Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 7(10): 18837–18852.
DOI: 10.3934/math.20221037
Received: 19 July 2022
Revised: 15 August 2022
Accepted: 18 August 2022
Published: 25 August 2022

Research article

Finite-time stability of nonlinear stochastic ψ-Hilfer fractional systems with time delay

Qing Yang, Chuanzhi Bai and Dandan Yang*

Department of Mathematics, Huaiyin Normal University, Huaian, Jiangsu 223300, China

* **Correspondence:** Email: ydd@hytc.edu.cn, 457045776@qq.com.

Abstract: In this paper, we study the finite time stability of stochastic ψ-Hilfer fractional-order time-delay systems. Under the stochastic analysis techniques and the generalized Gronwall's inequality for ψ-fractional derivative, the criterion of finite time stability of the solution for nonlinear stochastic ψ-Hilfer fractional systems with time delay is obtained. An example is provided to illustrate the effectiveness of the proposed methods. Some known results in the literature are extended.

Keywords: ψ-Hilfer fractional derivative; generalized Gronwall's inequality; finite time stability
Mathematics Subject Classification: 26A33, 34A08, 34D20, 34F05, 60J65

1. Introduction

In control systems, neural networks, missile systems and many fields, finite-time stability is a more practical method which is much valuable to analyze the transient behavior of nature of a system within a finite interval of time [1–4]. We know that the traditional stability, asymptotical stability and exponential stability in the sense of Lyapunov, which the stable property of a system is considered in an infinite-time interval. Finite time stability just focuses on the behavior of a given system in a fixed time interval [5].

With the development of science and applied mathematics, people have found that some physical phenomenon systems are actually described by fractional differential equations, in which such systems cannot be effectively modeled by using the classical integer order differential equations. In recent decades, researchers have become more and more interested in the study of the stability of the solution for fractional-order system, see [6,7]. The finite-time stability analysis of fractional differential systems has received considerable attention, for instance [8–11] and the references therein. As we all known, either in nature or in artificial systems, noise or stochastic discomfort cannot be prohibited. Thus, stochastic differential equations have sparked interest as a result of their wide application in physical, pharmaceutical domains, scientific and engineering, one can check [12–15]. Considering fractional order and randomness, Mchin et al. in [16] investigated the finite-time stability of the solution for linear stochastic fractional-order time delay system as follows:

$$ {}^{C}D^{\alpha}_{0,t}y(t) = Ay(t) + By(t-\tau) + Cy(t-\nu)\frac{dW(t)}{dt}, \tag{1.1} $$

where the initial condition is $\{y(t), -\nu \le t \le 0\} = \overline{\phi}(t) \in \mathbb{R}^n$, ${}^{C}D^{\alpha}_{0,t}$ denotes the operator of the Caputo fractional derivative of order $\alpha \in (\frac{1}{2}, 1)$, $A, B, C \in \mathbb{R}^{n\times n}$, and $W(t)$ is a 1-dimensional Brownian motion defined on the probability space.

In the literature, there exist several definitions of fractional integrals and derivatives, in which the most popular are Riemann-Liouville and Caputo-type fractional derivatives. A generalization of both Riemann-Liouville and Caputo derivatives was given by Hilfer [17], which was known as the Hilfer fractional derivative. Theoretical simulations of thermoelastic in crystal compounds, engineering, rheological constitutive modelling, chemical processing and other domains have uncovered the usefulness and applicability of the Hilfer fractional derivative. Recently, Vanterler and Capelas de Oliveira [18] presented a fractional differential operator of a function with respect to another function, the so-called ψ-Hilfer fractional derivative. Obviously, the class of fractional derivatives derived from the ψ-Hilfer operator is making the fractional operator a generalization of the fractional operators. Some properties of this operator could be found in [18]. There are some authors who have worked on the existence, stability of solutions of ψ-Hilfer fractional-order differential equations as in [19–22].

From the above statement, it makes sense that Caputo fractional derivative of (1.1) is generalized to the ψ-Hilfer fractional derivative type. Moreover, (1.1) is a linear stochastic fractional-order time delay system, which is lack of nonlinear term. It is noted that the nonlinear term in system is more important and more general.

Motivated by the above works, we are concerned with the following nonlinear stochastic ψ-Hilfer fractional systems with time delay of the form

$$ \begin{cases} {}^{H}D^{\alpha,\beta,\psi}_{0^+}x(t) = Ax(t) + Bx(t-\tau) + f(t, x(t), x(t-\tau)) \\ \qquad\qquad +(Cx(t) + Dx(t-\tau))\frac{dW(t)}{dt}, & t \in [0, T], \\ x(t) = \phi(t), & t \in [-\tau, 0], \end{cases} \tag{1.2} $$

where ${}^{H}D^{\alpha,\beta,\psi}_{0^+}$ is the ψ-Hilfer fractional derivative of order $\frac{1}{2} < \alpha \le 1$, with respect to function ψ and $0 \le \beta \le 1$; $A, B, C, D \in R^{n\times n}$ are matrices; $f : J \times X \times X \to X$ ($X = C([-\tau, T], \mathbb{R}^n)$) is a function satisfying some specific assumptions given in (H1) and (H2). $W(t)$ is a 1-dimensional Brownian motion defined on the probability space.

The main contributions and advantages of this paper are as follows:

(1) For the first time in literature, the finite time stability for nonlinear stochastic ψ-Hilfer fractional-order time-delay systems is investigated.

(2) New set of sufficient conditions are established for the finite time stability for nonlinear stochastic ψ-Hilfer fractional-order time-delay systems (1.2). This work generalizes the main results of [16].

(3) Our main technique relies on generalized Gronwall's inequality for ψ-Hilfer derivative and stochastic analysis techniques is effectively used to establish the new results.

(4) A numerical example is presented to show the proposed theoretical results.

This paper will be organized as follows. In Section 2, we will briefly recall some notations, definitions and preliminaries. Section 3 is devoted to proving the finite-time stability for system (1.2). In Section 4, an example is given to illustrate our theoretical result. Finally, the paper is concluded in Section 5.

2. Preliminaries

In this section, we recall some basic definitions and lemmas which are used in the sequel.

Let $\{X, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}\}$ be a complete probability space with a filtration fulfilling the usual conditions. $W(t)$ is a 1-dimensional Brownian motion defined on the probability space.

Let $C([-\tau, 0], \mathbb{R}^n)$ be the space of the continuous functions $\phi : [-\tau, 0] \to \mathbb{R}^n$ with the norm $||\phi|| = \sup_{-\tau\leq s\leq 0} ||\phi(s)||$, and $C([-\tau, T], \mathbb{R}^n)$ be the space of the continuous functions $x : [-\tau, T] \to \mathbb{R}^n$ with the norm $||x|| = \sup_{-\tau\leq s\leq T} ||x(s)||$.

Definition 2.1. *[23] System (1.2) is finite-time stochastically stable (FTSS) w.r.t. $\{\delta, \varepsilon, T\}$, $\delta < \varepsilon$, if*

$$\mathbb{E}||\phi||^2 < \delta$$

implying

$$\mathbb{E}||x(t)||^2 < \varepsilon, \quad \forall t \in [0, T].$$

Now we present some definitions and properties of fractional calculus which will be used throughout this paper.

