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Research article

A comparative inference on reliability estimation for a multi-component stress-strength model under power Lomax distribution with applications

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Abstract: In this article, reliability estimation for a system of multi-component stress-strength model is considered. Working under progressively censored samples is of great advantage over complete and usual censoring samples, therefore Type-II right progressive censored sample is selected. The lifetime of the components and the stress and strength components are following the power Lomax distribution. Consequently, the problem of point and interval estimation has been studied from different points of view. The maximum likelihood estimate and the maximum product spacing of reliability are evaluated. Also approximate confidence intervals are constructed using the Fisher information matrix. For the traditional methods, bootstrap confidence intervals are calculated. Bayesian estimation is obtained under the squared error and linear-exponential loss functions, where the numerical techniques such as Newton-Raphson and the Markov Chain Monte Carlo algorithm are implemented. For dependability, the largest posterior density credible intervals are generated. Simulations are used to compare the results of the proposed estimation methods, where it shows that the Bayesian estimation method of the reliability function is significantly better than the other methods. Finally, a real data of the water capacity of the Shasta reservoir is examined for illustration.

Keywords: reliability analysis; power Lomax; multi-component stress-strength model; maximum likelihood; maximum product spacing; Bayes estimation; Markov Chain Monte Carlo; highest posterior density
Mathematics Subject Classification: 62N05, 62N02, 62N01, 62H12, 62F15, 62F10, 62F40

1. Introduction

The inference of the stress-strength parameter $R = P(Y < X)$ is an interesting topic in reliability analysis. The stress Y and the strength X are considered to be random variables. Let X represents the barrier's fire resistance, and Y represents the severity of the fire to which the barrier is exposed. If a unit's strength exceeds the stress applied to it, it works in the simplest stress-strength paradigm. Several authors have studied the reliability estimate of a single component stress-strength version using various lifetime distributions for the stress-strength random variate. The reliability and estimation of X and Y have been investigated under various distributional assumptions. There is a large literature on estimating R for various stress-strength distributions under various situations. For more examples see: Ghitany et al. [1], Chen and Cheng [2], Rezaei et al. [3] and Sharma [4], where the inference of reliability was performed using complete data as examples. The estimation of R under various filtering schemes was established by Genç [5], Krishna et al. [6] and Babayi and Khorram [7]. The inference for R based on upper or lower record data was introduced by Nadar and Kızılaslan [8], Tripathi et al. [9] and Asgharzadeh et al. [10]. Under ranked set sampling data, Akgül and Şenoğlu [11], Akgül et al. [12,13] and Safariyan et al. [14] investigated the R estimator. Al-Babtain et al. [15] introduced R estimation of stress-strength model for power-modified Lindley distribution. Sabry et al. [16] obtained stress-strength model and reliability estimation for extension of the exponential distribution. Yousef and Almetwally [17] obtained multi stress-strength R based on progressive first failure for Kumaraswamy model. Rezaei et al. [18] discussed estimation of $P[Y < X]$ for generalized Pareto distribution. Kundu and Gupta [19] estimated $P[Y < X]$ for Weibull distributions. Jose [20] discussed Estimation of stress-strength reliability using discrete phase type distribution. Almetwally et al. [21] discussed optimal plan of multi-stress-strength reliability Bayesian and non-Bayesian methods for the alpha power exponential model using progressive first failure. Kotz et al. [22], added a significant literature on the topic up to the year 2003, it can be consulted for more information. In all the previous studies there was a single component in the model, while in our proposed paper we are considering a multi-component system.

A multi-component system refers to a system with more than one component. This system, which is made up of k independent and identical strength components, operates if s $(1 \leq s \leq k)$ or more of the components operate at the same time. The system is subjected to stress Y in its operating environment. The component's strengths, or the minimal stress required to create failure, are random variables with a distribution function that is independent and identically distributed. The s-out-of-k: G system corresponds to this model. The s-out-of-k system is used in a variety of industrial and military systems.

In a multi-component system with k components, each component has independent and identically distributed (iid) random strengths $X_1, X_2, ..., X_k$ and each component is stressed randomly Y. The system would survive if and only if the strengths were greater than the stresses by at least s out of $k, (1 \leq s \leq k)$. Let $Y, X_1, X_2, ..., X_k$ be independent random variables, with $G(y)$ being the continuous cumulative density function (cdf) of Y and $F(x)$ being the common continuous cdf of $X_1, X_2, ..., X_k$. Bhattacharyya and Johnson [23] introduced the reliability in a multi-component stress-strength (MSS) model, which is given by

$$\begin{aligned} R_{s,k} &= P\left[at\ \ least\ s\ of\ the\ \ (X_1, X_2, ..., X_k)\, exceed\ Y\right] \\ &= \sum_{i=s}^{k} \binom{k}{i} \int_{-\infty}^{\infty} [1 - F(y)]^i \, [F(y)]^{k-i} \ dG(y). \end{aligned} \tag{1.1}$$

Although complete sample cases have been used to derive statistical inference on multi-component stress-strength models in the reliability literature, this subject has received little attention under censored data, particularly progressive Type-II censored sample. The following sources are closely relevant to the structure of our research.

Kohansal [24] recently discussed estimating dependability in a multi-component stress-strength model, where data are observed using progressive Type-II censoring for the Kumaraswamy distribution. For the general class of inverse exponentiated distributions, Kizilaslan [25] studied the classical and Bayesian estimate of reliability in a multi-component stress-strength model based on complete data. Gunasekera [26] evaluated the reliability of a multi-component system using progressively Type-II censored sample with uniformly random removals. When the common parameter is known, the author obtains different interval inferences. Estimation is evaluated in the cases in which the common parameter is either known or unknown. Gunasekera [26] performed inference for known common parameter cases. In the Bayesian situation, we develop a uniformly-minimum-variance-unbiased estimator and use the Tierney-Kadane approximation technique. It should be noted that Gunasekera [26] did not consider the case of an unknown common parameter.

Progressive Type-II censoring is commonly used in life-testing trials to analyze data under time and cost restrictions. Type-I and Type-II are the two most common censoring techniques. When using Type-I censoring, a test is almost finished at a specific time, however when using Type-II censoring, the test is finished after a certain amount of failure times have been logged. These censoring systems prevent live units from being removed between studies. The following is a description of the progressive Type II censoring scheme. On the life test, the experimenter first arranges N independent and identical units. When the first failure occurs, say at $t_{(1)}$, r_1 units are eliminated at random from the remaining $N - 1$ surviving units. When the second failure occurs at $t_{(2)}$, r_2 units are eliminated at random from the remaining $N - r_1 - 2$ surviving units. When the n^{th} failure occurs at time t_n, the experiment ends, and the $r_n = N - n - \sum_{i=1}^{n-1} r_i$ surviving units are eliminated from the test. For various uses of this censoring in lifespan analysis, see Balakrishnan and Aggarwala [27] and Balakrishnan and Cramer [28]. For some useful implications on this censoring scheme, see Raqab and Madi [29], Wu et al. [30] and Rastogi and Tripathi [31].

In this study, we attempt to estimate $R_{s,k}$ when the underlying distribution is the power Lomax distribution (POLO). According to Rady et al. [32], the power Lomax POLO distribution is obtained by the power transformation $X = Y^{\frac{1}{\beta}}$, where the random variable X has pdf in Eq (1.2). The pdf of the POLO distribution is defined by

$$f(x) = \frac{\alpha\,\beta}{\lambda} x^{\beta-1}(1 + \frac{x^{\beta}}{\lambda})^{-\alpha-1}. \tag{1.2}$$

The corresponding cumulative distribution function (CDF) and survival function of POLO distribution are given by

$$F(x) = 1 - (1 + \frac{x^{\beta}}{\lambda})^{-\alpha} \tag{1.3}$$

and

$$S(x) = (1 + \frac{x^{\beta}}{\lambda})^{-\alpha}, \tag{1.4}$$

where α and β are shape parameters and λ is a scale parameter.

Due to physical constraints in these sectors, such as limited power supply, maintenance resources, and/or system design life, we suggest the POLO distribution for modeling reliability and life-testing data sets. The survival function studies the possibility of breakdowns of organisms, technical units, and other systems failing beyond a certain point in time. The hazard rate is used to measure a unit's lifetime over the duration of its lifetime distribution. The hazard rate (HRF) is a significant criterion for determining lifetime distributions since it measures the probability of failing or dying based on the age attained.

When both stress and strength follow the POLO distribution, the major goal of this study is to estimate $R_{s,k}$ using both classical and Bayesian techniques. This paper studies the feasibility of estimating $R_{s,k}$ by using maximum likelihood under progressive Type-II censoring and constructing an asymptotic confidence interval when all the parameters are unknown, these are discussed in Section 1. In Section 2, the maximum product of spacing methodology is used to obtain the explicit estimator of $R_{s,k}$. Section 3 includes the construction of boot-p and boot-t confidence intervals. In Section 4, the Bayes estimates are determined under a squared error loss function (SELF) and a linear-exponential loss function (LINEX) using gamma informative priors. The Markov-Chain Monte-Carlo (MCMC) method is obtained for Bayesian computation. Also, the Bayesian credible and HPD credible intervals are constructed. A simulation study and real data set are analyzed in Sections 5 and 6 respectively. Finally, conclusion is presented in Section 7.

2. Estimation of $R_{s,k}$ when α is unknown

Let $X_1, X_2, ..., X_k$ and Y be an iid random samples taken from the general class of power Lomax POLO$(\alpha_1, \beta, \lambda)$ and POLO$(\alpha_2, \beta, \lambda)$ distributions, respectively, with a common shape parameter β and scale parameter λ. Under this setup, the reliability $R_{s,k}$ is as follows

$$R_{s,k} = \sum_{i=s}^{k} \binom{k}{i} \int_0^1 \left[(1+\frac{y^\beta}{\lambda})^{-\alpha_1}\right]^i \left[1-(1+\frac{y^\beta}{\lambda})^{-\alpha_1}\right]^{k-i} \frac{\alpha_2\,\beta}{\lambda} y^{\beta-1}(1+\frac{y^\beta}{\lambda})^{-\alpha_2} d\,y, \tag{2.1}$$

where $u = (1+\frac{y^\beta}{\lambda})^{-\alpha_2}$.

$$\begin{aligned} R_{s,k} &= \sum_{i=s}^{k} \binom{k}{i} \int_0^1 u^{\frac{i\,\alpha_1}{\alpha_2}} \left[1-u^{\frac{\alpha_1}{\alpha_2}}\right]^{k-i} d\,u \\ &= \sum_{i=s}^{k} \binom{k}{i} \frac{\alpha_2}{\alpha_1} B(\frac{\alpha_2}{\alpha_1}+i, k-i+1), \end{aligned} \tag{2.2}$$

since $B(.,.)$ is the standard Beta function and k and i are integers.

2.1. Maximum likelihood estimation of $R_{s,k}$

To achieve the desired MLE of $R_{s,k}$, the MLEs of $\alpha_1, \alpha_2, \beta$ and λ are evaluated assuming the progressive Type-II censoring scheme. Suppose N systems are employed in a life-testing experiment, with a progressive Type-II censored sample $\{X_{i1}, X_{i2}, ..., X_{ik}\}$, $i = 1, 2, \ldots, n$ is generated from the general class of power Lomax POLO$(\alpha_1, \beta, \lambda)$, where the progressive censoring scheme is

$\{K, k, r_1, \ldots, r_k\}$. Consider a progressively censored sample $\{Y_1, Y_2, ..., Y_n\}$ obtained from another broad class of power Lomax POLO$(\alpha_2, \beta, \lambda)$ using the censoring scheme $\{N, n, S_1, \ldots, S_n\}$.

Then likelihood function of $\alpha_1, \alpha_2, \beta$ and λ is obtained as

$$L(\alpha_1, \alpha_2, \beta, \lambda) = c_1 \prod_{i=1}^{n} \left[c_2 \prod_{j=1}^{k} f(x_{ij})[1 - F(x_{ij})]^{r_j} \right] f(y_i)[1 - F(y_i)]^{S_i}, \tag{2.3}$$

where the constants $c_1$and c_2 are given by

$$\begin{aligned} c_1 &= N(N - S_1 - 1)..........(N - S_1 - .... - S_{n-1} - n + 1), \\ c_2 &= K(K - r_1 - 1)..........(K - r_1 - .... - r_{k-1} - k + 1). \end{aligned}$$

As a result, the likelihood function is expressed as:

$$\begin{aligned} L(data|\alpha_1, \alpha_2, \beta, \lambda) &= c_1 c_2^n \prod_{i=1}^{n} \left[\prod_{j=1}^{k} \frac{\alpha_1 \beta}{\lambda} x_{ij}^{\beta-1} (1 + \frac{x_{ij}^{\beta}}{\lambda})^{-\alpha_1 - 1} [1 + \frac{x_{ij}^{\beta}}{\lambda}]^{-r_j \alpha_1} \right] \\ &\quad \prod_{i=1}^{n} \frac{\alpha_2 \beta}{\lambda} y_i^{\beta-1} (1 + \frac{y_i^{\beta}}{\lambda})^{-\alpha_2 - 1} [1 + \frac{y_i^{\beta}}{\lambda}]^{-S_i \alpha_2}. \end{aligned} \tag{2.4}$$

The log-likelihood function is as follows:

$$\begin{aligned} \ell(\alpha_1, \alpha_2, \beta, \lambda|data) &= nk \ln(\alpha_1) + nk \ln(\alpha_2) + n(k+1) \ln(\beta) + n(k+1) \ln(\lambda) \\ &\quad + (\beta - 1) \sum_{i=1}^{n} \sum_{j=1}^{k} \ln(x_{ij}) + (\beta - 1) \sum_{i=1}^{n} \ln(y_i) \\ &\quad - \sum_{i=1}^{n} \sum_{j=1}^{k} ((r_j + 1)\, \alpha_1 + 1) \ln \left[1 + \frac{x_{ij}^{\beta}}{\lambda} \right] - \sum_{i=1}^{n} ((S_i + 1)\, \alpha_2 + 1) \ln \left[1 + \frac{y_i^{\beta}}{\lambda} \right]. \end{aligned} \tag{2.5}$$

