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Research article

New analysis of fuzzy fractional Langevin differential equations in Caputo's derivative sense

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Abstract: The extraction of analytical solution of uncertain fractional Langevin differential equations involving two independent fractional-order is frequently complex and difficult. As a result, developing a proper and comprehensive technique for the solution of this problem is very essential. In this article, we determine the explicit and analytical fuzzy solution for various classes of the fuzzy fractional Langevin differential equations (FFLDEs) with two independent fractional-orders both in homogeneous and non-homogeneous cases. The potential solution of FFLDEs is also extracted using the fuzzy Laplace transformation technique. Furthermore, the solution of FFLDEs is defined in terms of bivariate and trivariate Mittag-Leffler functions both in the general and special forms. FFLDEs are a new topic having many applications in science and engineering then to grasp the novelty of this work, we connect FFLDEs with RLC electrical circuit to visualize and support the theoretical results.

Keywords: fuzzy fractional Langevin differential equations; Caputo-type fractional derivative; fuzzy Laplace transformation; bivariate and trivariate Mittag-Leffler function; RLC electrical circuit
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1. Introduction

Fuzzy sets are used to describe the dynamical model that does not involve randomness but has some uncertain parameters. Some of these models naturally result in fuzzy differential equations (FDEs). FDEs have been attaining much attention in the field of mathematics under the impact of applied scientific domains including physics, automated control theory, artificial intelligence, abnormal diffusion, medical sciences, random processes, and many more. It has been used to analyze the dynamical behavior of real-world physical problems. FDEs [1] were introduced in 1982 based on the fuzzy derivative called Dubois-Prade derivative. Subsequently, several definitions of fuzzy derivatives were introduced among which Hukuhara derivative (H-derivative) [2] (which is known as Puri Ralescu derivative) in 1983, Goetschel-Voxman [3] derivative in 1986, Seikkala [4] derivative in 1987, and Friedman-Ming-Kandel [5] derivative in 1996. Although many definitions of fuzzy derivatives have been defined in various forms, these definitions give the equivalent result of fuzzy function via α-cut. Puri-Ralescu derivative and Seikkala derivative are the most well-known derivatives. Puri and Ralescu derivatives are defined on the base of H-difference, whereas the Seikkala derivative is defined using the α-level set of the fuzzy function. The representation of the same and reverse order derivative based on Seikkala derivative [6] were another effort in 1998 to overcome the difficulty arising from the use of H-derivative or equivalently Seikkala derivative. This derivative is closely related to ones that present a fuzzy derivative, called generalized H-derivative and generalized Seikkala derivative. In 2005, Bede and Gal [7] introduced the concept of strongly generalized Hukuhara derivative (SGHD). The structure of SGHD offers two types of differentiability of a fuzzy function, which are referred to as the first and second form of differentiability. The H-derivative corresponds to the first form and the non-decreasing diameter of a differentiable fuzzy function is the second form of differentiability if it exists.

The concept of the fractional-order appears to be universal and the integer-order looks like local. The revelation of fractional-order has been shown effective in symbolizing complicated systems whereas the computational effects are very vague. In many domains of the field of fractional calculus, calculation procedures are not straightforward; they are subject to complicated methodologies and approaches. Many authors attracted the attention for the solution of fractional differential equations and fractional integro-differential equations both in crisp and uncertain environment. Recently, Raja et al. [8] developed optimal control results for integro-differential equations; for example, for Sobolev-type fractional mixed Volterra-Fredholm with order belongs to $(0, 1)$, approximate controllability results for integro-differential equations [9], for existence and continuous dependence results [10], results on controllability for Sobolev type fractional differential equations [11] and the approximate controllability results for fractional integro-differential systems [12]. A comprehensive study of fuzzy fractional calculus makes us capable to deal with many problems in both theoretical and applied science. The idea of fuzzy derivatives has resulted in the introduction of several definitions of fuzzy fractional derivatives (FFDs). Agarwal et al. [13] initially proposed the concept of fuzzy fractional-order differential equations in 2010. The Riemann-Liouville (R-L) FFDs in the sense of H-derivative [14] and Seikkala derivative [15] were introduced in 2010 and 2011, respectively. After that, Salahshour et al. [16] introduced the concept of SGHD based on R-L FFD in 2012. SGHD presents two types of differentiability of fuzzy function. If the order belongs to $(0, 1)$, then we have two solutions of FFDEs that come from the first and second form of differentiability. The two-type solution of FFDEs [17, 18] in the sense of Caputo H-derivative was investigated in 2012.

Meanwhile, Agarwal et al. [13] proposed the solution of FFDEs using R-L differentiability. In 2013, Arshad [19] ensured the existence and uniqueness of FFDEs. Allahviranloo et al. [20, 21] introduced the solution procedure for FFDEs using generalized fuzzy Caputo-differentiability. In parallel, Khastan et al. [22] prove the existence and uniqueness theorem for the solution of FFDEs using the Schauder fixed point theorem under the R-L fuzzy fractional derivative. Hoa et al. [23] proposed the idea of generalized Hukuhara differentiability using interval numbers. Ngo et al. [24] introduced the fuzzy fractional initial-value problem using Caputo-differentiability. Several considerable investigations of

FFDEs in the sense of Caputo gH-derivative introduced in [24]. By using the fixed point approach, Vu et al. [25] studied the stability of both fuzzy fractional integral and fuzzy Ulam-Hyers-Rassias fractional differential equations. Vu and Hoa [26] develop the solution procedure for FFDEs on a time scale under granular differentiability. Ezadi and Allahviranloo [27] developed an artificial neural network approach for solving fuzzy fractional initial-value problems using gH-differentiability. Akram et al. [28, 29] developed the numerical technique for solving bipolar and m-polar fuzzy initial-value problems. Further, Akram et al. [30–32] presented methods for solving bipolar and m-polar fuzzy linear systems. Ghaffari et al. [33] introduced the solution of time-fractional FDEs using the fuzzy Laplace and Fourier transformation technique. Moreover, we refer to the reader for other different techniques and applications of fractional differential equations in an uncertain environment [34–37, 39–42].

Langevin [38] proposed the idea of Langevin differential equations (LDE) to describe the dynamical behavior of fluctuation and interpretation of Brownian motion. Indeed, LDE is a useful tool for investigating the dynamical features of a wide range of important systems in science and engineering [43, 44]. Kubo [45] later presented the concept of generalized LDE, where the fractional memory kernel was inserted into the equation to characterized the fractal and memory features. As a result, this gives the idea to study of fractional Langevin differential equation [46]. After that, various types of fractional Langevin differential equation were introduced and investigated [47–49] with different fractional order on the different interval and boundary points. Many researchers [50–52] studied the existence and uniqueness of solutions to fractional Langevin differential equations and anti-periodic fractional Langevin differential equations. Applications of fractional differential equations play a vital role in the study of electrical circuits. A considerable growing interest in electrical circuit theory and simulation has been seen in the last few years. Many researchers used three types of fractional differential operators to explain electrical circuit equations. Kaczorek [53] obtained the solutions of the fractional differential equation using the Laplace transformation technique as well as discussed their applications in electric circuit theory. But certain errors occur due to environmental conditions in circuit parameters that may lead to the account uncertainty and vagueness in circuit analysis. Devi and Ganesan [54] developed a circuit equation as a fuzzy differential equation with fuzzy variables using a conventional type of fuzzy derivative. We investigate an initial-value problem of inhomogeneous fuzzy fractional Langevin differential equations with same dynamics given in [55] with general fractional orders as shown in the following:

$$\begin{cases} {}^{C}\mathfrak{D}_{0^+}^{\tau_1}\upsilon(\mathfrak{m}) - \lambda {}^{C}\mathfrak{D}_{0^+}^{\tau_2}\upsilon(\mathfrak{m}) - \mu\upsilon(\mathfrak{m}) = \mathfrak{g}(\mathfrak{m}), \\ \upsilon_i(0) = \upsilon_i(0, \ltimes), \quad 0 \leq i \leq q-1, \end{cases} \tag{1.1}$$

where ${}^{C}\mathfrak{D}_{0^+}^{\tau_1}$ and ${}^{C}\mathfrak{D}_{0^+}^{\tau_2}$ Caputo type conventional fuzzy fractional derivative of order τ_1 and τ_2 in various intervals $q-2 < \tau_2 \leq q-1$, $q-1 < \tau_1 \leq q$ and $q \geq 2$ with $\chi_2 \geq 1$ (for short $\chi_2 = \tau_1 - \tau_2$) and λ, μ are the real parameters, $\ltimes \in [0, 1]$.

Although, several authors developed many interesting techniques and approaches to solve FFDEs with their applications. The current article asserts its novelty from the following perspectives:

(i) We determine the explicit and analytical fuzzy solutions for the various classes of fuzzy fractional Langevin differential equations involving two independent fractional-orders both in homogeneous and non-homogeneous cases.
(ii) We extract two possible potential solutions for general and special cases of fuzzy fractional

Langevin differential equations under the strongly generalized H-differentiability using fuzzy Laplace transformation tool.

(iii) Moreover, we discuss an application of fuzzy fractional Langevin differential equations in *RLC*-electrical circuit.

The remaining part of the article is organized as follows: Section 2 presents preliminary results of the R-L fractional derivative, Caputo fractional derivative, and integral in both crisp and uncertain environments. Solution procedure of multi-order FFLDEs is presented in Section 3. The Section 4 is devoted for the application of FFLDEs in *RLC*-electrical circuit. The conclusion and the future direction of this work are presented in Section 5.

2. Basic concepts

We begin this section with some basic definitions and terminologies of fuzzy sets and preliminaries of the fundamental structure of fractional calculus. Throughout this article, the class of all real numbers is denoted by $\mathbb{R}$ and the family of all fuzzy numbers (FNs) on $\mathbb{R}$ by $F^{\mathbb{R}}$. The $\ltimes$-cut of the fuzzy set υ is denoted by $\upsilon_{\ltimes}$.

Definition 2.1. [3] Suppose that the fuzzy set υ in a non-empty subset Ω of $\mathbb{R}$ identified with the rule of membership grade $\upsilon : \Omega \longrightarrow [0,1]$. Firmly, υ is convex because

$$\upsilon(\ltimes\mathfrak{m} + (1-\ltimes)\mathfrak{n}) \geq \min\{\upsilon(\mathfrak{m}), \upsilon(\mathfrak{n})\}, \quad \forall \ltimes, \mathfrak{m}, \mathfrak{n} \text{ with } \ltimes \in [0,1], \ \mathfrak{m}, \mathfrak{n} \in \mathbb{R};$$

υ is upper semi-continuous so that $\{\mathfrak{m} \in \mathbb{R} | \ \upsilon(\mathfrak{m}) \geq \lambda\}$ is closed for all $\lambda \in [0,1]$; and υ is normal because there exist $\mathfrak{m} \in \mathbb{R}$ such that $\upsilon(\mathfrak{m}) = 1$. The support of υ is $\{\mathfrak{m} \in \mathbb{R} | \ \upsilon(\mathfrak{m}) > 0\}$.
The $\ltimes$-cut of $\upsilon \in F^{\mathbb{R}}$ is closed and bounded interval $[\upsilon_1(\ltimes), \upsilon_2(\ltimes)]$ with $\upsilon_1(\ltimes)$ and $\upsilon_2(\ltimes)$ are called the left and right end points of $\upsilon(\ltimes)$. The triangular fuzzy number (TFN) $\upsilon \in F^{\mathbb{R}}$ is characterized in the form of parametric fuzzy number as $(\upsilon_1(\ltimes), \upsilon_2(\ltimes))$, $0 \leq \ltimes \leq 1$. The functions $\upsilon_1(\ltimes)$ and $\upsilon_2(\ltimes)$ have fulfil the following requirements:

(i) The function $\upsilon_1(\ltimes)$ is bounded, monotonically increasing and left continuous,
(ii) The function $\upsilon_2(\ltimes)$ is bounded, monotonically decreasing, left continuous,
(iii) $\upsilon_1(\ltimes) \leq \upsilon_2(\ltimes)$.

Definition 2.2. Let $\upsilon_1, \upsilon_2 \in F^{\mathbb{R}}$, if there exist $\upsilon_3 \in F^{\mathbb{R}}$ such that $\upsilon_1 = \upsilon_2 + \upsilon_3$. Then υ_3 is called Hukuhara difference (H-difference for short) of υ_1 and υ_2 and is defined as $\upsilon_1 \ominus \upsilon_2$.

Definition 2.3. [7] The mapping $\upsilon : [0,b] \longrightarrow F^{\mathbb{R}}$ is continuous and Lebesgue integrable fuzzy number-valued function on $[0,b]$, and $V(\mathfrak{m}) = \dfrac{1}{\Gamma(1-\tau)} \displaystyle\int_a^{\mathfrak{m}} \dfrac{\upsilon(q) \ominus \upsilon_0(0)}{(\mathfrak{m}-q)^{\tau}} dq$. Then $\upsilon(\mathfrak{m})$ is called Caputo fuzzy fractional differentiable function of order $0 < \tau < 1$ at $\mathfrak{m} \in (0,b)$ in the first form, if there exists ${}^{C}\mathfrak{D}^{\tau}\upsilon(\mathfrak{m}) \in F^{\mathbb{R}}$ such that:

(i) For every $\hbar$ positive, the expressions $V(\mathfrak{m}+\hbar) \ominus V(\mathfrak{m})$ and $V(\mathfrak{m}) \ominus V(\mathfrak{m}-\hbar)$ both exists such that

$$ {}^{C}\mathfrak{D}^{\tau}\upsilon(\mathfrak{m}) = \lim_{\hbar \searrow 0} \frac{V(\mathfrak{m}+\hbar) \ominus V(\mathfrak{m})}{\hbar} = \lim_{\hbar \searrow 0} \frac{V(\mathfrak{m}) \ominus V(\mathfrak{m}-\hbar)}{\hbar}. \tag{2.1}$$

Or, $\upsilon(\mathfrak{m})$ is called Caputo fuzzy fractional differentiable function of order $0 < \tau < 1$ at $\mathfrak{m} \in (0,b)$ in the second form, if $\exists\, {}^{C}\mathfrak{D}^{\tau}\upsilon(\mathfrak{m}) \in F^{\mathbb{R}}$ such that

(ii) For every $\hbar$ positive, the expressions $V(\mathfrak{m}) \ominus V(\mathfrak{m}+\hbar)$ and $V(\mathfrak{m}-\hbar) \ominus V(\mathfrak{m})$ both exists such that

$$^{C}\mathfrak{D}^{\tau}\upsilon(\mathfrak{m}) = \lim_{\hbar\searrow 0} \frac{V(\mathfrak{m}) \ominus V(\mathfrak{m}+\hbar)}{-\hbar} = \lim_{\hbar\searrow 0} \frac{V(\mathfrak{m}-\hbar) \ominus V(\mathfrak{m})}{-\hbar}. \tag{2.2}$$

Indeed, υ is differentiable on $\mathscr{T}$ if υ is differentiable for every $\mathfrak{m} \in \mathscr{T}$. The limits are taken in the metric space $(F^{\mathbb{R}}, \mathfrak{d}_1)$ where $\mathfrak{d}_1(\mathfrak{b}_1, \mathfrak{b}_2) = \sup\{\mathfrak{d}_H(\mathfrak{b}_1(\ltimes), \mathfrak{b}_2(\ltimes) \ : \ 0 \leq \ltimes \leq 1, \mathfrak{b}_1, \mathfrak{b}_2 \in F^{\mathbb{R}})\}$ and $\mathfrak{d}_H$ is Hausdorff distance.

Note 2.1. R-L fuzzy fractional derivative of $\upsilon(\mathfrak{m})$ at $\mathfrak{m}_0$ is denoted by $^{RL}\mathfrak{D}^{\tau}\upsilon(\mathfrak{m})$ can be defined as same as the above Definition 2.3.

Definition 2.4. [7] For a mapping $\upsilon : (0, b) \longrightarrow F^{\mathbb{R}}$ is a fuzzy number-valued function (FNVF) and $\upsilon(\mathfrak{m}, \ltimes) = (\upsilon_1(\mathfrak{m}, \ltimes), \upsilon_2(\mathfrak{m}, \ltimes))$, where $\upsilon_1(\mathfrak{m}, \ltimes)$ and $\upsilon_2(\mathfrak{m}, \ltimes)$ are the left and right membership function of $\ltimes$-plane, respectively.

(i) If $\upsilon(\mathfrak{m}, \ltimes)$ is Caputo fractional differentiable of first form, then the functions $\upsilon_1(\mathfrak{m}, \ltimes)$ and $\upsilon_2(\mathfrak{m}, \ltimes)$ are differentiable and

$$\mathfrak{D}^{\tau}_{0^+}\upsilon(\mathfrak{m}, \ltimes) = \Big[\mathfrak{D}^{\tau}_{0^+}\upsilon_1(\mathfrak{m}, \ltimes), \mathfrak{D}^{\tau}_{0^+}\upsilon_2(\mathfrak{m}, \ltimes)\Big].$$

(ii) If $\upsilon(\mathfrak{m}, \ltimes)$ is Caputo fractional differentiable of first form, then the functions $\upsilon_1(\mathfrak{m}, \ltimes)$ and $\upsilon_2(\mathfrak{m}, \ltimes)$ are differentiable and

$$\mathfrak{D}^{\tau}_{0^+}\upsilon(\mathfrak{m}, \ltimes) = \Big[\mathfrak{D}^{\tau}_{0^+}\upsilon_2(\mathfrak{m}, \ltimes), \mathfrak{D}^{\tau}_{0^+}\upsilon_1(\mathfrak{m}, \ltimes)\Big],$$

where,

$$^{C}\mathfrak{D}^{\tau}{}_{0^+}\upsilon_1(\mathfrak{m}, \ltimes) = \frac{1}{\Gamma(\tau)} \int_0^{\mathfrak{m}} (\mathfrak{m}-q)^{\tau-1} \upsilon_1^{'}(q, \ltimes) dq, \ \ \text{and} \ \ ^{C}\mathfrak{D}^{\tau}{}_{0^+}\upsilon_2(\mathfrak{m}, \ltimes) = \frac{1}{\Gamma(\tau)} \int_0^{\mathfrak{m}} (\mathfrak{m}-q)^{\tau-1} \upsilon_2^{'}(q, \ltimes) dq, \tag{2.3}$$

$\upsilon_1^{'}(q, \ltimes) = \dfrac{d\upsilon_1^{'}(q, \ltimes)}{dq}$, $\upsilon_2^{'}(q, \ltimes) = \dfrac{d\upsilon_2^{'}(q, \ltimes)}{dq}$, and $\tau \in (0, 1)$. The functions $\upsilon_1(q, \ltimes)$ and $\upsilon_2(q, \ltimes)$ are both strongly generalized differentiable function of the first and second form.

The Sobolev space of order one on $\mathscr{T}$ with a mapping $\upsilon : \mathscr{T} :\longrightarrow \mathbb{R}$ such that it can be defined as

$$\mathscr{S}(\mathscr{T}) = \{\upsilon \in L^2(S) : \ \ \upsilon, \upsilon^{'} \in L^2(S)\},$$

where $\mathscr{T} = (0, b)$.

