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Research article

Hermite-Hadamard inequality involving Caputo-Fabrizio fractional integrals and related inequalities via *s*-convex functions in the second sense

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Abstract: In this paper, firstly, Hermite-Hadamard inequality via s-convex functions in the second sense using Caputo-Fabrizio fractional integral operator is established. We also compare our results with the existing ones. It is also shown that the obtained results are a generalization of the existing results. Finally, we give their applications to special means.

Keywords: Hermite-Hadamard inequality; s-convex functions; Caputo-Fabrizio fractional integral
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1. Introduction

Fractional calculus has been a fascinating field in the last decades. Some researchers introduced various fractional derivatives, integrals, and their properties with singular or non-singular kernels to exploit its applications in real world problems [3–5, 9–15]. These fractional derivatives and fractional integrals have been used in inequalities as well. In inequalities, Hermite-Hadamard inequality is the renowned and is given in the following theorem.

Theorem 1.1. *[1] Let $\xi : I \to \mathbb{R}$ is a convex function on the interval $I \subseteq \mathbb{R}$ and $l_1, l_2 \in I$ with $l_1 < l_2$, then*

$$\xi\left(\frac{l_1 + l_2}{2}\right) \leq \frac{1}{l_2 - l_1} \int_{l_1}^{l_2} \xi(w)dw \leq \frac{\xi(l_1) + \xi(l_2)}{2}. \tag{1.1}$$

In [7, 8] this inequality was generalized for s-convex functions. In the field of fractional calculus, the main focus of a researcher is to give a definition of new operators and to develop their applications and solution to problems by involving these operators. Singularity and locality are the properties in that the operators are distinct from each other. Whereas, the kernel of operators is based on the type of functions including power law, the exponential function, and the Mittag-Lefler function. The distinct property of the Caputo-Fabrizio operator is that its kernel is non-singular. In this paper, we develop the

Hermite-Hadamard inequality for s-convex functions via Caputo-Fabrizio fractional integral. First we define Caputo-Fabrizio fracional derivative and integral [3].

Definition 1.1. *Let* $\xi \in H^1(l_1, l_2)$*(class of first order differentiable functions),* $l_1 < l_2$, $0 \leq \beta \leq 1$, *then the left fractional derivative in the Caputo-Fabrizio sense is*

$$({}_{l_1}^{CF}D^{\beta}\xi)(t) = \frac{B(\beta)}{1-\beta}\int_{l_1}^{t} \xi'(w)e^{\frac{-\beta(t-w)^{\beta}}{1-\beta}}dw, \tag{1.2}$$

and the fractional integral corresponding to this operator is

$$({}_{l_1}^{CF}I^{\beta}\xi)(t) = \frac{1-\beta}{B(\beta)}\xi(t) + \frac{\beta}{B(\beta)}\int_{l_1}^{t} \xi(w)dw. \tag{1.3}$$

Right fractional derivative is defined as

$$({}^{CF}D_{l_2}^{\beta}\xi)(t) = \frac{-B(\beta)}{1-\beta}\int_{t}^{l_2} \xi'(w)e^{\frac{-\beta(w-t)^{\beta}}{1-\beta}}dw, \tag{1.4}$$

and the fractional integral corresponding to this operator is

$$({}^{CF}I_{l_2}^{\beta}\xi)(t) = \frac{1-\beta}{B(\beta)}\xi(t) + \frac{\beta}{B(\beta)}\int_{t}^{l_2} \xi(w)dw. \tag{1.5}$$

The Caputo-Fabrizio derivative is obtained by changind the singular kernel $(t-w)^{-\beta}$ of Caputo fractional derivative by non-singular kernel $e^{\frac{-\beta(t-w)^{\beta}}{1-\beta}}$ and $\frac{1}{\Gamma(1-\beta)}$ by $\frac{B(\beta)}{1-\beta}$, where $B(\beta) > 0$ is a normalization function satisfying $B(0) = B(1) = 1$ given in [3].

Definition 1.2. *[6] A function* $\xi : I \subseteq \mathbb{R}^+ \to \mathbb{R}^+$ *is said to be s-convex function in second sense on I if*

$$\xi(tl_1 + (1-t)l_2) \leq t^s\xi(l_1) + (1-t)^s\xi(l_2),$$

holds for all $l_1, l_2 \in I$, *and* $t \in [0, 1]$, *for some fix* $s \in (0, 1]$.