The likelihood equations are constructed with respect to the variable of interest by calculating the derivatives of Eq (1.5) in the following forms

$$\frac{\partial \ell}{\partial \alpha_1} = \frac{nk}{\alpha_1} - \sum_{i=1}^{n} \sum_{j=1}^{k} (r_j + 1) \ln \left[1 + \frac{x_{ij}^{\beta}}{\lambda} \right] = 0, \tag{2.6}$$

$$\frac{\partial \ell}{\partial \alpha_2} = \frac{n}{\alpha_2} - \sum_{i=1}^{n} (S_i + 1) \ln \left[1 + \frac{y_i^{\beta}}{\lambda} \right] = 0, \tag{2.7}$$

$$\begin{aligned} \frac{\partial \ell}{\partial \beta} &= \frac{n(k+1)}{\beta} + \sum_{i=1}^{n} \sum_{j=1}^{k} \ln(x_{ij}) + \sum_{i=1}^{n} \ln(y_i) \\ &- \sum_{i=1}^{n} \sum_{j=1}^{k} ((r_j + 1)\, \alpha_1 + 1) \frac{x_{ij}^{\beta} \ln(x_{ij})}{\lambda \left[1 + \frac{x_{ij}^{\beta}}{\lambda} \right]} - \sum_{i=1}^{n} ((S_i + 1)\, \alpha_2 + 1) \frac{y_i^{\beta} \ln(y_i)}{\lambda \left[1 + \frac{y_i^{\beta}}{\lambda} \right]} = 0 \end{aligned} \tag{2.8}$$

and

$$\frac{\partial \ell}{\partial \lambda} = \frac{n(k+1)}{\lambda} + \sum_{i=1}^{n}\sum_{j=1}^{k}((r_j+1)\,\alpha_1+1)\frac{x_{ij}^{\beta}}{\lambda^2\left[1+\frac{x_{ij}^{\beta}}{\lambda}\right]}$$

$$\sum_{i=1}^{n}((S_i+1)\,\alpha_2+1)\frac{y_i^{\beta}}{\lambda^2\left[1+\frac{y_i^{\beta}}{\lambda}\right]} = 0. \tag{2.9}$$

The parameters α_1 and α_2 MLEs are derived from the solutions of Eqs (1.6) and (1.7), respectively:

$$\hat{\alpha}_1 = \frac{nk}{\sum_{i=1}^{n}\sum_{j=1}^{k}(r_j+1)\,\ln\left[1+\frac{x_{ij}^{\beta}}{\lambda}\right]} \tag{2.10}$$

and

$$\hat{\alpha}_2 = \frac{n}{\sum_{i=1}^{n}(S_i+1)\,\ln\left[1+\frac{y_i^{\beta}}{\lambda}\right]}. \tag{2.11}$$

By incorporating $\hat{\alpha}_1$ and $\hat{\alpha}_2$ into Eq (1.2), the MLE of $R_{s,k}$ becomes

$$\hat{R}_{s,k} = \sum_{i=s}^{k}\binom{k}{i}\frac{\hat{\alpha}_2}{\hat{\alpha}_1}B(\frac{\hat{\alpha}_2}{\hat{\alpha}_1}+i, k-i+1). \tag{2.12}$$

2.2. Asymptotic confidence intervals

We use the asymptotic distribution of MLE $\hat{R}_{s,k}$ to construct an asymptotic confidence interval for the multicomponent reliability $R_{s,k}$, also need to observe an asymptotic distribution of $\hat{\theta} = (\hat{\alpha}_1, \hat{\alpha}_2, \hat{\beta}, \hat{\lambda})$. In this regard, let $E[I(\theta)]$ denote the expected Fisher-information matrix, where

$$I(\theta) = I_{ij} = \left[-\frac{\partial^2 \ell}{\partial\theta_i \partial\theta_j}\right], i, j = 1, 2, 3, 4.$$

The elements of this matrix are obtained as

$$I_{11} = \frac{nk}{\alpha_1^2}, I_{12} = I_{21} = 0,$$

$$I_{13} = I_{31} = \sum_{i=1}^{n}\sum_{j=1}^{k}(r_j+1)\,\frac{x_{ij}^{\beta}\,\ln(x_{ij})}{\lambda\left[1+\frac{x_{ij}^{\beta}}{\lambda}\right]},$$

$$I_{14} = I_{41} = -\sum_{i=1}^{n}\sum_{j=1}^{k}(r_j+1)\,\frac{x_{ij}^{\beta}}{\lambda^2\left[1+\frac{x_{ij}^{\beta}}{\lambda}\right]},$$

$$I_{22} = \frac{n}{\alpha_2^2},$$

$$I_{23} = I_{32} = \sum_{i=1}^{n}(S_i + 1)\ \frac{y_i^{\beta}\ \ln(y_i)}{\lambda\left[1 + \frac{y_i^{\beta}}{\lambda}\right]},$$

$$I_{24} = I_{42} = -\sum_{i=1}^{n}(S_i + 1)\ \frac{y_i^{\beta}}{\lambda^2\left[1 + \frac{y_i^{\beta}}{\lambda}\right]},$$

$$\begin{aligned} I_{33} &= \frac{n(k+1)}{\beta^2} + \sum_{i=1}^{n}\sum_{j=1}^{k}((r_j + 1)\ \alpha_1 + 1)\frac{x_{ij}^{\beta}\ \left(\ln(x_{ij})\right)^2}{\lambda\left[1 + \frac{x_{ij}^{\beta}}{\lambda}\right]} - \sum_{i=1}^{n}\sum_{j=1}^{k}((r_j + 1)\ \alpha_1 + 1)\frac{\left(x_{ij}^{\beta}\ \ln(x_{ij})\right)^2}{\lambda^2\left[1 + \frac{x_{ij}^{\beta}}{\lambda}\right]^2} \\ &+ \sum_{i=1}^{n}((S_i + 1)\ \alpha_2 + 1)\frac{y_i^{\beta}\ (\ln(y_i))^2}{\lambda\left[1 + \frac{y_i^{\beta}}{\lambda}\right]} - \sum_{i=1}^{n}((S_i + 1)\ \alpha_2 + 1)\frac{\left(y_i^{\beta}\ \ln(y_i)\right)^2}{\lambda^2\left[1 + \frac{y_i^{\beta}}{\lambda}\right]^2}, \end{aligned}$$

$$\begin{aligned} I_{44} &= \frac{n(k+1)}{\lambda^2} + 2\sum_{i=1}^{n}\sum_{j=1}^{k}((r_j + 1)\ \alpha_1 + 1)\frac{x_{ij}^{\beta}}{\lambda^3\left[1 + \frac{x_{ij}^{\beta}}{\lambda}\right]} + \sum_{i=1}^{n}\sum_{j=1}^{k}((r_j + 1)\ \alpha_1 + 1)\frac{\left(x_{ij}^{\beta}\right)^2}{\lambda^4\left[1 + \frac{x_{ij}^{\beta}}{\lambda}\right]^2} \\ &+ 2\sum_{i=1}^{n}((S_i + 1)\ \alpha_2 + 1)\frac{y_i^{\beta}}{\lambda^3\left[1 + \frac{y_i^{\beta}}{\lambda}\right]} + \sum_{i=1}^{n}((S_i + 1)\ \alpha_2 + 1)\frac{\left(y_i^{\beta}\right)^2}{\lambda^4\left[1 + \frac{y_i^{\beta}}{\lambda}\right]^2} \end{aligned}$$

and

$$\begin{aligned} I_{34} &= I_{43} = -\sum_{i=1}^{n}\sum_{j=1}^{k}((r_j + 1)\ \alpha_1 + 1)\frac{x_{ij}^{\beta}\ \ln(x_{ij})}{\lambda^2\left[1 + \frac{x_{ij}^{\beta}}{\lambda}\right]} + \sum_{i=1}^{n}\sum_{j=1}^{k}((r_j + 1)\ \alpha_1 + 1)\frac{\left(x_{ij}^{\beta}\right)^2\ln(x_{ij})}{\lambda^3\left[1 + \frac{x_{ij}^{\beta}}{\lambda}\right]^2} \\ &- \sum_{i=1}^{n}((S_i + 1)\ \alpha_2 + 1)\frac{y_i^{\beta}\ \ln(y_i)}{\lambda^2\left[1 + \frac{y_i^{\beta}}{\lambda}\right]} + \sum_{i=1}^{n}((S_i + 1)\ \alpha_2 + 1)\frac{\left(y_i^{\beta}\right)^2\ln(y_i)}{\lambda^3\left[1 + \frac{y_i^{\beta}}{\lambda}\right]^2}. \end{aligned}$$

The asymptotic variances (AV) of $\hat{\alpha}_1$ and $\hat{\alpha}_2$ are calculated from Fisher information as given below

$$V(\hat{\alpha}_1) = \left(E[I_{ij}]\right)^{-1} = \left(E[-\frac{\partial^2\ell}{\partial^2\alpha_1}]\right)^{-1} = \frac{\alpha_1^2}{nk} \tag{2.13}$$

and

$$V(\hat{\alpha}_2) = \left(E[-\frac{\partial^2\ell}{\partial^2\alpha_{21}}]\right)^{-1} = \frac{\alpha_2^2}{n}. \tag{2.14}$$

The MLE of $R_{s,k}$ is asymptotically normal with mean $R_{s,k}$, and a corresponding asymptotic variance given by:

$$\begin{aligned} V(R_{s,k}) &= \sum_{j=1}^{4}\sum_{i=1}^{4}\frac{\partial R_{s,k}}{\partial\theta_i}\frac{\partial R_{s,k}}{\partial\theta_j}I_{ij}^{-1} \\ &= \left(\frac{\partial R_{s,k}}{\partial\alpha_1}\right)^2 I_{11}^{-1} + 2\frac{\partial R_{s,k}}{\partial\alpha_1}\frac{\partial R_{s,k}}{\partial\alpha_2}I_{12}^{-1} + \left(\frac{\partial R_{s,k}}{\partial\alpha_2}\right)^2 I_{22}^{-1}, \end{aligned} \tag{2.15}$$

for more details one may refer to Rao [33] It should be noticed that we obtain $R_{s,k}$ and its derivatives for $(s,k) = (1,3)$ and $(2,4)$, independently to avoid the difficulty in deriving $R_{s,k}$.

Therefore, $100(1-\gamma)\%$ confidence interval of $R_{s,k}$ is constructed as given below

$$\hat{R}_{s,k} \pm z_{\gamma/2}\sqrt{\hat{V}(\hat{R}_{s,k})},$$

where $z_{\gamma/2}$ denotes the upper $\gamma/2$th quantile of the standard normal distribution and $\hat{V}(\hat{R}_{s,k})$ is the MLE of $V(R_{s,k})$ which is obtained by replacing $(\alpha_1,\alpha_2,\beta,\lambda)$ in $V(R_{s,k})$ by their corresponding MLEs.

3. Maximum product of spacing estimation

Ng et al. [34] presented the MPS approach. MPS technique determines the parameter values that makes the observed data as uniform as possible, with respect to a given quantitative measure of uniformity and based on a progressively Type-II censored sample

$$S = \prod_{i=1}^{n+1}\left(F(t_i;\theta) - F(t_{i-1};\theta)\right)\prod_{i=1}^{n+1}\left(1 - F(t_i;\theta)\right)^{R_i}. \tag{3.1}$$

Cheng and Amin [35] defined as the geometric mean of the spacing as

$$G = \left(\prod_{i=1}^{n+1} D_i\right)^{\frac{1}{n+1}},$$

where

$$D_i = \begin{cases} D_1 = F(t_1), \\ D_i = F(t_i) - F(t_{i-1}) = F(t_{2n}), \quad i = 2,...,n, \\ D_{m+1} = 1 - F(t_n), \end{cases}$$

such that $\sum D_i = 1$, depending on MPS method that was introduced by Cheng and Amin [36] and progressive Type-II censored scheme that was discussed by Balakrishnan and Aggarwala [27] and Ng et al. [37]. For more application of MPS on complete samples see Abu El Azm et al. [38], Sabry et al. [39] and Singh et al. [40].

Then MPS of α_1,α_2,β, and λ are obtained as

$$L_{MPS} = C\left(F(x_{11})(1 - F(x_{n\,k}))\right)\prod_{i=2}^{n}\prod_{j=2}^{k}\left(F(x_{ij}) - F(x_{i-1\;j-1})\right)\prod_{i=1}^{n}\prod_{j=1}^{k}\left(1 - F(x_{ij})\right)^{r_j}$$

$$\begin{aligned}
&(F(y_1)(1-F(y_n)))\prod_{i=2}^{n}(F(y_i)-F(y_{i-1}))\prod_{i=1}^{n}(1-F(y_i))^{S_i}\\
&= C\left((1-(1+\frac{x_{11}^{\beta}}{\lambda})^{-\alpha_1})\,((1+\frac{x_{n\,k}^{\beta}}{\lambda})^{-\alpha_1})\right)\left((1-(1+\frac{y_1^{\beta}}{\lambda})^{-\alpha_2})\,((1+\frac{y_n^{\beta}}{\lambda})^{-\alpha_2})\right)\\
&\prod_{i=2}^{n}\prod_{j=2}^{k}\left((1+\frac{x_{i-1\,j-1}^{\beta}}{\lambda})^{-\alpha_1}-(1+\frac{x_{i\,j}^{\beta}}{\lambda})^{-\alpha_1}\right)\prod_{i=1}^{n}\prod_{j=1}^{k}(1+\frac{x_{i\,j}^{\beta}}{\lambda})^{-\alpha_1 r_j}\\
&\prod_{i=2}^{n}\left((1+\frac{y_{i-1}^{\beta}}{\lambda})^{-\alpha_2}-(1+\frac{y_i^{\beta}}{\lambda})^{-\alpha_2}\right)\prod_{i=1}^{n}(1+\frac{y_i^{\beta}}{\lambda})^{-\alpha_2 S_i}. \qquad (3.2)
\end{aligned}$$

The natural logarithmic likelihood functions are

$$\begin{aligned}
\ln L_{MPS} &= \ln(1-(1+\frac{x_{1\,1}^{\beta}}{\lambda})^{-\alpha_1})-\alpha_1\ln(1+\frac{x_{n\,k}^{\beta}}{\lambda})+\sum_{i=2}^{n}\sum_{j=2}^{k}\ln\left[(1+\frac{x_{i-1\,j-1}^{\beta}}{\lambda})^{-\alpha_1}-(1+\frac{x_{i\,j}^{\beta}}{\lambda})^{-\alpha_1}\right]\\
&-\alpha_1\sum_{i=2}^{n}\sum_{j=2}^{k}r_j\ln(1+\frac{x_{i\,j}^{\beta}}{\lambda})+\ln(1-(1+\frac{y_1^{\beta}}{\lambda})^{-\alpha_2})-\alpha_2\ln(1+\frac{y_n^{\beta}}{\lambda})\\
&+\sum_{i=2}^{n}\ln\left[(1+\frac{y_{i-1}^{\beta}}{\lambda})^{-\alpha_2}-(1+\frac{y_i^{\beta}}{\lambda})^{-\alpha_2}\right]-\alpha_2\sum_{i=1}^{n}S_i\ln(1+\frac{y_i^{\beta}}{\lambda}). \qquad (3.3)
\end{aligned}$$

We partially differentiate Eq (2.3) with respect to the parameters $\alpha_1, \alpha_2, \beta$ and λ, then equate them to zero to obtain the normal equations for the unknown parameters. The estimators for $\alpha_1, \alpha_2, \beta$ and λ can be found by solving the equations below.