Definition 2.5. [56–58] Let $\upsilon : \mathscr{T} \longrightarrow \mathbb{R}$, $\upsilon \in \mathscr{S}(\mathscr{T})$. Then the R-L fractional integral of order $\tau > 0$ is defined as:

$$(\mathfrak{I}^{\tau}_{a^+}\upsilon)(\mathfrak{m}) = \frac{1}{\Gamma(\tau)} \int_a^{\mathfrak{m}} (\mathfrak{m}-q)^{\tau-1} \upsilon(q) dq, \quad \text{for } \mathfrak{m} > a. \tag{2.4}$$

Definition 2.6. [56–58] Let $\upsilon : \mathscr{T} \longrightarrow \mathbb{R}$, $\upsilon \in \mathscr{S}(\mathscr{T})$. Then the R-L derivative of fractional order $\tau \geq 0$ is

$$({}^{RL}\mathfrak{D}^{\tau}_{a^+}\upsilon)(\mathfrak{m}) = \frac{\mathrm{d}^{\mathrm{k}}}{\mathrm{dx}^{\mathrm{k}}}(\mathfrak{I}^{k-\tau}_{a^+}\upsilon)(\mathfrak{m}), \quad \text{for } \mathfrak{m} > a, \tag{2.5}$$

where $k \in \mathbb{N}$ and $k-1 \leq \tau < k$. In particular, if $\tau \in (0,1)$ and $a = 0$ then the aforementioned definition takes the following form

$$({}^{RL}\mathfrak{D}^{\tau}_{a^+}\upsilon)(\mathfrak{m}) = \frac{1}{\Gamma(1-\tau)}\frac{\mathrm{d}}{\mathrm{d}\mathfrak{m}}\int_0^{\mathfrak{m}}(\mathfrak{m}-q)^{-\tau}\upsilon(q)dq, \quad \text{for } \mathfrak{m} > 0. \tag{2.6}$$

Definition 2.7. [56–58] Let $\upsilon : \mathscr{T} \longrightarrow \mathbb{R}$, $\upsilon \in \mathscr{S}(\mathscr{T})$. Then the Caputo-derivative of fractional order $\tau \geq 0$ is

$$({}^{C}\mathfrak{D}^{\tau}_{a^+}\upsilon)(\mathfrak{m}) = \Big(\mathfrak{I}^{k-\tau}_{a^+}(\frac{\mathrm{d}^{\mathrm{k}}}{\mathrm{dx}^{\mathrm{k}}}\upsilon)\Big)(\mathfrak{m}), \quad \text{for } \mathfrak{m} > a, \tag{2.7}$$

where $k \in \mathbb{N}$ and $k-1 \leq \tau < k$. In particular, if $\tau \in (0,1)$ and $a = 0$ then the above definition is

$$({}^{C}\mathfrak{D}^{\tau}_{a^+}\upsilon)(\mathfrak{m}) = \frac{1}{\Gamma(1-\tau)}\int_0^{\mathfrak{m}}(\mathfrak{m}-q)^{-\tau}\upsilon^{'}(q)dq, \quad \text{for } \mathfrak{m} > 0. \tag{2.8}$$

Definition 2.8. Let $\upsilon : \mathscr{T} \longrightarrow \mathbb{R}$, $\upsilon \in \mathscr{S}(\mathscr{T})$. The fractional integral [59] (Section 3.1, Definition 2) in terms of bivariate Mittag-Leffler function in the univariate form is defined as

$$({}_a\mathfrak{I}^{\epsilon;\eta_1,\eta_2}_{\alpha,\beta,\gamma}\upsilon)(\mathfrak{m}) = \int_a^{\mathfrak{m}}(\mathfrak{m}-q)^{\gamma-1}E^{\epsilon}_{\alpha,\beta,\gamma}(\eta_1(\mathfrak{m}-q)^{\alpha},\eta_2(\mathfrak{m}-q)^{\beta})\upsilon(q)dq, \;\; u > a, \tag{2.9}$$

where $\alpha,\beta,\gamma > 0$ and ϵ,η_1,η_2 are real parameters. The integral $({}_a\mathfrak{I}^{\epsilon;\eta_1,\eta_2}_{\alpha,\beta,\gamma}\upsilon)(\mathfrak{m})$ can be written in summation form as

$$({}_a\mathfrak{I}^{\epsilon;\eta_1,\eta_2}_{\alpha,\beta,\gamma}\upsilon)(\mathfrak{m}) = \sum_{k=0}^{+\infty}\sum_{l=0}^{+\infty}\frac{(\epsilon)_{k+l}\eta_1^k\eta_2^l}{k!l!}({}^{RL}_{a}I^{\alpha k+\beta l+\gamma}\upsilon)(\mathfrak{m}). \tag{2.10}$$

If $\epsilon = 0$, then the integral in Eq (2.10) is coincide with R-L integral as

$$({}_a\mathfrak{I}^{\epsilon;\eta_1,\eta_2}_{\alpha,\beta,\gamma}\upsilon)(\mathfrak{m}) = ({}^{RL}_{a}I^{\gamma}\upsilon)(\mathfrak{m}). \tag{2.11}$$

Definition 2.9. Let $\upsilon : \mathscr{T} \longrightarrow \mathbb{R}$, $\upsilon \in \mathscr{S}(\mathscr{T})$. The fractional integral [60] in terms of trivariate Mittag-Leffler function is defined as

$$({}_a\mathfrak{I}^{\epsilon;\eta_1,\eta_2,\eta_3}_{\alpha,\beta,\gamma,\delta}\upsilon)(\mathfrak{m}) = \int_a^{\mathfrak{m}}(\mathfrak{m}-q)^{\delta-1}E^{\epsilon}_{\alpha,\beta,\gamma,\delta}(\eta_1(\mathfrak{m}-q)^{\alpha},\eta_2(\mathfrak{m}-q)^{\beta},\eta_2(\mathfrak{m}-q)^{\gamma})\upsilon(q)dq, \;\; u > a, \tag{2.12}$$

where $\alpha,\beta,\gamma,\delta > 0$ and $\epsilon,\eta_1,\eta_2,\eta_3$ are real parameters. The integral $({}_a\mathfrak{I}^{\epsilon;\eta_1,\eta_2,\eta_3}_{\alpha,\beta,\gamma,\delta}\upsilon)(\mathfrak{m})$ can be written in summation form as

$$({}_a\mathfrak{I}^{\epsilon;\eta_1,\eta_2,\eta_3}_{\alpha,\beta,\gamma,\delta}\upsilon)(\mathfrak{m}) = \sum_{k=0}^{+\infty}\sum_{l=0}^{+\infty}\sum_{m=0}^{+\infty}\frac{(\epsilon)_{k+l+m}\eta_1^k\eta_2^l\eta_3^m}{k!l!m!}({}^{RL}_{a}I^{\alpha k+\beta l+\gamma m+\delta}\upsilon)(\mathfrak{m}). \tag{2.13}$$

If $\epsilon = 0$, then the integral in Eq (2.13) is coincide with R-L integral as

$$({}_a\mathfrak{I}^{\epsilon;\eta_1,\eta_2,\eta_3}_{\alpha,\beta,\gamma,\delta}\upsilon)(\mathfrak{m}) = ({}^{RL}_{a}I^{\delta}\upsilon)(\mathfrak{m}) \tag{2.14}$$

Fuzzy fractional calculus

In this subsection, we present some fundamental concepts related to the R-L and Caputo fuzzy fractional derivative and integral. Let $\mathbb{C}^{F^{\mathbb{R}}}(\mathscr{T})$ and $L^{F^{\mathbb{R}}}(\mathscr{T})$ be the space of all continuous and Lebesgue integrable fuzzy number-valued functions on $\mathscr{T}$, respectively. We understand these definitions from the following perspective:

Definition 2.10. Let $\upsilon : \mathscr{T} \longrightarrow \mathbb{R}$, $\upsilon \in \mathbb{C}^{F^{\mathbb{R}}}(\mathscr{T}) \cap L^{F^{\mathbb{R}}}(\mathscr{T})$. Then the R-L integral [16] for fuzzy number-valued function υ (in the form of $\ltimes$-cut representation) of fractional order $\tau > 0$ is defined as:

$$\mathfrak{I}^{\tau}_{a^+}\upsilon(\mathfrak{m}, \ltimes) = \frac{1}{\Gamma(\tau)} \int_a^{\mathfrak{m}} (\mathfrak{m} - q)^{\tau-1} \upsilon(q, \ltimes) dq.$$

Moreover, $\upsilon(\mathfrak{m}, \ltimes)$ is called the R-L fractional differentiable of first form, if

$$\mathfrak{I}^{\tau}_{a^+}\upsilon(\mathfrak{m}, \ltimes) = \left[\mathfrak{I}^{\tau}_{a^+}\upsilon_1(\mathfrak{m}, \ltimes), \mathfrak{I}^{\tau}_{a^+}\upsilon_2(\mathfrak{m}, \ltimes)\right], \quad \ltimes \in [0, 1], \ \mathfrak{m} > a, \tag{2.15}$$

and, $\upsilon(\mathfrak{m}, \ltimes)$ is called the R-L fractional differentiable of second form, if

$$\mathfrak{I}^{\tau}_{a^+}\upsilon(\mathfrak{m}, \ltimes) = \left[\mathfrak{I}^{\tau}_{a^+}\upsilon_2(\mathfrak{m}, \ltimes), \mathfrak{I}^{\tau}_{a^+}\upsilon_1(\mathfrak{m}, \ltimes)\right], \quad \ltimes \in [0, 1], \ \mathfrak{m} > a, \tag{2.16}$$

where the lower and upper fuzzy number-valued functions defined as following

$$\mathfrak{I}^{\tau}_{a^+}\upsilon_1(\mathfrak{m}, \ltimes) = \frac{1}{\Gamma(\tau)} \int_a^{\mathfrak{m}} (\mathfrak{m} - q)^{\tau-1} \upsilon_1(q, \ltimes) dq, \ \ \mathfrak{I}^{\tau}_{a^+}\upsilon_2(\mathfrak{m}, \ltimes) = \frac{1}{\Gamma(\tau)} \int_a^{\mathfrak{m}} (\mathfrak{m} - q)^{\tau-1} \upsilon_2(q, \ltimes) dq.$$

On the based of above definition, the fuzzy R-L derivative of fractional order is defined as:

Definition 2.11. Let $\upsilon : \mathscr{T} \longrightarrow \mathbb{R}$, $\upsilon \in \mathbb{C}^{F^{\mathbb{R}}}(\mathscr{T}) \cap L^{F^{\mathbb{R}}}(\mathscr{T})$. Suppose υ is fuzzy fractional differentiable of first form. Then the fuzzy R-L derivative [16] of fractional order $\tau \geq 0$ is

$${}^{RL}\mathfrak{D}^{\tau}_{a^+}\upsilon(\mathfrak{m}, \ltimes) = \left[\frac{\mathrm{d^k}}{\mathrm{dx^k}} \mathfrak{I}^{k-\tau}_{a^+}\upsilon_1(\mathfrak{m}, \ltimes), \frac{\mathrm{d^k}}{\mathrm{dx^k}} \mathfrak{I}^{k-\tau}_{a^+}\upsilon_2(\mathfrak{m}, \ltimes)\right], \quad \text{for } \mathfrak{m} > a. \tag{2.17}$$

Suppose υ is fuzzy fractional differentiable of the second form. Then the fuzzy R-L derivative of fractional order $\tau \geq 0$ is

$${}^{RL}\mathfrak{D}^{\tau}_{a^+}\upsilon(\mathfrak{m}, \ltimes) = \left[\frac{\mathrm{d^k}}{\mathrm{dx^k}} \mathfrak{I}^{k-\tau}_{a^+}\upsilon_2(\mathfrak{m}, \ltimes), \frac{\mathrm{d^k}}{\mathrm{dx^k}} \mathfrak{I}^{k-\tau}_{a^+}\upsilon_1(\mathfrak{m}, \ltimes)\right], \quad \text{for } \mathfrak{m} > a, \tag{2.18}$$

where, the entire integral is defined in Definition 2.10 and $k \in \mathbb{N}$ such that $k-1 \leq \tau < k$. In particular, if $\tau \in (0, 1)$ and $a = 0$ then the definition takes the following form for the first and second differentiability

$${}^{RL}\mathfrak{D}^{\tau}_{a^+}\upsilon(\mathfrak{m}, \ltimes) = \left[\frac{1}{\Gamma(1-\tau)} \frac{\mathrm{d}}{\mathrm{d}\mathfrak{m}} \int_a^{\mathfrak{m}} (\mathfrak{m} - q)^{-\tau} \upsilon_1(q, \ltimes) dq, \frac{1}{\Gamma(1-\tau)} \frac{\mathrm{d}}{\mathrm{d}\mathfrak{m}} \int_a^{\mathfrak{m}} (\mathfrak{m} - q)^{-\tau} \upsilon_2(q, \ltimes) dq\right], \quad \text{for } \mathfrak{m} > 0$$

and

$${}^{RL}\mathfrak{D}^{\tau}_{a^+}\upsilon(\mathfrak{m}, \ltimes) = \left[\frac{1}{\Gamma(1-\tau)} \frac{\mathrm{d}}{\mathrm{d}\mathfrak{m}} \int_a^{\mathfrak{m}} (\mathfrak{m} - q)^{-\tau} \upsilon_2(q, \ltimes) dq, \frac{1}{\Gamma(1-\tau)} \frac{\mathrm{d}}{\mathrm{d}\mathfrak{m}} \int_a^{\mathfrak{m}} (\mathfrak{m} - q)^{-\tau} \upsilon_1(q, \ltimes) dq\right], \quad \text{for } \mathfrak{m} > 0.$$

Definition 2.12. Let $\upsilon : \mathscr{T} \longrightarrow \mathbb{R}$, $\upsilon \in \mathbb{C}^{F^{\mathbb{R}}}(\mathscr{T}) \cap L^{F^{\mathbb{R}}}(\mathscr{T})$. Suppose υ is fuzzy fractional differentiable. Then the fuzzy Caputo-derivative [57] of fractional order $\tau \geq 0$ is

$$ {}^{C}\mathfrak{D}^{\tau}_{a^+}\upsilon(\mathfrak{m}, \ltimes) = \left[\mathfrak{I}^{k-\tau}_{a^+}\frac{\mathrm{d}^{\mathrm{k}}}{\mathrm{dx}^{\mathrm{k}}}\upsilon_1(\mathfrak{m}, \ltimes), \mathfrak{I}^{k-\tau}_{a^+}\frac{\mathrm{d}^{\mathrm{k}}}{\mathrm{dx}^{\mathrm{k}}}\upsilon_2(\mathfrak{m}, \ltimes)\right], \quad \text{for } \mathfrak{m} > a. \tag{2.19} $$

Suppose υ is fuzzy fractional differentiable of second form. Then the fuzzy Caputo-derivative of fractional order $\tau \geq 0$ is

$$ {}^{C}\mathfrak{D}^{\tau}_{a^+}\upsilon(\mathfrak{m}, \ltimes) = \left[\mathfrak{I}^{k-\tau}_{a^+}\frac{\mathrm{d}^{\mathrm{k}}}{\mathrm{dx}^{\mathrm{k}}}\upsilon_2(\mathfrak{m}, \ltimes), \mathfrak{I}^{k-\tau}_{a^+}\frac{\mathrm{d}^{\mathrm{k}}}{\mathrm{dx}^{\mathrm{k}}}\upsilon_1(\mathfrak{m}, \ltimes)\right], \quad \text{for } \mathfrak{m} > a, \tag{2.20} $$

where, the integral is defined in Definition 2.10 and $k \in \mathbb{N}$ such that $k - 1 \leq \tau < k$. In particular, if $\tau \in (0, 1)$ and $a = 0$ then the definition takes the following form for the first and second differentiability as

$$ {}^{C}\mathfrak{D}^{\tau}_{a^+}\upsilon(\mathfrak{m}, \ltimes) = \left[\frac{1}{\Gamma(1-\tau)}\int_a^{\mathfrak{m}}(\mathfrak{m}-q)^{-\tau}\upsilon_1'(q, \ltimes)dq, \frac{1}{\Gamma(1-\tau)}\int_a^{\mathfrak{m}}(\mathfrak{m}-q)^{-\tau}\upsilon_2'(q, \ltimes)dq\right], \text{ for } \mathfrak{m} > 0, $$

and

$$ {}^{C}\mathfrak{D}^{\tau}_{a^+}\upsilon(\mathfrak{m}, \ltimes) = \left[\frac{1}{\Gamma(1-\tau)}\int_a^{\mathfrak{m}}(\mathfrak{m}-q)^{-\tau}\upsilon_2'(q, \ltimes)dq, \frac{1}{\Gamma(1-\tau)}\int_a^{\mathfrak{m}}(\mathfrak{m}-q)^{-\tau}\upsilon_1'(q, \ltimes)dq\right], \text{ for } \mathfrak{m} > 0. $$

The following characterization theorem explains how to convert the FFDEs into a system of ordinary FFDEs.

Theorem 2.1. [18] Consider the following FFDEs

$$ {}^{C}\mathfrak{D}^{\tau_1}_{0^+}\upsilon(\mathfrak{m}) = \mathscr{Z}(\mathfrak{m}, \upsilon(\mathfrak{m})), \tag{2.21} $$

subject to the initial-condition

$$ \upsilon_0(0) = \upsilon(0, \ltimes), \tag{2.22} $$

where $\mathscr{Z} : [a, b] \times F^{\mathbb{R}} \longrightarrow F^{\mathbb{R}}$ such that

(i) $[\mathscr{Z}(\mathfrak{m}, \upsilon(\mathfrak{m}))]^{\ltimes} = [\mathscr{Z}_1(\mathfrak{m}, \upsilon_*(\mathfrak{m}, \ltimes), \upsilon_{**}(\mathfrak{m}, \ltimes)), \mathscr{Z}_2(\mathfrak{m}, \upsilon_*(\mathfrak{m}, \ltimes), \upsilon_{**}(\mathfrak{m}, \ltimes))]$.

(ii) For any $\epsilon > 0$, there exist a $\delta > 0$ such that

$$ \mid \mathscr{Z}_1(\mathfrak{m}, r, s) - \mathscr{Z}_1(\mathfrak{m}_1, r_1, s_1) \mid < \epsilon \text{ and } \mid \mathscr{Z}_2(\mathfrak{m}, r, s) - \mathscr{Z}_2(\mathfrak{m}_1, r_1, s_1) \mid < \epsilon, $$

whenever $(\mathfrak{m}, r, s), (\mathfrak{m}_1, r_1, s_1) \in [a, b] \times \mathbb{R} \times \mathbb{R}$, $\| (\mathfrak{m}, r, s) - (\mathfrak{m}_1, r_1, s_1) \|_{\mathbb{R}\times\mathbb{R}\times\mathbb{R}} < \delta$ and $\mathscr{Z}_1, \mathscr{Z}_2$ are uniformly bounded on some bounded set.

(iii) There is a real number $c > 0$ (say) such that

$$ \mid \mathscr{Z}_1(\mathfrak{m}_2, r_2, s_2) - \mathscr{Z}_1(\mathfrak{m}_1, r_1, s_1) \mid \leq c.\max\{\mid r_2 - r_1 \mid, \mid s_2 - s_1 \mid\} \tag{2.23} $$

and

$$ \mid \mathscr{Z}_2(\mathfrak{m}_2, r_2, s_2) - \mathscr{Z}_2(\mathfrak{m}_1, r_1, s_1) \mid \leq c.\max\{\mid r_2 - r_1 \mid, \mid s_2 - s_1 \mid\}, \tag{2.24} $$

for $\ltimes \in [0, 1]$. As a result, two systems of ordinary FFDEs are equivalent to FFDEs (2.21) and (2.22) as follows:

(iv) If υ is fuzzy Caputo-fractional differentiable of first form, then

$$^{C}\mathfrak{D}_{a^+}^{\tau_1}\upsilon_1(\mathfrak{m},\ltimes) = \mathscr{Z}_1(\mathfrak{m},\upsilon_*(\mathfrak{m},\ltimes),\upsilon_{**}(\mathfrak{m},\ltimes)) \tag{2.25}$$

and

$$^{C}\mathfrak{D}_{a^+}^{\tau_1}\upsilon_2(\mathfrak{m},\ltimes) = \mathscr{Z}_2(\mathfrak{m},\upsilon_*(\mathfrak{m},\ltimes),\upsilon_{**}(\mathfrak{m},\ltimes)) \tag{2.26}$$

with $\upsilon(0,\ltimes) = (\upsilon_{(1)_0}(0,\ltimes),\upsilon_{(2)_0}(0,\ltimes))$.

(v) If υ is fuzzy Caputo-fractional differentiable of second form, then

$$^{C}\mathfrak{D}_{0^+}^{\tau_1}\upsilon_1(\mathfrak{m},\ltimes) = \mathscr{Z}_2(\mathfrak{m},\upsilon_*(\mathfrak{m},\ltimes),\upsilon_{**}(\mathfrak{m},\ltimes)) \tag{2.27}$$

and

$$^{C}\mathfrak{D}_{0^+}^{\tau_1}\upsilon_2(\mathfrak{m},\ltimes) = \mathscr{Z}_1(\mathfrak{m},\upsilon_*(\mathfrak{m},\ltimes),\upsilon_{**}(\mathfrak{m},\ltimes)) \tag{2.28}$$

with $\upsilon(0,\ltimes) = (\upsilon_{(1)_0}(0,\ltimes),\upsilon_{(2)_0}(0,\ltimes))$.

Definition 2.13. Let $\upsilon : \mathscr{T} \longrightarrow F^{\mathbb{R}}$, $\upsilon \in \mathbb{C}^{F^{\mathbb{R}}}(\mathscr{T}) \cap L^{F^{\mathbb{R}}}(\mathscr{T})$. Assume that $e^{-qu}\upsilon(\mathfrak{m})$ is improper fuzzy Riemann integrable on $[0,+\infty)$, then the integral $\int_0^{+\infty} e^{-qu}\upsilon(u)du$ is said to be the fuzzy Laplace transformation [61] of function υ and its symbolic representation

$$\mathscr{L}[\upsilon(\mathfrak{m})] = \int_0^{+\infty} e^{-q\mathfrak{m}}\upsilon(\mathfrak{m})d\mathfrak{m}, \quad q > 0. \tag{2.29}$$

The Laplace transform can be written in $\ltimes$-cut representation as

$$\int_0^{+\infty} e^{-q\mathfrak{m}}\upsilon(\mathfrak{m},\ltimes)d\mathfrak{m} = \Big[\int_0^{+\infty} e^{-q\mathfrak{m}}\upsilon_1(\mathfrak{m},\ltimes)d\mathfrak{m}, \int_0^{+\infty} e^{-q\mathfrak{m}}\upsilon_2(\mathfrak{m},\ltimes)d\mathfrak{m}\Big].$$

Or equivalently

$$\mathcal{L}[\upsilon(\mathfrak{m},\ltimes)] = \Big[\mathbb{L}(\upsilon_1(\mathfrak{m},\ltimes)), \mathbb{L}(\upsilon_2(\mathfrak{m},\ltimes))\Big],$$

where, $\mathbb{L}(\upsilon_1(\mathfrak{m},\ltimes))$ and $\mathbb{L}(\upsilon_2(\mathfrak{m},\ltimes))$ are called the lower and upper fuzzy number-valued functions, respectively.

In the section below, we present an analytical approach to solve multi-order fuzzy fractional Langevin differential equations with two independent orders.