Definition 2.2. *[18] Let $\alpha > 0$, f be an integrable function defined on $[a, b]$ and $\psi \in C^1([a, b])$ be an increasing function with $\psi'(t) \neq 0$ for all $t \in [a, b]$. The left ψ-Riemann-Liouville fractional integral operator of order α of a function f is defined by*

$$I^{\alpha,\psi}_{a^+} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} f(s)ds. \tag{2.1}$$

Definition 2.3. *[18] Let $n-1 < \alpha < n$, $f \in C^n([a, b])$ and $\psi \in C^n([a, b])$ be an increasing function with $\psi'(t) \neq 0$ for all $t \in [a, b]$. The left ψ-Riemann-Liouville fractional derivative of order α of a function f is defined by*

$$D^{\alpha,\psi}_{a^+} f(t) = \left(\frac{1}{\psi'(t)}\frac{d}{dt}\right) I^{n-\alpha,\psi}_{a^+} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n \int_a^t (\psi(t) - \psi(s))^{n-\alpha-1} f(s)\psi'(s)ds. \tag{2.2}$$

Definition 2.4. *[18] Let $n-1 < \alpha < n$, $f \in C^n([a, b])$ and $\psi \in C^n([a, b])$ be an increasing function with $\psi'(t) \neq 0$ for all $t \in [a, b]$. The left ψ-Caputo fractional derivative of order α of a function f is defined by*

$${}^C D^{\alpha,\psi}_{a^+} f(t) = (I^{n-\alpha,\psi}_{a^+} f^{[n]})(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (\psi(t) - \psi(s))^{n-\alpha-1} f^{[n]}(s)\psi'(s)ds, \tag{2.3}$$

where $n = [\alpha] + 1$ and $f^{[n]}(t) := \left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n f(t)$ on $[a, b]$.

Definition 2.5. *[18] Let $n-1<\alpha<n$ with $n\in\mathbb{N}$, $f,\psi\in C^n([a,b],\mathbb{R})$ two functions such that ψ is increasing and $\psi'(x)\neq 0$, for all $x\in[a,b]$. The left-sided ψ-Hilfer fractional derivative ${}^HD_{a^+}^{\alpha,\beta;\psi}$ of function of order α and type $0\leq\beta\leq 1$, are defined by*

$$ {}^HD_{a^+}^{\alpha,\beta;\psi}f(x)=I_{a^+}^{\beta(n-\alpha);\psi}\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n I_{a^+}^{(1-\beta)(n-\alpha);\psi}f(x). \tag{2.4}$$

In the following, we will give some properties of the combinations of the fractional integral and the fractional derivatives of a function with respect to another function.

Lemma 2.1. *[14] Let $f\in C^n([a,b])$, $n-1<\alpha<n$ and $0\leq\beta\leq 1$. Then we have*

$$I_{a^+}^{\alpha;\psi}{}^HD_{a^+}^{\alpha,\beta;\psi}f(x)=f(x)-\sum_{k=1}^{n}\frac{(\psi(x)-\psi(a))^{\gamma-k}}{\Gamma(\gamma-k+1)}f_{\psi}^{[n-k]}I_{a^+}^{(1-\beta)(n-\alpha);\psi}f(a), \tag{2.5}$$

where $\gamma=\alpha+\beta(n-\alpha)$.

Lemma 2.2. *[14] Let $f\in C^1([a,b])$, $\alpha>0$ and $0\leq\beta\leq 1$. Then we have*

$${}^HD_{a^+}^{\alpha,\beta;\psi}I_{a^+}^{\alpha;\psi}f(x)=f(x). \tag{2.6}$$

Lemma 2.3. *[24] Suppose that $\alpha>0$ and $f\in C[0,b]$ is nonnegative and nondecreasing. Then $F(t)=\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f(s)ds$ is nondecreasing on $[0,b]$.*

From Lemma 2.3, we can give and prove the following lemma.

Lemma 2.4. *Assume that $\alpha>0$, $\psi\in C^1[0,b]$ and $\psi'\in C[0,b]$ with $\psi'(t)>0$ on $[0,b]$ a.e. If $f\in C[0,b]$ is nonnegative and nondecreasing, then $F(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(\psi(t)-\psi(s))^{\alpha-1}\psi'(s)f(s)ds$ is nondecreasing on $[0,b]$.*

Proof. We first suppose that $f\in C^1[0,b]$ is nonnegative and nondecreasing. Setting $w=\psi(t)-\psi(s)$, we get

$$F(t)=\frac{1}{\Gamma(\alpha)}\int_0^{\psi(t)-\psi(0)}w^{\alpha}f(\psi^{-1}(\psi(t)-w))dw.$$

Thus, we have

$$\begin{aligned}F'(t)&=\frac{1}{\Gamma(\alpha)}(\psi(t)-\psi(0))^{\alpha}f(\psi^{-1}(\psi(0)))\\&\quad+\frac{1}{\Gamma(\alpha)}\int_0^{\psi(t)-\psi(0)}w^{\alpha}f'(\psi^{-1}(\psi(t)-w))\frac{1}{\psi'(s)}\psi'(t)dw\\&=\frac{1}{\Gamma(\alpha)}(\psi(t)-\psi(0))^{\alpha}f(0)+\frac{1}{\Gamma(\alpha)}\int_0^t(\psi(t)-\psi(s))^{\alpha}\psi'(t)f'(s)ds\geq 0,\end{aligned}$$

for all $t\in[0,b]$ which implies that F is nondecreasing on $[0,b]$.

For the general case that $f\in C[0,b]$, we choose a sequence of nonnegative and nondecreasing functions $\{f_n\}\in C^1[0,b]$ such that $f_n\to f\in C[0,b]$. Since each $F_n(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(\psi(t)-\psi(s))^{\alpha-1}\psi'(s)f_n(s)ds$ is nondecreasing, we obtain that the limit $F=\lim_n F_n$ is therefore nondecreasing.

The Gronwall inequality plays an important role in the study of qualitative and qualitative properties of solution of fractional differential and integral equations [25–27]. In order to work with continuous dependence of differential equations via ψ-Hilfer fractional derivative, the authors in [28] gave the generalized Gronwall inequality by means of the fractional integral with respect to another function ψ as follows :

Theorem 2.1. *[28] Let u, v be two integrable functions and g be continuous defined on domain $[a, b]$. Let $\psi \in C^1[a, b]$ be an increasing function such that $\psi'(t) \neq 0$, $\forall t \in [a, b]$. Assume that*

(1) u and v are nonnegative;

(2) g is nonnegative and nondecreasing.

If

$$u(t) \leq v(t) + g(t) \int_a^t \psi'(\tau)(\psi(t) - \psi(\tau))^{\alpha-1} u(\tau) d\tau,$$

then

$$u(t) \leq v(t) + \int_a^t \sum_{k=1}^{\infty} \frac{[g(t)\Gamma(\alpha)]^k}{\Gamma(\alpha k)} \psi'(\tau)(\psi(t) - \psi(\tau))^{\alpha-1} v(\tau) d\tau, \quad \forall t \in [a, b].$$

Corollary 2.1. *[28] Under the hypothesis of Theorem 2.1, let v be a nondecreasing function on $[a, b]$. Then*

$$u(t) \leq v(t) E_\alpha(g(t)\Gamma(\alpha)(\psi(t) - \psi(a))^\alpha), \quad \forall t \in [a, b],$$

where E_α is the Mittag-Leffler function given by the series

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}.$$

In [29], Lin provided several generalizations of the Gronwall inequality for fractional derivative. Seemab et al. in [30] generalized Gronwalls inequality for ψ-fractional derivative as follows:

Theorem 2.2. *[30] Let u, v be two integrable functions, with domain $[a, b]$. Let $\psi \in C^1[a, b]$ an increasing function such that $\psi'(t) \neq 0$. $\forall t \in [a, b]$. Assume that*

(i) u and v are nonnegative;

(ii) The function $(g_i)_{i=1,...,n}$ are the bounded and monotonic functions on $[a, b]$;

(iii) The constants $\rho_i > 0$ ($i = 1, 2, ..., n$). If

$$u(t) \leq v(t) + \sum_{i=1}^{n} g_i(t) \int_a^t \psi'(\tau)(\psi(t) - \psi(\tau))^{\rho_i - 1} u(\tau) d\tau,$$

then

$$u(t) \leq v(t) + \sum_{k=1}^{\infty} \left(\sum_{1',2',...,k'=1} \frac{\prod_{i=1}^{k}(g_{i'}(t)\Gamma(\rho_{i'}))}{\Gamma(\sum_{i=1}^{k} \rho_{i'})} \int_a^t [\psi'(\tau)(\psi(t) - \psi(\tau))^{\sum_{i=1}^{k} \rho_{i'} - 1}] v(\tau) d\tau \right).$$