Lemma 1.1. *[2, Lemma 1] Let I be a real interval such that* $ł_1, ł_2 \in I^{\circ}$, *the interior of I with* $ł_1 < ł_2$. *Let* $\xi : I^0 \subseteq \mathbb{R} \to \mathbb{R}$ *be a differentiable mapping on* $I^0, l_1, l_2 \in I$ *with* $l_1 < l_2$. *If* $\xi' \in L([l_1, l_2])$, *then the following equality holds:*

$$\frac{\xi(l_1) + \xi(l_2)}{2} - \frac{1}{l_2 - l_1}\int_{l_1}^{l_2} \xi(w)dw = \frac{l_2 - l_1}{2}\int_{0}^{1} (1-2t)\xi'(tl_1 + (1-t)l_2)dt. \tag{1.6}$$

Lemma 1.2. *[2, Lemma 2] Let I be a real interval such that* $ł_1, ł_2 \in I^{\circ}$, *the interior of I with* $ł_1 < ł_2$. *Let* $\xi : I \subseteq \mathbb{R} \to \mathbb{R}$ *be a differentiable function on* I^0, $l_1, l_2 \in I$ *with* $l_1 < l_2$. *If* $\xi' \in L([l_1, l_2])$ *and* $0 \leq \beta \leq 1$, *then following inequality holds:*

$$\begin{aligned}&\frac{l_2 - l_1}{2}\int_{0}^{1} (1-2t)\xi'(tl_1 + (1-t)l_2)dt - \frac{2(1-\beta)}{\beta(l_2 - l_1)}\xi(k)\\ &= \frac{\xi(l_1) + \xi(l_2)}{2} - \frac{B(\beta)}{\beta(l_2 - l_1)}\Big(({}_{l_1}^{CF}I^{\beta}\xi)(k) + ({}^{CF}I_{l_2}^{\beta}\xi)(k)\Big)\end{aligned} \tag{1.7}$$

where $k \in [l_1, l_2]$ *and* $B(\beta) > 0$ *is normalization function.*

2. Generalization of Hermite-Hadamard inequality for s-convex function via Caputo-Fabrizio fractional operator

Here we present Hermite-Hadamard inequality for s-convex functions via Caputo-Fabrizio fractional operator which is stated in the following theorem.

Theorem 2.1. *Let I be a real interval such that $\text{ł}_1, \text{ł}_2 \in I^\circ$; the interior of I with $\text{ł}_1 < \text{ł}_2$. Let a function $\xi : [l_1, l_2] \subseteq \mathbb{R} \to \mathbb{R}$ be s-convex on $[l_1, l_2]$ for $s \in (0, 1)$ and $\xi \in L([l_1, l_2])$. If $0 \leq \beta \leq 1$, then we have the following double inequality:*

$$\begin{aligned} 2^{s-1}\xi\left(\frac{l_1 + l_2}{2}\right) &\leq \frac{B(\beta)}{\beta(l_2 - l_1)}\left[({}^{CF}_{l_1}I^{\beta}\xi)(k) + ({}^{CF}I^{\beta}_{l_2}\xi)(k) - \frac{2(1-\beta)}{B(\beta)}\xi(k)\right] \\ &\leq \frac{\xi(l_1) + \xi(l_2)}{2}. \end{aligned} \tag{2.1}$$

where $k \in [l_1, l_2]$ and $B(\beta) > 0$ is normalization function.

Proof. Since ξ is s-convex, taking $\lambda = \frac{1}{2}$ we get

$$2^s\xi\left(\frac{x+y}{2}\right) \leq \xi(x) + \xi(y).$$

On making change of variables and integrating over $[0, 1]$, we get

$$2^{s-1}\xi\left(\frac{l_1 + l_2}{2}\right) \leq \frac{1}{l_2 - l_1}\int_{l_1}^{l_2} \xi(x)dx.$$

Multiplying both sides by $\frac{\beta(l_2-l_1)}{B(\beta)}$ we get

$$\frac{2^{s-1}\beta(l_2 - l_1)}{B(\beta)}\xi\left(\frac{l_1 + l_2}{2}\right) \leq \frac{\beta}{B(\beta)}\int_{l_1}^{l_2} \xi(x)dx$$

and

$$\frac{2^{s-1}\beta(l_2 - l_1)}{B(\beta)}\xi\left(\frac{l_1 + l_2}{2}\right) \leq \frac{\beta}{B(\beta)}\left(\int_{l_1}^{k} \xi(x)dx + \int_{k}^{l_2} \xi(x)dx\right).$$