3. Fuzzy fractional Langevin differential equations

In this section, first we develop some categorial frame work for the closed form solution of fuzzy fractional Langevin differential equations with two independent fractional orders. For this purpose, we prove the following results, which are required to extract the Laplace transform of fuzzy fractional derivative.

Theorem 3.1. Let $\upsilon : \mathscr{T} \longrightarrow F^{\mathbb{R}}$, $\upsilon \in \mathbb{C}^{F^{\mathbb{R}}}(\mathscr{T}) \cap L^{F^{\mathbb{R}}}(\mathscr{T})$. If $\upsilon(\mathfrak{m})$ is fuzzy Caputo-fractional differentiable of first form. Then the Laplace transform of fuzzy Caputo-fractional derivative of order $k-1 < \tau \leq k$ is defined as

$$\mathscr{L}\Big[{}^{C}\mathfrak{D}^{\tau}_{a^+}G(\mathfrak{m})\Big] = \Big[q^{\tau}G(q)\Big] \ominus \Big[\sum_{i=0}^{k-1} q^{\tau-i-1}\upsilon_k(0)\Big]. \tag{3.1}$$

If $\upsilon(\mathfrak{m})$ is fuzzy Caputo-fractional differentiable of second form. Then the Laplace transform of fuzzy Caputo-fractional derivative of order $k-1 < \tau \leq k$ is defined as

$$\mathscr{L}\Big[{}^{C}\mathfrak{D}^{\tau}_{a^+}G(\mathfrak{m})\Big] = -\Big[\sum_{i=0}^{k-1} q^{\tau-i-1}\upsilon_k(0)\Big] \ominus \Big[-\Big(q^{\tau}G(q)\Big)\Big]. \tag{3.2}$$

Proof. On the base of [57]. We prove the following result as:
Suppose $\upsilon(\mathfrak{m})$ is fuzzy Caputo-fractional differentiable of first form. Then, by using the status first of Definition 2.12, we have

$$\begin{aligned}{}^{C}\mathfrak{D}^{\tau}_{a^+}G(\mathfrak{m},\ltimes) &= [{}^{C}\mathfrak{D}^{\tau}_{a^+}G_1(\mathfrak{m},\ltimes), {}^{C}\mathfrak{D}^{\tau}_{a^+}G_2(\mathfrak{m},\ltimes)] \\ &= [\mathfrak{D}^{(k-\tau)}_{a^+}G_1^{(k)}(\mathfrak{m},\ltimes), \mathfrak{D}^{(k-\tau)}_{a^+}G_2^{(k)}(\mathfrak{m},\ltimes)],\end{aligned}$$

where, $k-1 < \tau \leq k$. Using Laplace transform on both sides

$$\mathscr{L}\Big[{}^{C}\mathfrak{D}^{\tau}_{a^+}G(\mathfrak{m},\ltimes)\Big] = \mathscr{L}\Big[(\mathfrak{D}^{(\tau-k)}_{a^+}G_1^{(k)}(\mathfrak{m},\ltimes)), (\mathfrak{D}^{(\tau-k)}_{a^+}G_2^{(k)}(\mathfrak{m},\ltimes))\Big].$$

Since the operator $\mathscr{L}$ is linear. So, we have

$$\begin{aligned}\mathscr{L}\Big[{}^{C}\mathfrak{D}^{\tau}_{a^+}G(\mathfrak{m},\ltimes)\Big] &= \Big[\mathbb{L}(\mathfrak{D}^{(\tau-k)}_{a^+}G_1^{(k)}(\mathfrak{m},\ltimes)), \mathbb{L}(\mathfrak{D}^{(\tau-k)}_{a^+}G_2^{(k)}(\mathfrak{m},\ltimes))\Big] \\ &= \Big[q^{(\tau-k)}G_1^{(k)}(q,\ltimes), q^{(\tau-k)}G_2^{(k)}(q,\ltimes)\Big] \\ &= \Big[q^{(\tau-k)}\Big(q^{k}G_1(q,\ltimes) - \sum_{i=0}^{k-1} q^{k-i-1}\upsilon_{(1)_k}(0,\ltimes)\Big), q^{(\tau-k)}\Big(q^{k}G_2(q,\ltimes) - \sum_{i=0}^{k-1} q^{k-i-1}\upsilon_{(1)_k}(0,\ltimes)\Big)\Big] \\ &= \Big[q^{\tau}G_1(q,\ltimes) - \sum_{i=0}^{k-1} q^{\tau-i-1}\upsilon_{(1)_k}(0,\ltimes), q^{\tau}G_2(q,\ltimes) - \sum_{i=0}^{k-1} q^{\tau-i-1}\upsilon_{(2)_k}(0,\ltimes)\Big] \\ &= \Big[q^{\tau}G_1(q,\ltimes), q^{\tau}G_2(q,\ltimes)\Big] \ominus \Big[\sum_{i=0}^{k-1} q^{\tau-i-1}\upsilon_{(1)_k}(0,\ltimes), \sum_{i=0}^{k-1} q^{\tau-i-1}\upsilon_{(2)_k}(0,\ltimes)\Big] \\ &= \Big[q^{\tau}G(q,\ltimes)\Big] \ominus \Big[\sum_{i=0}^{k-1} q^{\tau-i-1}\upsilon_k(0,\ltimes)\Big].\end{aligned}$$

Now for the second-differentiability. Suppose $\upsilon(\mathfrak{m})$ is fuzzy Caputo-fractional differentiable of second form. Then, by using second status of Definition 2.12, we have

$$\begin{aligned}({}^{C}\mathfrak{D}^{\tau}_{a^+}G)(\mathfrak{m},\ltimes) &= [{}^{C}\mathfrak{D}^{\tau}_{a^+}G_2(\mathfrak{m},\ltimes), {}^{C}\mathfrak{D}^{\tau}_{a^+}G_1(\mathfrak{m},\ltimes)] \\ &= [\mathfrak{D}^{(k-\tau)}_{a^+}G_2^{(k)}(\mathfrak{m},\ltimes), \mathfrak{D}^{(k-\tau)}_{a^+}G_1^{(k)}(\mathfrak{m},\ltimes)],\end{aligned}$$

where, $k-1<\tau\leq k$. Using Laplace transform on both sides, we have

$$\mathscr{L}\Big[{}^{C}\mathfrak{D}_{a^+}^{\tau}G(\mathfrak{m},\ltimes)\Big]=\mathscr{L}\Big[(\mathfrak{D}_{a^+}^{(\tau-k)}G_2^{(k)}(\mathfrak{m},\ltimes)),(\mathfrak{D}_{a^+}^{(\tau-k)}G_1^{(k)}(\mathfrak{m},\ltimes))\Big].$$

Since the operator $\mathscr{L}$ is linear. So, we have

$$\begin{aligned}\mathscr{L}\Big[{}^{C}\mathfrak{D}_{a^+}^{\tau}G(\mathfrak{m},\ltimes)\Big]&=\Big[\mathbb{L}(\mathfrak{D}_{a^+}^{(\tau-k)}G_2^{(k)}(\mathfrak{m},\ltimes)),\mathbb{L}(\mathfrak{D}_{a^+}^{(\tau-k)}G_1^{(k)}(\mathfrak{m},\ltimes))\Big]\\&=\Big[q^{(\tau-k)}G_2^{(k)}(q,\ltimes),q^{(\tau-k)}G_1^{(k)}(q,\ltimes)\Big]\\&=\Big[q^{(\tau-k)}\Big(q^kG_2(q,\ltimes)-\sum_{i=0}^{k-1}q^{k-i-1}\upsilon_{(2)_k}(0,\ltimes)\Big),q^{(\tau-k)}\Big(q^kG_1(q,\ltimes)-\sum_{i=0}^{k-1}q^{k-i-1}\upsilon_{(1)_k}(0,\ltimes)\Big)\Big]\\&=\Big[q^{\tau}G_2(q,\ltimes)-\sum_{i=0}^{k-1}q^{\tau-i-1}\upsilon_{(2)_k}(0,\ltimes),q^{\tau}G_1(q,\ltimes)-\sum_{i=0}^{k-1}q^{\tau-i-1}\upsilon_{(1)_k}(0,\ltimes)\Big]\\&=\Big[-\sum_{i=0}^{k-1}q^{\tau-i-1}\upsilon_{(2)_k}(0,\ltimes),-\sum_{i=0}^{k-1}q^{\tau-i-1}\upsilon_{(1)_k}(0,\ltimes)\Big]\ominus\Big[-q^{\tau}G_2(q,\ltimes),-q^{\tau}G_1(q,\ltimes)\Big]\\&=-\Big[\sum_{i=0}^{k-1}q^{\tau-i-1}\upsilon_{(1)_k}(0,\ltimes),\sum_{i=0}^{k-1}q^{\tau-i-1}\upsilon_{(2)_k}(0,\ltimes)\Big]\ominus\Big[-\Big(q^{\tau}G_1(q,\ltimes),q^{\tau}G_2(q,\ltimes)\Big)\Big]\\&=-\Big[\sum_{i=0}^{k-1}q^{\tau-i-1}\upsilon_k(0,\ltimes)\Big]\ominus\Big[-\Big(q^{\tau}G(q,\ltimes)\Big)\Big].\end{aligned}$$

This completes the proof. □

General procedure to solve fuzzy fractional Langevin differential equations

In this subsection, we study an initial-value problem of inhomogeneous fuzzy linear Langevin fractional differential equations involving general orders as

$$\begin{cases}{}^{C}\mathfrak{D}_{0^+}^{\tau_1}\upsilon(\mathfrak{m})-\lambda^{C}\mathfrak{D}_{0^+}^{\tau_2}\upsilon(\mathfrak{m})-\mu\upsilon(\mathfrak{m})=\mathfrak{g}(\mathfrak{m}),\\ \upsilon_i(0)=\upsilon_i(0,\ltimes),\ \ 0\leq i\leq q-1,\end{cases}\tag{3.3}$$

where $q-2<\tau_2\leq q-1$, $q-1<\tau_1\leq q$ and $q\geq 2$ with $\chi_2\geq 1$.

Theorem 3.2. Let $\upsilon:\mathscr{T}\longrightarrow F^{\mathbb{R}}$, $\upsilon\in\mathbb{C}^{F^{\mathbb{R}}}(\mathscr{T})\cap L^{F^{\mathbb{R}}}(\mathscr{T})$. If υ is fuzzy Caputo-fractional differentiable of first form. Then the system (3.3) have the following solution

$$\begin{aligned}\upsilon(\mathfrak{m})&=\upsilon_0(0)+\upsilon_0(0)\mu\mathfrak{m}^{\tau_1}E_{\tau_1,\chi_2,\tau_1+1}(\mu\mathfrak{m}^{\tau_1},\lambda\mathfrak{m}^{\chi_2})+\upsilon_1(0)\mathfrak{m}+\upsilon_1(0)\mu\mathfrak{m}^{\tau_1+1}E_{\tau_1,\chi_2,\tau_1+2}(\mu\mathfrak{m}^{\tau_1},\lambda\mathfrak{m}^{\chi_2})\\&\quad+\cdots+\upsilon_{(k-2)}(0)\mu\mathfrak{m}^{\tau_1+k-2}E_{\tau_1,\chi_2,\tau_1+k-1}(\mu\mathfrak{m}^{\tau_1},\lambda\mathfrak{m}^{\chi_2})+\upsilon_{(k-2)}(0)\frac{\mathfrak{m}^{k-2}}{\Gamma(k-1)}+\upsilon_{(k-1)}(0)\mu\mathfrak{m}^{k-1}E_{\tau_1,\chi_2,k}(\mu\mathfrak{m}^{\tau_1},\lambda\mathfrak{m}^{\chi_2})\\&\quad+\int_0^{\mathfrak{m}}(\mathfrak{m}-u)^{\tau_1-1}E_{\tau_1,\chi_2,\tau_1}(\mu(\mathfrak{m}-u)^{\tau_1},\lambda(\mathfrak{m}-u)^{\chi_2})\mathfrak{g}(u)du.\end{aligned}$$

If υ is fuzzy Caputo-fractional differentiable of second form. Then the system (3.3) have the following solution

$$\begin{aligned}\upsilon(\mathfrak{m}) =& -\lambda\upsilon_0(0)\mathfrak{m}^{\chi_2}E_{\chi_1,\chi_2,\chi_3,\chi_2+1}(x,y,z) - \lambda\mathfrak{m}^{\chi_3}E_{\chi_1,\chi_2,\chi_3,\chi_3+1}(x,y,z)\\ &-\lambda\upsilon_1(0)\mathfrak{m}^{\chi_2+1}E_{\chi_1,\chi_2,\chi_3,\chi_2+2}(x,y,z) - \lambda\mathfrak{m}^{\chi_3+1}E_{\chi_1,\chi_2,\chi_3,\chi_3+2}(x,y,z) - \cdots\\ &-\lambda\upsilon_{(k-2)}(0)\mathfrak{m}^{\chi_2+k-2}E_{\chi_1,\chi_2,\chi_3,\chi_2+k-1}(x,y,z) - \lambda\mathfrak{m}^{\chi_3+k-2}E_{\chi_1,\chi_2,\chi_3,\chi_3+k-1}(x,y,z)\\ &-\lambda\upsilon_{(k-1)}(0)\mathfrak{m}^{\chi_2+k-1}E_{\chi_1,\chi_2,\chi_3,\chi_2+k}(x,y,z) - \lambda\int_0^{\mathfrak{m}}(\mathfrak{m}-\vartheta)^{2\chi_2-1}E_{\chi_1,\chi_2,\chi_3,2\chi_2}(x^*,y^*,z^*)\mathfrak{g}(\vartheta)d\vartheta\\ &+\upsilon_0(0)E_{\chi_1,\chi_2,\chi_3,1}(x,y,z) - \lambda\mathfrak{m}^{\chi_2}E_{\chi_1,\chi_2,\chi_3,\chi_2+1}(x,y,z) + \upsilon_1(0)\mathfrak{m}E_{\chi_1,\chi_2,\chi_3,2}(x,y,z)\\ &-\lambda\mathfrak{m}^{\chi_2+1}E_{\chi_1,\chi_2,\chi_3,\chi_2+2}(x,y,z) + \cdots + \upsilon^{(k-2)}\mathfrak{m}^{k-2}E_{\chi_1,\chi_2,\chi_3,k-1}(x,y,z)\\ &-\lambda\mathfrak{m}^{\chi_2+k-2}E_{\chi_1,\chi_2,\chi_3,\chi_2+k-1}(x,y,z) + \upsilon_{(k-1)}(0)\mathfrak{m}^{k-1}E_{\chi_1,\chi_2,\chi_3,k}(x,y,z)\\ &+\int_0^{\mathfrak{m}}(\mathfrak{m}-\vartheta)^{\tau_1-1}E_{\chi_1,\chi_2,\chi_3,\tau_1}(x^*,y^*,z^*)\mathfrak{g}(\vartheta)d\vartheta \ominus (-\mu\upsilon_0(0)\mathfrak{m}^{\tau_1}E_{\chi_1,\chi_2,\chi_3,\tau_1+1}(x,y,z)\\ &-\lambda\mathfrak{m}^{2\chi_2}E_{\chi_1,\chi_2,\chi_3,2\chi_2+1}(x,y,z)) \ominus (-\mu\upsilon_1(0)\mathfrak{m}^{\tau_1+1}E_{\chi_1,\chi_2,\chi_3,\tau_1+2}(x,y,z)\\ &-\lambda\mathfrak{m}^{2\chi_2+1}E_{\chi_1,\chi_2,\chi_3,2\chi_2+2}(x,y,z)) + \cdots \ominus (-\mu\upsilon^{(k-2)}\mathfrak{m}^{\tau_1+k-2}E_{\chi_1,\chi_2,\chi_3,\tau_1+k-1}(x,y,z)\\ &-\lambda\mathfrak{m}^{2\chi_2+k-2}E_{\chi_1,\chi_2,\chi_3,2\chi_2+k-1}(x,y,z)) \ominus (-\mu\upsilon^{(k-1)}\mathfrak{m}^{\tau_1+k-1}E_{\chi_1,\chi_2,\chi_3,\tau_1+k}(x,y,z))\\ &+\int_0^{\mathfrak{m}}(\mathfrak{m}-\vartheta)^{\chi_1-1}E_{\chi_1,\chi_2,\chi_3,\chi_1}(x^*,y^*,z^*)\mathfrak{g}(\vartheta)d\vartheta,\end{aligned}$$

where $\chi_1 = 2\tau_1, \chi_2 = \tau_1 - \tau_2, \chi_3 = 2\tau_1 - 2\tau_2$ and $(x,y,z) = (\mu^2\mathfrak{m}^{\chi_1}, 2\lambda\mathfrak{m}^{\chi_2}, -\lambda^2\mathfrak{m}^{\chi_3})$ and $(x^*,y^*,z^*) = (\mu^2(\mathfrak{m}-\vartheta)^{\chi_1}, 2\lambda(\mathfrak{m}-\vartheta)^{\chi_2}, -\lambda^2(\mathfrak{m}-\vartheta)^{\chi_3})$.

Proof. Applying fuzzy Laplace transform for Caputo derivative on both sides of the system (3.3), we have

$$\mathscr{L}[{}^C\mathfrak{D}_{0^+}^{\tau_1}\upsilon(\mathfrak{m})] - \mathscr{L}[\lambda{}^C\mathfrak{D}_{0^+}^{\tau_2}\upsilon(\mathfrak{m})] - \mathscr{L}[\mu\upsilon(\mathfrak{m})] = \mathscr{L}[\mathfrak{g}(\mathfrak{m})]. \tag{3.4}$$

If υ is fuzzy Caputo-fractional differentiable of first form, then by using status first of Theorem 3.1, Eq (3.4) gives

$$\Big[q^{\tau_1}\mathscr{L}[\upsilon(\mathfrak{m})]\ominus\sum_{i=0}^{k-1}q^{\tau_1-i-1}\upsilon_i(0,\ltimes)\Big] - \lambda\Big[q^{\tau_2}\mathscr{L}[\upsilon(\mathfrak{m})]\ominus\sum_{i=0}^{k-2}q^{\tau_2-i-1}\upsilon_i(0,\ltimes)\Big] - \mu\mathscr{L}[\upsilon(\mathfrak{m})] = \mathscr{L}[\mathfrak{g}(\mathfrak{m})]. \tag{3.5}$$

The above expression in the form of lower and upper fuzzy number-valued functions can be written as

$$q^{\tau_1}\mathbb{L}[\upsilon_1(\mathfrak{m},\ltimes)] - \sum_{i=0}^{k-1}q^{\tau_1-i-1}\upsilon_{(1)_i}(0,\ltimes) - \lambda q^{\tau_2}\mathbb{L}[\upsilon_1(\mathfrak{m},\ltimes)] + \lambda\sum_{i=0}^{k-2}q^{\tau_1-i-1}\upsilon_{(1)_i}(0,\ltimes) - \mu\mathbb{L}[\upsilon_1(\mathfrak{m},\ltimes)] = \mathcal{G}(q) \tag{3.6}$$

and

$$q^{\tau_1}\mathbb{L}[\upsilon_2(\mathfrak{m},\ltimes)] - \sum_{i=0}^{k-1}q^{\tau_1-i-1}\upsilon_{(2)_i}(0,\ltimes) - \lambda q^{\tau_2}\mathbb{L}[\upsilon_2(\mathfrak{m},\ltimes)] + \lambda\sum_{i=0}^{k-2}q^{\tau_1-i-1}\upsilon_{(2)_i}(0,\ltimes) - \mu\mathbb{L}[\upsilon_2(\mathfrak{m},\ltimes)] = \mathcal{G}(q). \tag{3.7}$$

We write the above Eqs (3.6) and (3.7) in the given explicit form

$$(q^{\tau_1} - \lambda q^{\tau_2} - \mu)\mathbb{L}[\upsilon_1(\mathfrak{m}, \ltimes)] = (q^{\tau_1-1} - \lambda q^{\tau_2-1})\upsilon_{(1)_0}(0, \ltimes) + (q^{\tau_1-2} - \lambda q^{\tau_2-2})\upsilon_{(1)_1}(0, \ltimes) + \cdots + (q^{\tau_1-k+1} - \lambda q^{\tau_2-k+1})\upsilon_{(1)_{(k-2)}}(0, \ltimes) + q^{\tau_1-k}\upsilon_{(1)_{(k-1)}}(0, \ltimes) + \mathcal{G}(q) \quad (3.8)$$

and

$$(q^{\tau_1} - \lambda q^{\tau_2} - \mu)\mathbb{L}[\upsilon_2(\mathfrak{m}, \ltimes)] = (q^{\tau_1-1} - \lambda q^{\tau_2-1})\upsilon_{(2)_0}(0, \ltimes) + (q^{\tau_1-2} - \lambda q^{\tau_2-2})\upsilon_{(2)_1}(0, \ltimes) + \cdots + (q^{\tau_1-k+1} - \lambda q^{\tau_2-k+1})\upsilon_{(2)_{(k-2)}}(0, \ltimes) + q^{\tau_1-k}\upsilon_{(2)_{(k-1)}}(0, \ltimes) + \mathcal{G}(q). \quad (3.9)$$