For $n = 2$ in Theorem 2.2, let $v(t)$ be a nondecreasing function on $a \leq t < b$. Then we have

Corollary 2.2. *Let u, v be two nonnegative integrable functions defined on domain $[a, b]$. And let $v(t)$ be a nondecreasing function. Assume that $\psi \in C^1[a, b]$ is an increasing function such that $\psi'(t) \neq 0$. $\forall t \in [a, b]$. If*

$$u(t) \leq v(t) + \sum_{i=1}^{2} g_i(t) \int_a^t \psi'(\tau)(\psi(t) - \psi(\tau))^{\rho_i - 1} u(\tau) d\tau,$$

where functions g_1 and g_2 are the bounded and monotonic functions on $[a,b]$, and constants $\rho_i > 0$ $(i = 1,2)$. Then we have

$$u(t) \leq v(t)[E_{\rho_1}(g_1(t)\Gamma(\rho_1)(\psi(t) - \psi(a))^{\rho_1}) + E_{\rho_2}(g_2(t)\Gamma(\rho_2)(\psi(t) - \psi(a))^{\rho_2}) - 1].$$

Proof. Since v is nondecreasing, we have $v(\tau) \leq v(t)$ for any $\tau \in [a,t]$. Thus, by Theorem 2.2, we obtain

$$\begin{aligned}
u(t) \leq{}& v(t) + \int_a^t \sum_{k=1}^{\infty} \frac{[g_1(t)\Gamma(\rho_1)]^k}{\Gamma(\rho_1 k)} \psi'(\tau)[\psi(t) - \psi(\tau)]^{\rho_1 k - 1} v(\tau) d\tau \\
&+ \int_a^t \sum_{k=1}^{\infty} \frac{[g_2(t)\Gamma(\rho_2)]^k}{\Gamma(\rho_2 k)} \psi'(\tau)[\psi(t) - \psi(\tau)]^{\rho_2 k - 1} v(\tau) d\tau \\
\leq{}& v(t)\left[1 + \int_a^t \sum_{k=1}^{\infty} \frac{[g_1(t)\Gamma(\rho_1)]^k}{\Gamma(\rho_1 k)} \psi'(\tau)[\psi(t) - \psi(\tau)]^{\rho_1 k - 1} d\tau \right.\\
&\left. + \int_a^t \sum_{k=1}^{\infty} \frac{[g_2(t)\Gamma(\rho_2)]^k}{\Gamma(\rho_2 k)} \psi'(\tau)[\psi(t) - \psi(\tau)]^{\rho_2 k - 1} d\tau \right] \\
={}& v(t)\left[1 + \sum_{k=1}^{\infty} \frac{[g_1(t)\Gamma(\rho_1)]^k}{\Gamma(\rho_1 k)} \frac{[\psi(t) - \psi(a)]^{\rho_1 k}}{\rho_1 k} + \sum_{k=1}^{\infty} \frac{[g_2(t)\Gamma(\rho_2)]^k}{\Gamma(\rho_2 k)} \frac{[\psi(t) - \psi(a)]^{\rho_2 k}}{\rho_2 k} \right] \\
={}& v(t)\left[1 + \sum_{k=1}^{\infty} \frac{[g_1(t)\Gamma(\rho_1)(\psi(t) - \psi(a))^{\rho_1}]^k}{\Gamma(\rho_1 k + 1)} + \sum_{k=1}^{\infty} \frac{[g_2(t)\Gamma(\rho_2)(\psi(t) - \psi(a))^{\rho_2}]^k}{\Gamma(\rho_2 k + 1)} \right] \\
={}& v(t)[E_{\rho_1}(g_1(t)\Gamma(\rho_1)(\psi(t) - \psi(a))^{\rho_1}) + E_{\rho_2}(g_2(t)\Gamma(\rho_2)(\psi(t) - \psi(a))^{\rho_2}) - 1].
\end{aligned}$$

3. Main results

This section is devoted to the finite-time stability problem for system (1.2).

Lemma 3.1. *If x is a solution of system (1.2), then it is a solution of stochastic Volterra integral system*

$$x(t) = \begin{cases} \dfrac{\phi(0)}{\Gamma(\gamma)\Gamma(2-\gamma)} + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (\psi(t) - \psi(s))^{\alpha-1} \psi'(s) A x(s) ds \\ \quad + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (\psi(t) - \psi(s))^{\alpha-1} \psi'(s) B x(s-\tau) ds \\ \quad + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (\psi(t) - \psi(s))^{\alpha-1} \psi'(s) f(s, x(s), x(s-\tau)) ds \\ \quad + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (\psi(t) - \psi(s))^{\alpha-1} \psi'(s) C x(s) dW(s) \\ \quad + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (\psi(t) - \psi(s))^{\alpha-1} \psi'(s) D x(s-\tau) dW(s), \quad t \in [0,T], \\ \phi(t), \qquad t \in [-\tau, 0], \end{cases} \tag{3.1}$$

where $\gamma = \alpha + \beta(1-\alpha)$.

Proof. If $x \in X$ satisfies the problem (1.2), then by applying the operator $I_{0^+}^{\alpha,\psi}$ to both sides of the first equation of (1.2), one has

$$\begin{aligned} I_{0^+}^{\alpha,\psi\,H}D_{0^+}^{\alpha,\beta,\psi}x(t) &= I_{0^+}^{\alpha,\psi}Ax(t) + I_{0^+}^{\alpha,\psi}Bx(t-\tau) \\ &\quad + I_{0^+}^{\alpha,\psi}f(t,x(t),x(t-\tau)) + I_{0^+}^{\alpha,\psi}(Cx(t)+Dx(t-\tau))\frac{dW(t)}{dt}. \end{aligned} \tag{3.2}$$

By properties of ψ-Hilfer fractional derivative and using Lemma 2.1, we obtain the left hand side of (3.2) as

$$\begin{aligned} I_{0^+}^{\alpha,\psi\,H}D_{0^+}^{\alpha,\beta,\psi}x(t) &= x(t) - \frac{(\psi(t)-\psi(0))^{\gamma-1}}{\Gamma(\gamma)} I_{0^+}^{(1-\beta)(1-\alpha),\psi}x(0) \\ &= x(t) - \frac{(\psi(t)-\psi(0))^{\gamma-1}}{\Gamma(\gamma)}\frac{1}{\Gamma(1-\gamma)}\int_0^t (\psi(t)-\psi(s))^{-\gamma}\psi'(s)x(0)ds \\ &= x(t) - \frac{\phi(0)}{\Gamma(\gamma)\Gamma(2-\gamma)}. \end{aligned} \tag{3.3}$$

Combining (3.2) and (3.3), we get that x is a solution of the stochastic Volterra integral system (3.1).

To study the finite-time stability of system (1.2), we need the following assumptions.

(H1) For $f : C([0,T]\times\mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}^n$, there exist two positive constants L_1 and L_2 such that

$$\|f(t,x(t),x(t-\tau)) - f(t,y(t),y(t-\tau))\| \le L_1\|x(t)-y(t)\| + L_2\|x(t-\tau)-y(t-\tau)\|,$$

for any $t \in [0,T]$, $x, y \in \mathbb{R}^n$.

(H2) $f(t,0,0) = \underbrace{[0,0,\cdots,0]}_{n}$.

Let $T > \tau$ and $m \in \mathbb{N}$ such that $(m+1)\tau < T \le (m+2)\tau$.

