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Research article

On the application of G_α integral transform to nonlinear dynamical models with non-integer order derivatives

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Abstract: The current study uses an essential and integrated form of Laplace-type integral transform coupled with the Adomian's approach to study nonlinear evolution equations endowed with non-integer derivatives. More so, of particular interest here is to demonstrate the application of this transform to a wider class of nonlinear problems. Three test models have been examined by the presented method, and their closed-form solutions have been reported iteratively. Lastly, the variational effect of the non-integer order derivatives on the evolution of these models has been studied via the two and three-dimensional depictions.

Keywords: integral transform method; dynamical models; evolution equations; non-integer derivatives
Mathematics Subject Classification: 44A05, 47J35

1. Introduction

Symmetry analysis method by Lie has been used extensively in the recent literature to study diverse forms of nonlinear evolution equations. However, as the determination of the symmetries is computationally demanding most at times, the present study resort to using a mixture of an integral transform method, and an iterative procedure. Integral transform method is an old efficient analytical method that is used to solve linear differential and integral equations. Thus, in favor of its amazing applications in tackling different mathematical physics problems, various integral transforms starting with the notable works of P. S. Laplace in the 1780s and J. Fourier in 1822 [1], respectively, have been continuously introduced till today. More so, recent years have witnessed the emergence of various integral transforms like the Sumudu transform in 1993 [2], Natural transform in 2008 [3], Elzaki transform in 2011 [4], Aboodh transform in 2013 [5], $\mathbb{M}$-transform in 2015 [6], ZZ [7] and

Ramadan group transforms [8] in 2016, and the G_α transform in 2017 [9], just to mention a few. However, of particular concern in this study is to further demonstrate the application of the G_α transform [9], which is alternatively called the generalized Laplace transform [10,11] in tackling certain nonlinear dynamical models with non-integer order; having unanimously represented multiple integral transforms in different values of α. Since the G_α transform is a generalized Laplace transform, it means that properties established in this transform are established in each transform having a specific alpha value. In this respect, this study is meaningful. The integral representation of the G_α integral transform reads

$$G_\alpha\{f(t)\} = F_\alpha(s) = s^\alpha \int_0^\infty f(t)e^{-t/s}dt, \qquad t \geq 0, \tag{1.1}$$

where s is the G_α transform parameter, and α is a generalized real parameter; we explain the role of this parameter for certain values of interest in the subsequent Section.

Furthermore, to demonstrate the applicability of this transform, we couple the G_α transform with Adomian's approach [12,13] to tackle a class of nonlinear evolution equations in the presence of non-integer order derivatives based on Caputo's fractional derivative definition [14–16]. Nonlinear evolution equations play an important role in many nonlinear sciences. More so, different approaches have been used in the past and present literature to tackle diverse classes of evolution equations. So, in order not to go too far in mentioning these known methods for solving different forms of nonlinear evolution equations, we briefly recall approaches like integration schemes [17–20], numerical approaches [21,22], and semi-analytical approaches [23–27]. On the other hand, we also mention some significant works on the theory and development of fractional calculus here to lay a solid foundation with regard to the non-integer order derivative [28–32]. It is also vital to point out that the theory fractional calculus is not something developed lately, as it goes back to the year of 1695.

In addition, this study considers three types of nonlinear evolution equations as tests problems to portray the application of the present methodology. These equations of interest are as follows

1) Burger's equation [13,17]

$$f_t + f f_x = f_{xx}, \tag{1.2}$$

2) Schrödinger differential equation [33]

$$i f_t + f_{xx} + 2f^2 f^* = 0, \tag{1.3}$$

3) Coupled singular inhomogeneous Burger's equation [34,35]

$$\begin{aligned} f_t &= x^{-1}\partial_x(x f_x) + 2f f_x - (fg)_x + h_1(x,t), \\ g_t &= x^{-1}\partial_x(x g_x) + 2g g_x - (fg)_x + h_2(x,t), \end{aligned} \tag{1.4}$$

where f^* is the conjugate of f in Eq (1.3), and $i = \sqrt{-1}$; while in the coupled equation given in Eq (1.4), $h_j(x,t)$ for $j = 1, 2$ are the prescribed source functions.

Moreover, the choice of this integral transform is to further reveal some of its salient and untapped properties. This of course is associated with the transform's ability to collectively represent multiple integral transforms for different values of α. Indeed, to our knowledge, this transform has never been coupled in the literature to study nonlinear dynamical equations; let alone, nonlinear dynamical models with non-integer order derivatives. Furthermore, Caputo's definition of the non-integer derivative is chosen in this study owing to its wide practicality. We also state here that the *Wolfram Mathematica 9* software will be utilized for the computational and graphical purposes. Besides, the current manuscript is organized in the following manner: Section 2 gives some preliminaries with regard to the G_α transform and non-integer order derivative. Section 3 presents the G_α decomposition approach; while Section 4 gives the application of this very approach. Section 5 discusses the obtained results; while Section 6 gives some concluding points.

2. Preliminaries

The present section outlines some important definitions related to the G_α integral transform, and on the other hand, the non-integer order derivative, which is alternatively known as the fractional derivative. The section systematically recalls some conceptual backgrounds of these two concepts.

2.1. G_α integral transform

This subsection recollects some vital definitions and concepts associated with the G_α transform to be utilized in the study.

Definition 1. *The G_α transform of the function $f(t)$ belonging to a set F defined by:*

$$F = \{f : |f(t)| < Me^{k_j|t|},\ if\ t \in (-1)^j \times [0, \infty),\ j = 1, 2;\ (M, k_1, k_2 > 0\ )\},$$

where M is a finite constant number; k_1, k_2 may be finite or infinite is defined by [9–11]

$$G_\alpha\{f(t)\} = F_\alpha(s) = s^\alpha \int_0^\infty f(t)e^{-t/s}dt,\quad t \geq 0,\quad k_1 \leq s \leq k_2. \tag{2.1}$$

Equally, one may alternatively express G_α transform as follows

$$G_\alpha\{f(t)\} = F_{\alpha+1}(s) = s^{\alpha+1} \int_0^\infty f(st)e^{-t}dt,\quad t \geq 0,\quad k_1 \leq s \leq k_2. \tag{2.2}$$

The Laplace transform by the logarithm function can be expressed as

$$s^{-\alpha} \int_1^\infty f(\ln x)x^{-s-1}dx, \qquad t = \ln x,$$

where in this case the transform reduces to each Laplace-type transform upon varying the value of α as follows. Of course, dozens of Laplace-type transforms not described here can also be expressed by changing the value of α. Moreover, some of the Kernel values of recent Laplace-type transforms are given in Table 1.