The solution of Eqs (3.8) and (3.9) in the form of $\mathbb{L}[\upsilon_1(\mathfrak{m}, \ltimes)]$ and $\mathbb{L}[\upsilon_2(\mathfrak{m}, \ltimes)]$

$$\mathbb{L}[\upsilon_1(\mathfrak{m}, \ltimes)] = \frac{q^{\tau_1-1} - \lambda q^{\tau_2-1}}{q^{\tau_1} - \lambda q^{\tau_2} - \mu}\upsilon_{(1)_0}(0, \ltimes) + \frac{q^{\tau_1-2} - \lambda q^{\tau_2-2}}{q^{\tau_1} - \lambda q^{\tau_2} - \mu}\upsilon_{(1)_1}(0, \ltimes) + \cdots + \frac{q^{\tau_1-k+1} - \lambda q^{\tau_2-k+1}}{q^{\tau_1} - \lambda q^{\tau_2} - \mu}\upsilon_{(1)_{(k-2)}}(0, \ltimes) + \frac{q^{\tau_1-k}}{q^{\tau_1} - \lambda q^{\tau_2} - \mu}\upsilon_{(1)_{(k-1)}}(0, \ltimes) + \frac{\mathcal{G}(q)}{q^{\tau_1} - \lambda q^{\tau_2} - \mu} \quad (3.10)$$

and

$$\mathbb{L}[\upsilon_2(\mathfrak{m}, \ltimes)] = \frac{q^{\tau_1-1} - \lambda q^{\tau_2-1}}{q^{\tau_1} - \lambda q^{\tau_2} - \mu}\upsilon_{(2)_0}(0, \ltimes) + \frac{q^{\tau_1-2} - \lambda q^{\tau_2-2}}{q^{\tau_1} - \lambda q^{\tau_2} - \mu}\upsilon_{(2)_1}(0, \ltimes) + \cdots + \frac{q^{\tau_1-k+1} - \lambda q^{\tau_2-k+1}}{q^{\tau_1} - \lambda q^{\tau_2} - \mu}\upsilon_{(2)_{(k-2)}}(0, \ltimes) + \frac{q^{\tau_1-k}}{q^{\tau_1} - \lambda q^{\tau_2} - \mu}\upsilon_{(2)_{(k-1)}}(0, \ltimes) + \frac{\mathcal{G}(q)}{q^{\tau_1} - \lambda q^{\tau_2} - \mu}. \quad (3.11)$$

After simplification of Eqs (3.24) and (3.25), it gives

$$\mathbb{L}[\upsilon_1(\mathfrak{m}, \ltimes)] = q^{-1}\Big[1 + \frac{\mu}{q^{\tau_1} - \lambda q^{\tau_2} - \mu}\Big]\upsilon_{(1)_0}(0, \ltimes) + q^{-2}\Big[1 + \frac{\mu}{q^{\tau_1} - \lambda q^{\tau_2} - \mu}\Big]\upsilon_{(1)_1}(0, \ltimes) + \cdots + q^{-(k-1)}\Big[1 + \frac{\mu}{q^{\tau_1} - \lambda q^{\tau_2} - \mu}\Big]\upsilon_{(1)_{(k-2)}}(0, \ltimes) + \frac{q^{\tau_1-k}}{q^{\tau_1} - \lambda q^{\tau_2} - \mu}\upsilon_{(1)_{(k-1)}}(0, \ltimes) + \frac{\mathcal{G}(q)}{q^{\tau_1} - \lambda q^{\tau_2} - \mu} \quad (3.12)$$

and

$$\mathbb{L}[\upsilon_2(\mathfrak{m}, \ltimes)] = q^{-1}\Big[1 + \frac{\mu}{q^{\tau_1} - \lambda q^{\tau_2} - \mu}\Big]\upsilon_{(2)_0}(0, \ltimes) + q^{-2}\Big[1 + \frac{\mu}{q^{\tau_1} - \lambda q^{\tau_2} - \mu}\Big]\upsilon_{(2)_1}(0, \ltimes) + \cdots + q^{-(k-1)}\Big[1 + \frac{\mu}{q^{\tau_1} - \lambda q^{\tau_2} - \mu}\Big]\upsilon_{(2)_{(k-2)}}(0, \ltimes) + \frac{q^{\tau_1-k}}{q^{\tau_1} - \lambda q^{\tau_2} - \mu}\upsilon_{(2)_{(k-1)}}(0, \ltimes) + \frac{\mathcal{G}(q)}{q^{\tau_1} - \lambda q^{\tau_2} - \mu}. \quad (3.13)$$

The solution of aforementioned equations after taking inverse fuzzy Laplace transform and using Definition 2.8, we have

$$\upsilon_1(\mathfrak{m}, \ltimes) = \upsilon_{(1)_0}(0, \ltimes) + \upsilon_{(1)_0}(0, \ltimes)\mu\mathfrak{m}^{\tau_1}E_{\tau_1,\chi_2,\tau_1+1}(\mu\mathfrak{m}^{\tau_1}, \lambda\mathfrak{m}^{\chi_2}) + \upsilon_{(1)_1}(0, \ltimes)\mathfrak{m} + \upsilon_{(1)_1}(0, \ltimes)\mu\mathfrak{m}^{\tau_1+1}E_{\tau_1,\chi_2,\tau_1+2}(\mu\mathfrak{m}^{\tau_1}, \lambda\mathfrak{m}^{\chi_2})$$

$$+\cdots+\upsilon_{(1)_{(k-2)}}(0,\ltimes)\frac{\mathfrak{m}^{k-2}}{\Gamma(k-1)}+\upsilon_{(1)_{(k-2)}}(0,\ltimes)\mu\mathfrak{m}^{\tau_1+k-2}E_{\tau_1,\chi_2,\tau_1+k-1}(\mu\mathfrak{m}^{\tau_1},\lambda\mathfrak{m}^{\chi_2})$$
$$+\upsilon_{(1)_{(k-1)}}(0,\ltimes)\mu\mathfrak{m}^{k-1}E_{\tau_1,\chi_2,k}(\mu\mathfrak{m}^{\tau_1},\lambda\mathfrak{m}^{\chi_2})$$
$$+\int_0^{\mathfrak{m}}(\mathfrak{m}-u)^{\tau_1-1}E_{\tau_1,\chi_2,\tau_1}(\mu(\mathfrak{m}-u)^{\tau_1},\lambda(\mathfrak{m}-u)^{\chi_2})\mathfrak{g}(u)du \tag{3.14}$$

and

$$\upsilon_2(\mathfrak{m},\ltimes)=\upsilon_{(2)_0}(0,\ltimes)+\upsilon_{(2)_0}(0,\ltimes)\mu\mathfrak{m}^{\tau_1}E_{\tau_1,\chi_2,\tau_1+1}(\mu\mathfrak{m}^{\tau_1},\lambda\mathfrak{m}^{\chi_2})+\upsilon_{(2)_1}(0,\ltimes)\mathfrak{m}+\upsilon_{(2)_1}(0,\ltimes)\mu\mathfrak{m}^{\tau_1+1}E_{\tau_1,\chi_2,\tau_1+2}(\mu\mathfrak{m}^{\tau_1},\lambda\mathfrak{m}^{\chi_2})$$
$$+\cdots+\upsilon_{(2)_{(k-2)}}(0,\ltimes)\frac{\mathfrak{m}^{k-2}}{\Gamma(k-1)}+\upsilon_{(2)_{(k-2)}}(0,\ltimes)\mu\mathfrak{m}^{\tau_1+k-2}E_{\tau_1,\chi_2,\tau_1+k-1}(\mu\mathfrak{m}^{\tau_1},\lambda\mathfrak{m}^{\chi_2})$$
$$+\upsilon_{(2)_{(k-1)}}(0,\ltimes)\mu\mathfrak{m}^{k-1}E_{\tau_1,\chi_2,k}(\mu\mathfrak{m}^{\tau_1},\lambda\mathfrak{m}^{\chi_2})$$
$$+\int_0^{\mathfrak{m}}(\mathfrak{m}-u)^{\tau_1-1}E_{\tau_1,\chi_2,\tau_1}(\mu(\mathfrak{m}-u)^{\tau_1},\lambda(\mathfrak{m}-u)^{\chi_2})\mathfrak{g}(u)du. \tag{3.15}$$

Or

$$[\upsilon_1(\mathfrak{m},\ltimes),\upsilon_2(\mathfrak{m},\ltimes)]=(\upsilon_{(1)_0}(0,\ltimes),\upsilon_{(2)_0}(0,\ltimes))+(\upsilon_{(1)_0}(0,\ltimes),\upsilon_{(2)_0}(0,\ltimes))[\mu\mathfrak{m}^{\tau_1}E_{\tau_1,\chi_2,\tau_1+1}(\mu\mathfrak{m}^{\tau_1},\lambda\mathfrak{m}^{\chi_2})]$$
$$+(\upsilon_{(1)_1}(0,\ltimes),\upsilon_{(2)_1}(0,\ltimes))[\mathfrak{m}]+(\upsilon_{(1)_1}(0,\ltimes),\upsilon_{(2)_1}(0,\ltimes))[\mu\mathfrak{m}^{\tau_1+1}E_{\tau_1,\chi_2,\tau_1+2}(\mu\mathfrak{m}^{\tau_1},\lambda\mathfrak{m}^{\chi_2})]$$
$$+\cdots+(\upsilon_{(1)_{(k-2)}}(0,\ltimes),\upsilon_{(2)_{(k-2)}}(0,\ltimes))[\mu\mathfrak{m}^{\tau_1+k-2}E_{\tau_1,\chi_2,\tau_1+k-1}(\mu\mathfrak{m}^{\tau_1},\lambda\mathfrak{m}^{\chi_2})]$$
$$+(\upsilon_{(1)_{(k-2)}}(0,\ltimes),\upsilon_{(2)_{(k-2)}}(0,\ltimes))[\frac{\mathfrak{m}^{k-2}}{\Gamma(k-1)}]$$
$$+(\upsilon_{(1)_{(k-1)}}(0,\ltimes),\upsilon_{(2)_{(k-1)}}(0,\ltimes))[\mu\mathfrak{m}^{k-1}E_{\tau_1,\chi_2,k}(\mu\mathfrak{m}^{\tau_1},\lambda\mathfrak{m}^{\chi_2})]$$
$$+\int_0^{\mathfrak{m}}(\mathfrak{m}-u)^{\tau_1-1}E_{\tau_1,\chi_2,\tau_1}(\mu(\mathfrak{m}-u)^{\tau_1},\lambda(\mathfrak{m}-u)^{\chi_2})\mathfrak{g}(u)du. \tag{3.16}$$

The above expression can be written in the form of $\ltimes$-level values as

$$[\upsilon(\mathfrak{m})]^{\ltimes}=[\upsilon_0(0)]^{\ltimes}+\Big[\upsilon_0(0)\mu\mathfrak{m}^{\tau_1}E_{\tau_1,\chi_2,\tau_1+1}(\mu\mathfrak{m}^{\tau_1},\lambda\mathfrak{m}^{\chi_2})\Big]^{\ltimes}+[\upsilon_1(0)\mathfrak{m}]^{\ltimes}$$
$$+\Big[\upsilon_1(0)\mu\mathfrak{m}^{\tau_1+1}E_{\tau_1,\chi_2,\tau_1+2}(\mu\mathfrak{m}^{\tau_1},\lambda\mathfrak{m}^{\chi_2})\Big]^{\ltimes}+\cdots$$
$$+\Big[\upsilon_{(k-2)}(0)\mu\mathfrak{m}^{\tau_1+k-2}E_{\tau_1,\chi_2,\tau_1+k-1}(\mu\mathfrak{m}^{\tau_1},\lambda\mathfrak{m}^{\chi_2})\Big]^{\ltimes}$$
$$+\Big[\upsilon_{(k-2)}(0)\frac{\mathfrak{m}^{k-2}}{\Gamma(k-1)}\Big]^{\ltimes}+\Big[\upsilon_{(k-1)}(0)\mu\mathfrak{m}^{k-1}E_{\tau_1,\chi_2,k}(\mu\mathfrak{m}^{\tau_1},\lambda\mathfrak{m}^{\chi_2})\Big]^{\ltimes}$$
$$+\int_0^{\mathfrak{m}}(\mathfrak{m}-u)^{\tau_1-1}E_{\tau_1,\chi_2,\tau_1}(\mu(\mathfrak{m}-u)^{\tau_1},\lambda(\mathfrak{m}-u)^{\chi_2})\mathfrak{g}(u)du. \tag{3.17}$$

Since $\ltimes$-values are arbitrary, so above expression can be written as

$$\upsilon(\mathfrak{m})=\upsilon_0(0)+\upsilon_0(0)\mu\mathfrak{m}^{\tau_1}E_{\tau_1,\chi_2,\tau_1+1}(\mu\mathfrak{m}^{\tau_1},\lambda\mathfrak{m}^{\chi_2})$$
$$+\upsilon_1(0)\mathfrak{m}+\upsilon_1(0)\mu\mathfrak{m}^{\tau_1+1}E_{\tau_1,\chi_2,\tau_1+2}(\mu\mathfrak{m}^{\tau_1},\lambda\mathfrak{m}^{\chi_2})+\cdots$$
$$+\upsilon_{(k-2)}(0)\mu\mathfrak{m}^{\tau_1+k-2}E_{\tau_1,\chi_2,\tau_1+k-1}(\mu\mathfrak{m}^{\tau_1},\lambda\mathfrak{m}^{\chi_2})$$

$$+\upsilon_{(k-2)}(0)\frac{\mathfrak{m}^{k-2}}{\Gamma(k-1)}+\upsilon_{(k-1)}(0)\mu\mathfrak{m}^{k-1}E_{\tau_1,\chi_2,k}(\mu\mathfrak{m}^{\tau_1},\lambda\mathfrak{m}^{\chi_2})$$
$$+\int_0^{\mathfrak{m}}(\mathfrak{m}-u)^{\tau_1-1}E_{\tau_1,\chi_2,\tau_1}(\mu(\mathfrak{m}-u)^{\tau_1},\lambda(\mathfrak{m}-u)^{\chi_2})\mathfrak{g}(u)du. \tag{3.18}$$

If υ is a fuzzy Caputo-fractional differentiable of the second form, then by using status second of Theorem 3.1 to the Eq (3.4), we have

$$\Big[-\sum_{i=0}^{k-1}q^{\tau_1-i-1}\upsilon_i(0,\ltimes)\ominus\Big(-q^{\tau_1}\mathscr{L}[\upsilon(\mathfrak{m})]\Big)\Big]-\lambda\Big[-\sum_{i=0}^{k-2}q^{\tau_2-i-1}\upsilon_i(0,\ltimes)\ominus\Big(-q^{\tau_2}\mathscr{L}[\upsilon(\mathfrak{m})]\Big)\Big]-\mu\mathscr{L}[\upsilon(\mathfrak{m})]=\mathscr{L}[\mathfrak{g}(\mathfrak{m})]. \tag{3.19}$$

Above expression in the form of upper and lower fuzzy valued function is follows as

$$q^{\tau_1}\mathbb{L}[\upsilon_2(\mathfrak{m},\ltimes)]-\sum_{i=0}^{k-1}q^{\tau_1-i-1}\upsilon_{(2)_i}(0,\ltimes)-\lambda q^{\tau_2}\mathbb{L}[\upsilon_2(\mathfrak{m},\ltimes)]+\lambda\sum_{i=0}^{k-2}q^{\tau_1-i-1}\upsilon_{(2)_i}(0,\ltimes)-\mu\mathbb{L}[\upsilon_1(\mathfrak{m},\ltimes)]=\mathcal{G}(q) \tag{3.20}$$

and

$$q^{\tau_1}\mathbb{L}[\upsilon_1(\mathfrak{m},\ltimes)]-\sum_{i=0}^{k-1}q^{\tau_1-i-1}\upsilon_{(1)_i}(0,\ltimes)-\lambda q^{\tau_2}\mathbb{L}[\upsilon_1(\mathfrak{m},\ltimes)]+\lambda\sum_{i=0}^{k-2}q^{\tau_1-i-1}\upsilon_{(1)_i}(0,\ltimes)-\mu\mathbb{L}[\upsilon_2(\mathfrak{m},\ltimes)]=\mathcal{G}(q). \tag{3.21}$$

We write the Eqs (3.20) and (3.21) in explicit form

$$-\mu\mathbb{L}[\upsilon_1(\mathfrak{m},\ltimes)]+(q^{\tau_1}-\lambda q^{\tau_2})\mathbb{L}[\upsilon_2(\mathfrak{m},\ltimes)]=(q^{\tau_1-1}-\lambda q^{\tau_2-1})\upsilon_{(2)_0}(0,\ltimes)+(q^{\tau_1-2}-\lambda q^{\tau_2-2})\upsilon_{(2)_1}(0,\ltimes)$$
$$+\cdots+(q^{\tau_1-k+1}-\lambda q^{\tau_2-k+1})\upsilon_{(2)_{(k-2)}}(0,\ltimes)$$
$$+q^{\tau_1-k}\upsilon_{(2)_{(k-1)}}(0,\ltimes)+\mathcal{G}(q) \tag{3.22}$$

and

$$(q^{\tau_1}-\lambda q^{\tau_2})\mathbb{L}[\upsilon_1(\mathfrak{m},\ltimes)]-\mu\mathbb{L}[\upsilon_2(\mathfrak{m},\ltimes)]=(q^{\tau_1-1}-\lambda q^{\tau_2-1})\upsilon_{(1)_0}(0,\ltimes)+(q^{\tau_1-2}-\lambda q^{\tau_2-2})\upsilon_{(1)_1}(0,\ltimes)$$
$$+\cdots+(q^{\tau_1-k+1}-\lambda q^{\tau_2-k+1})\upsilon_{(1)_{(k-2)}}(0,\ltimes)$$
$$+q^{\tau_1-k}\upsilon_{(1)_{(k-1)}}(0,\ltimes)+\mathcal{G}(q). \tag{3.23}$$

The solution of Eqs (3.22) and (3.23) in the form of $\mathbb{L}[\upsilon_1(\mathfrak{m},\ltimes)]$ and $\mathbb{L}[\upsilon_2(\mathfrak{m},\ltimes)]$ is follows as

$$\mathbb{L}[\upsilon_1(\mathfrak{m},\ltimes)]=-\lambda\Big[\frac{q^{\tau_1+\tau_2-1}-\lambda q^{2\tau_2-1}}{\mu^2-q^{\chi_1}-\lambda^2q^{2\tau_2}+2\lambda q^{\tau_1+\tau_2}}\upsilon_{(1)_0}(0,\ltimes)+\frac{q^{\tau_1+\tau_2-2}-\lambda q^{2\tau_2-2}}{\mu^2-q^{\chi_1}-\lambda^2q^{2\tau_2}+2\lambda q^{\tau_1+\tau_2}}\upsilon_{(1)_1}(0,\ltimes)$$
$$+\cdots+\frac{q^{\tau_1+\tau_2-k+1}-\lambda q^{2\tau_2-k+1}}{\mu^2-q^{\chi_1}-\lambda^2q^{2\tau_2}+2\lambda q^{\tau_1+\tau_2}}\upsilon_{(1)_{(k-2)}}(0,\ltimes)+\frac{q^{\tau_1+\tau_2-k}}{\mu^2-q^{\chi_1}-\lambda^2q^{2\tau_2}+2\lambda q^{\tau_1+\tau_2}}\upsilon_{(1)_{(k-1)}}(0,\ltimes)$$
$$+\frac{q^{\tau_2}\mathcal{G}(q)}{\mu^2-q^{\chi_1}-\lambda^2q^{2\tau_2}+2\lambda q^{\tau_1+\tau_2}}\Big]-\Big[\frac{q^{\chi_1-1}-\lambda q^{\tau_1+\tau_2-1}}{\mu^2-q^{\chi_1}-\lambda^2q^{2\tau_2}+2\lambda q^{\tau_1+\tau_2}}\upsilon_{(1)_0}(0,\ltimes)$$

$$+\frac{q^{\chi_1-2}-\lambda q^{\tau_1+\tau_2-2}}{\mu^2-q^{\chi_1}-\lambda^2q^{2\tau_2}+2\lambda q^{\tau_1+\tau_2}}\upsilon_{(1)_1}(0,\ltimes)+\cdots+\frac{q^{\chi_1-k+1}-\lambda q^{\tau_1+\tau_2-k+1}}{\mu^2-q^{\chi_1}-\lambda^2q^{2\tau_2}+2\lambda q^{\tau_1+\tau_2}}\upsilon_{(1)_{(k-2)}}(0,\ltimes)$$
$$+\frac{q^{\chi_1-k}}{\mu^2-q^{\chi_1}-\lambda^2q^{2\tau_2}+2\lambda q^{\tau_1+\tau_2}}\upsilon_{(1)_{(k-1)}}(0,\ltimes)+\frac{q^{\tau_1}\mathcal{G}(q)}{\mu^2-q^{\chi_1}-\lambda^2q^{2\tau_2}+2\lambda q^{\tau_1+\tau_2}}\Big]$$
$$-\mu\Big[\frac{q^{\tau_1-1}-\lambda q^{\tau_2-1}}{\mu^2-q^{\chi_1}-\lambda^2q^{2\tau_2}+2\lambda q^{\tau_1+\tau_2}}\upsilon_{(2)_0}(0,\ltimes)+\frac{q^{\tau_1-2}-\lambda q^{\tau_2-2}}{\mu^2-q^{\chi_1}-\lambda^2q^{2\tau_2}+2\lambda q^{\tau_1+\tau_2}}\upsilon_{(2)_1}(0,\ltimes)$$
$$+\cdots+\frac{q^{\tau_1-k+1}-\lambda q^{\tau_2-k+1}}{\mu^2-q^{\chi_1}-\lambda^2q^{2\tau_2}+2\lambda q^{\tau_1+\tau_2}}\upsilon_{(2)_{(k-2)}}(0,\ltimes)+\frac{q^{\tau_1-k}}{\mu^2-q^{\chi_1}-\lambda^2q^{2\tau_2}+2\lambda q^{\tau_1+\tau_2}}\upsilon_{(2)_{(k-1)}}(0,\ltimes)$$
$$+\frac{\mathcal{G}(q)}{\mu^2-q^{\chi_1}-\lambda^2q^{2\tau_2}+2\lambda q^{\tau_1+\tau_2}}\Big] \tag{3.24}$$