Theorem 3.1. *Assume that (H1) and (H2) hold. System (1.2) is FTSS with respect to* (δ,ε,T), *if the following condition is satisfied:*

$$l_T(\tau) \le \frac{\varepsilon}{\delta}, \tag{3.4}$$

where

$$\begin{aligned} l_T(\tau) = \Big(&M_1 + \frac{M_3}{\alpha}l_{m+1}(\tau)(\psi(T)-\psi(0))^{2\alpha} + \frac{\mu_T M_5}{2\alpha-1}l_{m+1}(\tau)(\psi(T)-\psi(0))^{2\alpha-1}\Big) \\ &\cdot\Big[E_\alpha(M_2\Gamma(\alpha)(\psi(T)-\psi(0))^{2\alpha}) \\ &\quad + E_{2\alpha-1}(M_4\mu_T\Gamma(2\alpha-1)(\psi(T)-\psi(0))^{2\alpha-1}) - 1\Big]\mathbb{E}\|\phi\|^2, \\ l_{k+1}(\tau) = \Big(&M_1 + \frac{M_3}{\alpha}l_k(\tau)(\psi((k+1)\tau)-\psi(0))^{2\alpha} + \frac{\mu_{(k+1)\tau}M_5}{2\alpha-1}l_k(\tau)(\psi((k+1)\tau)-\psi(0))^{2\alpha-1}\Big) \\ &\cdot\Big[E_\alpha(M_2\Gamma(\alpha)(\psi((k+1)\tau)-\psi(0))^{2\alpha}) \\ &\quad + E_{2\alpha-1}(M_4\mu_{(k+1)\tau}\Gamma(2\alpha-1)(\psi((k+1)\tau)-\psi(0))^{2\alpha-1}) - 1\Big]\mathbb{E}\|\phi\|^2, \end{aligned}$$

and $\mu_T = \sup_{t\in[0,T]}\psi'(t)$, $\mu_{(k+1)\tau} = \sup_{t\in[0,(k+1)\tau]}\psi'(t)$, $k \in [0,m]$, $l_0(\tau) = 1$,

$$\begin{aligned} M_1 &= \frac{6}{\Gamma^2(\gamma)\Gamma^2(2-\gamma)}, \quad M_2 = \frac{6(\|A\|^2+2L_1^2)}{\Gamma(\alpha)\Gamma(\alpha+1)}, \\ M_3 &= \frac{6(\|B\|^2+2L_2^2)}{\Gamma(\alpha)\Gamma(\alpha+1)}, \quad M_4 = \frac{6\|C\|^2}{\Gamma^2(\alpha)}, \quad M_5 = \frac{6\|D\|^2}{\Gamma^2(\alpha)}. \end{aligned} \tag{3.5}$$

Proof. According to Lemma 3.1, we know that the solution of the system (1.2) satisfies stochastic Volterra integral system (3.1). By using the Cauchy-Schwartz inequality, (H1) and (H2), we obtain

$$
\begin{aligned}
\|x(t)\|^2 \leq\ & \frac{6\|\phi\|^2}{\Gamma^2(\gamma)\Gamma^2(2-\gamma)} \\
& +\frac{6}{\Gamma^2(\alpha)}\left(\int_0^t (\psi(t)-\psi(s))^{\alpha-1}\|A\|\|x(s)\|\psi'(s)ds\right)^2 \\
& +\frac{6}{\Gamma^2(\alpha)}\left(\int_0^t (\psi(t)-\psi(s))^{\alpha-1}\|B\|\|x(s-\tau)\|\psi'(s)ds\right)^2 \\
& +\frac{6}{\Gamma^2(\alpha)}\left(\int_0^t (\psi(t)-\psi(s))^{\alpha-1}\|f(s,x(s),x(s-\tau))\|\psi'(s)ds\right)^2 \\
& +\frac{6}{\Gamma^2(\alpha)}\left\|\int_0^t (\psi(t)-\psi(s))^{\alpha-1}\psi'(s)Cx(s)dW(s)\right\|^2 \\
& +\frac{6}{\Gamma^2(\alpha)}\left\|\int_0^t (\psi(t)-\psi(s))^{\alpha-1}\psi'(s)Dx(s-\tau)dW(s)\right\|^2 \\
\leq\ & \frac{6\|\phi\|^2}{\Gamma^2(\gamma)\Gamma^2(2-\gamma)} \\
& +\frac{6}{\Gamma^2(\alpha)}\left(\int_0^t (\psi(t)-\psi(s))^{\alpha-1}\psi'(s)ds\right)\left(\int_0^t (\psi(t)-\psi(s))^{\alpha-1}\psi'(s)\|A\|^2\|x(s)\|^2ds\right) \\
& +\frac{6}{\Gamma^2(\alpha)}\left(\int_0^t (\psi(t)-\psi(s))^{\alpha-1}\psi'(s)ds\right)\left(\int_0^t (\psi(t)-\psi(s))^{\alpha-1}\psi'(s)\|B\|^2\|x(s-\tau)\|^2ds\right) \\
& +\frac{6}{\Gamma^2(\alpha)}\left(\int_0^t (\psi(t)-\psi(s))^{\alpha-1}\psi'(s)ds\right) \\
& \quad\times\left(\int_0^t (\psi(t)-\psi(s))^{\alpha-1}\psi'(s)(L_1\|x(s)\|+L_2\|x(s-\tau)\|)^2ds\right) \\
& +\frac{6}{\Gamma^2(\alpha)}\left\|\int_0^t (\psi(t)-\psi(s))^{\alpha-1}\psi'(s)Cx(s)dW(s)\right\|^2 \\
& +\frac{6}{\Gamma^2(\alpha)}\left\|\int_0^t (\psi(t)-\psi(s))^{\alpha-1}\psi'(s)Dx(s-\tau)dW(s)\right\|^2 \\
\leq\ & \frac{6\|\phi\|^2}{\Gamma^2(\gamma)\Gamma^2(2-\gamma)} \\
& +\frac{6\|A\|^2}{\Gamma(\alpha)\Gamma(\alpha+1)}(\psi(t)-\psi(0))^{\alpha}\int_0^t (\psi(t)-\psi(s))^{\alpha-1}\|x(s)\|^2\psi'(s)ds \\
& +\frac{6\|B\|^2}{\Gamma(\alpha)\Gamma(\alpha+1)}(\psi(t)-\psi(0))^{\alpha}\int_0^t (\psi(t)-\psi(s))^{\alpha-1}\|x(s-\tau)\|^2\psi'(s)ds \\
& +\frac{12L_1^2}{\Gamma(\alpha)\Gamma(\alpha+1)}(\psi(t)-\psi(0))^{\alpha}\int_0^t (\psi(t)-\psi(s))^{\alpha-1}\|x(s)\|^2\psi'(s)ds \\
& +\frac{12L_2^2}{\Gamma(\alpha)\Gamma(\alpha+1)}(\psi(t)-\psi(0))^{\alpha}\int_0^t (\psi(t)-\psi(s))^{\alpha-1}\|x(s-\tau)\|^2\psi'(s)ds \\
& +\frac{6}{\Gamma^2(\alpha)}\left\|\int_0^t (\psi(t)-\psi(s))^{\alpha-1}\psi'(s)Cx(s)dW(s)\right\|^2
\end{aligned}
$$

$$+\frac{6}{\Gamma^2(\alpha)}\left\|\int_0^t(\psi(t)-\psi(s))^{\alpha-1}\psi'(s)Dx(s-\tau)dW(s)\right\|^2. \tag{3.6}$$