$$\begin{aligned}
\frac{\partial \ln L_{MPS}}{\partial\alpha_1} &= \frac{(1+\frac{x_{11}^{\beta}}{\lambda})^{-\alpha_1}\ln(1+\frac{x_{11}^{\beta}}{\lambda})}{1-(1+\frac{x_1^{\beta}}{\lambda})^{-\alpha_1}}-\ln(1+\frac{x_{n\,k}^{\beta}}{\lambda})-\sum_{i=1}^{n}\sum_{j=1}^{k}r_j\ln(1+\frac{x_{i\,j}^{\beta}}{\lambda})\\
&+\sum_{i=2}^{n}\sum_{j=2}^{k}\frac{(1+\frac{x_{i\,j}^{\beta}}{\lambda})^{-\alpha_1}\ln(1+\frac{x_{i\,j}^{\beta}}{\lambda})-(1+\frac{x_{i-1\,j-1}^{\beta}}{\lambda})^{-\alpha_1}\ln(1+\frac{x_{i-1\,j-1}^{\beta}}{\lambda})}{(1+\frac{x_{i-1\,j-1}^{\beta}}{\lambda})^{-\alpha_1}-(1+\frac{x_{i\,j}^{\beta}}{\lambda})^{-\alpha_1}},
\end{aligned} \qquad (3.4)$$

$$\begin{aligned}
\frac{\partial \ln L_{MPS}}{\partial\alpha_2} &= \frac{(1+\frac{y_1^{\beta}}{\lambda})^{-\alpha_2}\ln(1+\frac{y_1^{\beta}}{\lambda})}{1-(1+\frac{y_1^{\beta}}{\lambda})^{-\alpha_2}}-\ln(1+\frac{y_n^{\beta}}{\lambda})-\sum_{i=1}^{n}S_i\ln(1+\frac{y_i^{\beta}}{\lambda})\\
&+\sum_{i=2}^{n}\frac{(1+\frac{y_i^{\beta}}{\lambda})^{-\alpha_2}\ln(1+\frac{y_i^{\beta}}{\lambda})-(1+\frac{y_{i-1}^{\beta}}{\lambda})^{-\alpha_2}\ln(1+\frac{y_{i-1}^{\beta}}{\lambda})}{(1+\frac{y_{i-1}^{\beta}}{\lambda})^{-\alpha_2}-(1+\frac{y_i^{\beta}}{\lambda})^{-\alpha_2}},
\end{aligned} \qquad (3.5)$$

$$\begin{aligned}\frac{\partial \ln L_{MPS}}{\partial \beta} = & \frac{\alpha_1\, x_{11}^{\beta} \ln(x_{11})(1+\frac{x_{11}^{\beta}}{\lambda})^{-\alpha_1-1}}{\lambda\left[1-(1+\frac{x_1^{\beta}}{\lambda})^{-\alpha_1}\right]} - \frac{\alpha_1\, x_{nk}^{\beta} \ln(x_{nk})}{\lambda(1+\frac{x_{n\,k}^{\beta}}{\lambda})} - \sum_{i=1}^{n}\sum_{j=1}^{k} r_j \frac{\alpha_1 x_{ij}^{\beta}\, \ln(x_{i\,j})}{\lambda(1+\frac{x_{i\,j}^{\beta}}{\lambda})} \\ & - \sum_{i=2}^{n}\sum_{j=2}^{k} \frac{\alpha_1\left[x_{ij}^{\beta}\, \ln(x_{i\,j})(1+\frac{x_{i\,j}^{\beta}}{\lambda})^{-\alpha_1-1} - x_{i-1\ j-1}^{\beta} \ln(x_{i-1\ j-1})(1+\frac{x_{i-1\ j-1}^{\beta}}{\lambda})^{-\alpha_1-1}\right]}{\lambda\left[(1+\frac{x_{i-1\ j-1}^{\beta}}{\lambda})^{-\alpha_1} - (1+\frac{x_{i\,j}^{\beta}}{\lambda})^{-\alpha_1}\right]} \\ & + \frac{\alpha_2\, y_1^{\beta} \ln(y_1)(1+\frac{y_1^{\beta}}{\lambda})^{-\alpha_2-1}}{\lambda\left[1-(1+\frac{y_1^{\beta}}{\lambda})^{-\alpha_2}\right]} - \frac{\alpha_2\, y_n^{\beta} \ln(y_n)}{\lambda(1+\frac{y_n^{\beta}}{\lambda})} - \sum_{i=1}^{n} S_i \frac{\alpha_2 y_i^{\beta}\, \ln(y_i)}{\lambda(1+\frac{y_i^{\beta}}{\lambda})} \\ & - \sum_{i=2}^{n} \frac{\alpha_2\left[y_i^{\beta}\, \ln(y_i)(1+\frac{y_i^{\beta}}{\lambda})^{-\alpha_2-1} - y_{i-1}^{\beta} \ln(y_{i-1})(1+\frac{y_{i-1}^{\beta}}{\lambda})^{-\alpha_2-1}\right]}{\lambda\left[(1+\frac{y_{i-1}^{\beta}}{\lambda})^{-\alpha_2} - (1+\frac{y_i^{\beta}}{\lambda})^{-\alpha_2}\right]} \end{aligned} \tag{3.6}$$

and

$$\begin{aligned}\frac{\partial \ln L_{MPS}}{\partial \lambda} = & \frac{-\alpha_1\, x_{11}^{\beta}(1+\frac{x_{11}^{\beta}}{\lambda})^{-\alpha_1-1}}{\lambda^2\left[1-(1+\frac{x_1^{\beta}}{\lambda})^{-\alpha_1}\right]} + \frac{\alpha_1\, x_{nk}^{\beta}}{\lambda^2(1+\frac{x_{n\,k}^{\beta}}{\lambda})} + \sum_{i=1}^{n}\sum_{j=1}^{k} r_j \frac{\alpha_1 x_{ij}^{\beta}}{\lambda^2(1+\frac{x_{i\,j}^{\beta}}{\lambda})} \\ & + \sum_{i=2}^{n}\sum_{j=2}^{k} \frac{\alpha_1\left[x_{i-1\ j-1}^{\beta}(1+\frac{x_{i-1\ j-1}^{\beta}}{\lambda})^{-\alpha_1-1} - x_{ij}^{\beta}\,(1+\frac{x_{i\,j}^{\beta}}{\lambda})^{-\alpha_1-1}\right]}{\lambda^2\left[(1+\frac{x_{i-1\ j-1}^{\beta}}{\lambda})^{-\alpha_1} - (1+\frac{x_{i\,j}^{\beta}}{\lambda})^{-\alpha_1}\right]} \\ & - \frac{\alpha_2\, y_1^{\beta}(1+\frac{y_1^{\beta}}{\lambda})^{-\alpha_2-1}}{\lambda^2\left[1-(1+\frac{y_1^{\beta}}{\lambda})^{-\alpha_2}\right]} - \frac{\alpha_2\, y_n^{\beta}}{\lambda^2(1+\frac{y_n^{\beta}}{\lambda})} + \sum_{i=1}^{n} S_i \frac{\alpha_2 y_i^{\beta}}{\lambda^2(1+\frac{y_i^{\beta}}{\lambda})} \\ & - \sum_{i=2}^{n} \frac{\alpha_2\left[y_{i-1}^{\beta}(1+\frac{y_{i-1}^{\beta}}{\lambda})^{-\alpha_2-1} - y_i^{\beta}\,(1+\frac{y_i^{\beta}}{\lambda})^{-\alpha_2-1}\right]}{\lambda^2\left[(1+\frac{y_{i-1}^{\beta}}{\lambda})^{-\alpha_2} - (1+\frac{y_i^{\beta}}{\lambda})^{-\alpha_2}\right]}. \end{aligned} \tag{3.7}$$

The MPS $\hat{\alpha}_{1(MPS)}$, $\hat{\alpha}_{2(MPS)}$, $\hat{\beta}_{(MPS)}$ and $\hat{\lambda}_{(MPS)}$ can be obtained by solving simultaneously the likelihood equations

$$\frac{\partial \ln L_{MPS}}{\partial \alpha_1} = 0,\ \frac{\partial \ln L_{MPS}}{\partial \alpha_2} = 0,\ \frac{\partial \ln L_{MPS}}{\partial \beta} = 0 \text{ and } \frac{\partial \ln L_{MPS}}{\partial \lambda} = 0.$$

The Eqs (2.4)–(2.7), on the other hand, must be solved numerically using a nonlinear optimization approach. By incorporating $\hat{\alpha}_{1(MPS)}$ and $\hat{\alpha}_{2(MPS)}$ into Eq (2.2), the MPS estimator of $R_{s,k}$ becomes

$$\tilde{R}_{s,k(MPS)} = \sum_{i=s}^{k} \binom{k}{i} \frac{\hat{\alpha}_{2(MPS)}}{\hat{\alpha}_{1(MPS)}} B(\frac{\hat{\alpha}_{2(MPS)}}{\hat{\alpha}_{1(MPS)}} + i, k-i+1). \tag{3.8}$$

4. Bootstrap confidence intervals (CIs)

According to point estimation, a parametric bootstrap interval informs us a lot about the population values of the quantity of our interest. Furthermore, CIs based on asymptotic results clearly make errors for small sample size. To determine the bootstrap CIs of $\alpha_1, \alpha_2, \beta$ and λ, two parametric bootstrap methods are explained. The percentile bootstrap (Boot-p) CIs, introduced by Efron [41], and the CIs known as the bootstrap-t (Boot-t), which was presented by Hall [42]. Boot-t was developed using a studentized 'pivot' and it requires a variance estimator for the MLE of $\alpha_1, \alpha_2, \beta$, and λ.

4.1. Parametric Boot-p

Step 1: Generate a bootstrap sample of size nk, $\left\{x_{i1}^*, x_{i2}^*, ..., x_{ik}^*\right\}$ from $\{x_{i1}, x_{i2}, ..., x_{ik}\}, i = 1, 2, ..., n,$ and generate a bootstrap sample of size n, $\{y_1^*, y_2^*, ..., y_n^*\}$ from $\{y_1, y_2, ..., y_n\}$. Compute the bootstrap estimate of $R_{s,k}$, say $\hat{R}_{s,k}^*$, using Eq (1.2).

Step 2: Repeat Step 1, *NBoot* times.

Step 3: Let $G_1(z) = P(\hat{R}_{s,k}^* \leq z)$ be the cumulative distribution function of $\hat{R}_{s,k}^*$. Define $\hat{R}_{s,k(boot-p)}^* = G_1^{-1}(z)$ for given z. The approximate bootstrap-p $100(1-\gamma)\%$ CI of $\hat{R}_{s,k}$, is given by

$$\left[\hat{R}_{s,k(boot-p)}^*\left(\frac{\gamma}{2}\right), \hat{R}_{s,k(boot-p)}^*\left(1-\frac{\gamma}{2}\right)\right]. \tag{4.1}$$

4.2. Parametric Boot-t

Step 1: From the samples $\{x_{i1}, x_{i2}, ..., x_{ik}\}, i = 1, 2, ..., n$ and $\{y_1, y_2, ..., y_n\},$ then compute $\hat{R}_{s,k}$.

Step 2: The same as the parametric Boot-p in Step 1.

Step 3: Compute the T^* statistic defined as

$$T^* = \frac{\left(\hat{R}_{s,k}^* - \hat{R}_{s,k}\right)}{\sqrt{\widehat{V\left(\hat{R}_{s,k}^*\right)}}},$$

where $V\left(\hat{R}_{s,k}^*\right)$can compute as in Eq (1.15).

Step 4: Repeat Steps 1–3, *NBoot* times.

Step 5: Let $G_2(z) = P(T^* \leq z)$ be the cumulative distribution function of T^* for given z. Define $\hat{R}_{s,k(boot-t)}^* = \hat{R}_{s,k} + G_2^{-1}(z)\sqrt{\widehat{\sigma^2\left(\hat{R}_{s,k}^*\right)}}$.

Then, the approximate bootstrap-t $100(1-\gamma)\%$ CI of $R_{s,k}$, is given by

$$\left[\hat{R}_{s,k(boot-t)}^*\left(\frac{\gamma}{2}\right), \hat{R}_{s,k(boot-t)}^*\left(1-\frac{\gamma}{2}\right)\right]. \tag{4.2}$$

5. Bayes estimation

In this section, Bayesian estimates are obtained for the parameters that are assumed to be random, and the uncertainties in the parameters are described by a joint prior distribution, which has been developed before the collected failure data. The Bayesian approach is highly useful in reliability analysis because it may incorporate previous knowledge into the analysis. Bayesian estimates of the unknown parameters $\alpha_1, \alpha_2, \beta$ and λ, as well as some lifetime parameter $R_{s,k}$ under the SELF and LINEX loss function are developed. It is assumed here that the parameters $\alpha_1, \alpha_2, \beta$ and λ are independent and follow the gamma prior distributions,

$$\begin{cases} \pi_1(\alpha_1) = \alpha_1^{a_1-1}\exp(-b_1\alpha_1) & , \alpha_1 > 0, \\ \pi_2(\alpha_2) = \alpha_2^{a_2-1}\exp(-b_2\alpha_2) & , \alpha_2 > 0, \\ \pi_3(\beta) = \beta^{a_3-1}\exp(-b_3\beta) & , \beta > 0, \\ \pi_4(\lambda) = \lambda^{a_4-1}\exp(-b_4\lambda) & , \lambda > 0, \end{cases} \tag{5.1}$$

where all the hyperparameters a_i and b_i, $i = 1, 2, 3, 4$ are assumed to be known non negative numbers. To determine the elicit hyper-parameters of the independent joint prior (5.1), we can use ML estimates and variance-covariance matrix of MLE method. By equating mean and variance of gamma priors, the estimated of hyper-parameters can be written as

$$a_j = \frac{\left[\frac{1}{L}\sum_{i=1}^{L}\hat{\Omega}_j^i\right]^2}{\frac{1}{L-1}\sum_{i=1}^{L}\left[\hat{\Omega}_j^i - \frac{1}{L}\sum_{i=1}^{L}\hat{\Omega}_j^i\right]^2};\ j = 1, ..., 4,$$

$$b_j = \frac{\frac{1}{L}\sum_{i=1}^{L}\hat{\Omega}_j^i}{\frac{1}{L-1}\sum_{i=1}^{L}\left[\hat{\Omega}_j^i - \frac{1}{L}\sum_{i=1}^{L}\hat{\Omega}_j^i\right]^2};\ j = 1, ..., 4,$$

where, L is the number of iteration and Ω is a vector of parameters.