and

$$\mathbb{L}[\upsilon_2(\mathfrak{m},\ltimes)]=-\lambda\Big[\frac{q^{\tau_1+\tau_2-1}-\lambda q^{2\tau_2-1}}{\mu^2-q^{\chi_1}-\lambda^2q^{2\tau_2}+2\lambda q^{\tau_1+\tau_2}}\upsilon_{(2)_0}(0,\ltimes)+\frac{q^{\tau_1+\tau_2-2}-\lambda q^{2\tau_2-2}}{\mu^2-q^{\chi_1}-\lambda^2q^{2\tau_2}+2\lambda q^{\tau_1+\tau_2}}\upsilon_{(2)_1}(0,\ltimes)$$
$$+\cdots+\frac{q^{\tau_1+\tau_2-k+1}-\lambda q^{2\tau_2-k+1}}{\mu^2-q^{\chi_1}-\lambda^2q^{2\tau_2}+2\lambda q^{\tau_1+\tau_2}}\upsilon_{(2)_{(k-2)}}(0,\ltimes)+\frac{q^{\tau_1+\tau_2-k}}{\mu^2-q^{\chi_1}-\lambda^2q^{2\tau_2}+2\lambda q^{\tau_1+\tau_2}}\upsilon_{(2)_{(k-1)}}(0,\ltimes)$$
$$+\frac{q^{\tau_2}\mathcal{G}(q)}{\mu^2-q^{\chi_1}-\lambda^2q^{2\tau_2}+2\lambda q^{\tau_1+\tau_2}}\Big]-\Big[\frac{q^{\chi_1-1}-\lambda q^{\tau_1+\tau_2-1}}{\mu^2-q^{\chi_1}-\lambda^2q^{2\tau_2}+2\lambda q^{\tau_1+\tau_2}}\upsilon_{(2)_0}(0,\ltimes)$$
$$+\frac{q^{\chi_1-2}-\lambda q^{\tau_1+\tau_2-2}}{\mu^2-q^{\chi_1}-\lambda^2q^{2\tau_2}+2\lambda q^{\tau_1+\tau_2}}\upsilon_{(2)_1}(0,\ltimes)+\cdots+\frac{q^{\chi_1-k+1}-\lambda q^{\tau_1+\tau_2-k+1}}{\mu^2-q^{\chi_1}-\lambda^2q^{2\tau_2}+2\lambda q^{\tau_1+\tau_2}}\upsilon_{(2)_{(k-2)}}(0,\ltimes)$$
$$+\frac{q^{\chi_1-k}}{\mu^2-q^{\chi_1}-\lambda^2q^{2\tau_2}+2\lambda q^{\tau_1+\tau_2}}\upsilon_{(2)_{(k-1)}}(0,\ltimes)+\frac{q^{\tau_1}\mathcal{G}(q)}{\mu^2-q^{\chi_1}-\lambda^2q^{2\tau_2}+2\lambda q^{\tau_1+\tau_2}}\Big]$$
$$-\mu\Big[\frac{q^{\tau_1-1}-\lambda q^{\tau_2-1}}{\mu^2-q^{\chi_1}-\lambda^2q^{2\tau_2}+2\lambda q^{\tau_1+\tau_2}}\upsilon_{(1)_0}(0,\ltimes)+\frac{q^{\tau_1-2}-\lambda q^{\tau_2-2}}{\mu^2-q^{\chi_1}-\lambda^2q^{2\tau_2}+2\lambda q^{\tau_1+\tau_2}}\upsilon_{(1)_1}(0,\ltimes)$$
$$+\cdots+\frac{q^{\tau_1-k+1}-\lambda q^{\tau_2-k+1}}{\mu^2-q^{\chi_1}-\lambda^2q^{2\tau_2}+2\lambda q^{\tau_1+\tau_2}}\upsilon_{(1)_{(k-2)}}(0,\ltimes)+\frac{q^{\tau_1-k}}{\mu^2-q^{\chi_1}-\lambda^2q^{2\tau_2}+2\lambda q^{\tau_1+\tau_2}}\upsilon_{(1)_{(k-1)}}(0,\ltimes)$$
$$+\frac{\mathcal{G}(q)}{\mu^2-q^{\chi_1}-\lambda^2q^{2\tau_2}+2\lambda q^{\tau_1+\tau_2}}\Big]. \tag{3.25}$$

The explicit form of the solution after taking inverse fuzzy Laplace transform on the Eqs (3.24) and (3.25) and applying the Definition 2.8, we have

$$\upsilon_1(\mathfrak{m},\ltimes)=-\lambda\Big[\Big(\mathfrak{m}^{\chi_2}E_{\chi_1,\chi_2,\chi_3,\chi_2+1}(x,y,z)-\lambda\mathfrak{m}^{\chi_3}E_{\chi_1,\chi_2,\chi_3,\chi_3+1}(x,y,z)\Big)\upsilon_{(1)_0}(0,\ltimes)$$
$$+\Big(\mathfrak{m}^{\chi_2+1}E_{\chi_1,\chi_2,\chi_3,\chi_2+2}(x,y,z)-\lambda\mathfrak{m}^{\chi_3+1}E_{\chi_1,\chi_2,\chi_3,\chi_3+2}(x,y,z)\Big)\upsilon_{(1)_1}(0,\ltimes)$$
$$+\cdots+\Big(\mathfrak{m}^{\chi_2+k-2}E_{\chi_1,\chi_2,\chi_3,\chi_2+k-1}(x,y,z)-\lambda\mathfrak{m}^{\chi_3+k-2}E_{\chi_1,\chi_2,\chi_3,\chi_3+k-1}(x,y,z)\Big)\upsilon_{(1)_{(k-2)}}(0,\ltimes)$$
$$+\mathfrak{m}^{\chi_2+k-1}E_{\chi_1,\chi_2,\chi_3,\chi_2+k}(x,y,z)\upsilon_{(1)_{(k-1)}}(0,\ltimes)+\int_0^{\mathfrak{m}}(\mathfrak{m}-\vartheta)^{2\chi_2-1}E_{\chi_1,\chi_2,\chi_3,2\chi_2}(x^*,y^*,z^*)\mathfrak{g}(\vartheta)d\vartheta\Big]$$
$$+\Big[\Big(\mathfrak{m}^{1-1}E_{\chi_1,\chi_2,\chi_3,1}(x,y,z)-\lambda\mathfrak{m}^{\chi_2}E_{\chi_1,\chi_2,\chi_3,\chi_2+1}(x,y,z)\Big)\upsilon_{(1)_0}(0,\ltimes)+\Big(\mathfrak{m}E_{\chi_1,\chi_2,\chi_3,2}(x,y,z)$$
$$-\lambda\mathfrak{m}^{\chi_2+1}E_{\chi_1,\chi_2,\chi_3,\chi_2+2}(x,y,z)\Big)\upsilon_{(1)_1}(0,\ltimes)+\cdots+\Big(\mathfrak{m}^{k-2}E_{\chi_1,\chi_2,\chi_3,k-1}(x,y,z)$$

$$
\begin{aligned}
&-\lambda\mathrm{m}^{\chi_2+k-2}E_{\chi_1,\chi_2,\chi_3,\chi_2+k-1}(x,y,z)\Big)\upsilon_{(1)_{(k-2)}}(0,\ltimes)+\mathrm{m}^{k-1}E_{\chi_1,\chi_2,\chi_3,k}(x,y,z)\upsilon_{(1)_{(k-1)}}(0,\ltimes)\\
&+\int_0^{\mathrm{m}}(\mathrm{m}-\vartheta)^{\tau_1-1}E_{\chi_1,\chi_2,\chi_3,\tau_1}(\mu^2(\mathrm{m}-\vartheta)^{\chi_1},2\lambda(\mathrm{m}-\vartheta)^{\chi_2},-\lambda^2(\mathrm{m}-\vartheta)^{\chi_3})\mathrm{g}(\vartheta)d\vartheta\Big]\\
&+\mu\Big[\Big(\mathrm{m}^{\tau_1}E_{\chi_1,\chi_2,\chi_3,\tau_1+1}(x,y,z)-\lambda\mathrm{m}^{2\chi_2}E_{\chi_1,\chi_2,\chi_3,2\chi_2+1}(x,y,z)\Big)\upsilon_{(2)_0}(0,\ltimes)\\
&+\Big(\mathrm{m}^{\tau_1+1}E_{\chi_1,\chi_2,\chi_3,\tau_1+2}(x,y,z)-\lambda\mathrm{m}^{2\chi_2+1}E_{\chi_1,\chi_2,\chi_3,2\chi_2+2}(x,y,z)\Big)\upsilon_{(2)_1}(0,\ltimes)\\
&+\cdots+\Big(\mathrm{m}^{\tau_1+k-2}E_{\chi_1,\chi_2,\chi_3,\tau_1+k-1}(x,y,z)-\lambda\mathrm{m}^{2\chi_2+k-2}E_{\chi_1,\chi_2,\chi_3,2\chi_2+k-1}(x,y,z)\Big)\upsilon_{(2)_{(k-2)}}(0,\ltimes)\\
&+\mathrm{m}^{\tau_1+k-1}E_{\chi_1,\chi_2,\chi_3,\tau_1+k}(x,y,z)\upsilon_{(2)_{(k-1)}}(0,\ltimes)+\int_0^{\mathrm{m}}(\mathrm{m}-\vartheta)^{\chi_1-1}E_{\chi_1,\chi_2,\chi_3,\chi_1}(x^*,y^*,z^*)\mathrm{g}(\vartheta)d\vartheta\Big]
\end{aligned}
\tag{3.26}
$$

and

$$
\begin{aligned}
\upsilon_2(\mathrm{m},\ltimes)=&-\lambda\Big[\Big(\mathrm{m}^{\chi_2}E_{\chi_1,\chi_2,\chi_3,\chi_2+1}(x,y,z)-\lambda\mathrm{m}^{\chi_3}E_{\chi_1,\chi_2,\chi_3,\chi_3+1}(x,y,z)\Big)\upsilon_{(2)_0}(0,\ltimes)\\
&+\Big(\mathrm{m}^{\chi_2+1}E_{\chi_1,\chi_2,\chi_3,\chi_2+2}(x,y,z)-\lambda\mathrm{m}^{\chi_3+1}E_{\chi_1,\chi_2,\chi_3,\chi_3+2}(x,y,z)\Big)\upsilon_{(2)_1}(0,\ltimes)\\
&+\cdots+\Big(\mathrm{m}^{\chi_2+k-2}E_{\chi_1,\chi_2,\chi_3,\chi_2+k-1}(x,y,z)-\lambda\mathrm{m}^{\chi_3+k-2}E_{\chi_1,\chi_2,\chi_3,\chi_3+k-1}(x,y,z)\Big)\upsilon_{(2)_{(k-2)}}(0,\ltimes)\\
&+\mathrm{m}^{\chi_2+k-1}E_{\chi_1,\chi_2,\chi_3,\chi_2+k}(x,y,z)\upsilon_{(2)_{(k-1)}}(0,\ltimes)+\int_0^{\mathrm{m}}(\mathrm{m}-\vartheta)^{2\chi_2-1}E_{\chi_1,\chi_2,\chi_3,2\chi_2}(x^*,y^*,z^*)\mathrm{g}(\vartheta)d\vartheta\Big]\\
&+\Big[\Big(\mathrm{m}^{1-1}E_{\chi_1,\chi_2,\chi_3,1}(x,y,z)-\lambda\mathrm{m}^{\chi_2}E_{\chi_1,\chi_2,\chi_3,\chi_2+1}(x,y,z)\Big)\upsilon_{(2)_0}(0,\ltimes)+\Big(\mathrm{m}E_{\chi_1,\chi_2,\chi_3,2}(x,y,z)\\
&-\lambda\mathrm{m}^{\chi_2+1}E_{\chi_1,\chi_2,\chi_3,\chi_2+2}(x,y,z)\Big)\upsilon_{(2)_1}(0,\ltimes)+\cdots+\Big(\mathrm{m}^{k-2}E_{\chi_1,\chi_2,\chi_3,k-1}(x,y,z)\\
&-\lambda\mathrm{m}^{\chi_2+k-2}E_{\chi_1,\chi_2,\chi_3,\chi_2+k-1}(x,y,z)\Big)\upsilon_{(2)_{(k-2)}}(0,\ltimes)+\mathrm{m}^{k-1}E_{\chi_1,\chi_2,\chi_3,k}(x,y,z)\upsilon_{(2)_{(k-1)}}(0,\ltimes)\\
&+\int_0^{\mathrm{m}}(\mathrm{m}-\vartheta)^{\tau_1-1}E_{\chi_1,\chi_2,\chi_3,\tau_1}(\mu^2(\mathrm{m}-\vartheta)^{\chi_1},2\lambda(\mathrm{m}-\vartheta)^{\chi_2},-\lambda^2(\mathrm{m}-\vartheta)^{\chi_3})\mathrm{g}(\vartheta)d\vartheta\Big]\\
&+\mu\Big[\Big(\mathrm{m}^{\tau_1}E_{\chi_1,\chi_2,\chi_3,\tau_1+1}(x,y,z)-\lambda\mathrm{m}^{2\chi_2}E_{\chi_1,\chi_2,\chi_3,2\chi_2+1}(x,y,z)\Big)\upsilon_{(2)_0}(0,\ltimes)\\
&+\Big(\mathrm{m}^{\tau_1+1}E_{\chi_1,\chi_2,\chi_3,\tau_1+2}(x,y,z)-\lambda\mathrm{m}^{2\chi_2+1}E_{\chi_1,\chi_2,\chi_3,2\chi_2+2}(x,y,z)\Big)\upsilon_{(1)_1}(0,\ltimes)\\
&+\cdots+\Big(\mathrm{m}^{\tau_1+k-2}E_{\chi_1,\chi_2,\chi_3,\tau_1+k-1}(x,y,z)-\lambda\mathrm{m}^{2\chi_2+k-2}E_{\chi_1,\chi_2,\chi_3,2\chi_2+k-1}(x,y,z)\Big)\upsilon_{(1)_{(k-2)}}(0,\ltimes)\\
&+\mathrm{m}^{\tau_1+k-1}E_{\chi_1,\chi_2,\chi_3,\tau_1+k}(x,y,z)\upsilon_{(1)_{(k-1)}}(0,\ltimes)+\int_0^{\mathrm{m}}(\mathrm{m}-\vartheta)^{\chi_1-1}E_{\chi_1,\chi_2,\chi_3,\chi_1}(x^*,y^*,z^*)\mathrm{g}(\vartheta)d\vartheta\Big],
\end{aligned}
\tag{3.27}
$$

where, $(x,y,z)=(\mu^2\mathrm{m}^{\chi_1},2\lambda\mathrm{m}^{\chi_2},-\lambda^2\mathrm{m}^{\chi_3})$ and $(x^*,y^*,z^*)=(\mu^2(\mathrm{m}-\vartheta)^{\chi_1},2\lambda(\mathrm{m}-\vartheta)^{\chi_2},-\lambda^2(\mathrm{m}-\vartheta)^{\chi_3})$. In the form of $\ltimes$-level values, we have

$$
\upsilon(\mathrm{m})=\big[-\lambda\upsilon_0(0)\mathrm{m}^{\chi_2}E_{\chi_1,\chi_2,\chi_3,\chi_2+1}(x,y,z)-\lambda\mathrm{m}^{\chi_3}E_{\chi_1,\chi_2,\chi_3,\chi_3+1}(x,y,z)\big]^{\ltimes}
$$

$$
\begin{aligned}
&-[\lambda\upsilon_1(0)\mathfrak{m}^{\chi_2+1}E_{\chi_1,\chi_2,\chi_3,\chi_2+2}(x,y,z)-\lambda\mathfrak{m}^{\chi_3+1}E_{\chi_1,\chi_2,\chi_3,\chi_3+2}(x,y,z)]^{\ltimes}\\
&-\cdots-[\lambda\upsilon_{(k-2)}(0)\mathfrak{m}^{\chi_2+k-2}E_{\chi_1,\chi_2,\chi_3,\chi_2+k-1}(x,y,z)-\lambda\mathfrak{m}^{\chi_3+k-2}E_{\chi_1,\chi_2,\chi_3,\chi_3+k-1}(x,y,z)]^{\ltimes}\\
&-[\lambda\upsilon_{(k-1)}(0)\mathfrak{m}^{\chi_2+k-1}E_{\chi_1,\chi_2,\chi_3,\chi_2+k}(x,y,z)]^{\ltimes}-\lambda\int_0^{\mathfrak{m}}(\mathfrak{m}-\vartheta)^{2\chi_2-1}E_{\chi_1,\chi_2,\chi_3,2\chi_2}(x^*,y^*,z^*)\mathfrak{g}(\vartheta)d\vartheta\\
&+[\upsilon_0(0)E_{\chi_1,\chi_2,\chi_3,1}(x,y,z)-\lambda\mathfrak{m}^{\chi_2}E_{\chi_1,\chi_2,\chi_3,\chi_2+1}(x,y,z)]^{\ltimes}+[\upsilon_1(0)\mathfrak{m}E_{\chi_1,\chi_2,\chi_3,2}(x,y,z)\\
&-\lambda\mathfrak{m}^{\chi_2+1}E_{\chi_1,\chi_2,\chi_3,\chi_2+2}(x,y,z)]^{\ltimes}+\cdots+[\upsilon^{(k-2)}\mathfrak{m}^{k-2}E_{\chi_1,\chi_2,\chi_3,k-1}(x,y,z)\\
&-\lambda\mathfrak{m}^{\chi_2+k-2}E_{\chi_1,\chi_2,\chi_3,\chi_2+k-1}(x,y,z)]^{\ltimes}+[\upsilon_{(k-1)}(0)\mathfrak{m}^{k-1}E_{\chi_1,\chi_2,\chi_3,k}(x,y,z)]^{\ltimes}\\
&+\int_0^{\mathfrak{m}}(\mathfrak{m}-\vartheta)^{\tau_1-1}E_{\chi_1,\chi_2,\chi_3,\tau_1}(x^*,y^*,z^*)\mathfrak{g}(\vartheta)d\vartheta\ominus[-\mu\upsilon_0(0)\mathfrak{m}^{\tau_1}E_{\chi_1,\chi_2,\chi_3,\tau_1+1}(x,y,z)\\
&-\lambda\mathfrak{m}^{2\chi_2}E_{\chi_1,\chi_2,\chi_3,2\chi_2+1}(x,y,z)]^{\ltimes}\ominus[-\mu\upsilon_1(0)\mathfrak{m}^{\tau_1+1}E_{\chi_1,\chi_2,\chi_3,\tau_1+2}(x,y,z)\\
&-\lambda\mathfrak{m}^{2\chi_2+1}E_{\chi_1,\chi_2,\chi_3,2\chi_2+2}(x,y,z)]^{\ltimes}+\cdots\ominus[-\mu\upsilon^{(k-2)}\mathfrak{m}^{\tau_1+k-2}E_{\chi_1,\chi_2,\chi_3,\tau_1+k-1}(x,y,z)\\
&-\lambda\mathfrak{m}^{2\chi_2+k-2}E_{\chi_1,\chi_2,\chi_3,2\chi_2+k-1}(x,y,z)]^{\ltimes}\ominus[-\mu\upsilon^{(k-1)}\mathfrak{m}^{\tau_1+k-1}E_{\chi_1,\chi_2,\chi_3,\tau_1+k}(x,y,z)]^{\ltimes}\\
&+\int_0^{\mathfrak{m}}(\mathfrak{m}-\vartheta)^{\chi_1-1}E_{\chi_1,\chi_2,\chi_3,\chi_1}(x^*,y^*,z^*)\mathfrak{g}(\vartheta)d\vartheta.
\end{aligned}
\tag{3.28}
$$