Taking the expectation on the two sides of (3.6), we have

$$\begin{aligned}
\mathbb{E}\|x(t)\|^2 \le\ & \frac{6}{\Gamma^2(\gamma)\Gamma^2(2-\gamma)}\mathbb{E}\|\phi\|^2\\
&+\frac{6\|A\|^2}{\Gamma(\alpha)\Gamma(\alpha+1)}(\psi(t)-\psi(0))^\alpha\int_0^t(\psi(t)-\psi(s))^{\alpha-1}\mathbb{E}\|x(s)\|^2\psi'(s)ds\\
&+\frac{6\|B\|^2}{\Gamma(\alpha)\Gamma(\alpha+1)}(\psi(t)-\psi(0))^\alpha\int_0^t(\psi(t)-\psi(s))^{\alpha-1}\mathbb{E}\|x(s-\tau)\|^2\psi'(s)ds\\
&+\frac{12L_1^2}{\Gamma(\alpha)\Gamma(\alpha+1)}(\psi(t)-\psi(0))^\alpha\int_0^t(\psi(t)-\psi(s))^{\alpha-1}\mathbb{E}\|x(s)\|^2\psi'(s)ds\\
&+\frac{12L_2^2}{\Gamma(\alpha)\Gamma(\alpha+1)}(\psi(t)-\psi(0))^\alpha\int_0^t(\psi(t)-\psi(s))^{\alpha-1}\mathbb{E}\|x(s-\tau)\|^2\psi'(s)ds\\
&+\frac{6}{\Gamma^2(\alpha)}\int_0^t(\psi(t)-\psi(s))^{2\alpha-2}(\psi'(s))^2\|C\|^2\mathbb{E}\|x(s)\|^2ds\\
&+\frac{6}{\Gamma^2(\alpha)}\int_0^t(\psi(t)-\psi(s))^{2\alpha-2}(\psi'(s))^2\|D\|^2\mathbb{E}\|x(s-\tau)\|^2ds\\
\le\ & \frac{6}{\Gamma^2(\gamma)\Gamma^2(2-\gamma)}\mathbb{E}\|\phi\|^2\\
&+\frac{6(\|A\|^2+2L_1^2)}{\Gamma(\alpha)\Gamma(\alpha+1)}(\psi(t)-\psi(0))^\alpha\left(\int_0^t(\psi(t)-\psi(s))^{\alpha-1}\mathbb{E}\|x(s)\|^2\psi'(s)ds\right)\\
&+\frac{6(\|B\|^2+2L_2^2)}{\Gamma(\alpha)\Gamma(\alpha+1)}(\psi(t)-\psi(0))^\alpha\left(\int_0^t(\psi(t)-\psi(s))^{\alpha-1}\mathbb{E}\|x(s-\tau)\|^2\psi'(s)ds\right)\\
&+\frac{6\|C\|^2}{\Gamma^2(\alpha)}\int_0^t(\psi(t)-\psi(s))^{2\alpha-2}(\psi'(s))^2\mathbb{E}\|x(s)\|^2ds\\
&+\frac{6\|D\|^2}{\Gamma^2(\alpha)}\int_0^t(\psi(t)-\psi(s))^{2\alpha-2}(\psi'(s))^2\mathbb{E}\|x(s-\tau)\|^2ds\\
=\ & M_1\mathbb{E}\|\phi\|^2+M_2(\psi(t)-\psi(0))^\alpha\int_0^t(\psi(t)-\psi(s))^{\alpha-1}\mathbb{E}\|x(s)\|^2\psi'(s)ds\\
&+M_3(\psi(t)-\psi(0))^\alpha\int_0^t(\psi(t)-\psi(s))^{\alpha-1}\mathbb{E}\|x(s-\tau)\|^2\psi'(s)ds\\
&+M_4\int_0^t(\psi(t)-\psi(s))^{2\alpha-2}(\psi'(s))^2\mathbb{E}\|x(s)\|^2ds\\
&+M_5\int_0^t(\psi(t)-\psi(s))^{2\alpha-2}(\psi'(s))^2\mathbb{E}\|x(s-\tau)\|^2ds,
\end{aligned} \tag{3.7}$$

where M_1, M_2, M_3 and M_4 are as in (3.5).

For $t\in[0,\tau]$, by (3.7), we get

$$\begin{aligned}
\mathbb{E}\|x(t)\|^2 \le\ & M_1\mathbb{E}\|\phi\|^2+M_2(\psi(\tau)-\psi(0))^\alpha\int_0^t(\psi(t)-\psi(s))^{\alpha-1}\mathbb{E}\|x(s)\|^2\psi'(s)ds\\
&+M_3(\psi(\tau)-\psi(0))^\alpha\int_0^t(\psi(t)-\psi(s))^{\alpha-1}\mathbb{E}\|\phi\|^2\psi'(s)ds\\
&+M_4\mu_\tau\int_0^t(\psi(t)-\psi(s))^{2\alpha-2}\psi'(s)\mathbb{E}\|x(s)\|^2ds
\end{aligned}$$

$$
\begin{aligned}
&+M_5\mu_\tau\int_0^t(\psi(t)-\psi(s))^{2\alpha-2}\psi'(s)\mathbb{E}\|\phi\|^2ds\\
&\le\left(M_1+\frac{M_3}{\alpha}(\psi(\tau)-\psi(0))^{2\alpha}+\frac{\mu_\tau M_5}{2\alpha-1}(\psi(\tau)-\psi(0))^{2\alpha-1}\right)\mathbb{E}\|\phi\|^2\\
&+M_2(\psi(\tau)-\psi(0))^{\alpha}\int_0^t(\psi(t)-\psi(s))^{\alpha-1}\mathbb{E}\|x(s)\|^2\psi'(s)ds\\
&+M_4\mu_\tau\int_0^t(\psi(t)-\psi(s))^{2\alpha-2}\psi'(s)\mathbb{E}\|x(s)\|^2ds,
\end{aligned}
$$

where $\mu_\tau=\sup_{t\in[0,\tau]}\psi'(t)$. By Corollary 2.2, one has

$$
\begin{aligned}
\mathbb{E}\|x(t)\|^2&\le\left(M_1+\frac{M_3}{\alpha}(\psi(\tau)-\psi(0))^{2\alpha}+\frac{\mu_\tau M_5}{2\alpha-1}(\psi(\tau)-\psi(0))^{2\alpha-1}\right)\\
&\quad\cdot\Big[E_\alpha(M_2\Gamma(\alpha)(\psi(\tau)-\psi(0))^{2\alpha})\\
&\qquad+E_{2\alpha-1}(M_4\mu_\tau\Gamma(2\alpha-1)(\psi(\tau)-\psi(0))^{2\alpha-1})-1\Big]\mathbb{E}\|\phi\|^2\\
&=l_1(\tau)\mathbb{E}\|\phi\|^2,\quad\forall t\in[0,\tau],
\end{aligned}
$$

where

$$
\begin{aligned}
l_1(\tau)&=\left(M_1+\frac{M_3}{\alpha}(\psi(\tau)-\psi(0))^{2\alpha}+\frac{\mu_\tau M_5}{2\alpha-1}(\psi(\tau)-\psi(0))^{2\alpha-1}\right)\\
&\quad\cdot\Big[E_\alpha(M_2\Gamma(\alpha)(\psi(\tau)-\psi(0))^{2\alpha})\\
&\qquad+E_{2\alpha-1}(M_4\mu_\tau\Gamma(2\alpha-1)(\psi(\tau)-\psi(0))^{2\alpha-1})-1\Big].
\end{aligned}
$$