Table 1. Kernel values of recent Laplace-type transforms.

S	-1
E	1
L	0
M	-2

Where from Table 1, the letters S, E, L and M stand for Sumudu, Elzaki, Laplace, and Mohand transforms, respectively.

Using

$$G(f) = s^{\alpha} \cdot F(1/s),$$

for Laplace transform $\pounds(f) = F(s)$, we can obtain the detailed relationship with Laplace transform as illustrated on page 2 of [9] and the introduction of [10]. In order to have value among many Laplace-type transforms, the form should be simple and the formula of Laplace transform should be available through a simple relational expression.

Lemma 2. *For G_α transform, the following properties holds [11]:*

1) $G_\alpha\{1\} = s^{\alpha+1}$,
2) $G_\alpha\{t^n\} = n! s^{n+\alpha+1},\ n > 1$,
3) $G_\alpha\{\sin(at)\} = \frac{as^{\alpha+2}}{1+s^2a^2}$,
4) $G_\alpha\{\cos(at)\} = \frac{s^{\alpha+1}}{1+s^2a^2}$,
5) *for any functions $x(t)$ and $y(t)$ defined over the set F, and constants a and b, then*

$$G_\alpha\{ax(t) \pm bw(t)\} = aG_\alpha\{x(t)\} \pm bG_\alpha\{y(t)\}.$$

Lemma 3. *Given the integrable function $f(t)$, the G_α transform of the $n^{th}(n \in \mathbb{N})$ derivative of $f(t)$ is given by [10]*

$$G_\alpha\{f^{(n)}(t)\} = s^{-n} G_\alpha\{f(t)\} - s^{\alpha} \sum_{k=1}^{n} s^{k-n} f^{(k-1)}(0). \tag{2.3}$$

2.2. Non-integer order derivative

Non-integer order calculus or rather the fractional order calculus is as old as the integer order calculus. However, there have been tremendous discoveries in recent times; having abandoned the area in the past. Thus, the present subsection recalls certain important definitions specifically based on Caputo's fractional derivative [14–16].

Definition 4. *The Caputo's fractional derivative of the fractional order $\mu > 0$ for a function $f(t)$ is defined by [16]*

$$D_t^{\mu} f(t) = \frac{1}{\Gamma(n-\mu)} \int_0^{\infty} \frac{f^{(n)}(s)}{(t-s)^{-n+\mu+1}} ds, \quad n-1 < \mu \leq n, \tag{2.4}$$

where $f(t) = 0$ for $t < 0$; while $\Gamma(.)$ is the gamma function given for $n(> -1) \in \mathbb{R}$ as $\Gamma(n+1) = n!$.

Definition 5. *The Caputo's fractional derivative satisfies the following useful properties:*

1) $D_t^{\mu}(a) = 0$, *a constant,*
2) $D_t^{\mu}(t^n) = \frac{\Gamma(1+n)}{\Gamma(1+n-\mu)} t^{n-\mu}$,
3) $D_t^{\mu}(ax(t)) = aD_t^{\mu} x(t)$, *a constant,*
4) $D_t^{\mu}(ax(t) \pm by(t)) = aD_t^{\mu}(x(t)) \pm bD_t^{\mu}(y(t))$.

Definition 6. *The Mittag-Leffler function for one parameter μ is defined as [16]*

$$E_\mu(t) = \sum_{n=0}^{\infty} \frac{t^n}{\Gamma(1+\mu n)}, \quad \mu > 0,\ t \in C. \tag{2.5}$$

Lemma 7. *The G_α transform for the Caputo's fractional derivative defined in Eq (2.4) is given by*

$$G_\alpha\{D_t^\mu(f(t))\} = s^{-\mu}G_\alpha\{f(t)\} - s^\alpha \sum_{k=1}^{n} s^{k-\mu} f^{(k-1)}(0), \quad n-1<\mu\leq n. \tag{2.6}$$

Proof. Without loss of generality, the proof follows directly from *Lemma 3* upon considering a fractional order $\mu > 0$ in the interval $n-1<\mu\leq n,\ n\in\mathbb{N}$. □

3. G_α decomposition approach

This section makes use of the G_α integral transform earlier discussed and further couples it with the famous decomposition approach by Adomian [12] to present the methodology of the current study. In short, the section derives iterative closed-form solutions to a generalized nonlinear dynamical model with an arbitrary non-integer (fractional) order derivative in mathematical physics. Thus, to present the methodology, let us consider the following theorem:

Theorem 8. *Considering the generalized nonlinear non-integer order Initial-Value Problem (IVP)*

$$\begin{aligned} D_t^\mu f(x,t) &= Lf(x,t) + Ff(x,t) + Nf(x,t) + h(x,t),\\ f(x,0) &= g(x), \end{aligned} \tag{3.1}$$

where L is the highest linear differential operator, F is also a linear operator, but with degree less that of L, N is the nonlinear differential operator, $h(x,t)$ is the inhomogeneous term, $g(x)$ is the prescribed initial data; while D_t^μ is the fractional order derivative defined in the Caputo's fractional sense for $0<\mu\leq 1$, *such that [16]*

$$D_t^\mu f(t) = J_{0+}^{1-\mu}\frac{d}{dt}f(t), \qquad J_{0+}^{\mu}f(t) = \frac{1}{\Gamma(\mu)}\int_0^t (t-s)^{\mu-1} f(s)ds, \tag{3.2}$$

then, the non-integer order IVP given in Eq (3.1) admits the following iterative G_α decomposition solution

$$\begin{cases} f_0(x,t) = g(x) + G_\alpha^{-1}\{s^\mu G_\alpha\{h(x,t)\}\}, \\ \\ f_{k+1}(x,t) = G_\alpha^{-1}\{s^\mu G_\alpha\{Lf_k(x,t) + Ff_k(x,t) + R_k\}\}, \quad k\geq 0, \end{cases} \tag{3.3}$$