Since $\ltimes$-level values are arbitrary, so above expression can be written as

$$
\begin{aligned}
\upsilon(\mathfrak{m})=&-\lambda\upsilon_0(0)\mathfrak{m}^{\chi_2}E_{\chi_1,\chi_2,\chi_3,\chi_2+1}(x,y,z)-\lambda\mathfrak{m}^{\chi_3}E_{\chi_1,\chi_2,\chi_3,\chi_3+1}(x,y,z)\\
&-\lambda\upsilon_1(0)\mathfrak{m}^{\chi_2+1}E_{\chi_1,\chi_2,\chi_3,\chi_2+2}(x,y,z)-\lambda\mathfrak{m}^{\chi_3+1}E_{\chi_1,\chi_2,\chi_3,\chi_3+2}(x,y,z)\\
&-\cdots-\lambda\upsilon_{(k-2)}(0)\mathfrak{m}^{\chi_2+k-2}E_{\chi_1,\chi_2,\chi_3,\chi_2+k-1}(x,y,z)-\lambda\mathfrak{m}^{\chi_3+k-2}E_{\chi_1,\chi_2,\chi_3,\chi_3+k-1}(x,y,z)\\
&-\lambda\upsilon_{(k-1)}(0)\mathfrak{m}^{\chi_2+k-1}E_{\chi_1,\chi_2,\chi_3,\chi_2+k}(x,y,z)-\lambda\int_0^{\mathfrak{m}}(\mathfrak{m}-\vartheta)^{2\chi_2-1}E_{\chi_1,\chi_2,\chi_3,2\chi_2}(x^*,y^*,z^*)\mathfrak{g}(\vartheta)d\vartheta\\
&+\upsilon_0(0)E_{\chi_1,\chi_2,\chi_3,1}(x,y,z)-\lambda\mathfrak{m}^{\chi_2}E_{\chi_1,\chi_2,\chi_3,\chi_2+1}(x,y,z)+\upsilon_1(0)\mathfrak{m}E_{\chi_1,\chi_2,\chi_3,2}(x,y,z)\\
&-\lambda\mathfrak{m}^{\chi_2+1}E_{\chi_1,\chi_2,\chi_3,\chi_2+2}(x,y,z)+\cdots+\upsilon^{(k-2)}\mathfrak{m}^{k-2}E_{\chi_1,\chi_2,\chi_3,k-1}(x,y,z)\\
&-\lambda\mathfrak{m}^{\chi_2+k-2}E_{\chi_1,\chi_2,\chi_3,\chi_2+k-1}(x,y,z)+\upsilon_{(k-1)}(0)\mathfrak{m}^{k-1}E_{\chi_1,\chi_2,\chi_3,k}(x,y,z)\\
&+\int_0^{\mathfrak{m}}(\mathfrak{m}-\vartheta)^{\tau_1-1}E_{\chi_1,\chi_2,\chi_3,\tau_1}(x^*,y^*,z^*)\mathfrak{g}(\vartheta)d\vartheta\ominus-\mu\upsilon_0(0)\mathfrak{m}^{\tau_1}E_{\chi_1,\chi_2,\chi_3,\tau_1+1}(x,y,z)\\
&-\lambda\mathfrak{m}^{2\chi_2}E_{\chi_1,\chi_2,\chi_3,2\chi_2+1}(x,y,z)\ominus-\mu\upsilon_1(0)\mathfrak{m}^{\tau_1+1}E_{\chi_1,\chi_2,\chi_3,\tau_1+2}(x,y,z)\\
&-\lambda\mathfrak{m}^{2\chi_2+1}E_{\chi_1,\chi_2,\chi_3,2\chi_2+2}(x,y,z)+\cdots\ominus-\mu\upsilon^{(k-2)}\mathfrak{m}^{\tau_1+k-2}E_{\chi_1,\chi_2,\chi_3,\tau_1+k-1}(x,y,z)\\
&-\lambda\mathfrak{m}^{2\chi_2+k-2}E_{\chi_1,\chi_2,\chi_3,2\chi_2+k-1}(x,y,z)\ominus-\mu\upsilon^{(k-1)}\mathfrak{m}^{\tau_1+k-1}E_{\chi_1,\chi_2,\chi_3,\tau_1+k}(x,y,z)\\
&+\int_0^{\mathfrak{m}}(\mathfrak{m}-\vartheta)^{\chi_1-1}E_{\chi_1,\chi_2,\chi_3,\chi_1}(x^*,y^*,z^*)\mathfrak{g}(\vartheta)d\vartheta.
\end{aligned}
$$

This completes the proof. □

Now we consider the homogeneous FFLDEs with uncertain initial-conditions

$$
\begin{cases}
{}^{C}\mathfrak{D}_{0^+}^{\tau_1}\upsilon(\mathfrak{m})-\lambda{}^{C}\mathfrak{D}_{0^+}^{\tau_2}\upsilon(\mathfrak{m})-\mu\upsilon(\mathfrak{m})=0,\\
\upsilon_i(0)=\upsilon_i(0,\ltimes),
\end{cases}
\tag{3.29}
$$

where $q-2<\tau_2\leq q-1$, $q-1<\tau_1\leq q$ and $q\geq 2$ with $\chi_2\geq 1$.

Theorem 3.3. Let $\upsilon : \mathscr{T} \longrightarrow F^{\mathbb{R}}$, $\upsilon \in \mathbb{C}^{F^{\mathbb{R}}}(\mathscr{T}) \cap L^{F^{\mathbb{R}}}(\mathscr{T})$. If υ is fuzzy Caputo-fractional differentiable of the first form. Then the system (3.29) have the following solution

$$\upsilon(\mathfrak{m}) = \upsilon_0(0) + \upsilon_0(0)\mu\mathfrak{m}^{\tau_1}E_{\tau_1,\chi_2,\tau_1+1}(\mu\mathfrak{m}^{\tau_1}, \lambda\mathfrak{m}^{\chi_2}) + \upsilon_1(0)\mathfrak{m} + \upsilon_1(0)\mu\mathfrak{m}^{\tau_1+1}E_{\tau_1,\chi_2,\tau_1+2}(\mu\mathfrak{m}^{\tau_1}, \lambda\mathfrak{m}^{\chi_2})$$
$$+\cdots+\upsilon_{(k-2)}(0)\mu\mathfrak{m}^{\tau_1+k-2}E_{\tau_1,\chi_2,\tau_1+k-1}(\mu\mathfrak{m}^{\tau_1}, \lambda\mathfrak{m}^{\chi_2}) + \upsilon_{(k-2)}(0)\frac{\mathfrak{m}^{k-2}}{\Gamma(k-1)} + \upsilon_{(k-1)}(0)\mu\mathfrak{m}^{k-1}E_{\tau_1,\chi_2,k}(\mu\mathfrak{m}^{\tau_1}, \lambda\mathfrak{m}^{\chi_2}).$$

If υ is the fuzzy Caputo-fractional differentiable of second form. Then the system (3.3) have the following solution

$$\begin{aligned}
\upsilon(\mathfrak{m}) =& -\lambda\upsilon_0(0)\mathfrak{m}^{\chi_2}E_{\chi_1,\chi_2,\chi_3,\chi_2+1}(x,y,z) - \lambda\mathfrak{m}^{\chi_3}E_{\chi_1,\chi_2,\chi_3,\chi_3+1}(x,y,z)\\
&-\lambda\upsilon_1(0)\mathfrak{m}^{\chi_2+1}E_{\chi_1,\chi_2,\chi_3,\chi_2+2}(x,y,z) - \lambda\mathfrak{m}^{\chi_3+1}E_{\chi_1,\chi_2,\chi_3,\chi_3+2}(x,y,z)\\
&-\cdots-\lambda\upsilon_{(k-2)}(0)\mathfrak{m}^{\chi_2+k-2}E_{\chi_1,\chi_2,\chi_3,\chi_2+k-1}(x,y,z) - \lambda\mathfrak{m}^{\chi_3+k-2}E_{\chi_1,\chi_2,\chi_3,\chi_3+k-1}(x,y,z)\\
&-\lambda\upsilon_{(k-1)}(0)\mathfrak{m}^{\chi_2+k-1}E_{\chi_1,\chi_2,\chi_3,\chi_2+k}(x,y,z) + \upsilon_0(0)E_{\chi_1,\chi_2,\chi_3,1}(x,y,z)\\
&-\lambda\mathfrak{m}^{\chi_2}E_{\chi_1,\chi_2,\chi_3,\chi_2+1}(x,y,z) + \upsilon_1(0)\mathfrak{m}E_{\chi_1,\chi_2,\chi_3,2}(x,y,z)\\
&-\lambda\mathfrak{m}^{\chi_2+1}E_{\chi_1,\chi_2,\chi_3,\chi_2+2}(x,y,z) + \cdots + \upsilon^{(k-2)}\mathfrak{m}^{k-2}E_{\chi_1,\chi_2,\chi_3,k-1}(x,y,z)\\
&-\lambda\mathfrak{m}^{\chi_2+k-2}E_{\chi_1,\chi_2,\chi_3,\chi_2+k-1}(x,y,z) + \upsilon_{(k-1)}(0)\mathfrak{m}^{k-1}E_{\chi_1,\chi_2,\chi_3,k}(x,y,z)\\
&\ominus(-\mu\upsilon_0(0)\mathfrak{m}^{\tau_1}E_{\chi_1,\chi_2,\chi_3,\tau_1+1}(x,y,z) - \lambda\mathfrak{m}^{2\chi_2}E_{\chi_1,\chi_2,\chi_3,2\chi_2+1}(x,y,z))\\
&\ominus(-\mu\upsilon_1(0)\mathfrak{m}^{\tau_1+1}E_{\chi_1,\chi_2,\chi_3,\tau_1+2}(x,y,z) - \lambda\mathfrak{m}^{2\chi_2+1}E_{\chi_1,\chi_2,\chi_3,2\chi_2+2}(x,y,z))\\
&\ominus\cdots\ominus(-\mu\upsilon^{(k-2)}\mathfrak{m}^{\tau_1+k-2}E_{\chi_1,\chi_2,\chi_3,\tau_1+k-1}(x,y,z) - \lambda\mathfrak{m}^{2\chi_2+k-2}E_{\chi_1,\chi_2,\chi_3,2\chi_2+k-1}(x,y,z))\\
&\ominus(-\mu\upsilon^{(k-1)}\mathfrak{m}^{\tau_1+k-1}E_{\chi_1,\chi_2,\chi_3,\tau_1+k}(x,y,z)).
\end{aligned}$$

Proof. We can proof the Theorem 3.3 easily by using the above general approach, so we omit it here. □

As a spacial case, when $0 < \tau_2 \leq 1$, $1 < \tau_1 \leq 2$. We consider the following inhomogeneous fuzzy fractional Langevin differential equations

$$\begin{cases}{}^C\mathfrak{D}_{0^+}^{\tau_1}\upsilon(\mathfrak{m}) - \lambda{}^C\mathfrak{D}_{0^+}^{\tau_2}\upsilon(\mathfrak{m}) - \mu\upsilon(\mathfrak{m}) = \mathfrak{g}(\mathfrak{m}),\\ \upsilon_0(0) = (\upsilon_{(1)_0}(0,\ltimes), \upsilon_{(2)_0}(0,\ltimes)),\quad \upsilon_1(0) = (\upsilon_{(1)_1}(0,\ltimes), \upsilon_{(2)_1}(0,\ltimes)),\end{cases} \tag{3.30}$$

as well as homogeneous fuzzy fractional Langevin differential equations

$$\begin{cases}{}^C\mathfrak{D}_{0^+}^{\tau_1}\upsilon(\mathfrak{m}) - \lambda{}^C\mathfrak{D}_{0^+}^{\tau_2}\upsilon(\mathfrak{m}) - \mu\upsilon(\mathfrak{m}) = 0,\\ \upsilon_0(0) = (\upsilon_{(1)_0}(0,\ltimes), \upsilon_{(2)_0}(0,\ltimes)),\quad \upsilon_1(0) = (\upsilon_{(1)_1}(0,\ltimes), \upsilon_{(2)_1}(0,\ltimes)).\end{cases} \tag{3.31}$$

Theorem 3.4. Let $\upsilon : \mathscr{T} \longrightarrow F^{\mathbb{R}}$, $\upsilon \in \mathbb{C}^{F^{\mathbb{R}}}(\mathscr{T}) \cap L^{F^{\mathbb{R}}}(\mathscr{T})$. If υ is fuzzy Caputo-fractional differentiable of first form. Then the system (4.3) have the following analytic solution

$$\begin{aligned}
\upsilon(\mathfrak{m}) =& \upsilon_0(0) + \upsilon_0(0)\mu\mathfrak{m}^{\tau_1}E_{\tau_1,\chi_2,\tau_1+1}(\mu\mathfrak{m}^{\tau_1}, \lambda\mathfrak{m}^{\chi_2})\\
&+\upsilon_1(0)\mathfrak{m} + \upsilon_1(0)\mu\mathfrak{m}^{\tau_1+1}E_{\tau_1,\chi_2,\tau_1+2}(\mu\mathfrak{m}^{\tau_1}, \lambda\mathfrak{m}^{\chi_2})\\
&+\int_0^{\mathfrak{m}}(\mathfrak{m}-u)^{\tau_1-1}E_{\tau_1,\chi_2,\tau_1}(\mu(\mathfrak{m}-u)^{\tau_1}, \lambda(\mathfrak{m}-u)^{\chi_2})\mathfrak{g}(u)du.
\end{aligned}$$

If υ is fuzzy Caputo-fractional differentiable of second form. Then the system (4.3) have the following solution

$$\begin{aligned}\upsilon(\mathfrak{m}) =& -\lambda\upsilon_0(0)\mathfrak{m}^{\chi_2}E_{\chi_1,\chi_2,\chi_3,\chi_2+1}(x,y,z) - \lambda\mathfrak{m}^{\chi_3}E_{\chi_1,\chi_2,\chi_3,\chi_3+1}(x,y,z)\\ &-\lambda\upsilon_1(0)\mathfrak{m}^{\chi_2+1}E_{\chi_1,\chi_2,\chi_3,\chi_2+2}(x,y,z) - \lambda\mathfrak{m}^{\chi_3+1}E_{\chi_1,\chi_2,\chi_3,\chi_3+2}(x,y,z)\\ &-\lambda\int_0^{\mathfrak{m}}(\mathfrak{m}-\vartheta)^{2\chi_2-1}E_{\chi_1,\chi_2,\chi_3,2\chi_2}(x^*,y^*,z^*)\mathfrak{g}(\vartheta)d\vartheta + \upsilon_0(0)E_{\chi_1,\chi_2,\chi_3,1}(x,y,z)\\ &-\lambda\mathfrak{m}^{\chi_2}E_{\chi_1,\chi_2,\chi_3,\chi_2+1}(x,y,z) + \upsilon_1(0)\mathfrak{m}E_{\chi_1,\chi_2,\chi_3,2}(x,y,z)\\ &-\lambda\mathfrak{m}^{\chi_2+1}E_{\chi_1,\chi_2,\chi_3,\chi_2+2}(x,y,z) + \int_0^{\mathfrak{m}}(\mathfrak{m}-\vartheta)^{\tau_1-1}E_{\chi_1,\chi_2,\chi_3,\tau_1}(x^*,y^*,z^*)\mathfrak{g}(\vartheta)d\vartheta\\ &\ominus(-\mu\upsilon_0(0)\mathfrak{m}^{\tau_1}E_{\chi_1,\chi_2,\chi_3,\tau_1+1}(x,y,z) - \lambda\mathfrak{m}^{2\chi_2}E_{\chi_1,\chi_2,\chi_3,2\chi_2+1}(x,y,z))\\ &\ominus(-\mu\upsilon_1(0)\mathfrak{m}^{\tau_1+1}E_{\chi_1,\chi_2,\chi_3,\tau_1+2}(x,y,z) - \lambda\mathfrak{m}^{2\chi_2+1}E_{\chi_1,\chi_2,\chi_3,2\chi_2+2}(x,y,z))\\ &+\int_0^{\mathfrak{m}}(\mathfrak{m}-\vartheta)^{\chi_1-1}E_{\chi_1,\chi_2,\chi_3,\chi_1}(x^*,y^*,z^*)\mathfrak{g}(\vartheta)d\vartheta.\end{aligned}$$

Theorem 3.5. Let $\upsilon : \mathscr{T} \longrightarrow F^{\mathbb{R}}$, $\upsilon \in \mathbb{C}^{F^{\mathbb{R}}}(\mathscr{T}) \cap L^{F^{\mathbb{R}}}(\mathscr{T})$. If υ is fuzzy Caputo-fractional differentiable of first form. Then the system (3.31) have the following analytic solution

$$\upsilon(\mathfrak{m}) = \upsilon_0(0) + \upsilon_0(0)\mu\mathfrak{m}^{\tau_1}E_{\tau_1,\chi_2,\tau_1+1}(\mu\mathfrak{m}^{\tau_1},\lambda\mathfrak{m}^{\chi_2}) + \upsilon_1(0)\mathfrak{m} + \upsilon_1(0)\mu\mathfrak{m}^{\tau_1+1}E_{\tau_1,\chi_2,\tau_1+2}(\mu\mathfrak{m}^{\tau_1},\lambda\mathfrak{m}^{\chi_2}).$$

If υ is fuzzy Caputo-fractional differentiable of second form. Then the system (3.31) have the following solution

$$\begin{aligned}\upsilon(\mathfrak{m}) =& -\lambda\upsilon_0(0)\mathfrak{m}^{\chi_2}E_{\chi_1,\chi_2,\chi_3,\chi_2+1}(x,y,z) - \lambda\mathfrak{m}^{\chi_3}E_{\chi_1,\chi_2,\chi_3,\chi_3+1}(x,y,z)\\ &-\lambda\upsilon_1(0)\mathfrak{m}^{\chi_2+1}E_{\chi_1,\chi_2,\chi_3,\chi_2+2}(x,y,z) - \lambda\mathfrak{m}^{\chi_3+1}E_{\chi_1,\chi_2,\chi_3,\chi_3+2}(x,y,z)\\ &+\upsilon_0(0)E_{\chi_1,\chi_2,\chi_3,1}(x,y,z) - \lambda\mathfrak{m}^{\chi_2}E_{\chi_1,\chi_2,\chi_3,\chi_2+1}(x,y,z) + \upsilon_1(0)\mathfrak{m}E_{\chi_1,\chi_2,\chi_3,2}(x,y,z)\\ &-\lambda\mathfrak{m}^{\chi_2+1}E_{\chi_1,\chi_2,\chi_3,\chi_2+2}(x,y,z)\ominus(-\mu\upsilon_0(0)\mathfrak{m}^{\tau_1}E_{\chi_1,\chi_2,\chi_3,\tau_1+1}(x,y,z)\\ &-\lambda\mathfrak{m}^{2\chi_2}E_{\chi_1,\chi_2,\chi_3,2\chi_2+1}(x,y,z))\ominus(-\mu\upsilon_1(0)\mathfrak{m}^{\tau_1+1}E_{\chi_1,\chi_2,\chi_3,\tau_1+2}(x,y,z)\\ &-\lambda\mathfrak{m}^{2\chi_2+1}E_{\chi_1,\chi_2,\chi_3,2\chi_2+2}(x,y,z)).\end{aligned}$$

Proof. The proof of Theorems 3.4 and 3.5 are straightforward. □

4. Application of fuzzy fractional Langevin differential equations in *RLC*-electrical circuit

This section is concerned with the subject of real-world applications. Real-world applications such as electrical circuits that have a mathematical description may lead to the linear fuzzy fractional differential equations. Several parameters in these equations are determined via measurements, observations, or experiments. Because measurements are not always exact, these parameters become unclear. When such uncertainties are represented by fuzzy sets, fuzzy linear fractional differential equations become more important. We solve an *RLC*-electrical circuit problem which is commonly used in engineering science and physics. Many researchers [50–52] studied the existence and uniqueness of solutions to fractional Langevin differential equations and anti-periodic fractional Langevin differential equations. We consider a fuzzified version from classical circuit analysis [55].

Suppose that $\mathcal{I}(\mathfrak{m})$ represents the current in the *RLC* series circuit where the *RLC* circuit symbolized resistor, inductor, and capacitor, respectively are connected in series with voltage. Consider the fuzzified *RLC*-electrical circuit associated with the resistor, inductor, and capacitor shown in the following Figure 1 such that the voltage:

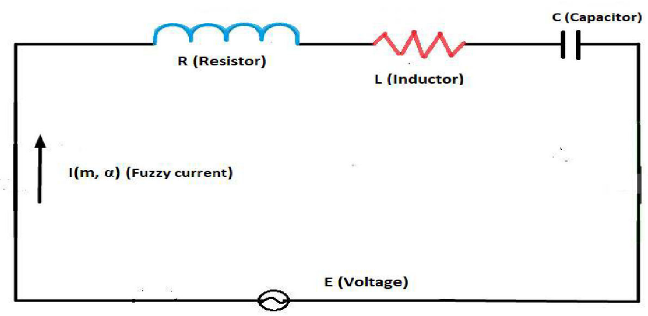

Figure 1. An RLC electrical network.

(i) Resistor: $V_R(\mathfrak{m}) = \mathcal{R}\mathcal{I}(\mathfrak{m})$ where $\frac{\mathrm{d}Q}{\mathrm{d}\mathfrak{m}} = \mathcal{I}$ and the charge Q.

(ii) Inductor: $V_L(\mathfrak{m}) = L\frac{\mathrm{d}\mathcal{I}(\mathfrak{m})}{\mathrm{d}\mathfrak{m}}$.

(iii) Capacitor: $V_C(\mathfrak{m}) = \frac{1}{C}\int_0^{\mathfrak{m}} \mathcal{I}(x)dx$.