Adding $\frac{2(1-\beta)}{B(\beta)}\xi(k)$ on both sides we have

$$\begin{aligned} &2^{s-1}\frac{\beta(l_2 - l_1)}{B(\beta)}\xi\left(\frac{l_1 + l_2}{2}\right) + \frac{2(1-\beta)}{B(\beta)}\xi(k) \\ &\leq \left(\frac{1-\beta}{B(\beta)}\xi(k) + \frac{\beta}{B(\beta)}\int_{l_1}^{k} \xi(x)dx\right) + \left(\frac{1-\beta}{B(\beta)}\xi(k) + \frac{\beta}{B(\beta)}\int_{k}^{l_2} \xi(x)dx\right), \end{aligned}$$

which means that

$$\frac{2^{s-1}\beta(l_2 - l_1)}{B(\beta)}\xi\left(\frac{l_1 + l_2}{2}\right) \leq ({}^{CF}_{l_1}I^{\beta}\xi)(k) + ({}^{CF}I^{\beta}_{l_2}\xi)(k) - \frac{2(1-\beta)}{B(\beta)}\xi(k),$$

i.e.,

$$2^{s-1}\xi\left(\frac{l_1+l_2}{2}\right) \leq \frac{B(\beta)}{\beta(l_2-l_1)}\left(({}_{l_1}^{CF}I^{\beta}\xi)(k) + ({}^{CF}I_{l_2}^{\beta}\xi)(k) - \frac{2(1-\beta)}{B(\beta)}\xi(k)\right). \tag{2.2}$$

which proves first inequality. Now, we prove second inequality, since

$$\frac{1}{l_2-l_1}\int_{l_1}^{l_2}\xi(x)dx = \int_0^1 \xi(tl_1+(1-t)l_2)dt.$$

As ξ is s-convex in the second sense, we get

$$\frac{1}{l_2-l_1}\int_{l_1}^{l_2}\xi(x)dx \leq \int_0^1 [t^s\xi(l_1)+(1-t)^s\xi(l_2)]dt.$$

Multiplying both sides by $\frac{\beta(l_2-l_1)}{B(\beta)}$ we get

$$\frac{\beta}{B(\beta)}\int_{l_1}^{l_2}\xi(x)dx \leq \frac{\beta(l_2-l_1)}{B(\beta)}\left(\frac{\xi(l_1)+\xi(l_2)}{s+1}\right).$$

Adding $\frac{2(1-\beta)}{B(\beta)}\xi(k)$ on both sides

$$\begin{aligned}&\left(\frac{\beta}{B(\beta)}\int_{l_1}^{k}\xi(x)dx + \frac{(1-\beta)}{B(\beta)}\xi(k)\right) + \left(\frac{\beta}{B(\beta)}\int_{k}^{l_2}\xi(x)dx + \frac{(1-\beta)}{B(\beta)}\xi(k)\right)\\ &\leq \frac{\beta(l_2-l_1)}{B(\beta)}\left(\frac{\xi(l_1)+\xi(l_2)}{s+1}\right) + \frac{2(1-\beta)}{B(\beta)}\xi(k),\end{aligned}$$

and this gives that

$$\frac{B(\beta)}{\beta(l_2-l_1)}\left(({}_{l_1}^{CF}I^{\beta}\xi)(k) + ({}^{CF}I_{l_2}^{\beta}\xi)(k) - \frac{2(1-\beta)}{B(\beta)}\xi(k)\right) \leq \frac{\xi(l_1)+\xi(l_2)}{s+1}. \tag{2.3}$$

From (9) and (10) we get (8), which is required. □

Remark 2.1. *For $s = 1$ we get Theorem 2 of [2].*

Theorem 2.2. *Let I be a real interval such that $ł_1, ł_2 \in I^{\circ}$; the interior of I with $ł_1 < ł_2$. Let $\xi_1 : [l_1, l2] \subseteq \mathbb{R} \to \mathbb{R}$ be a convex non-negative and $\xi_2 : [l_1, l_2] \subseteq \mathbb{R} \to \mathbb{R}$ be an s-convex functions on I. If $\xi_1\xi_2 \in L([l_1, l_2])$, then we have the following inequality:*

$$\begin{aligned}&\frac{B(\beta)}{\beta(l_2-l_1)}\left(({}_{l_1}^{CF}I^{\beta}\xi_1\xi_2)(k) + ({}^{CF}I_{l_2}^{\beta}\xi_1\xi_2)(k) - \frac{2(1-\beta)}{B(\beta)}\xi_1(k)\xi_2(k)\right)\\ &\leq \frac{M(l_1,l_2)}{s+2} + \frac{N(l_1,l_2)}{(s+1)(s+2)},\end{aligned} \tag{2.4}$$

where $M(l_1, l_2) = \xi_1(l_1)\xi_2(l_1) + \xi_1(l_2)\xi_2(l_2)$, $N(l_1, l_2) = \xi_1(l_1)\xi_2(l_2) + \xi_1(l_2)\xi_2(l_1)$.