For $t\in[\tau,2\tau]$, from (3.7), we have

$$
\begin{aligned}
\mathbb{E}\|x(t)\|^2&\le M_1\mathbb{E}\|\phi\|^2+M_2(\psi(2\tau)-\psi(0))^{\alpha}\int_0^t(\psi(t)-\psi(s))^{\alpha-1}\mathbb{E}\|x(s)\|^2\psi'(s)ds\\
&+M_3(\psi(2\tau)-\psi(0))^{\alpha}\int_0^t(\psi(t)-\psi(s))^{\alpha-1}l_1(\tau)\mathbb{E}\|\phi\|^2\psi'(s)ds\\
&+M_4\mu_{2\tau}\int_0^t(\psi(t)-\psi(s))^{2\alpha-2}\psi'(s)\mathbb{E}\|x(s)\|^2ds\\
&+M_5\mu_{2\tau}\int_0^t(\psi(t)-\psi(s))^{2\alpha-2}\psi'(s)l_1(\tau)\mathbb{E}\|\phi\|^2ds\\
&\le\left(M_1+\frac{M_3}{\alpha}l_1(\tau)(\psi(2\tau)-\psi(0))^{2\alpha}+\frac{\mu_{2\tau}M_5}{2\alpha-1}l_1(\tau)(\psi(2\tau)-\psi(0))^{2\alpha-1}\right)\mathbb{E}\|\phi\|^2\\
&+M_2(\psi(2\tau)-\psi(0))^{\alpha}\int_0^t(\psi(t)-\psi(s))^{\alpha-1}\mathbb{E}\|x(s)\|^2\psi'(s)ds\\
&+M_4\mu_{2\tau}\int_0^t(\psi(t)-\psi(s))^{2\alpha-2}\psi'(s)\mathbb{E}\|x(s)\|^2ds,
\end{aligned}
$$

where $\mu_{2\tau}=\sup_{t\in[0,2\tau]}\psi'(t)$. By using Corollary 2.2, we obtain

$$
\mathbb{E}\|x(t)\|^2\le l_2(\tau)\mathbb{E}\|\phi\|^2,\quad\forall t\in[0,2\tau],
$$

where

$$
\begin{aligned}
l_2(\tau)&=\left(M_1+\frac{M_3}{\alpha}l_1(\tau)(\psi(2\tau)-\psi(0))^{2\alpha}+\frac{\mu_{2\tau}M_5}{2\alpha-1}l_1(\tau)(\psi(2\tau)-\psi(0))^{2\alpha-1}\right)\\
&\quad\cdot\Big[E_\alpha(M_2\Gamma(\alpha)(\psi(2\tau)-\psi(0))^{2\alpha})\\
&\qquad+E_{2\alpha-1}(M_4\mu_{2\tau}\Gamma(2\alpha-1)(\psi(2\tau)-\psi(0))^{2\alpha-1})-1\Big].
\end{aligned}
$$

For $t\in[k\tau,(k+1)\tau]$, we get by (3.6) that

$$
\mathbb{E}\|x(t)\|^2\le\Big(M_1+\frac{M_3}{\alpha}l_k(\tau)(\psi((k+1)\tau)-\psi(0))^{2\alpha}
$$

$$+\frac{\mu_{(k+1)\tau}M_5}{2\alpha-1}l_k(\tau)(\psi((k+1)\tau)-\psi(0))^{2\alpha-1}\Big)\mathbb{E}\|\phi\|^2$$
$$+M_2(\psi((k+1)\tau)-\psi(0))^{\alpha}\int_0^t(\psi(t)-\psi(s))^{\alpha-1}\mathbb{E}\|x(s)\|^2\psi'(s)ds$$
$$+M_4\mu_{(k+1)\tau}\int_0^t(\psi(t)-\psi(s))^{2\alpha-2}\psi'(s)\mathbb{E}\|x(s)\|^2ds,$$

where $\mu_{(k+1)\tau}=\sup_{t\in[0,(k+1)\tau]}\psi'(t)$. By using Corollary 2.2 again, we get

$$\mathbb{E}\|x(t)\|^2\leq l_{k+1}(\tau)\mathbb{E}\|\phi\|^2,\quad \forall t\in[0,(k+1)\tau],$$

where

$$\begin{aligned}l_{k+1}(\tau)=&\Big(M_1+\frac{M_3}{\alpha}l_k(\tau)(\psi((k+1)\tau)-\psi(0))^{2\alpha}\\&+\frac{\mu_{(k+1)\tau}M_5}{2\alpha-1}l_k(\tau)(\psi((k+1)\tau)-\psi(0))^{2\alpha-1}\Big)\\&\cdot\Big[E_\alpha(M_2\Gamma(\alpha)(\psi((k+1)\tau)-\psi(0))^{2\alpha})\\&+E_{2\alpha-1}(M_4\mu_{(k+1)\tau}\Gamma(2\alpha-1)(\psi((k+1)\tau)-\psi(0))^{2\alpha-1})-1\Big].\end{aligned}$$

Finally, since $(m+1)\tau<T\leq(m+2)\tau$, for all $t\in[0,T]$, we can obtain

$$\begin{aligned}\mathbb{E}\|x(t)\|^2\leq&\Big(M_1+\frac{M_3}{\alpha}l_{m+1}(\tau)(\psi(T)-\psi(0))^{2\alpha}+\frac{\mu M_5}{2\alpha-1}l_{m+1}(\tau)(\psi(T)-\psi(0))^{2\alpha-1}\Big)\\&\cdot\Big[E_\alpha(M_2\Gamma(\alpha)(\psi(T)-\psi(0))^{2\alpha})\\&+E_{2\alpha-1}(M_4\mu_T\Gamma(2\alpha-1)(\psi(T)-\psi(0))^{2\alpha-1})-1\Big]\mathbb{E}\|\phi\|^2,\end{aligned}$$

where $\mu_T=\sup_{t\in[0,T]}\psi'(t)$, which completes the proof.

Theorem 3.2. *Assume that (H1) and (H2) hold. System (1.2) is FTSS with respect to (δ,ε,T), provided the following condition holds :*

$$\begin{aligned}&\overline{M}_1\Big[E_\alpha((\overline{M}_2+\overline{M}_3)\Gamma(\alpha)(\psi(T)-\psi(0))^{\alpha})\\&+E_{2\alpha-1}((\overline{M}_4+\overline{M}_5)\Gamma(2\alpha-1)(\psi(T)-\psi(0))^{2\alpha-1})-1\Big]<\frac{\varepsilon}{\delta},\end{aligned}\tag{3.8}$$

where

$$\overline{M}_1=M_1,\quad \overline{M}_2=M_2(\psi(T)-\psi(0))^{\alpha},\quad \overline{M}_3=M_3(\psi(T)-\psi(0))^{\alpha},\overline{M}_4=M_4\mu_T,\quad \overline{M}_5=M_5\mu_T.\tag{3.9}$$

Here M_i $(i=1,...,5)$ are as in (3.5).

Proof. For convenience, denote $h(t)=\mathbb{E}\|x(t)\|^2$. Then by inequality (3.7) and (3.9), one has

$$\begin{aligned}h(t)\leq&\overline{M}_1\mathbb{E}\|\phi\|^2+\overline{M}_2\int_0^t(\psi(t)-\psi(s))^{\alpha-1}h(s)\psi'(s)ds\\&+\overline{M}_3\int_0^t(\psi(t)-\psi(s))^{\alpha-1}h(s-\tau)\psi'(s)ds\\&+\overline{M}_4\int_0^t(\psi(t)-\psi(s))^{2\alpha-2}\psi'(s)h(s)ds\\&+\overline{M}_5\int_0^t(\psi(t)-\psi(s))^{2\alpha-2}\psi'(s)h(s-\tau)ds,\quad t\in[0,T].\end{aligned}$$

Let $g(t)=\sup_{\theta\in[-\tau,t]}h(\theta)$, for all $t\in[0,T]$. Obviously, $h(s)\leq g(s)$ and $h(s-\tau)\leq g(s)$, $\forall s\in[0,T]$. For $t\in[0,T]$, we get

$$h(t)\leq\overline{M}_1\mathbb{E}\|\phi\|^2+\overline{M}_2\int_0^t(\psi(t)-\psi(s))^{\alpha-1}g(s)\psi'(s)ds$$

$$
\begin{aligned}
&+\overline{M}_3\int_0^t(\psi(t)-\psi(s))^{\alpha-1}g(s)\psi'(s)ds\\
&+\overline{M}_4\int_0^t(\psi(t)-\psi(s))^{2\alpha-2}\psi'(s)g(s)ds\\
&+\overline{M}_5\int_0^t(\psi(t)-\psi(s))^{2\alpha-2}\psi'(s)g(s)ds.
\end{aligned}
$$