Combining the likelihood function in Eq (1.4) with the priors in Eq (4.1), resulted with the posterior distribution of the parameters $\alpha_1, \alpha_2, \beta$ and λ indicated by $\pi^*\left(\alpha_1, \alpha_2, \beta,\ \lambda \mid \underline{x}, \underline{y}\right)$, which can be expressed as

$$\pi^*\left(\alpha_1, \alpha_2, \beta,\ \lambda \mid \underline{x}, \underline{y}\right) = \frac{\pi_1(\alpha_1)\ \pi_2(\alpha_2)\ \pi_3(\beta)\ \pi_4(\lambda)\ L(\alpha_1, \alpha_2, \beta,\ \lambda \mid \underline{x}, \underline{y})}{\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty \pi_1(\alpha_1)\ \pi_2(\alpha_2)\ \pi_3(\beta)\ \pi_4(\lambda)\ L(\alpha_1, \alpha_2, \beta,\ \lambda \mid \underline{x}, \underline{y})\, d\alpha_1 d\alpha_2 d\beta d\lambda}. \tag{5.2}$$

A commonly used loss function is the SELF, which is a symmetrical loss function that assigns equal losses to overestimation and underestimation. If ϕ is the parameter to be estimated by an estimator $\hat{\phi}$, then the square error loss function is defined as:

$$L\left(\phi, \hat{\phi}\right) = \left(\hat{\phi} - \phi\right)^2.$$

Therefore, the Bayes estimate of any function of $\alpha_1, \alpha_2, \beta$ and λ, say $g(\alpha_1, \alpha_2, \beta,\ \lambda)$ under the SELF can be obtained as

$$\hat{g}_{BS}\left(\alpha_1, \alpha_2, \beta,\ \lambda \mid \underline{x}, \underline{y}\right) = E_{\alpha,\beta,\lambda|\underline{x},\underline{y}}\ (g(\alpha_1, \alpha_2, \beta,\ \lambda)),$$

where

$$E_{\alpha,\beta,\lambda|\underline{x},\underline{y}}\,(g(\alpha_1, \alpha_2, \beta,\ \lambda))$$

$$= \frac{\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty g(\alpha_1,\alpha_2,\beta,\ \lambda)\ \pi_1(\alpha_1)\ \pi_2(\alpha_2)\ \pi_3(\beta)\ \pi_4(\lambda)\ L(\alpha_1,\alpha_2,\beta,\ \lambda \mid \underline{x},\underline{y})d\alpha_1 d\alpha_2 d\beta d\lambda}{\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty \pi_1(\alpha_1)\ \pi_2(\alpha_2)\ \pi_3(\beta)\ \pi_4(\lambda)\ L(\alpha_1,\alpha_2,\beta,\ \lambda \mid \underline{x},\underline{y})\, d\alpha_1 d\alpha_2 d\beta d\lambda}. \tag{5.3}$$

Varian [43] considered the LINEX loss function $L(\triangle)$ for a parameter ϕ is given by

$$L(\triangle) = \left(e^{c\triangle} - c\triangle - 1\right),\ c \neq 0,\ \triangle = \hat{\phi} - \phi, \tag{5.4}$$

This loss function is suitable for situations where overestimation of is more costly than its underestimation. Zellner [44]. discussed Bayesian estimation and prediction using LINEX loss. Hence, under LINEX loss function in Eq (4.3), the Bayes estimate of a function $g(\alpha_1,\alpha_2,\beta,\ \lambda)$ is

$$\hat{g}_{BL}\left(\alpha_1,\alpha_2,\beta,\lambda \mid \underline{x},\underline{y}\right) = -\frac{1}{c}\log\left[E\left(e^{-cg(\alpha_1,\alpha_2,\beta,\lambda)} \mid \underline{x},\underline{y}\right)\right],\ c \neq 0, \tag{5.5}$$

where

$$E\left(e^{-cg(\alpha_1,\alpha_2,\beta,\lambda)} \mid \underline{x},\underline{y}\right)$$
$$= \frac{\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty e^{-cg(\alpha_1,\alpha_2,\beta,\lambda)}\ \pi_1(\alpha_1)\ \pi_2(\alpha_2)\ \pi_3(\beta)\ \pi_4(\lambda)\ L(\alpha_1,\alpha_2,\beta,\ \lambda \mid \underline{x},\underline{y})d\alpha_1 d\alpha_2 d\beta d\lambda}{\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty \pi_1(\alpha_1)\ \pi_2(\alpha_2)\ \pi_3(\beta)\ \pi_4(\lambda)\ L(\alpha_1,\alpha_2,\beta,\ \lambda \mid \underline{x},\underline{y})\, d\alpha_1 d\alpha_2 d\beta d\lambda}. \tag{5.6}$$

The multiple integrals in Eqs (4.3) and (4.6) can not be obtained analytically. Thus, the MCMC technique can be used to generate samples from the joint posterior density function in Eq (4.2). In order to be able to implement the MCMC technique, we consider the Gibbs within the Metropolis-Hasting samplers procedure. The Metropolis-Hasting and Gibbs sampling are two useful MCMC methods that have been widely used in statistics.

The joint posterior density function of $\alpha_1, \alpha_2, \beta$ and λ is obtained as follows:

$$\begin{aligned}\pi^*\left(\alpha_1,\alpha_2,\beta,\ \lambda \mid \underline{x},\underline{y}\right) \ \propto\ & \alpha_1^{nk+a_1-1}\alpha_2^{n+a_2-1}\beta^{nk+n+a_3-1}\lambda^{-nk-n+a_4-1}\exp\left(-b_1\alpha_1 - b_2\alpha_2 - b_3\beta - b_4\lambda\right)\\ & \prod_{i=1}^{n}\left[\prod_{j=1}^{k} x_{ij}^{\beta-1}[1+\frac{x_{ij}^{\beta}}{\lambda}]^{-(r_j+1)\alpha_1-1}\right]\prod_{i=1}^{n} y_i^{\beta-1}[1+\frac{y_i^{\beta}}{\lambda}]^{-(S_i+1)\ \alpha_2-1},\end{aligned} \tag{5.7}$$

Under the SELF and LINEX loss function, the Bayesian estimation of $R_{s,k}$ is the mean of the posterior function in Eq (4.7), which can be written as shown below

$$\tilde{R}_{s,k} = \int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty R_{s,k}\pi^*\left(\alpha_1,\alpha_2,\beta,\ \lambda \mid \underline{x},\underline{y}\right)d\alpha_1 d\alpha_2 d\beta d\lambda. \tag{5.8}$$

The integral given in Eq (4.8) is obviously impossible to be calculated analytically. As a result, the Bayesian estimator of $R_{s,k}$, specifically the Gibbs sampling methods, is obtained using this approach. The next subsection get across the specifics of these strategies.

5.1. Gibbs sampling

The Gibbs sampling method is employed, which is a sub type of Monte-Carlo Markov Chain (MCMC) method, to create the Bayesian estimate of $R_{s,k}$ and the related credible interval. The idea behind this method is to use posterior conditional density functions to generate posterior samples of parameters of interest. The posterior density function of the parameters of interest is produced by Eq (4.7). The posterior conditional density functions of $\alpha_1, \alpha_2, \beta$ and λ can be expressed as follows using this equation:

$$\pi_1^*\left(\alpha_1 \mid \alpha_2, \beta, \lambda, \underline{x}, \underline{y}\right) \propto \alpha_1^{nk+a_1-1} \exp\left(-b_1\alpha_1\right) \prod_{i=1}^{n} \prod_{j=1}^{k} x_{ij}^{\beta-1}[1 + \frac{x_{ij}^{\beta}}{\lambda}]^{-(r_j+1)\ \alpha_1-1}, \tag{5.9}$$

$$\pi_2^*\left(\alpha_2 \mid \alpha_1, \beta, \lambda, \underline{x}, \underline{y}\right) \propto \alpha_2^{n+a_2-1} \exp\left(-b_2\alpha_2\right) \prod_{i=1}^{n} y_i^{\beta-1}[1 + \frac{y_i^{\beta}}{\lambda}]^{-(S_i+1)\ \alpha_2-1}, \tag{5.10}$$

$$\pi_3^*\left(\beta \mid \alpha_1, \alpha_2, \lambda, \underline{x}, \underline{y}\right) \propto \beta^{nk+n+a_3-1} \exp\left(-b_3\beta\right) \prod_{i=1}^{n} \prod_{j=1}^{k} x_{ij}^{\beta-1}[1 + \frac{x_{ij}^{\beta}}{\lambda}]^{-(r_j+1)\ \alpha_1-1} \prod_{i=1}^{n} y_i^{\beta-1}[1 + \frac{y_i^{\beta}}{\lambda}]^{-(S_i+1)\ \alpha_2-1} \tag{5.11}$$

and

$$\pi_4^*\left(\lambda \mid \alpha_1, \alpha_2, \beta, \underline{x}, \underline{y}\right) \propto \lambda^{-nk-n+a_4-1} \exp\left(-b_4\lambda\right) \prod_{i=1}^{n} \prod_{j=1}^{k} x_{ij}^{\beta-1}[1 + \frac{x_{ij}^{\beta}}{\lambda}]^{-(r_j+1)\ \alpha_1-1} \prod_{i=1}^{n} y_i^{\beta-1}[1 + \frac{y_i^{\beta}}{\lambda}]^{-(S_i+1)\ \alpha_2-1}. \tag{5.12}$$

The conditional density function of $\alpha_1, \alpha_2, \beta$ and λ cannot be obtained in the form of the well-known density functions, as shown by Eqs (4.9)–(4.12). In this case, we can utilize the Metropolis-Hasting (MH) technique, developed by Metropolis et al. [45], to create random-samples from the posterior density of $\alpha_1, \alpha_2, \beta$ and λ using a normal proposal distribution.

The steps of Gibbs sampling are described as follows:

(1) Start with initial guess $\left(\alpha_1^{(0)}, \alpha_2^{(0)}, \beta^{(0)}, \lambda^{(0)}\right)$.

(2) Set $l = 1$.

(3) Using the following M-H algorithm, generate $\alpha_1^{(l)}, \alpha_2^{(l)}, \beta^{(l)}$ and $\lambda^{(l)}$ from $\pi_1^*\left(\alpha_1^{(l)} \mid \alpha_2^{(l-1)}, \beta^{(l-1)}, \lambda^{(l-1)}, \underline{x}, \underline{y}\right)$, $\pi_2^*\left(\alpha_2^{(l)} \mid \alpha_1^{(l)}, \beta^{(l-1)}, \lambda^{(l-1)}, \underline{x}, \underline{y}\right)$, $\pi_3^*\left(\beta^{(l)} \mid \alpha_1^{(l)}, \alpha_2^{(l)}, \lambda^{(l-1)}, \underline{x}, \underline{y}\right)$ and $\pi_4^*\left(\lambda^{(l)} \mid \alpha_1^{(l)}, \alpha_2^{(l)}, \beta^{(l)}, \underline{x}, \underline{y}\right)$ with the normal proposal distributions

$$N\left(\alpha_1^{(l-1)}, V\left(\alpha_1\right)\right),\ N\left(\alpha_2^{(l-1)}, V\left(\alpha_2\right)\right),\ N\left(\beta^{(l-1)}, V\left(\beta\right)\right) \text{ and } N\left(\lambda^{(l-1)}, V\left(\lambda\right)\right),$$

where $V\left(\alpha_1\right)$, $V\left(\alpha_2\right)$, $V\left(\beta\right)$ and $V\left(\lambda\right)$ can be obtained from the main diagonal in the inverse Fisher information matrix.

(4) Generate a proposal α_1^* from $N\left(\alpha_1^{(l-1)}, V(\alpha_1)\right)$, α_2^* from $N\left(\alpha_2^{(l-1)}, V(\alpha_2)\right)$, β^* from $N\left(\beta^{(l-1)}, V(\beta)\right)$ and λ^* from $N\left(\lambda^{(l-1)}, V(\lambda)\right)$.

(i) Evaluate the acceptance probabilities

$$\left.\begin{aligned}
\eta_{\alpha_1} &= \min\left[1, \frac{\pi_1^*\left(\alpha_1^*|\alpha_2^{(l-1)},\beta^{(l-1)},\lambda^{(l-1)},\underline{x},\underline{y}\right)}{\pi_1^*\left(\alpha_1^{(l)}|\alpha_2^{(l-1)},\beta^{(l-1)},\lambda^{(l-1)},\underline{x},\underline{y}\right)}\right],\\
\eta_{\alpha_2} &= \min\left[1, \frac{\pi_2^*\left(\alpha_2^*|\alpha_1^{(l)},\beta^{(l-1)},\lambda^{(l-1)},\underline{x},\underline{y}\right)}{\pi_2^*\left(\alpha_2^{(l)}|\alpha_1^{(l)},\beta^{(l-1)},\lambda^{(l-1)},\underline{x},\underline{y}\right)}\right]\\
\eta_{\beta} &= \min\left[1, \frac{\pi_3^*\left(\beta^*|\alpha_1^{(l)},\alpha_2^{(l)},\lambda^{(l-1)},\underline{x},\underline{y}\right)}{\pi_3^*\left(\beta^{(l)}|\alpha_1^{(l)},\alpha_2^{(l)},\lambda^{(l-1)},\underline{x},\underline{y}\right)}\right],\\
\eta_{\lambda} &= \min\left[1, \frac{\pi_4^*\left(\lambda^*|\alpha_1^{(l)},\alpha_2^{(l)},\beta^{(l)},\underline{x},\underline{y}\right)}{\pi_4^*\left(\lambda^{(l)}|\alpha_1^{(l)},\alpha_2^{(l)},\beta^{(l)},\underline{x},\underline{y}\right)}\right].
\end{aligned}\right\}$$

(ii) Generate a u_1, u_2, u_3 and u_4 from a uniform $(0, 1)$ distribution.

(iii) If $u_1 < \eta_{\alpha_1}$, accept the proposal and set $\alpha_1^{(l)} = \alpha_1^*$, else set $\alpha_1^{(l)} = \alpha_1^{(l-1)}$.

(iv) If $u_2 < \eta_{\alpha_2}$, accept the proposal and set $\alpha_2^{(l)} = \alpha_2^*$, else set $\alpha_2^{(l)} = \alpha_2^{(l-1)}$.
(iiv) If $u_3 < \eta_\beta$, accept the proposal and set $\beta^{(l)} = \beta^*$, else set $\beta^{(l)} = \beta^{(l-1)}$.
(v) If $u_4 < \eta_\lambda$,accept the proposal and set $\lambda^{(l)} = \lambda^*$, else set $\lambda^{(l)} = \lambda^{(l-1)}$.

(5) Compute $R_{s,k}^{(l)}$ at $(\alpha_1^{(l)},\ \alpha_2^{(l)},\ \beta^{(l)},\ \lambda^{(l)})$.

(6) Set $l = l + 1$.

(7) Repeat Steps (3)–(6), N times and obtain $\alpha_1^{(l)},\ \alpha_2^{(l)},\ \beta^{(l)},\ \lambda^{(l)}$ and $R_{s,k}^{(l)}, l = 1, 2, ...N$.

(8) To compute the CRs of $\alpha_1, \alpha_2,\ \beta,\ \lambda$ and $R_{s,k},\ \psi_k^{(l)}, k = 1, 2, 3, 4, 5$, $(\psi_1, \psi_2, \psi_3, \psi_4, \psi_5) = (\alpha_1, \alpha_2, \beta, \lambda, R_{s,k})$ as $\psi_k^{(1)} < \psi_k^{(2)} ... < \psi_k^{(N)}$, then the $100(1-\gamma)\%$CRIs of ψ_k is

$$\left(\psi_{k(N\ \gamma/2)}, \psi_{k(N\ (1-\gamma/2))}\right).$$

The first M simulated variants are discarded in order to ensure convergence and remove the affection of initial value selection. Then the selected samples are $\psi_k^{(i)}$, $j = M + 1, ...N$, for sufficiently large N.