Proof. Since ξ_1 is convex and ξ_2 is s-convex then we have

$$\xi_1(tl_1 + (1-t)l_2) \leq t\xi_1(l_1) + (1-t)\xi_2(l_2),$$

$$\xi_2(tl_1 + (1-t)l_2) \leq t^s\xi_2(l_1) + (1-t)^s\xi_2(l_2).$$

Multiplying these inequalities we get

$$\begin{aligned}&\xi_1(tl_1 + (1-t)l_2)\xi_2(tl_1 + (1-t)l_2)\\ &\leq t^{s+1}\xi_1(l_1)\xi_2(l_1) + (1-t)^{s+1}\xi_1(l_2)\xi_2(l_2) + t(1-t)^s\xi_1(l_1)\xi_2(l_2) + t^s(1-t)\xi_1(l_2)\xi_2(l_1).\end{aligned}$$

Integrating over $t \in [0,1]$, then making change of variables we get

$$\frac{1}{l_2 - l_1}\int_{l_1}^{l_2}\xi_1(x)\xi_2(x)dx \leq \frac{\xi_1(l_1)\xi_2(l_1) + \xi_1(l_2)\xi_2(l_2)}{s+2} + \frac{\xi_1(l_1)\xi_2(l_2) + \xi_1(l_2)\xi_2(l_1)}{(s+1)(s+2)}.$$

Multiplying both sides by $\frac{\beta(l_2-l_1)}{B(\beta)}$ and adding $\frac{2(1-\beta)}{B(\beta)}\xi_1(k)\xi_2(k)$, we get

$$\begin{aligned}&\frac{\beta}{B(\beta)}\int_{l_1}^{k}\xi_1(x)\xi_2(x)dx + \frac{(1-\beta)}{B(\beta)}\xi_1(k)\xi_2(k) + \frac{\beta}{B(\beta)}\int_{k}^{l_2}\xi_1(x)\xi_2(x)dx + \frac{(1-\beta)}{B(\beta)}\xi_1(k)\xi_2(k)\\ &\leq \frac{\beta(l_2-l_1)}{B(\beta)}\left(\frac{M(l_1,l_2)}{s+2}\right) + \frac{\beta(l_2-l_1)}{B(\beta)}\left(\frac{N(l_1,l_2)}{(s+1)(s+2)}\right) + \frac{(1-\beta)}{B(\beta)}\xi_1(k)\xi_2(k).\end{aligned}$$

On simplifying the last inequality, we get

$$\begin{aligned}&\frac{B(\beta)}{\beta(l_2-l_1)}\left(({}^{CF}_{l_1}I^{\beta}\xi_1\xi_2)(k) + ({}^{CF}I^{\beta}_{l_2}\xi_1\xi_2)(k) - \frac{2(1-\beta)}{B(\beta)}\xi_1(k)\xi_2(k)\right)\\ &\leq \frac{M(l_1,l_2)}{s+2} + \frac{N(l_1,l_2)}{(s+1)(s+2)}.\end{aligned}$$

□

Remark 2.2. *(i) For $\beta = 1, B(\beta) = B(1) = 1$, we get Theorem 5 of [6].*

(ii) For $s = 1$ we get Theorem 3 of [2].

Theorem 2.3. *Let I be a real interval such that $ł_1, ł_2 \in I^\circ$; the interior of I with $ł_1 < ł_2$. Let $\xi_1, \xi_2 : [l_1, l_2] \subseteq \mathbb{R} \to \mathbb{R}$, be non-negative functions such that ξ_1, ξ_2 and $\xi_1\xi_2$ are in $L([l_1, l_2])$. If ξ_1 is s_1-convex and ξ_2 is s_2-convex on $[l_1, l_2]$ for some fixed $s_1, s_2 \in (0, 1]$, then*

$$\begin{aligned}&\frac{B(\beta)}{b-a}\left(\left({}^{CF}_{l_1}I^{\beta}\xi_1\xi_2\right)(k) + \left({}^{CF}I^{\beta}_{l_2}\xi_1\xi_2\right)(k) - \frac{2(1-\beta)}{B(\beta)}\xi_1(k)\xi_2(k)\right)\\ &\leq \frac{1}{s_1+s_2+1}\left(M(l_1,l_2) + \frac{s_1s_2\Gamma s_1\Gamma s_2}{\Gamma(s_1+s_2+1)}N(l_1,l_2)\right),\end{aligned} \tag{2.5}$$

where $M(l_1, l_2) = \xi_1(l_1)\xi_2(l_1)+\xi_1(l_2)\xi_2(l_2)$, and $N(l_1, l_2) = \xi_1(l_1)\xi_2(l_2)+\xi_1(l_2)\xi_2(l_1)$, and $\Gamma.$ is the Gamma function.