Using Kirchoff's voltage law, sum of voltage on every loop in a circuit equivalent to the voltage $E(\mathfrak{m})$. Therefore, we have

$$V_R(\mathfrak{m}) + V_L(\mathfrak{m}) + V_C(\mathfrak{m}) = E(\mathfrak{m}),$$

or

$$\mathcal{R}\mathcal{I} + L\mathcal{I}' + \frac{1}{C}\int \mathcal{I}d\mathfrak{m} = E(\mathfrak{m}).$$

The second order non-homogeneous ODE is

$$\mathcal{R}\mathcal{I}'(\mathfrak{m}) + L\mathcal{I}''(\mathfrak{m}) + \frac{1}{C}\mathcal{I}(\mathfrak{m}) = E(\mathfrak{m}). \tag{4.1}$$

Now we develop a fuzzified version of the Eq (4.1). The initial-value problem for fuzzy fractional Langevin differential equations that represents the RLC series circuit in the following manner

$$\begin{cases} L^C\mathfrak{D}_{0^+}^{\tau_1}\mathcal{I}(\mathfrak{m}) + R^C\mathfrak{D}_{0^+}^{\tau_2}\mathcal{I}(\mathfrak{m}) + \frac{1}{C}\mathcal{I}(\mathfrak{m}) = E(\mathfrak{m}), \\ \mathcal{I}_0(0) = (\mathcal{I}_{(1)_0}(0,\ltimes), \mathcal{I}_{(2)_0}(0,\ltimes)), \quad \mathcal{I}_1(0) = (\mathcal{I}_{(1)_1}(0,\ltimes), \mathcal{I}_{(2)_1}(0,\ltimes)), \end{cases} \tag{4.2}$$

with the uncertain and vagueness initial condition $\mathcal{I}_0(0)$ and $\mathcal{I}_1(0)$ that represent the fuzzy numbers. The above initial-value problem is a special case of the system (3.3) where the fractional orders should be replaced with $0 < \tau_2 \leq 1$ and $1 < \tau_1 \leq 2$.

Now we determine the value of the current $\mathcal{I}(\mathfrak{m})$. If $\mathcal{I}$ is fuzzy Caputo-fractional differentiable of first form. Using the first status of Theorem 3.2, we have the following solution

$$\begin{aligned}\mathcal{I}_1(\mathfrak{m},\ltimes) &= \mathcal{I}_{(1)_0}(0,\ltimes) - \frac{\mathcal{I}_{(1)_0}(0,\ltimes)\mathfrak{m}^{\tau_1}}{CL}E_{\tau_1,\chi_2,\tau_1+1}\big(-\frac{\mathfrak{m}^{\tau_1}}{CL},-\frac{R\mathfrak{m}^{\chi_2}}{L}\big)\\ &+ \frac{\mathcal{I}_{(1)_1}(0,\ltimes)\mathfrak{m}}{L}E_{\tau_1,\chi_2,\tau_1+1}\big(-\frac{\mathfrak{m}^{\tau_1}}{CL},-\frac{R\mathfrak{m}^{\chi_2}}{L}\big)\\ &+ \frac{1}{L}\int_0^{\mathfrak{m}}(\mathfrak{m}-u)^{\tau_1-1}E_{\tau_1,\chi_2,\tau_1}\big(-\frac{(\mathfrak{m}-u)^{\tau_1}}{CL},-\frac{R(\mathfrak{m}-u)^{\chi_2}}{L}\big)E(u)du\end{aligned}$$

and

$$\begin{aligned}\mathcal{I}_2(\mathfrak{m},\ltimes) &= \mathcal{I}_{(2)_0}(0,\ltimes) - \frac{\mathcal{I}_{(2)_0}(0,\ltimes)\mathfrak{m}^{\tau_1}}{CL}E_{\tau_1,\chi_2,\tau_1+1}\big(-\frac{\mathfrak{m}^{\tau_1}}{CL},-\frac{R\mathfrak{m}^{\chi_2}}{L}\big)\\ &+ \frac{\mathcal{I}_{(2)_1}(0,\ltimes)\mathfrak{m}}{L}E_{\tau_1,\chi_2,\tau_1+1}\big(-\frac{\mathfrak{m}^{\tau_1}}{CL},-\frac{R\mathfrak{m}^{\chi_2}}{L}\big)\\ &+ \frac{1}{L}\int_0^{\mathfrak{m}}(\mathfrak{m}-u)^{\tau_1-1}E_{\tau_1,\chi_2,\tau_1}\big(-\frac{(\mathfrak{m}-u)^{\tau_1}}{CL},-\frac{R(\mathfrak{m}-u)^{\chi_2}}{L}\big)E(u)du.\end{aligned}$$

We assume the values for every parameter as $\tau_1 = 1.5$, $\tau_1 = 0.5$, $\mathcal{I}(0) = (\alpha, 2-\alpha)$, $\mathcal{I}^{(1)}(0) = (0.5\alpha, 1-0.5\alpha)$ with R, L and C are 20, 8 and $\frac{1}{200}$, respectively.

In Figure 2, the fuzzy solution for different values of α is strong. Moreover, the plots are smooth with different frequency (frequency taken as a fuzzy function).

As we can see from the Figures 2 and 3, the current has smoothness with the less frequency. The smoothness of the plots decreases as the fuzzy function increases.

If $\mathcal{I}$ is fuzzy Caputo-fractional differentiable of second form. Then the second status of Theorem 3.2 gives the following solution

$$\begin{aligned}\mathcal{I}_1(\mathfrak{m},\ltimes) &= \Big[E_{\chi_1,\chi_2,\chi_3,1}\Big(\frac{\mathfrak{m}^{\chi_1}}{C^2L^2},\frac{-2R\mathfrak{m}^{\chi_2}}{L},\frac{-R^2\mathfrak{m}^{\chi_3}}{L^2}\Big)\\ &+ \frac{R^2}{L^2}\mathfrak{m}^{\chi_3}E_{\chi_1,\chi_2,\chi_3,\chi_3+1}\Big(\frac{\mathfrak{m}^{\chi_1}}{C^2L^2},\frac{-2R\mathfrak{m}^{\chi_2}}{L},\frac{-R^2\mathfrak{m}^{\chi_3}}{L^2}\Big)\\ &+ \frac{2R}{L}\mathfrak{m}^{\chi_2}E_{\chi_1,\chi_2,\chi_3,\chi_2+1}\Big(\frac{\mathfrak{m}^{\chi_1}}{C^2L^2},\frac{-2R\mathfrak{m}^{\chi_2}}{L},\frac{-R^2\mathfrak{m}^{\chi_3}}{L^2}\Big)\Big]\mathcal{I}_{(1)_0}(0,\ltimes)\\ &+ \Big[\mathfrak{m}E_{\chi_1,\chi_2,\chi_3,2}\Big(\frac{\mathfrak{m}^{\chi_1}}{C^2L^2},\frac{-2R\mathfrak{m}^{\chi_2}}{L},\frac{-R^2\mathfrak{m}^{\chi_3}}{L^2}\Big)\\ &+ \frac{R^2}{L^2}\mathfrak{m}^{\chi_3+1}E_{\chi_1,\chi_2,\chi_3,\chi_3+2}\Big(\frac{\mathfrak{m}^{\chi_1}}{C^2L^2},\frac{-2R\mathfrak{m}^{\chi_2}}{L},\frac{-R^2\mathfrak{m}^{\chi_3}}{L^2}\Big)\\ &+ \frac{2R}{L}\mathfrak{m}^{\chi_2+1}E_{\chi_1,\chi_2,\chi_3,\chi_2+2}\Big(\frac{\mathfrak{m}^{\chi_1}}{C^2L^2},\frac{-2R\mathfrak{m}^{\chi_2}}{L},\frac{-R^2\mathfrak{m}^{\chi_3}}{L^2}\Big)\Big]\mathcal{I}_{(1)_1}(0,\ltimes)\\ &- \Big[\frac{1}{CL}\mathfrak{m}^{\tau_1}E_{\chi_1,\chi_2,\chi_3,\tau_1+1}\Big(\frac{\mathfrak{m}^{\chi_1}}{C^2L^2},\frac{-2R\mathfrak{m}^{\chi_2}}{L},\frac{-R^2\mathfrak{m}^{\chi_3}}{L^2}\Big)\\ &+ \frac{R}{CL^2}\mathfrak{m}^{2\chi_2}E_{\chi_1,\chi_2,\chi_3,2\chi_2+1}\Big(\frac{\mathfrak{m}^{\chi_1}}{C^2L^2},\frac{-2R\mathfrak{m}^{\chi_2}}{L},\frac{-R^2\mathfrak{m}^{\chi_3}}{L^2}\Big)\Big]\mathcal{I}_{(2)_0}(0,\ltimes)\\ &- \Big[\frac{1}{CL}\mathfrak{m}^{\tau_1+1}E_{\chi_1,\chi_2,\chi_3,\tau_1+2}\Big(\frac{\mathfrak{m}^{\chi_1}}{C^2L^2},\frac{-2R\mathfrak{m}^{\chi_2}}{L},\frac{-R^2\mathfrak{m}^{\chi_3}}{L^2}\Big)\end{aligned}$$

$$
\begin{aligned}
&+\frac{R}{CL^2}\mathfrak{m}^{2\chi_2+1}E_{\chi_1,\chi_2,\chi_3,2\chi_2+2}\Big(\frac{\mathfrak{m}^{\chi_1}}{C^2L^2},\frac{-2R\mathfrak{m}^{\chi_2}}{L},\frac{-R^2\mathfrak{m}^{\chi_3}}{L^2}\Big)\Big]\mathcal{I}_{(2)_1}(0,\ltimes)\\
&+\Big[\frac{1}{L}\int_0^{\mathfrak{m}}(\mathfrak{m}-\vartheta)^{\tau_1-1}E_{\chi_1,\chi_2,\chi_3,\tau_1}\Big(\frac{(\mathfrak{m}-\vartheta)^{\chi_1}}{C^2L^2},\frac{-2R(\mathfrak{m}-\vartheta)^{\chi_2}}{L},\frac{-R^2(\mathfrak{m}-\vartheta)^{\chi_3}}{L^2}\Big)E(\vartheta)d\vartheta\\
&+\frac{R}{L^2}\int_0^{\mathfrak{m}}(\mathfrak{m}-\vartheta)^{2\chi_2-1}E_{\chi_1,\chi_2,\chi_3,2\chi_2}\Big(\frac{(\mathfrak{m}-\vartheta)^{\chi_1}}{C^2L^2},\frac{-2R(\mathfrak{m}-\vartheta)^{\chi_2}}{L},\frac{-R^2(\mathfrak{m}-\vartheta)^{\chi_3}}{L^2}\Big)E(\vartheta)d\vartheta\Big]\\
&-\Big[\frac{1}{CL^2}\int_0^{\mathfrak{m}}(\mathfrak{m}-\vartheta)^{\chi_1-1}E_{\chi_1,\chi_2,\chi_3,\chi_1}\Big(\frac{(\mathfrak{m}-\vartheta)^{\chi_1}}{C^2L^2},\frac{-2R(\mathfrak{m}-\vartheta)^{\chi_2}}{L},\frac{-R^2(\mathfrak{m}-\vartheta)^{\chi_3}}{L^2}\Big)E(\vartheta)d\vartheta\Big],
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{I}_2(\mathfrak{m},\ltimes)=&\Big[E_{\chi_1,\chi_2,\chi_3,1}\Big(\frac{\mathfrak{m}^{\chi_1}}{C^2L^2},\frac{-2R\mathfrak{m}^{\chi_2}}{L},\frac{-R^2\mathfrak{m}^{\chi_3}}{L^2}\Big)\\
&+\frac{R^2}{L^2}\mathfrak{m}^{\chi_3}E_{\chi_1,\chi_2,\chi_3,\chi_3+1}\Big(\frac{\mathfrak{m}^{\chi_1}}{C^2L^2},\frac{-2R\mathfrak{m}^{\chi_2}}{L},\frac{-R^2\mathfrak{m}^{\chi_3}}{L^2}\Big)\\
&+\frac{2R}{L}\mathfrak{m}^{\chi_2}E_{\chi_1,\chi_2,\chi_3,\chi_2+1}\Big(\frac{\mathfrak{m}^{\chi_1}}{C^2L^2},\frac{-2R\mathfrak{m}^{\chi_2}}{L},\frac{-R^2\mathfrak{m}^{\chi_3}}{L^2}\Big)\Big]\mathcal{I}_{(2)_0}(0,\ltimes)\\
&+\Big[\mathfrak{m}E_{\chi_1,\chi_2,\chi_3,2}\Big(\frac{\mathfrak{m}^{\chi_1}}{C^2L^2},\frac{-2R\mathfrak{m}^{\chi_2}}{L},\frac{-R^2\mathfrak{m}^{\chi_3}}{L^2}\Big)\\
&+\frac{R^2}{L^2}\mathfrak{m}^{\chi_3+1}E_{\chi_1,\chi_2,\chi_3,\chi_3+2}\Big(\frac{\mathfrak{m}^{\chi_1}}{C^2L^2},\frac{-2R\mathfrak{m}^{\chi_2}}{L},\frac{-R^2\mathfrak{m}^{\chi_3}}{L^2}\Big)\\
&+\frac{2R}{L}\mathfrak{m}^{\chi_2+1}E_{\chi_1,\chi_2,\chi_3,\chi_2+2}\Big(\frac{\mathfrak{m}^{\chi_1}}{C^2L^2},\frac{-2R\mathfrak{m}^{\chi_2}}{L},\frac{-R^2\mathfrak{m}^{\chi_3}}{L^2}\Big)\Big]\mathcal{I}_{(2)_1}(0,\ltimes)\\
&-\Big[\frac{1}{CL}\mathfrak{m}^{\tau_1}E_{\chi_1,\chi_2,\chi_3,\tau_1+1}\Big(\frac{\mathfrak{m}^{\chi_1}}{C^2L^2},\frac{-2R\mathfrak{m}^{\chi_2}}{L},\frac{-R^2\mathfrak{m}^{\chi_3}}{L^2}\Big)\\
&+\frac{R}{CL^2}\mathfrak{m}^{2\chi_2}E_{\chi_1,\chi_2,\chi_3,2\chi_2+1}\Big(\frac{\mathfrak{m}^{\chi_1}}{C^2L^2},\frac{-2R\mathfrak{m}^{\chi_2}}{L},\frac{-R^2\mathfrak{m}^{\chi_3}}{L^2}\Big)\Big]\mathcal{I}_{(1)_0}(0,\ltimes)\\
&-\Big[\frac{1}{CL}\mathfrak{m}^{\tau_1+1}E_{\chi_1,\chi_2,\chi_3,\tau_1+2}\Big(\frac{\mathfrak{m}^{\chi_1}}{C^2L^2},\frac{-2R\mathfrak{m}^{\chi_2}}{L},\frac{-R^2\mathfrak{m}^{\chi_3}}{L^2}\Big)\\
&+\frac{R}{CL^2}\mathfrak{m}^{2\chi_2+1}E_{\chi_1,\chi_2,\chi_3,2\chi_2+2}\Big(\frac{\mathfrak{m}^{\chi_1}}{C^2L^2},\frac{-2R\mathfrak{m}^{\chi_2}}{L},\frac{-R^2\mathfrak{m}^{\chi_3}}{L^2}\Big)\Big]\mathcal{I}_{(1)_1}(0,\ltimes)\\
&+\Big[\frac{1}{L}\int_0^{\mathfrak{m}}(\mathfrak{m}-\vartheta)^{\tau_1-1}E_{\chi_1,\chi_2,\chi_3,\tau_1}\Big(\frac{(\mathfrak{m}-\vartheta)^{\chi_1}}{C^2L^2},\frac{-2R(\mathfrak{m}-\vartheta)^{\chi_2}}{L},\frac{-R^2(\mathfrak{m}-\vartheta)^{\chi_3}}{L^2}\Big)E(\vartheta)d\vartheta\\
&+\frac{R}{L^2}\int_0^{\mathfrak{m}}(\mathfrak{m}-\vartheta)^{2\chi_2-1}E_{\chi_1,\chi_2,\chi_3,2\chi_2}\Big(\frac{(\mathfrak{m}-\vartheta)^{\chi_1}}{C^2L^2},\frac{-2R(\mathfrak{m}-\vartheta)^{\chi_2}}{L},\frac{-R^2(\mathfrak{m}-\vartheta)^{\chi_3}}{L^2}\Big)E(\vartheta)d\vartheta\Big]\\
&-\Big[\frac{1}{CL^2}\int_0^{\mathfrak{m}}(\mathfrak{m}-\vartheta)^{\chi_1-1}E_{\chi_1,\chi_2,\chi_3,\chi_1}\Big(\frac{(\mathfrak{m}-\vartheta)^{\chi_1}}{C^2L^2},\frac{-2R(\mathfrak{m}-\vartheta)^{\chi_2}}{L},\frac{-R^2(\mathfrak{m}-\vartheta)^{\chi_3}}{L^2}\Big)E(\vartheta)d\vartheta\Big].
\end{aligned}
$$

The underlying notes can be seen in the graphs: All plots are nearly identical in their behaviors. The plots are in pretty close agreement with each other. Especially when considering the fuzzy Langevin linear fractional differential equations. The solution is a fuzzy function at every point in the domain and the fuzzy Langevin linear fractional differential equations have strong ties to the model profiles. To overcome the complexity of bivariate and trivariate Mittag-Leffler function, we draw the graphs using Definition 2.9 and taking $\epsilon = 0$ in Eq (2.13) with $E(m,\alpha) = (10+10\alpha, 30-10\alpha)m$ and $E(m,\alpha) = (10+10\alpha, 30-10\alpha)m^2$ as shown in Figures 2–5, respectively.

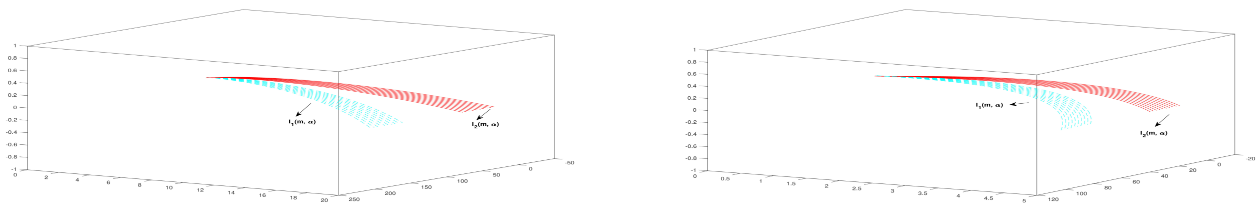

Figure 2. α-cut solution representation for upper and lower fuzzy number-valued functions for different values of $\alpha \in [0, 1]$.

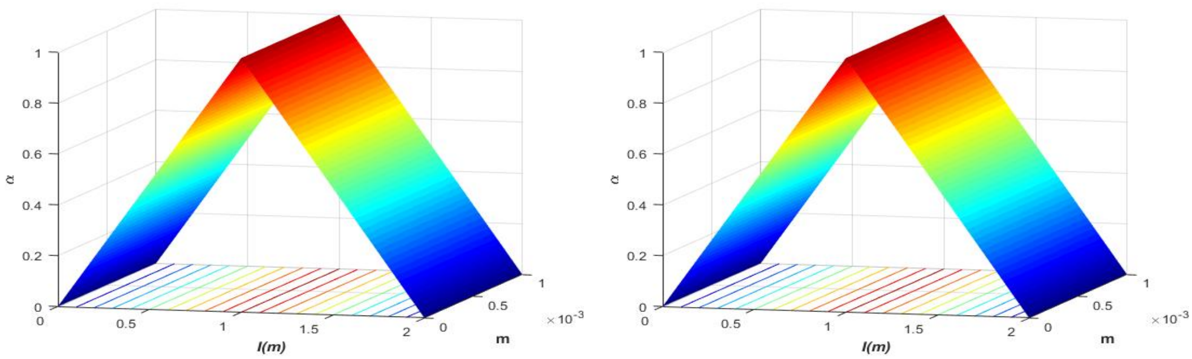

Figure 3. Plots of $\mathcal{I}(m)$ with $E(m, \alpha) = (10 + 10\alpha, 30 - 10\alpha)m$ and $E(m, \alpha) = (10 + 10\alpha, 30 - 10\alpha)m^2$, respectively.

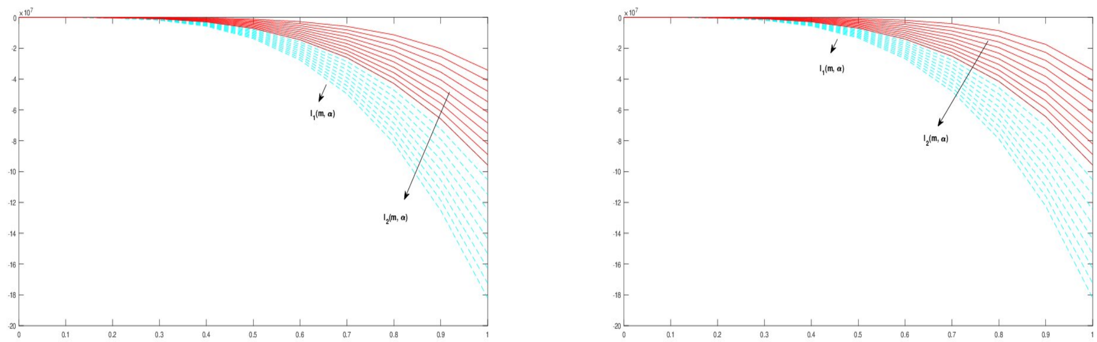

Figure 4. α-cut solution representation for upper and lower fuzzy number-valued functions for different values of $\alpha \in [0, 1]$.