Based on the SELF, the approximate Bayes estimates of ψ_k is given by

$$\hat{\psi}_k = \frac{1}{N-M}\sum_{j=M+1}^{N} \psi^{(j)}, k = 1, 2, 3, 4, 5,$$

the approximate Bayes estimates for ψ_k, under LINEX loss function, from Eq (4.6) is

$$\hat{\psi}_k = \frac{-1}{c}\log\left[\frac{1}{N-M}\sum_{j=M+1}^{N} e^{-c\ \psi^{(j)}}\right], k = 1, 2, 3, 4, 5.$$

6. Simulation

In this section random samples are generated from POLO distribution using the R-coding. The simulation experiment is carried out to determine the reliability coefficient and compare the suggested methods.

6.1. Simulation study

The performance of the parameters and $R_{s,k}$ is compared using different sample sizes based on Monte Carlo simulation, where k=5, and s=2,3, and 4. A total of 5,000 random samples of size are $n_1 = 10, n_2 = 15, n_3 = 15, n_4 = 10, n_5 = 12$ created from the stress and strength populations and the sample size of censored sample are chosen as $(m_1 = 7, m_2 = 10, m_3 = 10, m_4 = 8, m_5 = 9)$, and $(m_1 = 9, m_2 = 13, m_3 = 13, m_4 = 9, m_5 = 11)$. This section examines some empirical data derived from Monte-Carlo simulations to see how the proposed methods perform with different sample sizes. For (a_j, b_j); $j =, ..., 4$, we may use the estimate and variance-covariance matrix of the MLE approach to elicit hyper-parameters of the independent joint prior. The estimated hyper-parameters are calculated by equating the mean and variance of gamma priors. For the random variables generating, the values of the parameters $\alpha_1, \alpha_2, \lambda$, and β are chosen as follows:

$$\alpha_1 = 0.5, \lambda = 2, \alpha_2 = 3, \beta = 1.2;$$
$$\alpha_1 = 1.5, \lambda = 0.5, \alpha_2 = 0.5, \beta = 1.2;$$
$$\alpha_1 = 1.3, \lambda = 1.2, \alpha_2 = 2, \beta = 1.5.$$

Tables 1–6 show the simulation results of MLEs, MPS, Bayesian estimates, and interval estimations of $R_{s,k}$. All of the results are calculated using a total of 5000 simulated samples. The simulation methods are compared using the criteria of parameters estimation, the comparison is performed by calculating the Bias, the mean of square error (MSE), the length of asymptotic and bootstrap confidence intervals (L.CI) and coverage probability (CP) for each estimation method. In simulation results Tables 1–3, for each sample-size m_i, scheme (S), and estimator, the first four values represent the average bias and MSE of the parameters model, and the next three values represent the estimated risk for the corresponding stress-strength reliability when $k = 5$ and $s = 2, 3, 4$, respectively. In simulation results for CI Tables 4–6, for each sample size m_i, and scheme (S), in MLE, and MPS estimators, the first four values represent the average length of asymptotic CI (L.CI), CP, Boot-p (BP), and Boot-t (BT) of the parameters model, and the next three values represent the average length of delta CI of risk for the corresponding $R_{s,k}$ when $k = 5$ and $s = 2, 3, 4$, respectively. While, in Bayesian estimation, the average length of credible CI (L.CI) of the parameters model and risk for the corresponding stress-strength reliability when $k = 5$ and $s = 2, 3, 4$, respectively.

Numerical simulations, on the other hand, make it impossible to see in a basic sense how estimated dangers decrease with sample size. For probability, product spacing, and Bayesian estimates, we see this trend. In terms of estimated risks, the Bayesian estimates of $R_{s;k}$ perform significantly better than the MLE and MPS. We notice that the Bayesian estimate's predicted risks under the LINEX loss are often lower than those under the SELF. The Bayes estimates and their estimated risks are sometimes near to each other based on calculated findings. The average length of HPD intervals is found to be shorter than that of asymptotic confidence intervals. When the sample size is increased, the lengths of both intervals shrink. But, the bootstrap CI has the shortest length of CI.

Table 1. Bias, MSE for MLE, MPS and Bayesian when $\alpha_1 = 1.3,\ \lambda = 1.2,\ \alpha_2 = 2,\ \beta = 1.5$.

$\alpha_1 = 1.3, \lambda = 1.2, \alpha_2 = 2, \beta = 1.5$												
			MLE		MPS		SELF		LELF (c=0.5)		LELF (c=0.5)	
scheme	m_i		Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
I	7,10,10,8,9	α_1	0.1166	0.6711	-0.9288	0.9067	0.0195	0.0154	0.0173	0.0153	0.0128	0.0150
		α_2	0.4074	2.2733	-1.5609	2.5104	0.0082	0.0201	0.0054	0.0199	0.0000	0.0197
		λ	0.0360	1.4289	-1.1075	1.3954	-0.0260	0.0177	-0.0285	0.0180	-0.0336	0.0179
		β	0.5132	0.3687	0.2303	0.1479	0.1665	0.0382	0.1632	0.0368	0.1567	0.0341
		$R_{2,5}$	0.0108	0.0054	-0.0825	0.0105	-0.0035	0.0010	-0.0034	0.0009	-0.0032	0.0009
		$R_{3,5}$	0.0191	0.0081	-0.0878	0.0117	-0.0031	0.0013	-0.0030	0.0013	-0.0028	0.0013
		$R_{4,5}$	0.0224	0.0076	-0.0745	0.0084	-0.0021	0.0011	-0.0020	0.0011	-0.0018	0.0011
	9,13,13,9,11	α_1	0.2187	0.6604	-0.6722	0.5524	0.0307	0.0120	0.0293	0.0119	0.0263	0.0115
		α_2	0.3554	1.0477	-1.2310	1.6946	0.0001	0.0152	-0.0016	0.0152	-0.0050	0.0150
		λ	0.1044	1.3836	-0.9407	1.1079	-0.0350	0.0132	-0.0365	0.0134	-0.0397	0.0128
		β	0.4092	0.2326	0.2519	0.1337	0.1169	0.0202	0.1149	0.0196	0.1108	0.0184
		$R_{2,5}$	0.0093	0.0041	-0.0713	0.0083	-0.0068	0.0008	-0.0067	0.0008	-0.0066	0.0008
		$R_{3,5}$	0.0155	0.0059	-0.0763	0.0094	-0.0071	0.0011	-0.0070	0.0011	-0.0069	0.0010
		$R_{4,5}$	0.0176	0.0053	-0.0651	0.0068	-0.0059	0.0009	-0.0058	0.0009	-0.0057	0.0009
II	7,10,10,8,9	α_1	-0.9612	0.9661	-0.1381	0.4285	-0.1174	0.0281	-0.1203	0.0291	-0.1262	0.0283
		α_2	-1.6005	2.6358	-0.0855	1.1646	-0.0849	0.0309	-0.0882	0.0320	-0.0947	0.0314
		λ	-1.1641	1.5252	-0.2988	0.8392	0.1122	0.0311	0.1089	0.0299	0.1023	0.0276
		β	0.6415	0.6517	0.2548	0.1444	-0.0634	0.0142	-0.0652	0.0145	-0.0688	0.0144
		$R_{2,5}$	-0.0900	0.0130	0.0084	0.0044	0.0123	0.0013	0.0124	0.0014	0.0128	0.0012
		$R_{3,5}$	-0.0946	0.0142	0.0148	0.0064	0.0159	0.0020	0.0162	0.0020	0.0167	0.0018
		$R_{4,5}$	-0.0796	0.0101	0.0173	0.0057	0.0157	0.0017	0.0159	0.0018	0.0164	0.0017
	9,13,13,9,11	α_1	-0.6812	0.5859	-0.0313	0.4233	-0.0134	0.0110	-0.0149	0.0111	-0.0178	0.0112
		α_2	-1.2476	1.7928	-0.0332	1.1215	-0.0272	0.0150	-0.0290	0.0153	-0.0326	0.0158
		λ	-0.9660	1.2791	-0.2640	0.8265	0.0144	0.0119	0.0129	0.0118	0.0099	0.0117
		β	0.5348	0.4175	0.2008	0.0914	0.0358	0.0088	0.0345	0.0086	0.0319	0.0083
		$R_{2,5}$	-0.0781	0.0101	-0.0046	0.0034	-0.0013	0.0007	-0.0013	0.0007	-0.0011	0.0007
		$R_{3,5}$	-0.0829	0.0112	-0.0018	0.0046	-0.0008	0.0009	-0.0007	0.0009	-0.0006	0.0010
		$R_{4,5}$	-0.0703	0.0080	0.0009	0.0039	-0.0002	0.0008	-0.0001	0.0008	0.0000	0.0008

Table 2. MLE, MPS and Bayesian when $\alpha_1 = 1.5,\ \lambda = 0.5,\ \alpha_2 = 0.5,\ \beta = 1.2$.

$\alpha_1 = 1.5, \lambda = 0.5, \alpha_2 = 0.5, \beta = 1.2$												
			MLE		MPS		SELF		LELF (c=0.5)		LELF (c=0.5)	
scheme	m_i		Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
I	7,10,10,8,9	α_1	-1.1344	1.3358	-1.0874	1.2479	-0.0341	0.0161	-0.0364	0.0164	-0.0410	0.0169
		α_2	-0.3722	0.1427	-0.3401	0.1226	-0.0741	0.0175	-0.0763	0.0179	-0.0808	0.0187
		λ	-1.0672	1.1819	-1.0044	1.0842	0.0140	0.0172	0.0115	0.0171	0.0065	0.0169
		β	1.0407	1.2838	0.6646	0.5397	0.2643	0.0755	0.2606	0.0733	0.2530	0.0689
		$R_{2,5}$	0.0186	0.0046	0.0436	0.0060	-0.0359	0.0052	-0.0368	0.0053	-0.0386	0.0056
		$R_{3,5}$	0.0147	0.0026	0.0334	0.0035	-0.0250	0.0027	-0.0257	0.0027	-0.0269	0.0028
		$R_{4,5}$	0.0101	0.0011	0.0223	0.0015	-0.0157	0.0011	-0.0161	0.0011	-0.0169	0.0012
	9,13,13,9,11	α_1	-0.8252	0.7569	-0.8105	0.7274	0.0237	0.0118	0.0222	0.0116	0.0192	0.0113
		α_2	-0.2826	0.0868	-0.2493	0.0715	0.0105	0.0081	0.0092	0.0081	0.0066	0.0080
		λ	-0.9980	1.0273	-0.9458	0.9331	-0.0390	0.0135	-0.0408	0.0137	-0.0442	0.0142
		β	0.9210	0.9422	0.6533	0.4777	0.2694	0.0761	0.2655	0.0738	0.2574	0.0693
		$R_{2,5}$	-0.0033	0.0032	0.0254	0.0038	0.0003	0.0023	-0.0001	0.0024	-0.0009	0.0024
		$R_{3,5}$	-0.0016	0.0017	0.0195	0.0021	0.0007	0.0013	0.0005	0.0013	-0.0001	0.0013
		$R_{4,5}$	-0.0007	0.0007	0.0130	0.0009	0.0007	0.0005	0.0005	0.0005	0.0002	0.0005
II	7,10,10,8,9	α_1	0.1402	0.6173	0.0989	0.5211	0.0584	0.0192	0.0558	0.0187	0.0507	0.0177
		α_2	-0.0594	0.0195	0.0955	0.0192	0.0837	0.0195	0.0814	0.0188	0.0767	0.0175
		λ	-0.6457	0.5038	0.5071	0.4568	-0.0841	0.0258	-0.0874	0.0268	-0.0941	0.0288
		β	0.8474	0.7433	0.6247	0.4487	0.4021	0.1670	0.3946	0.1606	0.3788	0.1477
		$R_{2,5}$	-0.0241	0.0080	0.0476	0.0079	0.0292	0.0038	0.0286	0.0037	0.0273	0.0036
		$R_{3,5}$	-0.0157	0.0042	0.0368	0.0046	0.0223	0.0021	0.0218	0.0021	0.0209	0.0020
		$R_{4,5}$	-0.0094	0.0018	0.0248	0.0021	0.0148	0.0009	0.0145	0.0009	0.0139	0.0009
	9,13,13,9,11	α_1	0.1264	0.5814	0.0892	1.1950	0.0430	0.0103	0.0414	0.0100	0.0383	0.0096
		α_2	-0.0011	0.0140	0.0977	0.1798	0.0754	0.0138	0.0737	0.0134	0.0705	0.0126
		λ	-0.3463	0.4628	-0.4952	0.8521	-0.0601	0.0153	-0.0620	0.0157	-0.0657	0.0165
		β	0.8365	0.4767	0.5830	0.3726	0.3238	0.1086	0.3183	0.1048	0.3070	0.0973
		$R_{2,5}$	-0.0059	0.0051	0.0397	0.0061	0.0283	0.0028	0.0279	0.0028	0.0269	0.0027
		$R_{3,5}$	-0.0031	0.0027	0.0306	0.0035	0.0214	0.0016	0.0210	0.0016	0.0203	0.0015
		$R_{4,5}$	-0.0015	0.0011	0.0205	0.0016	0.0142	0.0007	0.0139	0.0007	0.0135	0.0007

Table 3. Bias, and MSE for Bias, and MSE for MLE, MPS and Bayesian when $\alpha_1 = 0.5,\ \lambda = 2,\ \alpha_2 = 3,\ \beta = 1.2$.