Proof. Taking the G_α integral transform in t of Eq (3.1) alongside utilizing *Lemma 7*, Eq (3.1) then transforms to the following

$$G_\alpha\{f(x,t)\} = s^{\alpha+1}g(x) + s^\mu G_\alpha\{Lf(x,t) + Ff(x,t) + Nf(x,t)\} + s^\mu G_\alpha\{h(x,t)\}. \tag{3.4}$$

Next, we apply the inverse G_α transform G_α^{-1} to the above equation in s to obtain

$$f(x,t) = g(x) + G_\alpha^{-1}\{s^\mu G_\alpha\{h(x,t)\}\} + G_\alpha^{-1}\{s^\mu G_\alpha\{Lf(x,t) + Ff(x,t) + Nf(x,t)\}\}. \tag{3.5}$$

More so, via the Adomian's approach, we decompose the unknown solution $f(x,t)$ using the following infinite sum [12,13]

$$f(x,t) = \sum_{n=0}^{\infty} f_n(x,t), \tag{3.6}$$

and the nonlinear termed operator $Nf(x,t)$ using the following infinite sum of Adomian's polynomials as follows

$$Nf(x,t) = \sum_{n=0}^{\infty} R_n, \tag{3.7}$$

where R_n's are the polynomials devised by Adomian [12,13], and to be computed via the following recurrent formula

$$R_n = \frac{1}{n!}\frac{d^n}{d\xi^n}\Big[N\Big(\sum_{j=0}^{\infty}\xi^j f_j\Big)\Big]_{\xi=0}, \quad n = 0,1,2,... \tag{3.8}$$

So, rewriting Eq (3.5) in terms of the summations given in Eqs (3.6) and (3.7), we get

$$f(x,t) = g(x) + G_\alpha^{-1}\{s^\mu G_\alpha\{h(x,t)\}\} + G_\alpha^{-1}\Big[s^\mu G_\alpha\Big[L\sum_{n=0}^{\infty} f_n(x,t) + F\sum_{n=0}^{\infty} f_n(x,t) + \sum_{n=0}^{\infty} R_n\Big]\Big]. \tag{3.9}$$

Finally, identifying the terms arising from the prescribed initial data and nonhomogeneous function with the first component $f_0(x,t)$, and the rest of the terms recurrently follow as suggested by the approach, we thus get the following recurrent scheme

$$\begin{cases} f_0(x,t) = g(x) + G_\alpha^{-1}\{s^\mu G_\alpha\{h(x,t)\}\}, \\ f_{k+1}(x,t) = G_\alpha^{-1}\{s^\mu G_\alpha\{Lf_k(x,t) + Ff_k(x,t) + R_k\}\}, \quad k \geq 0 \end{cases} \tag{3.10}$$

□

Moreover, upon taking the net sum of these components, we get the final closed-form solution

$$f(x,t) = \lim_{M\to\infty}\left(\sum_{n=0}^{M} f_n(x,t)\right). \tag{3.11}$$

□

Theorem 9. *Consider the following coupled system of generalized nonlinear non-integer order IVPs under the assumptions of Theorem 8*

$$\begin{aligned} D_t^\mu f_j(x,t) &= Lf_j(x,t) + Ff_j(x,t) + N\{f_1, f_2, ..., f_n\}(x,t) + h_j(x,t), \\ f_j(x,0) &= g_j(x), \qquad for \quad j = 1,2,...,n, \end{aligned} \tag{3.12}$$

where $h_j(x,t)$ are the inhomogeneous terms, $g_j(x)$ are the prescribed initial data for $j = 1,2,...,n$. Then, the non-integer order coupled system of IVPs given in Eq (3.12) admits the following iterative G_α decomposition solution

$$\begin{cases} f_{j_0}(x,t) = g_j(x) + G_\alpha^{-1}\{s^\mu G_\alpha\{h_j(x,t)\}\}, \qquad j = 1,2,...,n, \\ f_{j_{(k+1)}}(x,t) = G_\alpha^{-1}\{s^\mu G_\alpha\{Lf_{j_k}(x,t) + Ff_{j_k}(x,t) + R_{j_k}\}\}, \quad k \geq 0, \quad j = 1,2,...,n. \end{cases} \tag{3.13}$$

Proof. Without lost of generality, the current proof generalizes the proof of Theorem 8 by considering n coupled system of generalized nonlinear non-integer order IVPs. □

4. Applications of G_α decomposition approach

The current section makes consideration to non-integer order versions of the nonlinear evolution equations given in Eqs (1.2)–(1.4), and further employs the approach just introduced in the above Section (via Theorems 8 and 9) to scrutinize the non-integer order models.

4.1. Time non-integer order Burger's equation

Let us consider the following time non-integer order Burger's equation

$$D_t^\mu f + f f_x = f_{xx}, \qquad 0 < \mu \leq 1, \tag{4.1}$$

together with the following prescribed initial condition

$$f(x,0) = 2x, \tag{4.2}$$

where $Lf(x,t) = \frac{d^2 f}{dx^2}$, $Ff(x,t) = 0$, $Nf(x,t) = ff_x$, $h(x,t) = 0$, and $g(x,t) = 2x$. Therefore, without further delay, the proposed G_α decomposition methodology posed the following recurrent scheme for the time non-integer order Burger's equation

$$\begin{cases} f_0(x,t) = f(x,0), \\ f_{k+1}(x,t) = G_\alpha^{-1}\{s^\mu G_\alpha\{f_{k_{xx}} - R_k\}\}, \quad k \geq 0, \end{cases} \tag{4.3}$$

with R_k's denoting the polynomials by Adomian in favour of the following nonlinear term

$$Nf(x,t) = ff_x, \tag{4.4}$$

where we express some of the its few terms using the application of Eq (3.8) as follows

$$\begin{aligned} R_0 &= f_0 f_{0_x}, \\ R_1 &= f_0 f_{1_x} + f_{0_x} f_1, \\ R_2 &= f_0 f_{2_x} + f_1 f_{1_x} + f_2 f_{0_x}, \\ &\vdots \end{aligned} \tag{4.5}$$