Since g is nondecreasing, for $\forall\theta\in[0,t]$, we have by Lemma 2.4 that

$$
\begin{aligned}
h(\theta)\leq\ &\overline{M}_1\mathbb{E}\|\phi\|^2+\overline{M}_2\int_0^\theta(\psi(\theta)-\psi(s))^{\alpha-1}g(s)\psi'(s)ds\\
&+\overline{M}_3\int_0^\theta(\psi(\theta)-\psi(s))^{\alpha-1}g(s)\psi'(s)ds\\
&+\overline{M}_4\int_0^\theta(\psi(\theta)-\psi(s))^{2\alpha-2}\psi'(s)g(s)ds\\
&+\overline{M}_5\int_0^\theta(\psi(\theta)-\psi(s))^{2\alpha-2}\psi'(s)g(s)ds\\
\leq\ &\overline{M}_1\mathbb{E}\|\phi\|^2+\overline{M}_2\int_0^t(\psi(t)-\psi(s))^{\alpha-1}g(s)\psi'(s)ds\\
&+\overline{M}_3\int_0^t(\psi(t)-\psi(s))^{\alpha-1}g(s)\psi'(s)ds\\
&+\overline{M}_4\int_0^t(\psi(t)-\psi(s))^{2\alpha-2}\psi'(s)g(s)ds\\
&+\overline{M}_5\int_0^t(\psi(t)-\psi(s))^{2\alpha-2}\psi'(s)g(s)ds.
\end{aligned}
$$

Hence, for each $t\in[0,T]$, we obtain

$$
\begin{aligned}
g(t)=\ &\max\left\{\sup_{\theta\in[-\tau,0]}h(\theta),\ \sup_{\theta\in[0,t]}h(\theta)\right\}\\
\leq\ &\overline{M}_1\mathbb{E}\|\phi\|^2+(\overline{M}_2+\overline{M}_3)\int_0^t(\psi(t)-\psi(s))^{\alpha-1}g(s)\psi'(s)ds\\
&+(\overline{M}_4+\overline{M}_5)\int_0^t(\psi(t)-\psi(s))^{2\alpha-2}\psi'(s)g(s)ds.
\end{aligned}
$$

Using the generalized Gronwall inequality (Corollary 2.2), for $t\in[0,T]$, we have

$$
\begin{aligned}
g(t)\leq\ &\overline{M}_1\mathbb{E}\|\phi\|^2\Big[E_\alpha((\overline{M}_2+\overline{M}_3)\Gamma(\alpha)(\psi(t)-\psi(0))^\alpha)\\
&+E_{2\alpha-1}((\overline{M}_4+\overline{M}_5)\Gamma(2\alpha-1)(\psi(t)-\psi(0))^{2\alpha-1})-1\Big].
\end{aligned}\tag{3.10}
$$

Thus, for all $t\in[0,T]$, we get

$$
\begin{aligned}
\mathbb{E}\|x(t)\|^2=\ &h(t)\leq g(t)\\
\leq\ &\overline{M}_1\Big[E_\alpha((\overline{M}_2+\overline{M}_3)\Gamma(\alpha)(\psi(T)-\psi(0))^\alpha)\\
&+E_{2\alpha-1}((\overline{M}_4+\overline{M}_5)\Gamma(2\alpha-1)(\psi(T)-\psi(0))^{2\alpha-1})-1\Big]\mathbb{E}\|\phi\|^2.
\end{aligned}\tag{3.11}
$$

Thus, if $\mathbb{E}\|\phi\|^2<\delta$ and condition (3.8) holds, we have $\mathbb{E}\|x(t)\|^2<\varepsilon$, $\forall t\in[0,T]$. The proof is completed.

Remark 3.1. If $\psi(t)=t$, $\beta=1$, $f\equiv0$ and $C=0$, then system (1.2) reduces to system (1.1). Thus, Theorems 3.1 and 3.2 generalize the corresponding main results of [16].

4. Example

In this section, we give an example to show the usefulness of the main results.

Example 4.1. Consider the following nonlinear stochastic ψ-Hilfer fractional-order time-delay systems

$$\begin{aligned} {}^{H}D_{0^+}^{\frac{4}{5},\frac{9}{10},e^{\frac{t}{3}}} x(t) &= \begin{pmatrix} 0.6 & 0 \\ 0 & 0.5 \end{pmatrix} x(t) + \begin{pmatrix} 0.3 & 0 \\ 0 & 0.4 \end{pmatrix} x(t-\tau) + f(t, x(t), x(t-\tau)) \\ &\quad + \left[\begin{pmatrix} 0.7 & 0 \\ 0 & 0.6 \end{pmatrix} x(t) + \begin{pmatrix} 0.8 & 0 \\ 0 & 0.9 \end{pmatrix} x(t-\tau) \right] \frac{dW(t)}{dt}, \quad t \in [0, T], \end{aligned} \tag{4.1}$$

where the initial condition is

$$x(t) = \phi(t), \quad t \in [-\tau, 0].$$

Here

$$x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}, \quad x(t-\tau) = \begin{pmatrix} x_1(t-\tau) \\ x_2(t-\tau) \end{pmatrix}, \quad \phi(t) = \begin{pmatrix} \phi_1(t) \\ \phi_2(t) \end{pmatrix},$$

$\alpha = \frac{4}{5}, \beta = \frac{9}{10}, \psi(t) = e^{\frac{t}{3}}, \tau = 0.2$, and

$$f(t, x(t), x(t-\tau)) = \frac{1}{4} \begin{pmatrix} \sin x_1(t) \\ \sin x_2(t) \end{pmatrix} + \frac{1}{3} \begin{pmatrix} \arctan x_1(t-\tau) \\ \arctan x_2(t-\tau) \end{pmatrix}.$$

Obviously, $\gamma = \alpha + \beta(1-\alpha) = \frac{49}{50}$, f satisfies the conditions (H1) and (H2) with $L_1 = \frac{1}{4}$ and $L_2 = \frac{1}{3}$, $\|A\| = 0.6$, $\|B\| = 0.4$, $\|C\| = 0.7$ and $\|D\| = 0.9$.

Since $\psi'(t) = \frac{1}{3}e^{\frac{t}{3}}$, we have $\mu_\tau = 0.3563$, $\mu_{2\tau} = 0.3809$, $\mu_{3\tau} = 0.4071$, $\mu_{4\tau} = 0.4352$.

Let $\delta = 0.1$ and $\varepsilon = 10$. It is easy to verify that

$$M_1 = 5.9921, \quad M_2 = 2.6836, \quad M_3 = 2.1149, \quad M_4 = 2.1691, \quad \text{and } M_5 = 3.5856.$$

By Theorem 3.1, we can calculate that

$$l_1(\tau) = 8.8049, \quad l_2(\tau) = 23.4064, \quad l_3(\tau) = 80.9970 < 100 = \frac{\varepsilon}{\delta}.$$

The computed estimated time T in the system (4.1) is equal to 0.6. However, if $T = 0.7$, we can easily verify that

$$\overline{M}_1 = 5.9921, \quad \overline{M}_2 = 0.9213, \quad \overline{M}_3 = 0.7261, \quad \overline{M}_4 = 0.9130, \quad \overline{M}_5 = 1.5093,$$

and

$$\begin{aligned} &\overline{M}_1 \Big[E_\alpha(\overline{M}_2 + \overline{M}_3)\Gamma(\alpha)(\psi(T) - \psi(0))^\alpha) \\ &\qquad + E_{2\alpha-1}((\overline{M}_4 + \overline{M}_5)\Gamma(2\alpha-1)(\psi(T) - \psi(0))^{2\alpha-1}) - 1 \Big] \\ &= 5.9921 \cdot [E_{0.8}(0.6585) + E_{0.6}(1.6179) - 1] = 98.4220 < 100 = \frac{\varepsilon}{\delta}. \end{aligned}$$

By using Theorem 3.2, the calculated estimated time T of the system (4.1) is equal to 0.7. Thus, we conclude that System (4.1) is Finite time stochastic stable with respect to (δ, ε, T).