Proof. As ξ_1 is s_1-convex and ξ_2 is s_2-convex we have

$$\xi_1(tl_1 + (1-t)l_2) \leq t^{s_1}\xi_1(l_1) + (1-t)^{s_1}\xi_1(l_2),$$
$$\xi_2(tl_1 + (1-t)l_2) \leq t^{s_2}\xi_2(l_1) + (1-t)^{s_2}\xi_2(l_2).$$

Multiplying these inequalities we obtain

$$\begin{aligned}&\xi_1(tl_1 + (1-t)l_2)\xi_2(tl_1 + (1-t)l_2)\\ &\leq t^{s_1+s_2}\xi_1(l_1)\xi_2(l_1) + (1-t)^{s_1+s_2}\xi_1(l_2)\xi_2(l_2) + t^{s_1}(1-t)^{s_2}\xi_1(l_1)\xi_2(l_2) + t^{s_2}(1-t)^{s_1}\xi_1(l_2)\xi_2(l_1).\end{aligned}$$

Integrating with respect to t over $[0, 1]$ and making change of variables, we get

$$\begin{aligned}&\frac{1}{l_2 - l_1}\int_{l_1}^{l_2} \xi_1(x)\xi_2(x)dx\\ &\leq \frac{1}{s_1 + s_2 + 1}\Big(\xi_1(l_1)\xi_2(l_1) + \xi_1(l_2)\xi_2(l_2)\Big) + \frac{\Gamma(s_1+1)\Gamma(s_2+1)}{\Gamma(s_1+s_2+2)}\left(\xi_1(l_1)\xi_2(l_2) + \xi_1(l_2)\xi_2(l_1)\right).\end{aligned}$$

Multiplying both sides by $\frac{\beta(l_2-l_1)}{B(\beta)}$ and adding $\frac{2(1-\beta)}{B(\beta)}\xi_1(k)\xi_2(k)$ we get

$$\begin{aligned}&\frac{B(\beta)}{l_2 - l_1}\left(\left({}^{CF}_{l_1}I^{\beta}\xi_1\xi_2\right)(k) + \left({}^{CF}I^{\beta}_{l_2}\xi_1\xi_2\right)(k) - \frac{2(1-\beta)}{B(\beta)}\xi_1(k)\xi_2(k)\right)\\ &\leq \frac{1}{s_1+s_2+1}\left(M(l_1, l_2) + \frac{s_1 s_2 \Gamma s_1 \Gamma s_2}{\Gamma(s_1+s_2+1)}N(l_1, l_2)\right).\end{aligned}$$

□

Remark 2.3. *(i) For $\beta = 1, B(\beta) = B(1) = 1$, we get [5, Theorem 6].*

(ii) For $s_1 = s_2 = 1$ we get [2, Theorem 3].

Theorem 2.4. *Let I be a real interval such that $ł_1, ł_2 \in I^\circ$; the interior of I with $ł_1 < ł_2$. Let $\xi_1, \xi_2 : [l_1, l_2] \subseteq \mathbb{R} \to \mathbb{R}$, be the function such that $\xi_1, \xi_2, \xi_1\xi_2 \in L([l_1, l_2])$. If ξ_1 is convex and non-negative on $[l_1, l_2]$ and ξ_2 is s-convex on $[l_1, l_2]$ for some fixed $s \in (0, 1]$, then*

$$\begin{aligned}&\xi_1\left(\frac{l_1+l_2}{2}\right)\xi_2\left(\frac{l_1+l_2}{2}\right) - \frac{1}{2^s(l_2-l_1)}\left(\left({}^{CF}_{l_1}I^{\beta}\xi_1\xi_2\right)(k) + \left({}^{CF}I^{\beta}_{l_2}\xi_1\xi_2\right)(k) - \frac{(1-\beta)}{2^{s-1}\beta(l_2-l_1)}\xi_1(k)\xi_2(k)\right)\\ &\leq \frac{1}{2^s}\left(\frac{M(l_1, l_2)}{(s+1)(s+2)} + \frac{N(l_1, l_2)}{s+2}\right).\end{aligned} \tag{2.6}$$

Where $M(l_1, l_2) = \xi_1(l_1)\xi_2(l_1) + \xi_1(l_2)\xi_2(l_2)$, and $N(l_1, l_2) = \xi_1(l_1)\xi_2(l_2) + \xi_1(l_2)\xi_2(l_1)$.