Thus, from the recurrent scheme given in Eq (4.3) via Eq (4.5), we get some of the components explicitly as follows

$$
\begin{aligned}
f_0(x,t) &= 2x,\\
f_1(x,t) &= G_\alpha^{-1}\{s^\mu G_\alpha\{f_{0_{xx}} - R_0\}\},\\
&= -\frac{4xt^\mu}{\Gamma(1+\mu)},\\
f_2(x,t) &= G_\alpha^{-1}\{s^\mu G_\alpha\{f_{1_{xx}} - R_1\}\},\\
&= \frac{16xt^{2\mu}}{\Gamma(1+2\mu)},\\
f_3(x,t) &= G_\alpha^{-1}\{s^\mu G_\alpha\{f_{2_{xx}} - R_2\}\},\\
&= -\frac{96xt^{3\mu}}{\Gamma(1+3\mu)},\\
f_4(x,t) &= G_\alpha^{-1}\{s^\mu G_\alpha\{f_{3_{xx}} - R_3\}\},\\
&= \frac{768xt^{4\mu}}{\Gamma(1+4\mu)},\\
&\vdots
\end{aligned}
\tag{4.6}
$$

Therefore, the net sum of the above recurrent components gives the following

$$
\begin{aligned}
f(x,t) &= \lim_{M\to\infty}\left(\sum_{n=0}^{\infty} f_n(x,t)\right),\\
f(x,t) &= \sum_{n=0}^{\infty} f_n(x,t) = 2x\Big(1 - \frac{\Gamma(1+1)(2t^\mu)}{\Gamma(1+\mu)} + \frac{\Gamma(1+2)(2t^\mu)^2}{\Gamma(1+2\mu)} - \frac{\Gamma(1+3)(2t^\mu)^3}{\Gamma(1+3\mu)}\\
&\quad + \frac{\Gamma(1+4)(2t^\mu)^4}{\Gamma(1+4\mu)} + \ldots\Big),\\
&= 2x\left[\lim_{M\to\infty}\left(\sum_{n=0}^{M}(-1)^n\frac{\Gamma(1+n)}{\Gamma(1+n\mu)}(2t^\mu)^n\right)\right].
\end{aligned}
\tag{4.7}
$$

Note that, when $\mu = 1$ in the above equation, which corresponds to the corresponding integer order model, the obtained series solution further reduces to the following

$$f(x,t) = 2x\left[\lim_{M\to\infty}\left(\sum_{n=0}^{M}(-1)^n(2t)^n\right)\right], \tag{4.8}$$

which obviously converges to the following closed-form solution [36]

$$f(x,t) = \frac{2x}{1+2t}. \tag{4.9}$$

Additionally, we have shown in Figure 1(a) the three-dimensional (3D) and Figure 1(b) the two-dimensional (2D) graphical depictions of the obtained series solution in Eq (4.7) of the non-integer order Burger's equation. In Figure 1(a), we fix $M = 4$ and $\mu = 0.85$; while in Figure 1(b) we set $M = 4$, $x = 0.0125$ and for various values of μ.

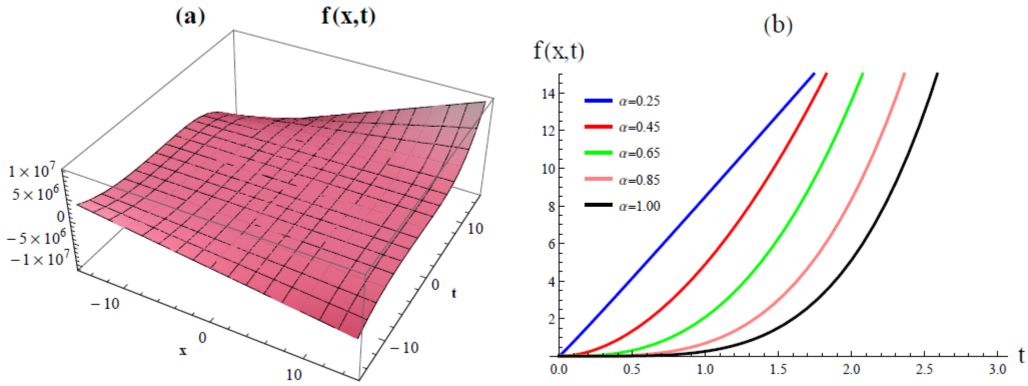

Figure 1. The (a) 3D and (b) 2D visualizations of the solution of non-integer order Burger's equation determined in Eq (4.7).

4.2. *Time non-integer order Schrödinger equation*

Let us consider the following time non-integer order nonlinear Schrödinger differential equation

$$iD_t^{\mu} f + f_{xx} + 2f^2 f^* = 0, \quad 0 < \mu \leq 1, \tag{4.10}$$

together with the following prescribed initial condition

$$f(x, 0) = e^{ix}, \tag{4.11}$$

where $Lf(x,t) = \frac{d^2 f}{dx^2}$, $Ff(x,t) = 0$, $Nf(x,t) = 2f^2 f^*$, $h(x,t) = 0$, and $g(x,t) = e^{ix}$. Therefore, without further delay, the proposed G_α decomposition methodology posed the following recurrent scheme for the problem

$$\begin{cases} f_0(x,t) = f(x,0), \\ \\ f_{k+1}(x,t) = iG_\alpha^{-1}\{s^\mu G_\alpha\{f_{k_{xx}} + 2R_k\}\}, \quad k \geq 0, \end{cases} \tag{4.12}$$

with R_k's denoting the polynomials by Adomian in favour of the nonlinear term given by

$$Nf(x,t) = f^2 f^*, \tag{4.13}$$

where we express some few terms as follows

$$\begin{aligned} R_0 &= f_0^2 f_0^*, \\ R_1 &= 2f_0 f_1 f_0^* + f_0^2 f_1^*, \\ R_2 &= 2f_0 f_2 f_0^* + u_1^2 f_0^* + 2f_0 f_1 f_1^* + f_0^2 f_2^*, \\ &\vdots \end{aligned} \tag{4.14}$$

