbol{P},\boldsymbol{Q},\boldsymbol{R},*,\circ)$ be a complete controlled neutrosophic metric-like space.

Suppose that

$|Л(\tau,\upsilon)\varpi(\tau) - Л(\tau,\upsilon)\nu(\tau)| \leq |\varpi(\tau) - \nu(\tau)|$ for $\varpi,\nu \in \mathfrak{C}$, $\theta \in (0,1)$ and $\forall\, \tau,\upsilon \in [\mathrm{c},\mathrm{a}]$. Also, let $Л(\tau,\upsilon)\left(\delta\int_{\mathrm{c}}^{\mathrm{a}} d\upsilon\right)^2 \leq \theta < 1$. Then integral Eq (11) has a unique solution.

Proof. Define $\xi: \mathfrak{C} \to \mathfrak{C}$ by

$$\xi\varpi(\tau) = \Lambda(\tau) + \delta \int_{c}^{a} Л(\tau, \upsilon)c(\tau)d\upsilon \text{ for all } \tau, \upsilon \in [c, a]$$

Now for all $\varpi, \nu \in \mathfrak{C}$, we deduce

$$\begin{aligned}
P(\xi\varpi(\tau), \xi\nu(\tau), \theta\,\hat{r}) &= \sup_{\tau\in[c,a]} \frac{\theta\hat{r}}{\theta\hat{r} + |\xi\varpi(\tau) - \xi\nu(\tau)|^2} \\
&= \sup_{\tau\in[c,a]} \frac{\theta\hat{r}}{\theta\hat{r} + \left|\Lambda(\tau) + \delta \int_{c}^{a} Л(\tau,\upsilon)c(\tau)d\upsilon - \Lambda(\tau) - \delta \int_{c}^{a} Л(\tau,\upsilon)c(\tau)d\upsilon\right|^2} \\
&= \sup_{\tau\in[c,a]} \frac{\theta\hat{r}}{\theta\hat{r} + \left|\delta \int_{c}^{a} Л(\tau,\upsilon)c(\tau)d\upsilon - \delta \int_{c}^{a} Л(\tau,\upsilon)c(\tau)d\upsilon\right|^2} \\
&= \sup_{\tau\in[c,a]} \frac{\theta\hat{r}}{\theta\hat{r} + |Л(\tau,\upsilon)\varpi(\tau) - Л(\tau,\upsilon)\nu(\tau)|^2\left(\delta \int_{c}^{a} d\upsilon\right)^2} \\
&\geq \sup_{\tau\in[c,a]} \frac{\hat{r}}{\hat{r} + |\varpi(\tau) - \nu(\tau)|^2} \\
&\geq P(\varpi(\tau), \nu(\tau), \hat{r}),
\end{aligned}$$

$$\begin{aligned}
Q(\xi\varpi(\tau), \xi\nu(\tau), \theta\,\hat{r}) &= 1 - \sup_{\tau\in[c,a]} \frac{\theta\hat{r}}{\theta\hat{r} + |\xi\varpi(\tau) - \xi\nu(\tau)|^2} \\
&= 1 - \sup_{\tau\in[c,a]} \frac{\theta\hat{r}}{\theta\hat{r} + \left|\Lambda(\tau) + \delta \int_{c}^{a} Л(\tau,\upsilon)c(\tau)d\upsilon - \Lambda(\tau) - \delta \int_{c}^{a} Л(\tau,\upsilon)c(\tau)d\upsilon\right|^2} \\
&= 1 - \sup_{\tau\in[c,a]} \frac{\theta\hat{r}}{\theta\hat{r} + \left|\delta \int_{c}^{a} Л(\tau,\upsilon)c(\tau)d\upsilon - \delta \int_{c}^{a} Л(\tau,\upsilon)c(\tau)d\upsilon\right|^2} \\
&= 1 - \sup_{\tau\in[c,a]} \frac{\theta\hat{r}}{\theta\hat{r} + |Л(\tau,\upsilon)\varpi(\tau) - Л(\tau,\upsilon)\nu(\tau)|^2\left(\delta \int_{c}^{a} d\upsilon\right)^2} \\
&\leq 1 - \sup_{\tau\in[c,a]} \frac{\hat{r}}{\hat{r} + |\varpi(\tau) - \nu(\tau)|^2} \\
&\leq Q(\varpi(\tau), \nu(\tau), \hat{r}),
\end{aligned}$$

and

$$\begin{aligned}
R(\xi\varpi(\tau), \xi\nu(\tau), \theta\,\hat{r}) &= \sup_{\tau\in[c,a]} \frac{|\xi\varpi(\tau) - \xi\nu(\tau)|^2}{\theta\hat{r}} \\
&= \sup_{\tau\in[c,a]} \frac{\left|\Lambda(\tau) + \delta \int_{c}^{a} Л(\tau,\upsilon)c(\tau)d\upsilon - \Lambda(\tau) - \delta \int_{c}^{a} Л(\tau,\upsilon)c(\tau)d\upsilon\right|^2}{\theta\hat{r}} \\
&= \sup_{\tau\in[c,a]} \frac{\left|\delta \int_{c}^{a} Л(\tau,\upsilon)c(\tau)d\upsilon - \delta \int_{c}^{a} Л(\tau,\upsilon)c(\tau)d\upsilon\right|^2}{\theta\hat{r}}
\end{aligned}$$

$$= \sup_{\tau\in[c,a]} \frac{|Л(\tau,\upsilon)\varpi(\tau) - Л(\tau,\upsilon)\nu(\tau)|^2\left(\delta \int_c^a d\upsilon\right)^2}{\theta\hat{r}}$$

$$\leq \sup_{\tau\in[c,a]} \frac{|\varpi(\tau) - \nu(\tau)|^2}{\hat{r}}$$

$$\leq R(\varpi(\tau), \nu(\tau), \hat{r}).$$

As a result, all of the conditions of Theorem 3.1 are satisfied and operator ξ has a unique fixed point. This indicates that an integral Eq (11) has a unique solution.

5. Conclusions

In this manuscript, we introduced the notion of controlled neutrosophic metric-like spaces as a generalization of a neutrosophic metric space and established some new type of fixed point theorems for contraction mappings in this new setting. Moreover, we provided the non-trivial examples with graphical analysis to demonstrate the viability of the proposed methods. Also, our structure is more general than the controlled fuzzy metric space and fuzzy metric like space and neutrosophic metric space. Also, our results and notions expand and generalize a number of previously published results.

Conflict of interest

The authors declare no conflict of interest.

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