Proof. As

$$\frac{l_1+l_2}{2} = \frac{tl_1 + (1-t)l_2}{2} + \frac{(1-t)l_1 + tl_2}{2},$$

and

$$\begin{aligned}&\xi_1\left(\frac{l_1+l_2}{2}\right)\xi_2\left(\frac{l_1+l_2}{2}\right)\\ &= \xi_1\left(\frac{tl_1 + (1-t)l_2}{2} + \frac{(1-t)l_1 + tl_2}{2}\right)\xi_2\left(\frac{tl_1 + (1-t)l_2}{2} + \frac{(1-t)l_1 + tl_2}{2}\right).\end{aligned}$$

Using convexity of ξ_1 and s-convexity of ξ_2, we get

$$\begin{aligned}
&\xi_1\left(\frac{l_1+l_2}{2}\right)\xi_2\left(\frac{l_1+l_2}{2}\right)\\
&\le \frac{1}{2}[\xi_1(tl_1+(1-t)l_2)+\xi_1((1-t)l_1+tl_2)]\frac{1}{2^s}[\xi_2(tl_1+(1-t)l_2)+\xi_2((1-t)l_1+tl_2)],\\
&\le \frac{1}{2^{s+1}}[\xi_1(tl_1+(1-t)l_2)\xi_2(tl_1+(1-t)l_2)+\xi_1((1-t)l_1+tl_2)\xi_2((1-t)l_1+tl_2)]\\
&+\frac{1}{2^{s+1}}[\{(1-t)\xi_1(l_1)+t\xi_1(l_2)\}\{t^s\xi_2(l_1)+(1-t)^s\xi_2(l_2)\}\\
&+\{t\xi_1(l_1)+(1-t)\xi_1(l_2)\}\{(1-t)^s\xi_2(l_1)+t^s\xi_2(l_2)\}]\\
&=\frac{1}{2^{s+1}}[\xi_1(tl_2+(1-t)l_2)\xi_2(tl_1+(1-t)l_2)+\xi_1((1-t)l_1+tl_2)\xi_2((1-t)l_1+tl_2)]\\
&+\frac{1}{2^{s+1}}[(t(1-t)^s+(1-t)t^s)\{\xi_1(l_1)\xi_2(l_1)+\xi_1(l_2)\xi_2(l_2)\}\\
&+(t^{s+1}+(1-t)^{s+1})\{\xi_1(l_1)\xi_2(l_2)+\xi_2(l_1)\xi_1(l_2)\}].
\end{aligned}$$

Integrating with respect to t over $[0,1]$ and making change of variable we get

$$\begin{aligned}
&\xi_1\left(\frac{l_1+l_2}{2}\right)\xi_2\left(\frac{l_1+l_2}{2}\right)\\
&\le \frac{1}{2^s(l_2-l_1)}\int_{l_1}^{l_2}\xi_1(x)\xi_2(x)dx+\frac{1}{2^{s+1}}\left(\frac{2}{(s+1)(s+2)}M(l_1,l_2)+\frac{2}{s+2}N(l_1,l_2)\right),\\
&2^s\xi_1\left(\frac{l_1+l_2}{2}\right)\xi_2\left(\frac{l_1+l_2}{2}\right)\\
&\le \frac{1}{(l_2-l_1)}\int_{l_1}^{l_2}\xi_1(x)\xi_2(x)dx+\frac{1}{(s+1)(s+2)}M(l_1,l_2)+\frac{1}{s+2}N(l_1,l_2).
\end{aligned}$$

Multiplying both sides by $\frac{\beta(l_2-l_1)}{B(\beta)}$ and subtracting $\frac{2(1-\beta)}{B(\beta)}\xi_1(k)\xi_2(k)$, we obtain

$$\begin{aligned}
&\frac{2^s\beta(l_2-l_1)}{B(\beta)}\xi_1\left(\frac{l_1+l_2}{2}\right)\xi_2\left(\frac{l_1+l_2}{2}\right)\\
&-\frac{\beta}{B(\beta)}\left[\int_{l_1}^{k}\xi_1(x)\xi_2(x)dx+\int_{k}^{l_2}\xi_1(x)\xi_2(x)dx\right]-\frac{2(1-\beta)}{B(\beta)}\xi_1(k)\xi_2(k)\\
&\le \frac{\beta(l_2-l_1)}{B(\beta)}\left(\frac{M(l_1,l_2)}{(s+1)(s+2)}+\frac{N(l_1,l_2)}{(s+2)}\right)-\frac{2(1-\beta)}{B(\beta)}\xi_1(k)\xi_2(k).
\end{aligned}$$