Thus, from the recurrent scheme given in Eq (4.12) via Eq (4.14), we get some of the components explicitly as follows

$$
\begin{aligned}
f_0(x,t) &= e^{ix},\\
f_1(x,t) &= iG_\alpha^{-1}\{s^\mu G_\alpha\{f_{0_{xx}} + 2R_0\}\},\\
&= i\frac{t^\mu}{\Gamma(1+\mu)}e^{ix},\\
f_2(x,t) &= iG_\alpha^{-1}\{s^\mu G_\alpha\{f_{1_{xx}} + 2R_1\}\},\\
&= -\frac{t^{2\mu}}{\Gamma(1+2\mu)}e^{ix},\\
f_3(x,t) &= iG_\alpha^{-1}\{s^\mu G_\alpha\{f_{2_{xx}} + 2R_2\}\},\\
&= -i\frac{t^{3\mu}}{\Gamma(1+3\mu)}e^{ix},\\
f_4(x,t) &= iG_\alpha^{-1}\{s^\mu G_\alpha\{f_{3_{xx}} + 2R_3\}\},\\
&= \frac{t^{4\mu}}{\Gamma(1+4\mu)}e^{ix},\\
&\ \vdots
\end{aligned}
\tag{4.15}
$$

Therefore, the net sum of the above recurrent components gives the following

$$
\begin{aligned}
f(x,t) &= \lim_{M\to\infty}\left(\sum_{n=0}^{\infty} f_n(x,t)\right),\\
&= \left(1 + \frac{it^\mu}{\Gamma(1+\mu)} + \frac{(it^\mu)^2}{\Gamma(1+2\mu)} + \frac{(it^\mu)^3}{\Gamma(1+3\mu)} + \frac{(it^\mu)^4}{\Gamma(1+4\mu)} + \ldots\right)e^{ix},\\
&= \left[\lim_{M\to\infty}\left(\sum_{n=0}^{M}\frac{(it^\mu)^n}{\Gamma(1+n\mu)}\right)\right]e^{ix},
\end{aligned}
\tag{4.16}
$$

which converges to the following closed-form solution

$$
f(x,t) = E_\mu(it^\mu)e^{ix}. \tag{4.17}
$$

where $E_\mu(t)$ is the one-parameter Mittag-Leffler function for $0 < \mu \leq 1$. More so, when $\mu = 1$, the above solution reduces to the following integer solution version [33]

$$
f(x,t) = e^{i(x+t)}. \tag{4.18}
$$

Additionally, we have shown in Figure 2(a),(b) the depictions of the 3D plots of the real and imaginary solution parts of the non-integer order Schrödinger equation determined, respectively in Eq (4.17) for $\mu = 0.85$. Also, we have shown in Figure 3(a),(b) the corresponding depictions of the 2D plots of the non-integer order Schrödinger equation determined in Eq (4.17) using $x = \pi/5$ and for various values of μ.

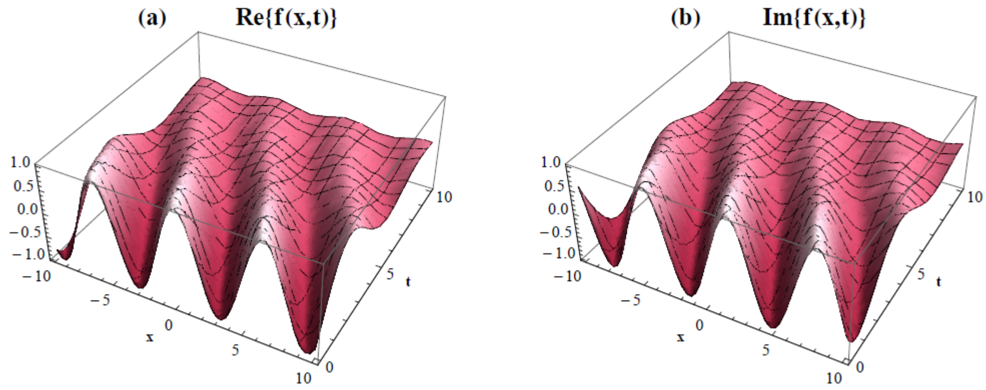

Figure 2. The 3D visualizations of the (a) real and (b) imaginary solution parts of the non-integer order Schr*ö*dinger equation determined in Eq (1.4).

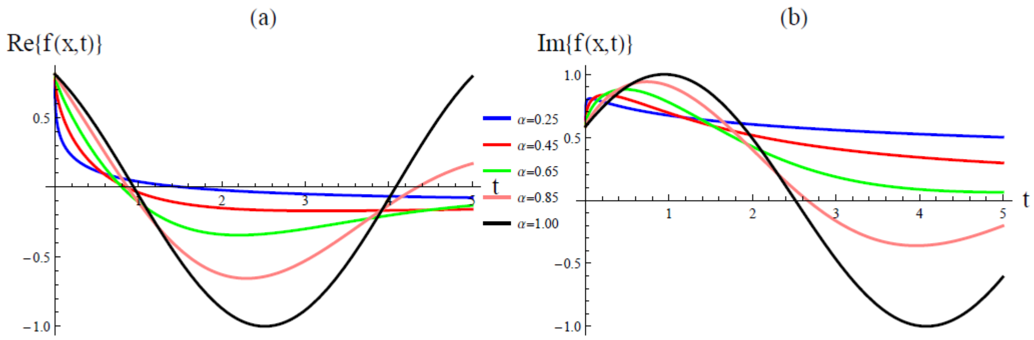

Figure 3. The 2D visualizations of the (a) real and (b) imaginary solution parts of the non-integer order Schr*ö*dinger equation determined in Eq (1.4) for $x = \pi/5$, and for various values of non-integer order μ.

4.3. *Time non-integer order coupled Burger's equation*

Here, we study two variants of the time non-integer order coupled system of Burger's equations comprising of homogeneous and inhomogeneous settings. In addition, *Theorem 9* will be utilized in this regard.