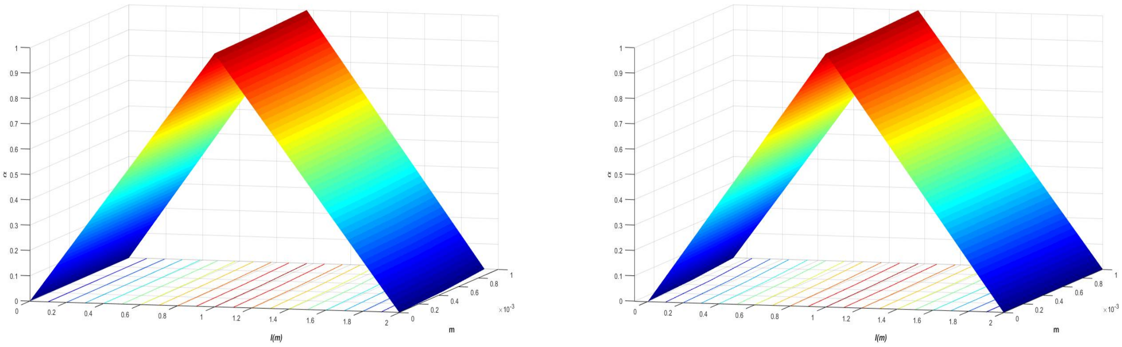

Figure 5. Graphical representation of the current $\mathcal{I}(m)$ involving with different functions $E(m,\alpha) = (10 + 10\alpha, 30 - 10\alpha)m$ and $E(m,\alpha) = (10 + 10\alpha, 30 - 10\alpha)m^2$, respectively.

Example 4.1. Now we present the example as an application of Theorem 3.5 mentioned in Section 3. Consider the following inhomogeneous fuzzy fractional Langevin differential equations

$$\begin{cases} {}^C\mathfrak{D}^{1.5}_{0^+}\upsilon(\mathfrak{m}) - 2\,{}^C\mathfrak{D}^{0.5}_{0^+}\upsilon(\mathfrak{m}) - \Gamma(1.5)\upsilon(\mathfrak{m}) = \mathfrak{g}(\mathfrak{m}), \\ \upsilon_0(0) = (\ltimes + 1, 3 - \ltimes), \quad \upsilon_1(0) = (\ltimes + 4, 6 - \ltimes), \end{cases} \tag{4.3}$$

For instance, let $\tau_1 = 1.5$, $\tau_2 = 0.5$, $\lambda = 2$, $\mu = \Gamma(1.5)$ and $\mathfrak{g}(\mathfrak{m}) = \sin(\mathfrak{m})$. If υ is fuzzy Caputo-fractional differentiable of first form. Then the system (4.3) have the following analytic solution

$$\begin{aligned} \upsilon(\mathfrak{m}) = {} & (\ltimes + 1, 3 - \ltimes) + (\ltimes + 1, 3 - \ltimes)\Gamma(1.5)\mathfrak{m}^{1.5}E_{1.5,1,2.5}(\Gamma(1.5)\mathfrak{m}^{1.5}, 2\mathfrak{m}^1) \\ & + (\ltimes + 4, 6 - \ltimes)\mathfrak{m} + (\ltimes + 4, 6 - \ltimes)\Gamma(1.5)\mathfrak{m}^{2.5}E_{1.5,1,3.5}(\Gamma(1.5)\mathfrak{m}^{1.5}, 2\mathfrak{m}^1) \\ & + \int_0^{\mathfrak{m}} (\mathfrak{m} - \vartheta)^{1.5-1}E_{1.5,1,1.5}(\Gamma(1.5)(\mathfrak{m} - \vartheta)^{1.5}, 2(\mathfrak{m} - u)^1)\sin(\vartheta)d\vartheta. \end{aligned}$$

If υ is fuzzy Caputo-fractional differentiable of second form. Then the system (4.3) have the following solution

$$\begin{aligned} \upsilon(\mathfrak{m}) = & - 2(\ltimes + 1, 3 - \ltimes)\mathfrak{m}^1E_{3,1,2,1+1}(x,y,z) - 2\mathfrak{m}^2E_{3,1,2,2+1}(x,y,z) \\ & - 2(\ltimes + 4, 6 - \ltimes)\mathfrak{m}^{1+1}E_{3,1,2,1+2}(x,y,z) - 2\mathfrak{m}^{2+1}E_{3,1,2,2+2}(x,y,z) \\ & - 2\int_0^{\mathfrak{m}} (\mathfrak{m} - \vartheta)^{21-1}E_{3,1,2,21}(x^*,y^*,z^*)\sin(\vartheta)d\vartheta + (\ltimes + 1, 3 - \ltimes)E_{3,1,2,1}(x,y,z) \\ & - 2\mathfrak{m}^1E_{3,1,2,1+1}(x,y,z) + (\ltimes + 4, 6 - \ltimes)\mathfrak{m}E_{3,1,2,2}(x,y,z) \\ & - 2\mathfrak{m}^{1+1}E_{3,1,2,1+2}(x,y,z) + \int_0^{\mathfrak{m}} (\mathfrak{m} - \vartheta)^{1.5-1}E_{3,1,2,1.5}(x^*,y^*,z^*)\sin(\vartheta)d\vartheta \\ & \ominus \big(- \Gamma(1.5)(\ltimes + 1, 3 - \ltimes)\mathfrak{m}^{1.5}E_{3,1,2,2.5}(x,y,z) - 2\mathfrak{m}^{21}E_{3,1,2,21+1}(x,y,z)\big) \\ & \ominus \big(- \Gamma(1.5)(\ltimes + 4, 6 - \ltimes)\mathfrak{m}^{2.5}E_{3,1,2,3.5}(x,y,z) - 2\mathfrak{m}^{21+1}E_{3,1,2,21+2}(x,y,z)\big) \\ & + \int_0^{\mathfrak{m}} (\mathfrak{m} - \vartheta)^{3-1}E_{3,1,2,3}(x^*,y^*,z^*)\sin(\vartheta)d\vartheta, \end{aligned}$$

with $(x,y,z) = ((\Gamma(1.5))^2\mathfrak{m}^3, 2(2)\mathfrak{m}^1, -(2)^2\mathfrak{m}^2)$ and $(x^*,y^*,z^*) = ((\Gamma(1.5))^2(\mathfrak{m} - \vartheta)^3, 2(2)(\mathfrak{m} - \vartheta)^1, -(2)^2(\mathfrak{m} - \vartheta)^2)$.

5. Conclusions

The fuzzy fractional Langevin differential equation is a significant topic with several applications in science and engineering. Using the aforementioned technique, this paper shows how to extend and find the solution of the fuzzy Langevin fractional differential equation analytically. The aim of this study is to determine the explicit and analytical fuzzy solution of both fuzzy fractional Langevin homogeneous differential equations and non-homogeneous fuzzy fractional Langevin differential equations. The potential solution of fuzzy fractional Langevin differential equations is extracted using the Laplace transformation technique. Furthermore, these solutions are defined in terms of bivariate and trivariate Mittage-Leffler function both in general and special cases. To grasp the novelty of this work, we connect fuzzy fractional Langevin differential equations with the RLC electrical circuit and analyze their graphs to visualize and support the theoretical results. In future, we plan to solve the system of fuzzy fractional differential equations using the proposed technique.

Conflicts of interest

The authors declare no conflicts of interest.

References

1. D. Dubios, H. Prade, Towards fuzzy differential calculus part 3: Differentiation, *Fuzzy Set. Syst.*, **8** (1982), 225–233.
2. M. L. Puri, D. A. Ralescu, Differentials of fuzzy functions, *J. Math. Anal. Appl.*, **91** (1983), 552–558. https://doi.org/10.1016/0022-247X(83)90169-5
3. R. G. Jr, W. Voxman, Elementary fuzzy calculus, *Fuzzy Set. Syst.*, **18** (1986), 31–43. https://doi.org/10.1016/0165-0114(86)90026-6
4. S. Seikkala, On the fuzzy initial value problem, *Fuzzy Set. Syst.*, **24** (1987), 319–330. https://doi.org/10.1016/0165-0114(87)90030-3
5. M. Friedman, M. Ming, A. Kandel, *Fuzzy derivatives and fuzzy Cauchy problems using LP metric*, In Fuzzy Logic Foundations and Industrial Applications, Springer, Boston, **8** (1996), 57–72. https://doi.org/10.1007/978-1-4613-1441-7_3
6. Z. Yue, W. Guangyuan, Time domain methods for the solutions of N-order fuzzy differential equations, *Fuzzy Set. Syst.*, **94** (1998), 77–92. https://doi.org/10.1016/S0165-0114(96)00235-7
7. B. Bede, S. G. Gal, Generalizations of the differentiability of fuzzy-number-valued functions with applications to fuzzy differential equations, *Fuzzy Set. Syst.*, **151** (2005), 581–599. https://doi.org/10.1016/j.fss.2004.08.001
8. M. M. Raja, V. Vijayakumar, Optimal control results for Sobolev-type fractional mixed Volterra-Fredholm type integrodifferential equations of order $1 < r < 2$ with sectorial operators, *Opt. Contr. Appl. Met.*, 2022. https://doi.org/10.1002/oca.2892
9. M. M. Raja, V. Vijayakumar, A. Shukla, K. S. Nisar, N. Sakthivel, K. Kaliraj, Optimal control and approximate controllability for fractional integrodifferential evolution equations with infinite delay of order $r \in (1, 2)$, *Opt. Contr. Appl. Met.*, 2022. https://doi.org/10.1002/oca.2867

10. Y. K. Ma, M. M. Raja, V. Vijayakumar, A. Shukla, W. Albalawi, K. S. Nisar, Existence and continuous dependence results for fractional evolution integrodifferential equations of order $r \in (1, 2)$, *Alex. Eng. J.*, **61** (2022), 9929–9939. https://doi.org/10.1016/j.aej.2022.03.010

11. Y. K. Ma, M. M. Raja, K. S. Nisar, A. Shukla, V. Vijayakumar, Results on controllability for Sobolev type fractional differential equations of order $1 < r < 2$ with finite delay, *AIMS Math.*, **7** (2022), 10215–10233. https://doi.org/10.3934/math.2022568

12. M. M. Raja, V. Vijayakumar, A. Shukla, K. S. Nisar, H. M. Baskonus, On the approximate controllability results for fractional integrodifferential systems of order $1 < r < 2$ with sectorial operators, *J. Comput. Appl. Math.*, **415** (2022), 114492. https://doi.org/10.1016/j.cam.2022.114492

13. R. P. Agarwal, V. Lakshmikantham, J. J. Nieto, On the concept of solution for fractional differential equations with uncertainty, *Nonlinear Anal.-Theor.*, **72** (2010), 2859–2862. https://doi.org/10.1016/j.na.2009.11.029

14. J. U. Jeong, Existence results for fractional order fuzzy differential equations with infinite delay, *Int. Math. Forum*, **5** (2010), 3221–3230.

15. S. Arshad, V. Lupulescu, On the fractional differential equations with uncertainty, *Nonlinear Anal.-Theor.*, **74** (2011), 3685–3693. https://doi.org/10.1016/j.na.2011.02.048

16. S. Salahshour, T. Allahviranloo, S. Abbasbandy, Solving fuzzy fractional differential equations by fuzzy Laplace transforms, *Commun. Nonlinear Sci.*, **17** (2012), 1372–1381. https://doi.org/10.1016/j.cnsns.2011.07.005

17. M. Mazandarani, A. V. Kamyad, Modified fractional Euler method for solving fuzzy fractional initial value problem, *Commun. Nonlinear Sci.*, **18** (2013), 12–21. https://doi.org/10.1016/j.cnsns.2012.06.008

18. S. Salahshour, T. Allahviranloo, S. Abbasbandy, D. Baleanu, Existence and uniqueness results for fractional differential equations with uncertainty, *Adv. Differ. Equ.*, **2012** (2012), 1–12. https://doi.org/10.1186/1687-1847-2012-112

19. S. Arshad, On existence and uniqueness of solution of fuzzy fractional differential equations, *Iran. J. Fuzzy Syst.*, **10** (2013), 137–151.

20. T. Allahviranloo, A. Armand, Z. Gouyandeh, Fuzzy fractional differential equations under generalized fuzzy Caputo derivative, *J. Intell. Fuzzy Syst.*, **26** (2014), 1481–1490. https://doi.org/10.3233/IFS-130831

21. T. Allahviranloo, *Fuzzy fractional differential operators and equations: Fuzzy fractional differential equations*, Studies in fuzziness and soft computing series, Springer Nature, Switzerland, 2020. https://doi.org/10.1007/978-3-030-51272-9

22. A. Khastan, J. J. Nieto, R. Rodríguez-López, Schauder fixed-point theorem in semilinear spaces and its application to fractional differential equations with uncertainty, *Fixed Point Theory A.*, **2014** (2014), 1–14. https://doi.org/10.1186/1687-1812-2014-21

23. N. V. Hoa, V. Lupulescu, D. O'Regan, Solving interval-valued fractional initial value problems under Caputo *gH*-fractional differentiability, *Fuzzy Set. Syst.*, **309** (2017), 1–34. https://doi.org/10.1016/j.fss.2016.09.015

24. H. V. Ngo, V. Lupulescu, D. O'Regan, A note on initial value problems for fractional fuzzy differential equations, *Fuzzy Set. Syst.*, **347** (2018), 54–69. https://doi.org/10.1016/j.fss.2017.10.002

25. S. Melliani, E. Arhrrabi, M. H. Elomari, L. S. Chadli, Ulam-Hyers-Rassias stability for fuzzy fractional integrodifferential equations under Caputo gH-differentiability, *Int. J. Optim. Appl.*, 2021, 51. https://doi.org/10.1007/s40306-017-0207-2

26. H. Vu, N. V. Hoa, Uncertain fractional differential equations on a time scale under granular differentiability concept, *Comput. Appl. Math.*, **38** (2019), 1–22. https://doi.org/10.1007/s40314-019-0873-x

27. S. Ezadi, T. Allahviranloo, Artificial neural network approach for solving fuzzy fractional order initial value problems under gH-differentiability, *Math. Method. Appl. Sci.*, 2020. https://doi.org/10.1002/mma.7287

28. M. Saqib, M. Akram, S. Bashir, T. Allahviranloo, Numerical solution of bipolar fuzzy initial value problem, *J. Intell. Fuzzy Syst.*, **40** (2021), 1309–1341. https://doi.org/10.3233/JIFS-201619

29. M. Akram, M. Saqib, S. Bashir, T. Allahviranloo, An efficient numerical method for solving *m*-polar fuzzy initial value problems, *Comput. Appl. Math.*, **41** (2022), 157. https://doi.org/10.1007/s40314-022-01841-2

30. M. Akram, M. Ali, T. Allahviranloo, A method for solving bipolar fuzzy complex linear systems with real and complex coefficients, *Soft Comput.*, **26** (2022), 2157–2178. https://doi.org/10.1007/s00500-021-06672-7

31. M. Akram, T. Allahviranloo, W. Pedrycz, M. Ali, Methods for solving LR-bipolar fuzzy linear systems, *Soft Comput.*, **25** (2021), 85–108. https://doi.org/10.1007/s00500-020-05460-z

32. A. N. A. Koam, M. Akram, G. Muhammad, N. Hussain, LU decomposition scheme for solving m-polar fuzzy system of linear equations, *Math. Probl. Eng.*, **2020** (2020), 8384593. https://doi.org/10.1155/2020/8384593

33. M. Ghaffari, T. Allahviranloo, S. Abbasbandy, M. Azhini, On the fuzzy solutions of time-fractional problems, *Iran. J. Fuzzy Syst.*, **18** (2021), 51–66. https://doi.org/10.22111/IJFS.2021.6081

34. M. Keshavarz, T. Allahviranloo, Fuzzy fractional diffusion processes and drug release, *Fuzzy Set. Syst.*, **436** (2022), 82–101. https://doi.org/10.1016/j.fss.2021.04.001

35. M. Keshavarz, T. Allahviranloo, S. Abbasbandy, M. H. Modarressi, A study of fuzzy methods for solving system of fuzzy differential equations, *New Math. Nat. Comput.*, **17** (2021), 1–27. https://doi.org/10.1142/S1793005721500010

36. D. Qiu, The generalized Hukuhara differentiability of interval-valued function is not fully equivalent to the one-sided differentiability of its end point functions, *Fuzzy Set. Syst.*, **419** (2021), 158–168. https://doi.org/10.1016/j.fss.2020.07.012

37. H. Wang, R. Rodriguez-Lopez, On the existence of solutions to boundary value problems for interval-valued differential equations under gH-differentiability, *Inform. Sci.*, **553** (2021), 225–246. https://doi.org/10.1016/j.ins.2020.10.052

38. P. Langevin, Sur la théorie du mouvement brownien, *Compt. Rendus*, **146** (1908), 530–533.

39. M. Z. Ahmad, M. K. Hassan, S. Abbasbanday, Solving fuzzy fractional differential equations using Zadeh's extension principle, *The Scientific World J.*, **2013** (2013). https://doi.org/10.1155/2013/454969
40. R. L. Magin, *Fractional calculus in bioengineering*, Begell House Publisher, Connecticut, 2006. https://doi.org/10.1615/CritRevBiomedEng.v32.i1.10
41. T. R. Prabhakar, A singular integral equation with a generalized Mittag-Leffler function in the kernel, *Yokohama Math. J.*, 1971.
42. M. Akram, T. Ihsan, T. Allahviranloo, Solving Pythagorean fuzzy fractional differential equations using Laplace transform, *Granular Comput.*, 2022. https://doi.org/10.1007/s41066-022-00344-z
43. D. Baleanu, J. A. T. Machado, A. C. J. Luo, *Fractional dynamics and control*, Springer Science & Business Media, 2011.
44. V. Lakshmikantham, S. Leela, J. V. Devi, *Theory of fractional dynamic systems*, Cambridge Scientific Publishers, Cambridge, 2009.
45. R. Kubo, The fluctuation-dissipation theorem, *Rep. Prog. Phys.*, **29** (1966), 255–284.
46. E. Lutz, Fractional Langevin equation, *Phys. Rev. E*, **64** (2001), 1–4. https://doi.org/10.1142/9789814340595_0012
47. Y. Adjabi, M. E. Samei, M. M. Matar, J. Alzabut, Langevin differential equation in frame of ordinary and Hadamard fractional derivatives under three point boundary conditions, *AIMS Math.*, **6** (2021), 2796–2843. https://doi.org/10.3934/math.2021171
48. B. Ahmad, A. Alsaedi, S. Salem, On a nonlocal integral boundary value problem of nonlinear Langevin equation with different fractional orders, *Adv. Differ. Equ.*, **2019** (2019), 57. https://doi.org/10.1186/s13662-019-2003-x
49. B. Ahmad, J. J. Nieto, A. Alsaedi, M. El-Shahed, A study of nonlinear Langevin equation involving two fractional orders in different intervals, *Nonlinear Anal.*, **13** (2012), 599–606. https://doi.org/10.1016/j.nonrwa.2011.07.052
50. H. Baghani, Existence and uniqueness of solutions to fractional Langevin equations involving two fractional orders, *J. Fix. Point Theory A.*, **20** (2018), 1–7. https://doi.org/10.1007/s11784-018-0540-7
51. Z. Kiyamehr, H. Baghani, Existence of solutions of BVPs for fractional Langevin equations involving Caputo fractional derivatives, *J. Appl. Anal.*, **27** (2021), 47–55. https://doi.org/10.1515/jaa-2020-2029
52. A. Salem, Existence results of solutions for anti-periodic fractional Langevin equation, *J. Appl. Anal. Comput.*, **10** (2020), 2557–2574. https://doi.org/10.11948/20190419
53. T. Kaczorek, Positive different orders fractional 2D linear systems, *Acta Mech. Automatica*, **2** (2008), 51–58.
54. S. S. Devi, K. Ganesan, Modelling electric circuit problem with fuzzy differential equations, *J. Phys. Conf. Ser.*, **1377** (2019), 012024. https://doi.org/10.1088/1742-6596/1377/1/012024
55. A. Ahmadova, N. I. Mahmudov, Langevin differential equations with general fractional orders and their applications to electric circuit theory, *J. Comput. Appl. Math.*, **388** (2021), 113299. https://doi.org/10.1016/j.cam.2020.113299

56. K. Diethelm, *The analysis of fractional differential equations: An application-oriented exposition using differential operators of Caputo type*, Springer, Berlin, 2004.

57. I. Podlubny, *Fractional differential equations*, Academic Press, San Diego, 1999.

58. S. G. Samko, A. A. Kilbas, O. I. Marichev, *Fractional integrals and derivatives: Theory and applications*, Gordon & Breach Science Publishers, Yverdon, 1993.

59. A. Fernandez, C. Kürt , M. A. Özarslan, A naturally emerging bivariate Mittag-Leffler function and associated fractional-calculus operators, *Comput. Appl. Math.*, **39** (2020), 1–27. https://doi.org/10.1007/s40314-020-01224-5

60. A. Ahmadova, I. T. Huseynov, A. Fernandez, N. I. Mahmudov, Trivariate Mittag-Leffler functions used to solve multi-order systems of fractional differential equations, *Commun. Nonlinear Sci.*, **97** (2021), 105735. https://doi.org/10.1016/j.cnsns.2021.105735

61. T. Allahviranloo, M. B. Ahmadi, Fuzzy Laplace transforms, *Soft Comput.*, **14** (2010) 235. https://doi.org/10.1007/s00500-008-0397-6