After simplification we get

$$\begin{aligned}
&\xi_1\left(\frac{l_1+l_2}{2}\right)\xi_2\left(\frac{l_1+l_2}{2}\right)-\frac{1}{2^s(l_2-l_1)}[\left({}_{l_1}^{CF}I^{\beta}\xi_1\xi_2\right)(k)\\
&+\left({}^{CF}I_{l_2}^{\beta}\xi_1\xi_2\right)(k)-\frac{(1-\beta)}{2^{s-1}\beta(l_2-l_1)}\xi_1(k)\xi_2(k)]\\
&\le \frac{1}{2^s}\left(\frac{M(l_1,l_2)}{(s+1)(s+2)}+\frac{N(l_1,l_2)}{s+2}\right).
\end{aligned}$$

□

Corollary 2.1. *For $\beta = 1$ we get Theorem 7 of [6].*

3. Some inequalities of Hermite-Hadamard type via Caputo-Fabrizio fractional operator

In this section, we present Hermite-Hadamard type inequalities for s-convex functions via Caputo-Fabrizio fractional operator.

Theorem 3.1. *Let I be a real interval such that $ł_1, ł_2 \in I^\circ$; the interior of I with $ł_1 < ł_2$. Let $\xi : I \subseteq \mathbb{R} \to \mathbb{R}$ be a positive differentiable mapping on I^0 and $|\xi'|$ is s-convex on $[l_1, l_2]$. If $\xi' \in L([l_1, l_2])$ and $\beta \in [0, 1]$, then we have the following inequality:*

$$\begin{aligned}&\left|\frac{\xi(l_1)+\xi(l_2)}{2} - \frac{B(\beta)}{\beta(l_2-l_1)}\left(({}^{CF}_{l_1}I^{\beta}\xi)(k) + ({}^{CF}I^{\beta}_{l_2}\xi)(k) - \frac{2(1-\beta)}{B(\beta)}\xi(k)\right)\right| \\ &\leq \frac{l_2-l_1}{2}\left(\frac{2^{-s}+s}{(s+1)(s+2)}\right)(|\xi'(l_1)| + |\xi'(l_2)|)\,.\end{aligned} \tag{3.1}$$

Proof. By Lemma 1.3 and the property of modulus we obtain,

$$\begin{aligned}&\left|\frac{\xi(l_1)+\xi(l_2)}{2} + \frac{2(1-\beta)}{\beta(l_2-l_1)}\xi(k) - \frac{B(\beta)}{\beta(l_2-l_1)}\left(({}^{CF}_{l_1}I\xi)(k) + ({}^{CF}I_{l_2}\xi)(k)\right)\right| \\ &\leq \frac{l_2-l_1}{2}\int_0^1 |1-2t||\xi'(tl_1+(1-t)l_2)|dt,\end{aligned}$$

Using s-convexity of $|\xi'|$ we get

$$\begin{aligned}&\left|\frac{\xi(l_1)+\xi(l_2)}{2} + \frac{2(1-\beta)}{\beta(l_2-l_1)}\xi(k) - \frac{B(\beta)}{\beta(l_2-l_1)}\left(({}^{CF}_{l_1}I\xi)(k) + ({}^{CF}I_{l_2}\xi)(k)\right)\right| \\ &\leq \frac{l_2-l_1}{2}\int_0^1 |1-2t|(t^s|\xi'(l_1)| + (1-t)^s|\xi'(l_2)|)dt, \\ &\qquad = \frac{l_2-l_1}{2}\left(\frac{2^{-s}+s}{(s+1)(s+2)}\right)(|\xi'(l_1)| + |\xi'(l_2)|).\end{aligned}$$

□

Remark 3.1. *Setting $s = 1$ in Theorem 3.1, we recapture Theorem 5 in [2].*

Next theorem gives new upper bound of the left-Hadamard inequality for s-convex function via Caputto-Fabrizio fractional integral.

Theorem 3.2. *Let I be a real interval such that $ł_1, ł_2 \in I^\circ$; the interior of I with $ł_1 < ł_2$. Let $\xi : I \subseteq \mathbb{R} \to \mathbb{R}$ be a positive differentiable mapping on I^0 and $|\xi'|^q$ is s-convex on $[l_1, l_2]$, where $p > 1, p^{-1} + q^{-1} = 1$. If $\xi' \in L([l_1, l_2])$ and $\beta \in [0, 1]$, then we have the following inequality:*

$$\begin{aligned}&\left|\frac{\xi(l_1)+\xi(l_2)}{2} + \frac{2(1-\beta)}{\beta(l_2-l_1)}\xi(k) - \frac{B(\beta)}{\beta(l_2-l_1)}\left(({}^{CF}_{l_1}I^{\beta}\xi)(k) + ({}^{CF}I^{\beta}_{l_2}\xi)(k)\right)\right| \\ &\leq \left(\frac{l_2-l_1}{2}\right)\left(\frac{1}{1+p}\right)^{\frac{1}{p}}\left(\frac{|\xi'(l_1)|^q + |\xi'(l_2)|^q}{s+1}\right)^{\frac{1}{q}}.\end{aligned} \tag{3.2}$$