4.3.1. Time non-integer order coupled homogeneous Burger's equation

Let us consider the following time non-integer order coupled nonsingular homogeneous Burger's equation

$$\begin{cases} D_t^{\mu} f = f_{xx} + 2 f f_x - (fg)_x, & 0 < \mu \leq 1, \\ D_t^{\mu} g = g_{xx} + 2 g g_x - (fg)_x, & 0 < \mu \leq 1, \end{cases} \tag{4.19}$$

together with the following prescribed initial condition

$$\begin{cases} f(x,0) = \sin(x), \\ g(x,0) = \sin(x), \end{cases} \tag{4.20}$$

where $L_1 f(x,t) = \frac{d^2 f}{dx^2},\ F_1 f(x,t) = 0,\ N_1 f(x,t) = 2ff_x,\ h_1(x,t) = 0$, and $g_1(x,t) = \sin(x)$, $L_2 g(x,t) = \frac{d^2 g}{dx^2},\ F_2 g(x,t) = 0$, $N_2 g(x,t) = 2gg_x$, $N_3(f,g) = (fg)_x$, $h_2(x,t) = 0$, and $g_2(x,t) = \sin(x)$. Therefore, as preceded, the following recurrent scheme for the time non-integer order coupled homogeneous Burger's equation is obtained

$$\begin{cases} f_0(x,t) = f(x,0), \\ g_0(x,t) = g(x,0), \\ \\ f_{k+1}(x,t) = G_\alpha^{-1}\{s^\mu G_\alpha\{f_{k_{xx}} + 2A_k - C_k\}\}, \quad k \geq 0, \\ g_{k+1}(x,t) = G_\alpha^{-1}\{s^\mu G_\alpha\{g_{k_{xx}} + 2B_k - C_k\}\}, \quad k \geq 0, \end{cases} \tag{4.21}$$

with A_k's, B_k's and C_k's denote the polynomials by Adomian in favour of the following nonlinear terms

$$N_1 f(x,t) = ff_x, \qquad N_2 g(x,t) = gg_x, \qquad N_3(f,g)(x,t) = (fg)_x, \tag{4.22}$$

where we express few terms from these nonlinear terms using the application of Eq (3.8) as follows

$$\begin{aligned} A_0 &= f_0 f_{0_x}, \\ A_1 &= f_0 f_{1_x} + f_{0_x} f_1, \\ A_2 &= f_0 f_{2_x} + f_1 f_{1_x} + f_2 f_{0_x}, \\ &\vdots \end{aligned} \tag{4.23}$$

$$\begin{aligned} B_0 &= g_0 g_{0_x}, \\ B_1 &= g_0 g_{1_x} + g_{0_x} g_1, \\ B_2 &= g_0 g_{2_x} + g_1 g_{1_x} + g_2 g_{0_x}, \\ &\vdots \end{aligned} \tag{4.24}$$

and

$$\begin{aligned} C_0 &= f_0 g_{0_x} + f_{0_x} g_0, \\ C_1 &= f_0 g_{1_x} + g_{0_x} f_1 + f_{1_x} g_0 + g_1 f_{0_x}, \\ C_2 &= f_0 g_{2_x} + f_1 g_{1_x} + f_2 g_{0_x} + f_{2_x} g_0 + f_{1_x} g_1 + f_{0_x} g_2, \\ &\vdots \end{aligned} \tag{4.25}$$

Thus, from the recurrent scheme given in Eq (4.21) via Eqs (4.23)–(4.25), we get some of the components explicitly as follows

$$\begin{aligned} f_0(x,t) &= \sin(x),\\ g_0(x,t) &= \sin(x),\\ f_1(x,t) &= -\frac{t^{\mu}}{\Gamma(1+\mu)}\sin(x),\\ g_1(x,t) &= -\frac{t^{\mu}}{\Gamma(1+\mu)}\sin(x),\\ f_2(x,t) &= \frac{t^{2\mu}}{\Gamma(1+2\mu)}\sin(x),\\ g_2(x,t) &= \frac{t^{2\mu}}{\Gamma(1+2\mu)}\sin(x),\\ f_3(x,t) &= -\frac{t^{3\mu}}{\Gamma(1+3\mu)}\sin(x),\\ g_3(x,t) &= -\frac{t^{3\mu}}{\Gamma(1+3\mu)}\sin(x),\\ f_4(x,t) &= \frac{t^{4\mu}}{\Gamma(1+4\mu)}\sin(x),\\ g_4(x,t) &= \frac{t^{4\mu}}{\Gamma(1+4\mu)}\sin(x),\\ &\vdots \end{aligned} \tag{4.26}$$

Then, the net sums of the above recurrent components obtained above gives

$$\begin{aligned} f(x,t) &= \lim_{M\to\infty}\left(\sum_{n=0}^{\infty} f_n(x,t)\right),\\ &= \left(1-\frac{t^{\mu}}{\Gamma(1+\mu)}+\frac{t^{2\mu}}{\Gamma(1+2\mu)}-\frac{t^{3\mu}}{\Gamma(1+3\mu)}+\frac{t^{4\mu}}{\Gamma(1+4\mu)}+\ldots\right)\sin(x),\\ &= \left[\lim_{M\to\infty}\left(\sum_{n=0}^{M}\frac{(-t^{\mu})^n}{\Gamma(1+n\mu)}\right)\right]\sin(x), \end{aligned} \tag{4.27}$$

and

$$\begin{aligned} g(x,t) &= \lim_{M\to\infty}\left(\sum_{n=0}^{\infty} f_n(x,t)\right),\\ &= \left(1-\frac{t^{\mu}}{\Gamma(1+\mu)}+\frac{t^{2\mu}}{\Gamma(1+2\mu)}-\frac{t^{3\mu}}{\Gamma(1+3\mu)}+\frac{t^{4\mu}}{\Gamma(1+4\mu)}+\ldots\right)\sin(x),\\ &= \left[\lim_{M\to\infty}\left(\sum_{n=0}^{M}\frac{(-t^{\mu})^n}{\Gamma(1+n\mu)}\right)\right]\sin(x), \end{aligned} \tag{4.28}$$

which subsequently converges to the following closed-form solution

$$\begin{aligned} f(x,t) &= E_\mu(t^\mu)\sin(x), \\ g(x,t) &= E_\mu(t^\mu)\sin(x). \end{aligned} \tag{4.29}$$

Equally, at $\mu = 1$. the above solution apparently corresponds to that of the corresponding integer order model that converges to the following exact solution [34,35]

$$\begin{aligned} f(x,t) &= e^t\sin(x), \\ g(x,t) &= e^t\sin(x). \end{aligned} \tag{4.30}$$

What's more, we have shown in Figure 4(a) the 3D and Figure 4(b) the 2D graphical depictions of the obtained closed-form solution in Eq (4.29) of the non-integer order coupled nonsingular homogeneous Burger's equation. In Figure 4(a) we fix $\mu = 0.85$; while in Figure 4(b), we set $x = \pi/15$, and for various values of non-integer order μ.