$\alpha_1 = 0.5, \lambda = 2, \alpha_2 = 3, \beta = 1.2$												
			MLE		MPS		SELF		LELF (c=0.5)		LELF (c=0.5)	
S	m_i		Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
I	7,10,10,8,9	α_1	-0.3231	0.1160	-0.3187	0.1116	-0.2883	0.0844	-0.2899	0.0853	-0.2928	0.0869
		α_2	-2.3918	5.9435	-2.4610	6.1722	-0.1364	0.0377	-0.1406	0.0396	-0.1487	0.0434
		λ	-0.5085	0.8239	-0.6595	0.7020	0.1527	0.0370	0.1488	0.0354	0.1410	0.0321
		β	-0.6135	0.5168	-0.8425	0.7978	-0.4011	0.1816	-0.4138	0.1937	-0.4380	0.2179
		$R_{2,5}$	-0.0546	0.0041	-0.0668	0.0055	0.0136	0.0002	0.0136	0.0002	0.0136	0.0002
		$R_{3,5}$	-0.1098	0.0156	-0.1319	0.0202	0.0489	0.0024	0.0491	0.0024	0.0494	0.0025
		$R_{4,5}$	-0.1594	0.0314	-0.1884	0.0397	0.1197	0.0146	0.1202	0.0147	0.1212	0.0149
	9,13,13,9,11	α_1	-0.1979	0.0580	-0.2102	0.0571	-0.2177	0.0493	-0.2191	0.0499	-0.2219	0.0512
		α_2	-1.7902	3.7333	-1.9899	4.1974	-0.0669	0.0166	-0.0689	0.0171	-0.0730	0.0182
		λ	-0.1231	0.6531	-0.3797	0.4140	0.0781	0.0167	0.0762	0.0162	0.0725	0.0153
		β	-0.5686	0.5054	-0.8537	0.7814	-0.2391	0.0702	-0.2442	0.0733	-0.2542	0.0797
		$R_{2,5}$	-0.0364	0.0021	-0.0465	0.0029	0.0116	0.0001	0.0117	0.0001	0.0118	0.0001
		$R_{3,5}$	-0.0770	0.0088	-0.0970	0.0119	0.0398	0.0016	0.0400	0.0017	0.0404	0.0017
		$R_{4,5}$	-0.1161	0.0190	-0.1446	0.0251	0.0912	0.0087	0.0917	0.0088	0.0928	0.0090
II	7,10,10,8,9	α_1	-0.0258	0.0328	-0.0717	0.0388	-0.1630	0.0299	-0.1643	0.0304	-0.1669	0.0312
		α_2	-0.3955	1.1937	-0.4753	2.1640	-0.0271	0.0214	-0.0299	0.0217	-0.0353	0.0224
		λ	0.8106	1.7667	0.4953	1.5316	0.0554	0.0230	0.0526	0.0224	0.0470	0.0213
		β	-0.6871	0.5263	-0.8098	0.7180	-0.2307	0.0704	-0.2361	0.0736	-0.2466	0.0800
		$R_{2,5}$	-0.0080	0.0002	-0.0070	0.0003	0.0094	0.0001	0.0095	0.0001	0.0096	0.0001
		$R_{3,5}$	-0.0186	0.0012	-0.0147	0.0016	0.0311	0.0011	0.0313	0.0011	0.0316	0.0011
		$R_{4,5}$	-0.0291	0.0037	-0.0204	0.0046	0.0687	0.0054	0.0691	0.0055	0.0700	0.0056
	9,13,13,9,11	α_1	-0.0368	0.0269	-0.0381	0.0353	-0.1543	0.0266	-0.1555	0.0269	-0.1578	0.0277
		α_2	-0.2684	0.5381	-0.4694	1.7106	-0.0234	0.0122	-0.0251	0.0123	-0.0284	0.0126
		λ	0.5297	0.6023	0.4497	1.0560	0.0447	0.0141	0.0430	0.0138	0.0398	0.0133
		β	-0.6739	0.5091	-0.8528	0.6536	-0.1712	0.0409	-0.1745	0.0425	-0.1811	0.0456
		$R_{2,5}$	-0.0029	0.0001	-0.0094	0.0003	0.0091	0.0001	0.0092	0.0001	0.0093	0.0001
		$R_{3,5}$	-0.0052	0.0009	-0.0211	0.0017	0.0299	0.0010	0.0300	0.0010	0.0304	0.0010
		$R_{4,5}$	-0.0049	0.0029	-0.0326	0.0046	0.0651	0.0048	0.0656	0.0049	0.0664	0.0050

Table 4. Length of CI for MLE, MPS and Bayesian when $\alpha_1 = 1.3,\ \lambda = 1.2,\ \alpha_2 = 2,\ \beta = 1.5$.

$\alpha_1 = 1.3, \lambda = 1.2, \alpha_2 = 2, \beta = 1.5$													
			MLE				MPS				Bayesian		
	m_i		L.CI	CP	BP	BT	L.CI	CP	BP	BT	L.CI	L.CI	L.CI
I	7,10,10,8,9	α_1	3.1817	0.9440	0.2955	0.1936	0.8224	96%	0.0581	0.0582	0.4513	0.4488	0.4471
		α_2	5.6962	0.9520	0.5681	0.3573	1.0666	96%	0.0755	0.0755	0.5389	0.5387	0.5333
		λ	4.6884	0.9440	0.4749	0.3172	1.6126	96%	0.0863	0.0837	0.5159	0.5188	0.5120
		β	1.2733	0.9520	0.0929	0.0954	1.2084	69%	0.0580	0.0580	0.3983	0.3914	0.3873
		$R_{2,5}$	0.2859	0.9440	0.0122	0.0121	0.2385	96%	0.0108	0.0109	0.1174	0.1181	0.1178
		$R_{3,5}$	0.3451	0.9520	0.0124	0.0124	0.2481	95%	0.0111	0.0113	0.1367	0.1383	0.1374
		$R_{4,5}$	0.3301	0.9440	0.0108	0.0108	0.2086	95%	0.0094	0.0092	0.1243	0.1257	0.1248
	9,13,13,9,11	α_1	3.0721	0.9517	0.1040	0.0705	1.2440	95%	0.0358	0.0347	0.4130	0.4126	0.4104
		α_2	4.7821	0.9571	0.1288	0.0923	1.6616	96%	0.0490	0.0489	0.4800	0.4819	0.4812
		λ	4.5989	0.9517	0.1631	0.1365	1.8531	95%	0.0745	0.0726	0.4028	0.4030	0.3994
		β	1.0024	0.9571	0.0614	0.0610	1.0397	80%	0.0444	0.0444	0.3027	0.2984	0.2907
		$R_{2,5}$	0.2479	0.9517	0.0111	0.0110	0.2230	95%	0.0100	0.0101	0.1038	0.1040	0.1024
		$R_{3,5}$	0.2958	0.9571	0.0122	0.0121	0.2359	95%	0.0109	0.0111	0.1203	0.1212	0.1206
		$R_{4,5}$	0.2768	0.9517	0.0099	0.0098	0.2006	95%	0.0090	0.0090	0.1088	0.1097	0.1088
II	7,10,10,8,9	α_1	1.3806	0.9314	0.1942	0.1499	2.5109	93%	0.1190	0.1191	0.4597	0.4629	0.4572
		α_2	1.0686	0.9580	0.1950	0.2615	4.2213	92%	0.1886	0.1897	0.5881	0.5931	0.5861
		λ	1.6186	0.9640	0.2098	0.2074	3.3981	97%	0.1570	0.1576	0.5437	0.5339	0.5224
		β	1.9230	0.9580	0.1392	0.1832	1.1058	93%	0.0511	0.0512	0.3834	0.3854	0.3907
		$R_{2,5}$	0.2735	0.9484	0.0685	0.0653	0.2585	95%	0.0120	0.0120	0.1281	0.1293	0.1231
		$R_{3,5}$	0.2853	0.9580	0.0823	0.0742	0.3077	95%	0.0141	0.0139	0.1532	0.1556	0.1585
		$R_{4,5}$	0.2413	0.9400	0.0792	0.0694	0.2890	95%	0.0124	0.0123	0.1422	0.1445	0.1482
	9,13,13,9,11	α_1	1.3702	0.9580	0.1492	0.1190	2.5499	98%	0.1096	0.1098	0.4036	0.4036	0.4051
		α_2	1.0091	0.9600	0.1590	0.1231	4.1534	96%	0.1741	0.1731	0.4666	0.4688	0.4688
		λ	1.3083	0.9696	0.1492	0.1743	3.4135	96%	0.1449	0.1459	0.4299	0.4255	0.4179
		β	1.4230	0.9600	0.0572	0.0571	0.8867	91%	0.0399	0.0400	0.3372	0.3362	0.3339
		$R_{2,5}$	0.2482	0.9580	0.0146	0.0138	0.2273	96%	0.0095	0.0094	0.1000	0.1011	0.1040
		$R_{3,5}$	0.2592	0.9600	0.0175	0.0165	0.2661	95%	0.0114	0.0115	0.1181	0.1194	0.1229
		$R_{4,5}$	0.2183	0.9580	0.0160	0.0158	0.2455	95%	0.0113	0.0111	0.1085	0.1096	0.1129

Table 5. Length of CI for MLE, MPS and Bayesian when $\alpha_1 = 0.5,\ \lambda = 2,\ \alpha_2 = 3,\ \beta = 1.2$.

$\alpha_1 = 0.5, \lambda = 2, \alpha_2 = 3, \beta = 1.2$														
			MLE				MPS				Bayesian			
S	m_i		L.CI	CP	BP	BT	L.CI	CP	BP	BT	L.CI	L.CI	L.CI
I	7,10,10,8,9	α_1	0.5388	95.84%	0.0236	0.0231	0.4229	95.80%	0.0197	0.0194	0.1330	0.1326	0.1317
		α_2	2.8524	97.40%	0.1238	0.1188	1.8529	97.80%	0.0887	0.0891	0.5233	0.5345	0.5504
		λ	3.1342	97.00%	0.1412	0.1411	2.9503	97.60%	0.0950	0.0937	0.4379	0.4330	0.4169
		β	1.4704	95.40%	0.0652	0.0631	1.4704	92.20%	0.0554	0.0556	0.5403	0.5621	0.6123
		$R_{2,5}$	0.1337	95.40%	0.0058	0.0057	0.1337	93.00%	0.0059	0.0059	0.0061	0.0061	0.0060
		$R_{3,5}$	0.2334	95.00%	0.0101	0.0102	0.2334	92.60%	0.0102	0.0101	0.0267	0.0267	0.0265
		$R_{4,5}$	0.3032	94.80%	0.0143	0.0142	0.3032	92.40%	0.0120	0.0117	0.0762	0.0767	0.0767
	9,13,13,9,11	α_1	0.4229	97.20%	0.0220	0.0195	0.4174	97.20%	0.0190	0.0181	0.1163	0.1164	0.1164
		α_2	1.8529	96.00%	0.0899	0.0839	1.7953	97.20%	0.0610	0.0574	0.4323	0.4336	0.4438
		λ	2.9503	96.60%	0.1302	0.1298	2.1869	98.00%	0.0893	0.0877	0.3929	0.3876	0.3805
		β	1.1279	95.20%	0.0524	0.0524	1.1005	90.40%	0.0429	0.0424	0.4281	0.4370	0.4542
		$R_{2,5}$	0.1125	95.60%	0.0051	0.0051	0.1123	92.60%	0.0050	0.0048	0.0031	0.0031	0.0030
		$R_{3,5}$	0.2110	95.60%	0.0085	0.0087	0.2126	92.40%	0.0083	0.0083	0.0172	0.0171	0.0169
		$R_{4,5}$	0.2918	95.60%	0.0126	0.0124	0.2974	92.20%	0.0112	0.0115	0.0607	0.0605	0.0602
II	7,10,10,8,9	α_1	0.7336	98.40%	0.5691	0.5516	0.7204	95.60%	0.0314	0.0314	0.2192	0.2185	0.2188
		α_2	4.1699	96.40%	0.4572	0.4431	5.4627	93.84%	0.2435	0.2422	0.5421	0.5431	0.5481
		λ	4.3129	97.20%	0.3872	0.3473	4.4502	94.60%	0.1981	0.1915	0.5373	0.5312	0.5286
		β	0.9525	96.00%	0.0504	0.0504	0.9795	96.20%	0.0444	0.0427	0.4991	0.5135	0.5350
		$R_{2,5}$	0.0434	95.80%	0.0021	0.0021	0.0613	90.80%	0.0027	0.0028	0.0099	0.0099	0.0099
		$R_{3,5}$	0.1162	95.80%	0.0059	0.0059	0.1459	89.80%	0.0065	0.0067	0.0381	0.0385	0.0387
		$R_{4,5}$	0.2187	96.40%	0.0118	0.0116	0.2533	87.80%	0.0116	0.0116	0.0974	0.0984	0.0988
	9,13,13,9,11	α_1	0.6389	99.80%	0.0646	0.0630	0.7216	98.20%	0.0312	0.0311	0.2010	0.2022	0.2044
		α_2	2.7270	99.80%	0.2178	0.1547	4.7902	83.40%	0.2177	0.2209	0.4051	0.4074	0.4149
		λ	2.2657	99.80%	0.1677	0.1597	3.6258	85.00%	0.1658	0.1653	0.4314	0.4325	0.4332
		β	0.9371	97.80%	0.0391	0.0376	0.7662	92.40%	0.0343	0.0337	0.4059	0.4136	0.4299
		$R_{2,5}$	0.0425	95.00%	0.0023	0.0023	0.0594	90.60%	0.0025	0.0025	0.0098	0.0097	0.0097
		$R_{3,5}$	0.1147	95.40%	0.0055	0.0055	0.1384	89.40%	0.0063	0.0063	0.0368	0.0370	0.0370
		$R_{4,5}$	0.2155	95.80%	0.0112	0.0112	0.2317	88.20%	0.0105	0.0111	0.0915	0.0917	0.0923

Table 6. Length of CI for MLE, MPS, Bayesian when $\alpha_1 = 1.5, \lambda = 0.5, \alpha_2 = 0.5, \beta = 1.2$.