5. Conclusions

Based on the generalized Gronwall's inequality for ψ-fractional derivative and stochastic calculus techniques, the finite-time stability of nonlinear stochastic ψ-Hilfer fractional systems with time delay has been investigated in this paper. The novelty of this study is ψ-Hilfer fractional stochastic systems has been considered. Moreover, an illustrative example is provided to demonstrate the effectiveness of the theoretical results. Our results obtained are new and extend the existing literature on this topic. In the future, this result could be extended to investigate the finite-time stability for a class of ψ-Hilfer fractional stochastic semi-Markov jump systems.

Acknowledgments

This work is supported by Natural Science Foundation of China (11571136).

Conflict of interest

The authors declare that there are no conflicts of interest.

References

1. F. Amato, R. Ambrosino, M. Ariola, C. Cosentino, *Finite-time stability and control*, London: Springer-Verlag, 2014. https://doi.org/10.1007/978-1-4471-5664-2
2. R. Yang, Y. Wang, Finite-time stability and stabilization of a class of nonlinear time-delay systems, *SIAM J. Control Optim.*, **50** (2012), 3113–3131. https://doi.org/10.1137/11083798X
3. L. V. Hien, D. T. Son, Finite-time stability of a class of non-autonomous neural networks with heterogeneous proportional delays, *Appl. Math. Comput.*, **251** (2015), 14–23. https://doi.org/10.1016/j.amc.2014.11.044
4. L. Lee, Y. Liu, J. Liang, X. Cai, Finite time stability of nonlinear impulsive systems and its applications in sampled-data systems, *ISA Trans.*, **57** (2015), 172–178. https://doi.org/10.1016/j.isatra.2015.02.001
5. X. Li, X. Yang, S. Song, Lyapunov conditions for finite-time stability of time-varying time-delay systems, *Automatica*, **103** (2019), 135–140. https://doi.org/10.1016/j.automatica.2019.01.031
6. A. Jmal, O. Naifar, A. B. Makhlouf, N. Derbel, M. A. Hammami, On observer design for nonlinear Caputo fractional order systems, *Asian J. Control*, **20** (2018), 1533–1540. https://doi.org/10.1002/asjc.1645
7. O. Naifar, A. B. Makhlouf, M. A. Hammami, Comments on "Mittag-Leffler stability of fractional order nonlinear dynamic systems", *Automatica*, **75** (2017), 329. https://doi.org/10.1016/j.automatica.2016.09.023
8. M. P. Lazarevic, A. M. Spasic, Finite-time stability analysis of fractional order time-delay systems: Gronwall approach, *Math. Comput. Model.*, **49** (2009), 475–481. https://doi.org/10.1016/j.mcm.2008.09.011
9. R. Wu, Y. Lu, L. Chen, Finite-time stability of fractional delayed neural networks, *Neurocomputing*, **149** (2015), 700–707. https://doi.org/10.1016/j.neucom.2014.07.060
10. M. Li, J. Wang, Exploring delayed Mittag-Leffler type matrix functions to study finite time stability of fractional delay differential equations, *Appl. Math. Comput.*, **324** (2018), 254–265. https://doi.org/10.1016/j.amc.2017.11.063
11. F. Du, J. Lu, Finite-time stability of neutral fractional order time delay systems with Lipschitz nonlinearities, *Appl. Math. Comput.*, **375** (2020), 125079. https://doi.org/10.1016/j.amc.2020.125079

12. P. Balasubramaniam, P. Tamilalagan, Approximate controllability of a class of fractional neutral stochastic integro-differential inclusions with infinite delay by using Mainardi's function, *Appl. Math. Comput.*, **256** (2015), 232–246. https://doi.org/10.1016/j.amc.2015.01.035

13. Y. Guo, M. Chen, X. B. Shu, F. Xu, The existence and Hyers-Ulam stability of solution for almost periodical fractional stochastic differential equation with fBm, *Stoch. Anal. Appl.*, **39** (2021), 643–666. https://doi.org/10.1080/07362994.2020.1824677

14. C. Wei, Parameter estimation for partially observed stochastic differential equations driven by fractional Brownian motion, *AIMS Math.*, **7** (2022), 12952–12961. https://doi.org/10.3934/math.2022717

15. Y. Liu, M. Wnag, J. L. Wang, Stabilization of stochastic highly non-linear multi-links systems via aperiodically intermittent control, *Automatica*, **142** (2022), 110405. https://doi.org/10.1016/j.automatica.2022.110405

16. L. Mchiri, A. B. Makhlouf, D. Baleanu, M. Rhaima, Finite-time stability of linear stochastic fractional-order systems with time delay, *Adv. Differ. Equ.*, **2021** (2021), 345. https://doi.org/10.1186/s13662-021-03500-y

17. R. Hilfer, *Applications of fractional calculus in physics*, World Scientific Publishing, 2000.

18. J. V. C. Sousa, E. C. Oliveira, On the ψ-Hilfer fractional derivative, *Commun. Nonlinear Sci.*, **60** (2018), 72–91. https://doi.org/10.1016/j.cnsns.2018.01.005

19. A. Suechoei, P. S. Ngiamsunthorn, Existence uniqueness and stability of mild solutions for semilinear ψ-Caputo fractional evolution equations, *Adv. Differ. Equ.*, **2020** (2020), 114. https://doi.org/10.1186/s13662-020-02570-8

20. W. Sudsutad, C. Thaiprayoon, S. K. Ntouyas, Existence and stability results for ψ-Hilfer fractional integro-differential equation with mixed nonlocal boundary conditions, *AIMS Math.*, **6** (2021), 4119–4141. https://doi.org/10.3934/math.2021244

21. D. Luo, K. Shah, Z. Luo, On the novel Ulam-Hyers stability for a class of nonlinear ψ-Hilfer fractional differential equation with time-varying delays, *Mediterr. J. Math.*, **16** (2019), 112. https://doi.org/10.1007/s00009-019-1387-x

22. A. M. Saeed, M. A. Almalahi, M. S. Abdo, Explicit iteration and unique solution for ϕ-Hilfer type fractional Langevin equations, *AIMS Math.*, **7** (2022), 3456–3476. https://doi.org/10.3934/math.2022192

23. M. P. Lazarevic, A. M. Spasic, Finite-time stability analysis of fractional-order time-delay systems: Gronwall's approach, *Math. Comput. Model.*, **49** (2009), 475–481. https://doi.org/10.1016/j.mcm.2008.09.011

24. Q. Dong, C. Liu, Z. Fan, Weighted fractional differential equations with infinite delay in Banach spaces, *Open Math.*, **14** (2016), 370–383. https://doi.org/10.1515/math-2016-0035

25. R. Almeida, A Gronwall inequality for a general Caputo fractional operator, *Math. Inequal. Appl.*, **20** (2017), 1089–1105. https://doi.org/10.7153/mia-2017-20-70

26. D. S. Oliveira, E. C. Oliveira, Hilfer-Katugampola fractional derivatives, *Comput. Appl. Math.*, **37** (2018), 3672–3690. https://doi.org/10.1007/s40314-017-0536-8

27. F. Wang, D. Chen, X. Zhang, Y. Wu, The existence and uniqueness theorem of the solution to a class of nonlinear fractional order system with time delay, *Appl. Math. Lett.*, **53** (2016), 45–51. https://doi.org/10.1016/j.aml.2015.10.001

28. J. Vanterler, J. V. V. Sousa, E. C. Oliveira, A Gronwall inequality and the Cauchy-type problem by means of ψ-Hilfer operator, *Differ. Equ. Appl.*, **11** (2019), 87–106. https://doi.org/10.7153/dea-2019-11-02

29. S. Lin, Generalized Gronwall inequalities and their applications to fractional differential equations, *J. Inequal. Appl.*, **2013** (2013), 549. https://doi.org/10.1186/1029-242x-2013-549

30. A. Seemab, J. Alzabut, M. Rehman, Y. Adjabi, M. S. Abdo, Langevin equation with nonlocal boundary conditions involving a ψ-Caputo fractional operator, 2020, arxiv: 2006.00391.

AIMS Press