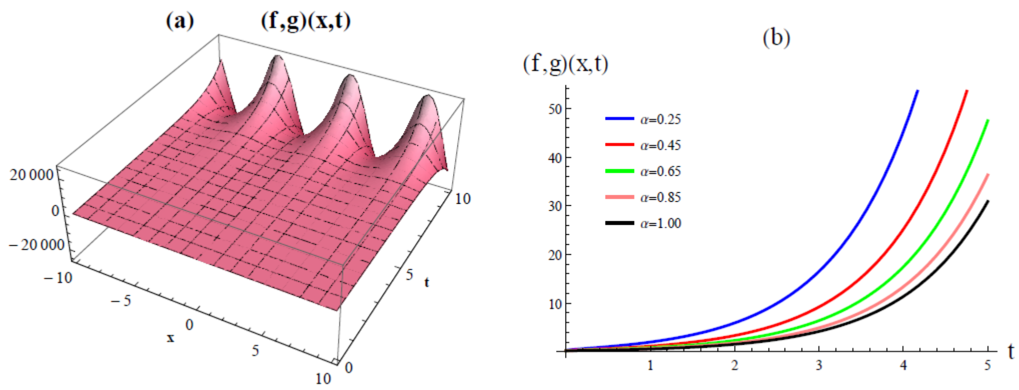

Figure 4. The (a) 3D and (b) 2D visualizations of the solution of non-integer order coupled nonsingular homogeneous Burger's equation determined in Eq (4.29).

4.3.2. Time non-integer order coupled inhomogeneous Burger's equation

Let us consider the time non-integer order coupled singular inhomogeneous Burger's equation as follows

$$\begin{cases} D_t^\mu f = x^{-1}\partial_x(xf_x) + 2ff_x - (fg)_x + (x^2-4)e^t, & 0 < \mu \le 1, \\ D_t^\mu g = x^{-1}\partial_x(xg_x) + 2gg_x - (fg)_x + (x^2-4)e^t, & 0 < \mu \le 1, \end{cases} \tag{4.31}$$

with the initial condition as follows

$$\begin{cases} f(x,0) = x^2, \\ g(x,0) = x^2, \end{cases} \tag{4.32}$$

where $L_1 f(x,t) = \frac{d^2 f}{dx^2}$, $F_1 f(x,t) = x^{-1}\frac{df}{dx}$, $N_1 f(x,t) = 2ff_x$, $h_1(x,t) = (x^2-4)e^t$, and $g_1(x,t) = x^2$, $L_2 g(x,t) = \frac{d^2 g}{dx^2}$, $F_2 g(x,t) = x^{-1}\frac{dg}{dx}$, $N_2 g(x,t) = 2gg_x$, $N_3(f,g) = (fg)_x$, $h_2(x,t) = (x^2-4)e^t$, and

$g_2(x,t) = x^2$. Therefore, as preceded, the following recurrent scheme is obtained

$$\begin{cases} f_0(x,t) = f(x,0) + G_\alpha^{-1}\{s^\mu G_\alpha\{(x^2-4)e^t\}\}, \\ g_0(x,t) = g(x,0) + G_\alpha^{-1}\{s^\mu G_\alpha\{(x^2-4)e^t\}\}, \\ \\ f_{k+1}(x,t) = G_\alpha^{-1}\{s^\mu G_\alpha\{x^{-1}\partial_x(xf_{k_x}) + 2A_k - C_k\}\}, \quad k \geq 0, \\ g_{k+1}(x,t) = G_\alpha^{-1}\{s^\mu G_\alpha\{x^{-1}\partial_x(xg_{k_x}) + 2B_k - C_k\}\}, \quad k \geq 0, \end{cases} \tag{4.33}$$

with A_k's, B_k's and C_k's the polynomials by Adomian as in the above homogeneous case via the application of Eq (3.8). Hence, some of the few components from the above scheme are explicitly revealed as follows

$$\begin{aligned} f_0(x,t) &= x^2 + \left(1 - \frac{\Gamma(\mu,t)}{\Gamma(\mu)}\right)\left(-4 + x^2\right)e^t, \\ g_0(x,t) &= x^2 + \left(1 - \frac{\Gamma(\mu,t)}{\Gamma(\mu)}\right)\left(-4 + x^2\right)e^t, \\ f_1(x,t) &= 4e^t\left(1 - \frac{\Gamma(\mu,t)}{\Gamma(\mu)}\right) + 4t, \\ g_1(x,t) &= 4e^t\left(1 - \frac{\Gamma(\mu,t)}{\Gamma(\mu)}\right) + 4t, \\ f_2(x,t) &= 0, \qquad g_2(x,t) = 0, \\ f_3(x,t) &= 0, \qquad g_3(x,t) = 0, \\ &\vdots \qquad\qquad \vdots \end{aligned} \tag{4.34}$$

Accordingly, the net sum reveals the following solution

$$\begin{aligned} f(x,t) &= x^2 + \gamma\left(-4 + x^2\right)e^t + 4\gamma e^t + 4t, \\ g(x,t) &= x^2 + \gamma\left(-4 + x^2\right)e^t + 4\gamma e^t + 4t, \end{aligned} \tag{4.35}$$

where

$$\gamma = 1 - \frac{1}{\Gamma(\mu)}\Gamma(\mu,t), \tag{4.36}$$

with $\Gamma(\mu,t)$ denoting the incomplete Gamma function. More so, it is apparent at $\mu = 1$ that the above solution corresponds to that of the corresponding integer order model, which converges to the following exact solution [37]

$$\begin{aligned} f(x,t) &= x^2e^t, \\ g(x,t) &= x^2e^t. \end{aligned} \tag{4.37}$$

Additionally, we have shown in Figure 5(a) the 3D and Figure 5(b) the 2D graphical depictions of the obtained closed-form solution in Eq (4.35) of the non-integer order coupled singular inhomogeneous Burger's equation. In Figure 5(a) we fix $\mu = 0.85$; while in Figure 5(b), we set $x = \pi/15$, and for various values of non-integer order μ.