Proof. Using similar arguments as the proof of Theorem 3.1, but this time we use Hölder inequality and s-convexity of $|\xi'|^q$ we have

$$\begin{aligned}
&\left|\frac{\xi(l_1)+\xi(l_2)}{2}+\frac{2(1-\beta)}{\beta(l_2-l_1)}\xi(k)-\frac{B(\beta)}{\beta(l_2-l_1)}\left(({}^{CF}_{l_1}I^{\beta}\xi)(k)+({}^{CF}I^{\beta}_{l_2}\xi)(k)\right)\right| \\
&\leq \frac{l_2-l_1}{2}\int_0^1 |1-2t||\xi'(tl_1+(1-t)l_2)|dt, \\
&\leq \frac{l_2-l_1}{2}\left(\int_0^1 |1-2t|^p dt\right)^{\frac{1}{p}}\left(\int_0^1 |\xi'(tl_1+(1-t)l_2)|^q dt\right)^{\frac{1}{q}}, \\
&\leq \frac{l_2-l_1}{2}\left(\int_0^1 |1-2t|^p dt\right)^{\frac{1}{p}}\left(\int_0^1 t^s|\xi'(l_1)|^q dt+\int_0^1 (1-t)^s|\xi'(l_2)|^q dt\right)^{\frac{1}{q}}, \\
&= \frac{l_2-l_1}{2}\left(\frac{1}{1+p}\right)^{\frac{1}{p}}\left(\frac{|\xi'(l_1)|^q+|\xi'(l_2)|^q}{s+1}\right)^{\frac{1}{q}}.
\end{aligned}$$

□

Remark 3.2. *For s=1 we get Theorem 6 of [2].*

4. Application to special means

In this section, first we recall some mathematical means of two real numbers, and then present application of previous results on these means of real numbers.

(i) The arithmetic mean

$$\mathcal{A}=\mathcal{A}(w_1,w_2)=\frac{w_1+w_2}{2}, w_1,w_2\in\mathbb{R}.$$

(ii) The generalized logarithmic mean

$$\mathcal{L}=\mathcal{L}_n^n(w_1,w_2)=\frac{w_2^{n+1}-w_1^{n+1}}{(n+1)(w_2-w_1)}, n\in\mathbb{R}-\{-1,0\}, w_1,w_2\in\mathbb{R}, w_1\neq w_2.$$

Proposition 1. *Let $w_1,w_2\in\mathbb{R}^+$, $w_1<w_2$, $0<s<1$, then*

$$|\mathcal{A}(w_1^s,w_2^s)-L_s^s(w_1,w_2)|\leq s\left(\frac{w_2-w_1}{2}\right)\left(\frac{2^{-s}+s}{(s+1)(s+2)}\right)\left(|w_1|^{s-1}+|w_2|^{s-1}\right). \tag{4.1}$$

Proof. By setting $\xi(z)=z^s$ with $\beta=1$ and $B(\beta)=1$ in Theorem 3.1 we get required result. □

Proposition 2. *Let $w_1,w_2\in\mathbb{R}^+$ with $w_1<w_2$, and $0<s<1$ and for all $q>1$, we have*

$$|\mathcal{A}(w_1^s,w_2^s)-L_s^s(w_1,w_2)|\leq s\left(\frac{w_2-w_1}{2}\right)\left(\frac{1}{1+p}\right)^{\frac{1}{p}}\left(\frac{1}{s+1}\right)^{\frac{1}{q}}\left[|w_1|^{(s-1)q}+|w_2|^{(s-1)q}\right]^{\frac{1}{q}}. \tag{4.2}$$

Proof. By setting $\xi(z)=z^s$ with $\beta=1$ and $B(\beta)=1$ in Theorem 3.2 we get required result. □

5. Conclusions

The theory of convex functions is considered an important tool in the theory of optimization. Fractional calculus has a wide range of applications in Engineering and Applied Sciences. Caputo-Fabrizio fractional integral, with a non-singular kernel, is one of the major operators which is used to develop different mathematical models. In this paper, Hermite-Hadamard type inequalities for s-convex functions in the second sense via Caputo-Fabrizio integral operator are established. The obtained results are generalizations. In literature, the existing results become particular cases for these results which are given in remarks. The results of this paper may stimulate further research for the researchers working in this field.

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Conflict of interest

The authors declare that they have no competing interests.

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