			$\alpha_1 = 1.5, \lambda = 0.5, \alpha_2 = 0.5, \beta = 1.2$										
			MLE				MPS				Bayesian		
S	m_i		L.CI	CP	BP	BT	L.CI	CP	BP	BT	L.CI	L.CI	L.CI
I	7,10,10,8,9	α_1	0.8685	97.00%	0.0513	0.0483	1.0049	96.80%	0.0474	0.0472	0.4759	0.4795	0.4832
		α_2	0.2534	95.60%	0.0152	0.0151	0.3274	94.40%	0.0175	0.0172	0.4061	0.4087	0.4093
		λ	0.8133	97.00%	0.0414	0.0375	1.0776	96.80%	0.0464	0.0448	0.4796	0.4816	0.4864
		β	1.7576	95.40%	0.0793	0.0817	1.2285	79.00%	0.0572	0.0568	0.2906	0.2879	0.2784
		$R_{2,5}$	0.2572	94.40%	0.0119	0.0119	0.2508	93.00%	0.0110	0.0110	0.2408	0.2415	0.2437
		$R_{3,5}$	0.1917	94.40%	0.0087	0.0087	0.1901	92.60%	0.0088	0.0088	0.1733	0.1738	0.1753
		$R_{4,5}$	0.1260	94.40%	0.0056	0.0060	0.1264	92.80%	0.0057	0.0055	0.1111	0.1114	0.1124
	9,13,13,9,11	α_1	1.0810	95.80%	0.0410	0.0393	1.0413	95.60%	0.0454	0.0457	0.4103	0.4083	0.4030
		α_2	0.3260	95.40%	0.0113	0.0109	0.3787	94.40%	0.0138	0.0140	0.3403	0.3401	0.3385
		λ	0.6944	95.20%	0.0300	0.0320	0.7712	94.40%	0.0326	0.0318	0.4114	0.4161	0.4177
		β	1.2031	96.40%	0.0539	0.0535	0.8853	75.80%	0.0396	0.0396	0.2294	0.2241	0.2149
		$R_{2,5}$	0.2229	95.00%	0.0098	0.0099	0.2189	92.40%	0.0100	0.0102	0.1847	0.1849	0.1865
		$R_{3,5}$	0.1633	95.20%	0.0072	0.0071	0.1635	92.20%	0.0070	0.0072	0.1352	0.1353	0.1362
		$R_{4,5}$	0.1060	95.20%	0.0049	0.0050	0.1075	92.20%	0.0045	0.0044	0.0876	0.0877	0.0882
II	7,10,10,8,9	α_1	3.4992	99.80%	0.4719	0.4704	3.2811	99.80%	0.3482	0.3690	0.4743	0.4728	0.4708
		α_2	0.5714	99.80%	0.3131	0.3061	0.5218	99.80%	0.2913	0.2534	0.3966	0.3918	0.3849
		λ	1.3347	99.80%	0.5013	0.5011	1.2631	99.80%	0.2216	0.2159	0.5294	0.5364	0.5405
		β	0.7185	95.40%	0.0579	0.0584	0.9483	87.80%	0.0436	0.0433	0.2862	0.2773	0.2519
		$R_{2,5}$	0.3899	94.00%	0.0146	0.0144	0.2950	90.00%	0.0134	0.0133	0.2034	0.2028	0.2016
		$R_{3,5}$	0.2852	94.80%	0.0103	0.0102	0.2243	90.40%	0.0099	0.0099	0.1522	0.1517	0.1502
		$R_{4,5}$	0.1849	95.00%	0.0064	0.0064	0.1496	90.80%	0.0068	0.0068	0.1002	0.0999	0.0985
	9,13,13,9,11	α_1	3.0538	99.80%	0.4511	0.4531	4.2752	94.30%	0.1875	0.1868	0.3560	0.3570	0.3513
		α_2	0.4794	99.80%	0.2197	0.2018	1.6189	93.90%	0.0797	0.0724	0.3448	0.3429	0.3385
		λ	1.1899	99.80%	0.3827	0.3715	3.0569	94.10%	0.1452	0.1393	0.4109	0.4145	0.4219
		β	0.6036	95.40%	0.0361	0.0361	0.7104	82.60%	0.0355	0.0352	0.2332	0.2267	0.2152
		$R_{2,5}$	0.2831	93.80%	0.0123	0.0121	0.2648	89.60%	0.0121	0.0120	0.1758	0.1750	0.1733
		$R_{3,5}$	0.2064	94.00%	0.0094	0.0093	0.2002	89.80%	0.0091	0.0090	0.1307	0.1300	0.1284
		$R_{4,5}$	0.1337	94.00%	0.0064	0.0064	0.1328	90.00%	0.0060	0.0062	0.0857	0.0852	0.0838

7. Application of real data

For demonstration purposes, the analysis of a pair of real data sets is shown. The idea is to figure out how we can create conditions in which there is a lot of droughts. We assert that there will be no excessive drought if the water capacity of a reservoir in an area in August for at least two years out of the next five years is more than the amount of water achieved in December of 2019. It's also feasible that in this case, rather than entire samples from both groups, one sees censored samples. To achieve this purpose, we used the monthly water capacity of the Shasta reservoir in California, as well as the months of August and December from 1975 to 2016. http://cdec.water.ca.gov/cgi-progs/queryMonthly/SHA contains the data. Some writers have previously used these data, including Nadar and Kizilaslan [46], and Kızılaslan and Nadar [47].

In the whole data case, assuming k = 5 and s = 2, Y_1 represents December 1975 capacity while $X_{11}, ..., X_{15}$ represents August capacities from 1976 to 1980. Also, Y_2 represents December 1981 capacity, while $X_{21}, ..., X_{25}$ represents August capabilities from 1982 to 1986. n = 7 data for Y are acquired by continuing this approach till 2016. The corrected data are listed in Table 7.

Table 7. Transformed data.

X_1	X_2	X_3	X_4	X_5	y
0.287785	0.126977	0.768563	0.703119	0.729986	0.667157
0.811159	0.829569	0.726164	0.423813	0.715158	0.767135
0.363359	0.463726	0.371904	0.291172	0.414087	0.640395
0.538082	0.744881	0.722613	0.561238	0.813964	0.650691
0.668612	0.524947	0.605979	0.71585	0.529518	0.709025
0.742025	0.468782	0.345075	0.425334	0.76707	0.82486
0.613911	0.461618	0.294834	0.392917	0.6881	0.679829

To begin, we make sure that the POLO distribution can be utilized to examine the data set in Table 7. We get the MLEs of unknown parameters in Table 8. The Kolmogorov-Smirnov distance (KSD) values are also reported, along with the appropriate p-values. We can see from this table that the POLO distribution fits the data pretty well. For the data sets X and Y, the empirical cdf distribution, estimated pdf with histogram plot, and the PP-plot are given in Figures 1–3, respectively. In Figure 4, we plot the empirical cdf distribution, estimated pdf with histogram plot, and the PP-plot for $X = (x_1, x_2, x_3, x_4, x_5)$. These figures confirmed that the data have been fitted for POLO distribution.

Table 8. MLE and KS-test.

	X_1	X_2	X_3	X_4	X_5	y
α	19.6968	17.8297	16.7605	12.2651	16.1808	1.1683
λ	3.3794	3.9210	3.0245	1.2313	1.5485	0.0030
β	3.8742	2.7228	3.4386	3.8406	6.9615	16.7439
KSD	0.1770	0.2777	0.2607	0.2635	0.2449	0.1849
P-value	0.9538	0.5602	0.6382	0.6252	0.7122	0.9366

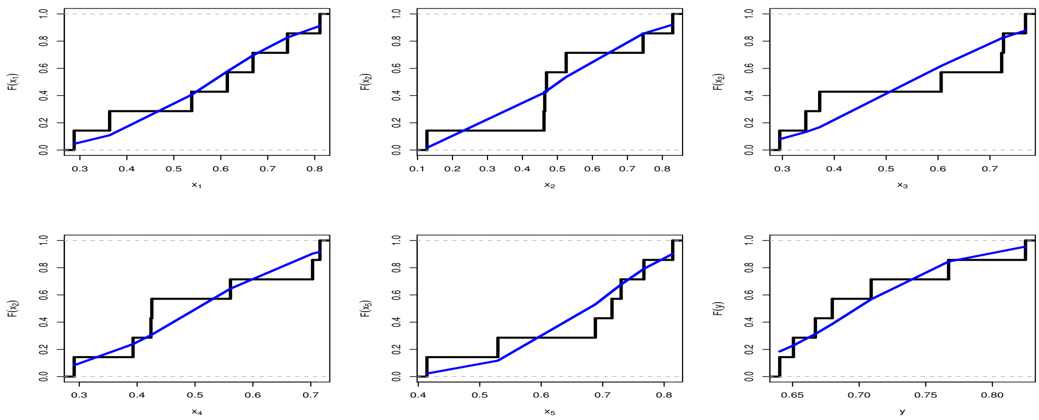

Figure 1. Estimated cdf for each variable of data.

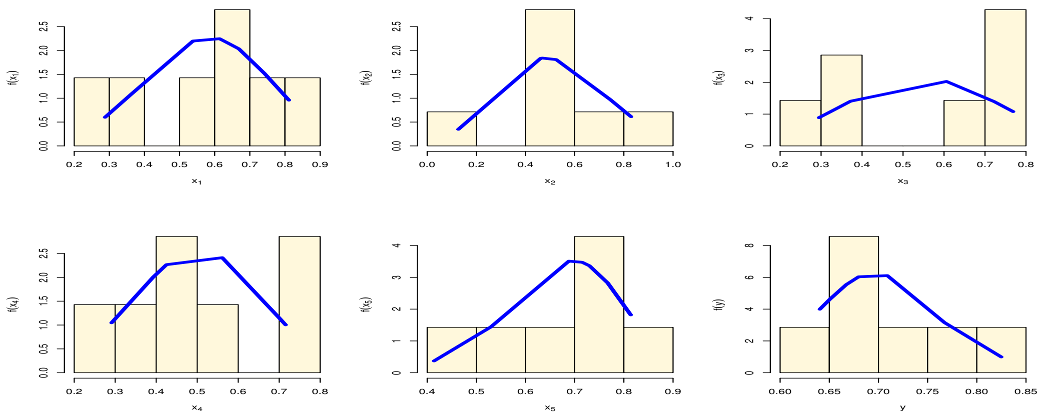

Figure 2. Estimated pdf for each variable of data.

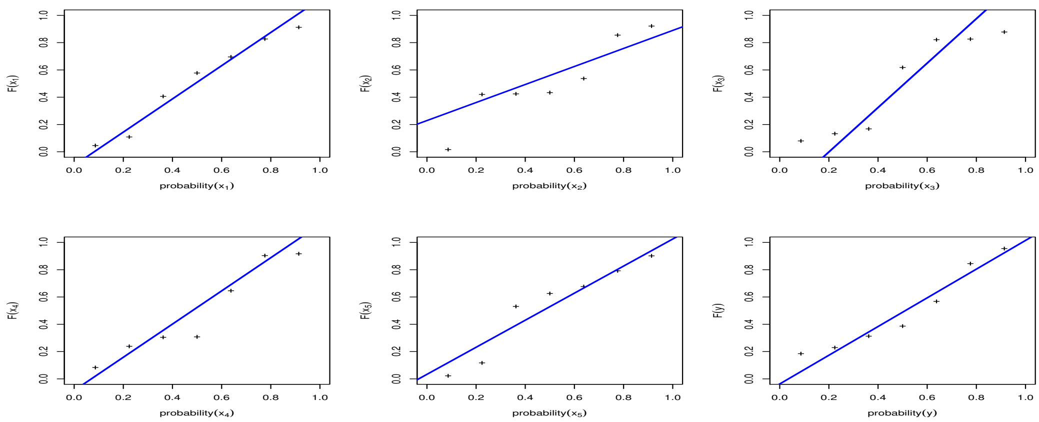

Figure 3. PP-plot for each variable of data.

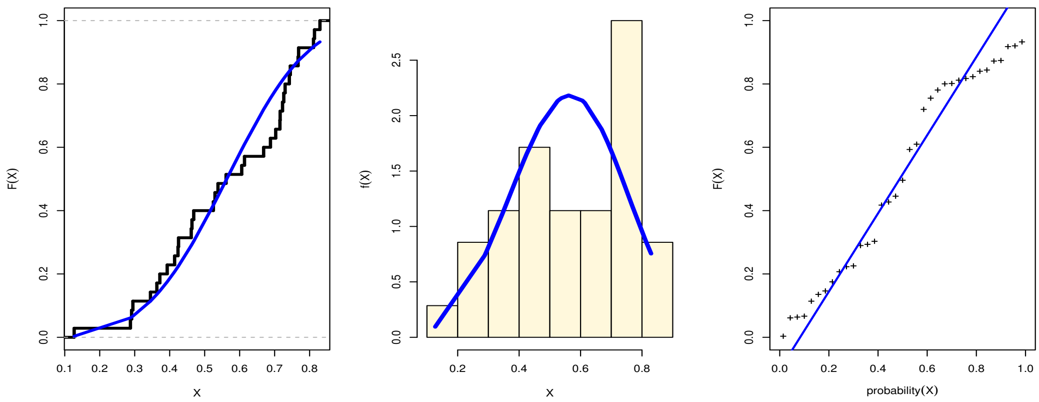

Figure 4. Estimated cdf, pdf and PP-plot for X variable.

Two distinct progressively censored samples are produced from the previous data sets for illustration purposes. In Table 9, the MLEs, MPS, and the Bayesian estimation of unknown parameters for the model have been obtained for this scheme. In Table 10, the estimation of reliability in a multi-component stress-strength model have been obtained. Figure 5 shows the trace and density plots for all parameters in the MCMC trace. Also it shows the trace and density plots for all parameters in an MCMC trace. The posterior density of MCMC results for each parameter is shown, which demonstrates a symmetric normal distribution that is identical to the proposed distribution. The convergence of the MCMC results is confirmed in Figure 5.

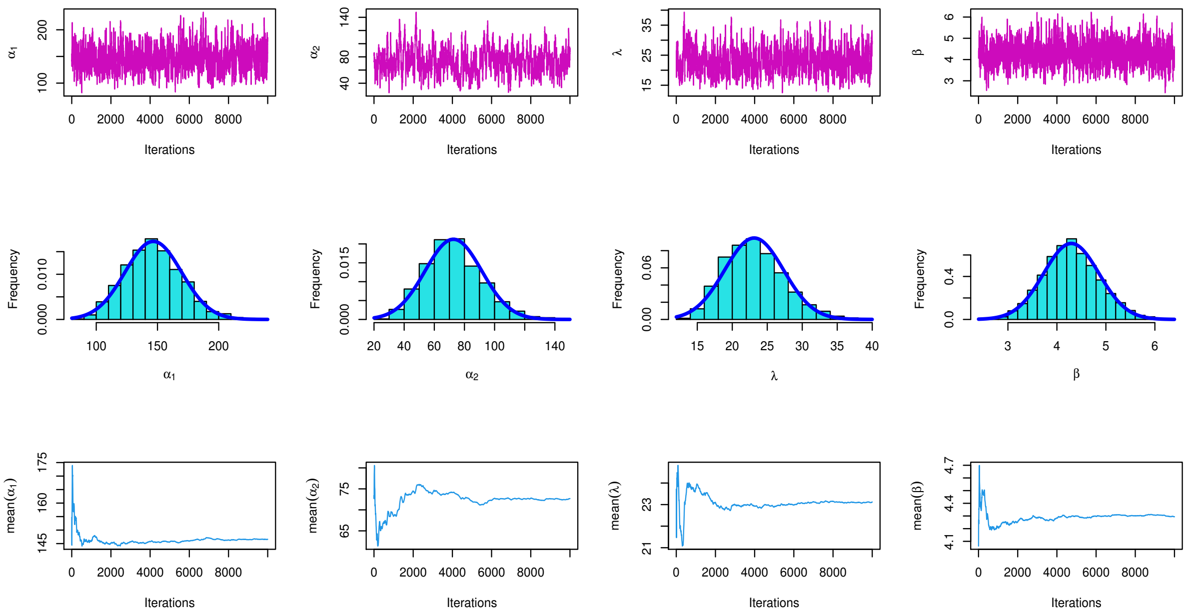

Figure 5. Convergence plots of MCMC for parameter estimates of the POLO base on multicomponent stress-strength model.

Table 9. MLE, MPS, and Bayesian estimation with different loss functions.