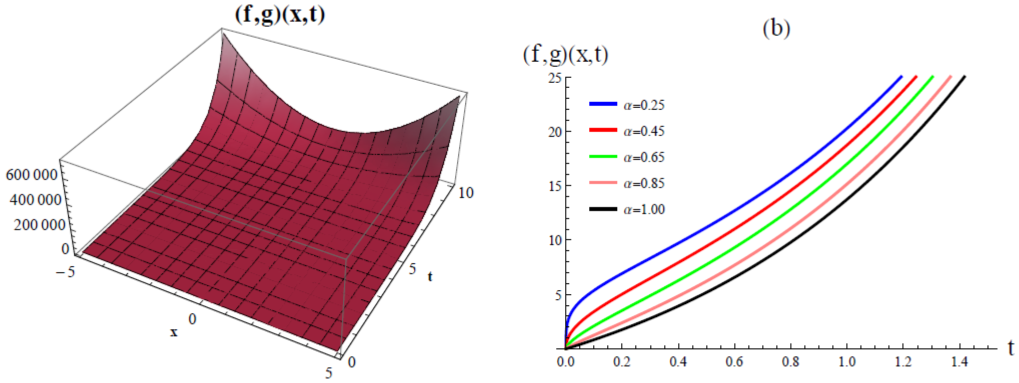

Figure 5. The (a) 3D and (b) 2D visualizations of the solution of non-integer order coupled singular inhomogeneous Burger's equation determined in Eq (4.35).

5. Discussion of results

The current study uses an integrated form of Laplace-type integral transform by the name G_α integral transform coupled with the Adomian's approach to examine nonlinear evolution equations endowed with non-integer order derivatives in time. The specific kinds of nonlinear evolution equations of interest in the present study include: non-integer Burger's equation, non-integer Schr$\ddot{o}$dinger equation, and the non-integer coupled Burger's equation. Thus, the present method rapidly revealed convergent closed-form solutions iteratively and efficiently. Additionally, in order to shed for more light on the obtained solutions, and the relevance of the non-integer order derivative on these models, we further make use of the *Wolfram Mathematica 9* software for the computational simulations, as well as the graphical illustrations. More specifically, Figure 1 portrays the 3D and 2D graphical illustrations of the non-integer Burger's equation; Figures 2 and 3 portray the 3D and 3D graphical illustrations of the non-integer Schr$\ddot{o}$dinger equation, respectively; while Figures 4 and 5 portray the respective 3D and 2D graphical illustrations of the coupled system of non-integer Burger's equations, correspondingly. In these plots, the 2D plots show the variational effects of the non-integer order derivative $0 < \mu < 1$ on the respective wave profiles in comparison with the integer order derivative at $\mu = 1$. For instance, in the case of the non-integer Burger's equation shown in Figure 1(b), the wave propagates linearly for smaller values of μ, and gradually increases exponentially with an increase in μ. However, a periodic behavior was observed in the case of the integer Schr$\ddot{o}$dinger equation at $\mu = 1$ as shown in Figure 2; while this behavior vanishes as μ decreases. Moreover, since both the wave profiles of the non-integer coupled Burger's equation are the same, that is, $f(x,t) = g(x,t)$ with regard to the nonsingular homogeneous and singular inhomogeneous cases, the wave profiles shown in Figures 4(b) and 5(b) decrease with an increase in μ, with the integer version being the least. Finally, Figures 1(a), 2, 4(a) and 5(a) show the corresponding 3D plots for the respective non-integer nonlinear models under consideration when $\mu = 0.85$. More comparatively, the integer-order solution determined in Subsection 4.1 corresponds to the solution obtained in [13] by Biazar and Aminikhah using variational iteration method; the same solution was equally reported by Nuruddeen at al. in [36] via the application of Sumudu decomposition method. The integer-order solution determined in Subsection 4.2 corresponds to the solution presented by Nuruddeen [33] using Elzaki decomposition approach; it also corresponds to the solution presented by Wazwaz [38] using Adomian's approach. Finally, the integer-order and non-integer-order solutions determined in Subsection 4.3 correspond to the respective results presented in [34,35,37,39] using iterative based decomposition methods. Indeed, the mentioned references confirmed the exactness of the obtained results.

6. Conclusions

To conclude the current manuscript, an essential and integrated form of Laplace-type integral transform by the name G_α integral transform coupled with the well-known Adomian's approach has been employed to study some important nonlinear evolution equations endowed with non-integer order derivatives in time. More specifically, we have examined the time-fractional Burger's equation, time-fractional Schrödinger equation, and the coupled system of time-fractional Burger's equation as the test of non-integer nonlinear models. Additionally, Caputo's definition of the non-integer derivative was chosen owing to the fact that the non-integer order does not affect the initial conditions, and on the other hand, owing to its practicality. Thus, the presented methodology rapidly revealed convergent closed-form solutions iteratively and efficiently. Lastly, we studied the variational effect of the non-integer order derivatives on the evolution of the three models under consideration in comparison with the integer-order derivative when $\mu = 1$, as graphically portrayed via the two and three-dimensional depictions. Moreover, the literature is well equipped with various coupling between the famous Adomian decomposition method and various integral transforms. Thus, considering the effectiveness of the derived techniques, the method is highly recommended for solving strongly nonlinear problems arising in nonlinear sciences, such as nonlinear evolution equations and Schrödinger equations. More so, appropriate initial and/or boundary conditions are expected to be supplied for the existence and uniqueness of the solutions.

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Conflict of interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

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