		Non-Bayesian				Bayesian			
		MLE		MPS		SELF		c=0.5	c=1.5
scheme		estimates	SE	estimates	SE	estimates	SE	estimates	estimates
Complelet	α_1	138.6205	399.0043	29.6983	55.2087	141.4085	21.7230	96.7112	87.4652
	α_2	85.6020	247.8306	19.7213	37.1186	88.6028	18.9293	42.9141	32.5260
	λ	22.3500	65.0217	7.7696	15.0518	22.6133	3.9472	19.4669	16.1599
	β	3.9753	0.5159	2.9737	0.4364	4.0554	0.5051	3.9927	3.8738
I	α_1	80.9614	329.1084	24.4695	69.4420	83.2840	18.8717	48.1163	37.1532
	α_2	66.4676	271.2264	19.6858	54.3504	69.4033	19.5470	37.0415	27.4366
	λ	27.2614	111.9030	12.9688	37.3678	27.7317	4.1830	24.1218	19.9190
	β	3.6505	0.5967	2.5016	0.4731	3.7391	0.5966	3.6531	3.4972
II	α_1	146.1876	453.5932	37.4513	87.7195	146.5873	23.0879	98.5567	87.8391
	α_2	68.5133	214.2732	19.9103	47.2777	72.6906	18.7809	39.4898	30.9711
	λ	22.5976	70.8227	10.7927	25.9588	23.1222	4.2110	19.8643	16.9361
	β	4.2557	0.6274	3.0077	0.5107	4.2941	0.5640	4.2159	4.0674

Table 10. Estimation of reliability in a multicomponent stress-strength model.

		Non-Bayesian		Bayesian		
scheme		mle	MPS	SELF	c=0.5	c=1.5
Complelet	$R_{1,5}$	0.6980	0.7203	0.7025	0.5917	0.5343
	$R_{2,5}$	0.5114	0.5346	0.5161	0.4105	0.3611
	$R_{3,5}$	0.3606	0.3801	0.3644	0.2798	0.2423
	$R_{4,5}$	0.2290	0.2429	0.2317	0.1732	0.1484
I	$R_{1,5}$	0.7821	0.7765	0.7863	0.7641	0.7520
	$R_{2,5}$	0.6033	0.5967	0.6081	0.5824	0.5689
	$R_{3,5}$	0.4404	0.4345	0.4448	0.4217	0.4098
	$R_{4,5}$	0.2873	0.2828	0.2906	0.2733	0.2645
II	$R_{1,5}$	0.6095	0.6504	0.6279	0.5585	0.5172
	$R_{2,5}$	0.4265	0.4646	0.4434	0.3815	0.3470
	$R_{3,5}$	0.2922	0.3223	0.3054	0.2576	0.2318
	$R_{4,5}$	0.1816	0.2021	0.1906	0.1585	0.1415

When the two schemes are compared using Bayesian and non-Bayesian, it is found that estimators in scheme 1 have lower standard errors than estimators in scheme 2. Also, it is found that estimators in scheme 1 have high reliability than estimators in scheme 2 and the whole scheme.

8. Conclusions

The study discusses the multi-component stress-strength model. The reliability has been investigated where both the stress and strength variables follow the POLO distribution. To calculate the multi-component stress-strength reliability $R_{s;k}$, we apply both classical and Bayesian methods.

We compute the Bayesian estimates of parameters and $R_{s;k}$ under symmetric and asymmetric loss functions by using the MH algorithm. We compute the classical estimates as MLE ad MPS of the parameter of the model and $R_{s;k}$ by using the Newton-Raphson (NR) and Markov Chain Monte Carlo algorithms. Based on the simulation analysis, we observe that the predicted risks of the proposed $R_{s;k}$ estimators show good behavior when the sample size increases. In general, as the sample size increases, the average length of the intervals decreases, therefor The average length of higher posterior density intervals is found to be shorter than that of asymptotic confidence intervals. Based on tabulated numerical results we find that the predicted risks of Bayesian estimation are often lower than the risks of the classical approaches. For illustrating the applicability of the multi-component stress strength model under POLO distribution we have examined a real life data taken from the monthly water capacity of the Shasta reservoir in California, and by using Kolmogorov-Smirnov distance (KSD)and its corresponding p-values we conclude that the above model fit the data very well. This study can further be extended by using different censoring schemes and different lifetime models.

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Conflict of interest

The authors declare no conflict of interest.

References

1. M. E. Ghitany, D. K. Al-Mutairi, S. M. Aboukhamseen, Estimation of the reliability of a stress-strength system from power Lindley distributions, *Commun. Stat.-Simul. Comput.*, **44** (2015), 118–136. https://doi.org/10.1080/03610918.2013.767910
2. J. Chen, C. Cheng, Reliability of stress-strength model for exponentiated Pareto distributions, *J. Stat. Comput. Simul.*, **87** (2017), 791–805. https://doi.org/10.1080/00949655.2016.1226309
3. A. Rezaei, M. Sharafi, J. Behboodian, A. Zamani, Inference on stress-streng the parameter based on GLD5 distribution, *Commun. Stat.-Simul. Comput.*, **47** (2018), 1251–1263. https://doi.org/10.1080/03610918.2017.1309666
4. V. K. Sharma, Bayesian analysis of head and neck cancer data using generalized inverse Lindley stress-strength reliability model, *Commun. Stat.-Theor. M.*, **47** (2018), 1155–1180. https://doi.org/10.1080/03610926.2017.1316858
5. A. I. Genç, Estimation of $P(X > Y)$ with Topp-Leone distribution, *J. Stat. Comput. Simul.*, **83** (2013), 326–339. https://doi.org/10.1080/00949655.2011.607821
6. H. Krishna, M. Dube, R. Garg, Estimation of $P(Y < X)$ for progressively first-failure-censored generalized inverted exponential distribution, *J. Stat. Comput. Simul.*, **87** (2017), 2274–2289. https://doi.org/10.1080/00949655.2017.1326119

7. S. Babayi, E. Khorram, Inference of stress-strength for the Type-II generalized logistic distribution under progressively Type-II censored samples, *Commun. Stat.-Simul. Comput.*, **47** (2018), 1975–1995. https://doi.org/10.1080/03610918.2017.1332214

8. M. Nadar, F. Kızılaslan, Classical and Bayesian estimation of $P(X < Y)$ using upper record values from Kumaraswamy's distribution, *Stat. Pap.*, **55** (2014), 751–783. https://doi.org/10.1007/s00362-013-0526-x

9. A. Tripathi, U. Singh, S. K. Singh, Estimation of $P(X < Y)$ for Gompertz distribution based on upper records, *Int. J. Model. Simul.*, **42** (2022), 388–399. https://doi.org/10.1080/02286203.2021.1923979

10. A. Asgharzadeh, R. Valiollahi, M. Z. Raqab, Estimation of $P(Y < X)$ for the two-parameter generalized exponential records, *Commun. Stat.-Simul. Comput.*, **46** (2017), 379–394. https://doi.org/10.1080/03610918.2014.964046

11. F. G. Akgül, B. Şenoğlu, Estimation of $P(X < Y)$ using ranked set sampling for Weibull distribution, *Qual. Technol. Quant. M.*, **14** (2017), 296–309. https://doi.org/10.1080/16843703.2016.1226590

12. F. G. Akgül, Ş. Acıtaş, B. Şenoğlu, Inference on stress-strength reliability based on ranked set sampling data in case of Lindley distribution, *J. Stat. Comput. Simul.*, **88** (2018), 3018–3032. https://doi.org/10.1080/00949655.2018.1498095

13. F. G. Akgül, B. Şenoğlu, Ş. Acıtaş, Interval estimation of the system reliability for Weibull distribution based on ranked set sampling data, *Hacet. J. Math. Stat.*, **47** (2018), 1404–1416. https://doi.org/10.15672/HJMS.2018.562

14. A. Safariyan, M. Arashi, R. A. Belaghi, Improved point and interval estimation of the stress strength reliability based on ranked set sampling, *Statistics*, **53** (2019), 101–125. https://doi.org/10.1080/02331888.2018.1547906

15. A. A. Al-Babtain, I. Elbatal, E. M. Almetwally, Bayesian and non-Bayesian reliability estimation of stress-strength model for power-modified Lindley distribution, *Comput. Intell. Neurosci.*, **2022** (2022). https://doi.org/10.1155/2022/1154705

16. M. A. Sabry, E. M. Almetwally, H. M. Almongy, Monte Carlo simulation of stress-strength model and reliability estimation for extension of the exponential distribution, *Thail. Statist.*, **20** (2022), 124–143.

17. M. M. Yousef, E. M. Almetwally, Multi stress-strength reliability based on progressive first failure for Kumaraswamy model: Bayesian and non-Bayesian estimation, *Symmetry*, **13** (2021), 2120. https://doi.org/10.3390/sym13112120

18. S. Rezaei, R. Tahmasbi, M. Mahmoodi, Estimation of $P[Y < X]$ for generalized Pareto distribution, *J. Stat. Plan. Infer.*, **140** (2010), 480–494. https://doi.org/10.1016/j.jspi.2009.07.024

19. D. Kundu, R. D. Gupta, Estimation of $P[Y < X]$ for Weibull distributions, *IEEE Trans. Reliab.*, **55** (2006), 270–280.

20. J. K. Jose, Estimation of stress-strength reliability using discrete phase type distribution, *Commun. Stat.-Theor. M.*, **51** (2022), 368–386. https://doi.org/10.1080/03610926.2020.1749663

21. E. M. Almetwally, R. Alotaibi, A. A. Mutairi, C. Park, H. Rezk, Optimal plan of multi-stress-strength reliability Bayesian and non-Bayesian methods for the alpha power exponential model using progressive first failure, *Symmetry*, **14** (2022), 1306. https://doi.org/10.3390/sym14071306
22. S. Kotz, Y. Lumelskii, M. Pensky, *The stress-strength model and its generalizations: Theory and applications*, Singapore, World Scientific, 2003.
23. G. K. Bhattacharyya, R. A. Johnson, Estimation of reliability in a multicomponent stress-strength model, *J. Am. Stat. Assoc.*, **69** (1974), 966–970. https://doi.org/10.1080/01621459.1974.10480238
24. A. Kohansal, On estimation of reliability in a multicomponent stress-strength model for a Kumaraswamy distribution based on progressively censored sample, *Stat. Pap.*, **60** (2017), 2185–2224. https://doi.org/10.1007/s00362-017-0916-6
25. F. Kızılaslan, Classical and Bayesian estimation of reliability in a multicomponent stress strength model based on the proportional reversed hazard rate mode, *Math. Comput. Simul.*, **136** (2017), 36–62. https://doi.org/10.1016/j.matcom.2016.10.011
26. S. Gunasekera, Classical, Bayesian, and generalized inferences of the reliability of a multicomponent system with censored data, *J. Stat. Comput. Simul.*, **88** (2018), 3455–3501. https://doi.org/10.1080/00949655.2018.1523410
27. N. Balakrishnan, R. Aggarwala, *Progressive censoring: Theory, methods, and applications*, Springer, Berlin, 2000.
28. N. Balakrishnan, E. Cramer, *The art of progressive censoring*, Springer, New York, 2014.
29. M. Z. Raqab, M. T. Madi, Inference for the generalized Rayleigh distribution based on progressively censored data, *J. Stat. Plan. Infer.*, **141** (2011), 3313–3322. https://doi.org/10.1016/j.jspi.2011.04.016
30. S. F. Wu, C. C. Wu, C. H. Chou, H. M. Lin, Statistical inferences of a two-parameter distribution with the bathtub shape based on progressive censored sample, *J. Stat. Comput. Simul.*, **81** (2011), 315–329. https://doi.org/10.1080/00949650903334221
31. M. K. Rastogi, Y. M. Tripathi, Estimating the parameters of a Burr distribution under progressive type II censoring, *Stat. Methodol.*, **9** (2012), 381–391. https://doi.org/10.1016/j.stamet.2011.10.002
32. E. H. A. Rady, W. A. Hassanein, T. A. Elhaddad, The power Lomax distribution with an application to bladder cancer data, *SpringerPlus*, **5** (2016), 1838. https://doi.org/10.1186/s40064-016-3464-y
33. C. R. Rao, *Linear statistical inference and its applications*, Wiley Eastern Limited, India, 1973.
34. H. K. T. Ng, L. Luo, Y. Hu, F. Duan, Parameter estimation of three parameter Weibull distribution based on progressively Type II censored samples, *J. Stat. Comput. Simul.*, **82** (2012), 1661–1678. https://doi.org/10.1080/00949655.2011.591797
35. R. C. H. Cheng, N. A. K. Amin, Estimating parameters in continuous univariate distributions with a shifted origin, *J. Roy. Stat. Soc. B*, **45** (1983), 394–403. https://doi.org/10.1111/j.2517-6161.1983.tb01268.x
36. R. C. H. Cheng, N. A. K. Amin, *Maximum product of spacings estimation with applications to the lognormal distribution*, Math. Report, University of Wales IST, 1979.
37. H. K. T. Ng, P. S. Chan, N. Balakrishnan, Optimal progressive censoring plans for the Weibull distribution, *Technometrics*, **46** (2004), 470–481. https://doi.org/10.1198/004017004000000482

38. W. S. Abu El Azm, E. M. Almetwally, A. S. Alghamdi, H. M. Aljohani, A. H. Muse, O. E. Abo-Kasem, Stress-strength reliability for exponentiated inverted Weibull distribution with application on breaking of Jute fiber and Carbon fibers, *Comput. Intel. Neurosc.*, **2021** (2021). https://doi.org/10.1155/2021/4227346

39. M. A. Sabry, E. M. Almetwally, O. A. Alamri, M. Yusuf, H. M. Almongy, A. S. Eldeeb, Inference of fuzzy reliability model for inverse Rayleigh distribution, *AIMS Math.*, **6** (2021), 9770–9785. https://doi.org/10.3934/math.2021568

40. U. Singh, S. K. Singh, R. K. Singh, A comparative study of traditional estimation methods and maximum product spacings method in generalized inverted exponential distribution, *J. Stat. Appl. Prob.*, **3** (2014), 153. https://doi.org/10.12785/jsap/030206

41. B. Efron, *The jacknife, the bootstrap and other resampling plans*, SIAM, Philadelphia, 1982.

42. R. E. Hall, Intertemporal substitution in consumption, *J. Polit. Econ.*, **96** (1988), 339–357.

43. H. R. Varian, *A Bayesian approach to real estate assessment*, North Holland, Amsterdam, 1975, 195–208.

44. A. Zellner, Bayesian estimation and prediction using asymmetric loss functions, *J. Am. Stat. Assoc.*, **81** (1986), 446–451. https://doi.org/10.1080/01621459.1986.10478289

45. N. Metropolis, A. W. Rosenbluth, M. N. Rosenbluth, A. H. Teller, Equations of state calculations by fast computing machines, *J. Chem. Phys.*, **21** (1953), 1087–1092. https://doi.org/10.1063/1.1699114

46. M. Nadar, F. Kızılaslan, Estimation of reliability in a multicomponent stress-strength model based on a Marshall-Olkin bivariate Weibull distribution, *IEEE T. Reliab.*, **65** (2015), 370–380. https://doi.org/10.1109/TR.2015.2433258

47. F. Kızılaslan, M. Nadar, Estimation of reliability in a multicomponent stress-strength model based on a bivariate Kumaraswamy distribution, *Stat. Pap.*, **59** (2018), 307–340. https://doi.org/10.1007/s00362-016-0765-8

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