



Research article

An efficient iterative algorithm for common solutions of three G -nonexpansive mappings in Banach spaces involving a graph with applications to signal and image restoration problems

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Abstract: In this paper, three G -nonexpansive mappings are implemented and analyzed using an efficient modified three-step iteration algorithm. Assuming coordinate-convexity in a uniformly convex Banach space endowed with a directed graph, conditions for the weak and strong convergence of the scheme are determined. We give numerical comparisons to back up our main theorem, and compare our algorithm's convergence behavior to that of the three-step Noor iteration and SP-iteration. We use our proposed algorithm to solve image deblurring problems as an application. In addition, we discuss a novel approach to signal recovery in situations where the type of noise is unknown.

Keywords: coordinate-convex; G -nonexpansive mapping; weak and strong convergence; signal recovering problem; image deblurring problem

Mathematics Subject Classification: 47E10, 47H09, 47H10, 54H25

1. Introduction

Fixed point theory is an extremely active research area because of its multifaceted applications. The typical situation is that a self-map on a set admits a fixed point under certain conditions. In a complete metric space, Banach [1] proved the existence of unique fixed point for contractions. The Banach contraction principle has been generalized and extended in several ways due to its applications in mathematics and other related disciplines. The most recent version of the theorem was proved in Banach spaces endowed with a graph G , where $G = (V(G), E(G))$ is a directed graph such that the set $V(G)$ of its vertices of a graph and the set $E(G)$ of its edges contains all loops.

In order to extend the Banach contraction principle, by combination of the concepts in fixed point theory and graph theory, Banach G -contractions were introduced by Jachymaski [2] in complete metric spaces accompanied with a graph G which has a convex subset of the underlying metric space as its set of vertices, see e.g., [3–10].

In the last few decades finding of fixed points by some iterative schemes for G -contraction, G -nonexpansive and G -monotone nonexpansive mappings have been studied extensively by various authors. In 2012, Aleomraninejad et al. [11] presented some iterative scheme for G -contraction and G -nonexpansive mappings in a metric space with a directed graph and stated the convergence for such mappings. In 2015, Alfuraidan and Khamsi [12] defined the concept of G -monotone nonexpansive multivalued mappings defined on a hyperbolic metric space with a graph. Alfuraidan [13] studied the existence of fixed points of monotone nonexpansive mappings on a Banach space endowed with a directed graph. Tiammee et al. [14] proved the Browder convergence theorem for G -nonexpansive mappings and studied the convergence of the Halpern iteration process to the projection of initial point where the projection is onto the set of fixed points of the G -nonexpansive mapping in Hilbert space with the directed graph. Tripak [15] defined the Ishikawa type iteration process for two G -nonexpansive mappings and gave some weak and strong convergence theorems of such iterations in a Banach space endowed with a graph.

The three-step iterative scheme, usually called the Noor iteration method, [16] is used to approximate common fixed point of three G -nonexpansive mappings via

$$\begin{aligned} z_n &= (1 - \gamma_n)x_n + \gamma_n\mathcal{T}_3x_n, \\ y_n &= (1 - \beta_n)x_n + \beta_n\mathcal{T}_2z_n, \\ x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n\mathcal{T}_1y_n, \quad n \geq 0, \end{aligned} \tag{1.1}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[0, 1]$.

Glowinski and Le Tallec [17] employed three-step iterative approach in elastoviscoplasticity, eigenvalue computation and theory of liquid crystals. In [17], it was shown that the three-step iterative process yields better numerical results than the estimated iterations in two and one steps. In 1998, Haubruge, Nguyen and Strodiot [18] investigated the convergence analysis of three-step methods of Glowinski and Le Tallec [17] and applied these methods to obtain new splitting-type algorithms for solving variation inequalities, separable convex programming and minimization of a sum of convex functions. They also proved that three-step iterations lead to highly parallelized algorithms under certain conditions.

Recently, Sridarat et al. [19] modified the SP-iteration process for three G -nonexpansive mappings \mathcal{T}_1 , \mathcal{T}_2 and \mathcal{T}_3 as follows:

$$\begin{aligned} z_n &= (1 - \gamma_n)x_n + \gamma_n\mathcal{T}_3x_n, \\ y_n &= (1 - \beta_n)z_n + \beta_n\mathcal{T}_2z_n, \\ x_{n+1} &= (1 - \alpha_n)y_n + \alpha_n\mathcal{T}_1y_n, \quad n \geq 0, \end{aligned} \tag{1.2}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are appropriate real sequences in $[0, 1]$. They studied the weak and strong convergence of the iterative scheme (1.2) under proper conditions in a uniformly convex Banach space endowed with a graph.

The main purpose of this paper is to construct an efficient modified three-step iteration algorithm for approximating common fixed points of three G -nonexpansive mappings in a uniformly convex Banach space endowed with a graph. This paper is organized as follows.

The notations, basic definitions, and some lemmas for proving our main result are given in Section 2. Our main result is presented in Section 3. In this section, we also prove weak and strong convergence results of the proposed method under mild conditions in a uniformly convex Banach space endowed with a graph. In Section 4, we study the convergence behavior of our method, and compare its efficiency with competing methods. In Section 5, we apply our proposed method to solve certain image deblurring and signal recovering problems.

2. Graph basic definitions

In this section, we recall a few basic notions concerning the connectivity of graphs. All of these notions can be found, e.g., in [20].

Let C be a nonempty subset of a real Banach space X . We identify the graph G with the pair $(V(G), E(G))$, where the set $V(G)$ of its vertices coincide with set C and the set of edges $E(G)$ contains $\{(x, x) : x \in C\}$. Also, G is such that no two edges are parallel. A mapping $\mathcal{T} : C \rightarrow C$ is said to be G -contraction if \mathcal{T} preserves edges of G (or \mathcal{T} is edge-preserving), i.e.,

$$(x, y) \in E(G) \Rightarrow (\mathcal{T}x, \mathcal{T}y) \in E(G),$$

and \mathcal{T} decreases weights of edges of G in the following way: there exists $\alpha \in (0, 1)$ such that

$$(x, y) \in E(G) \Rightarrow \|\mathcal{T}x - \mathcal{T}y\| \leq \alpha\|x - y\|.$$

A mapping $\mathcal{T} : C \rightarrow C$ is said to be G -nonexpansive (see [12], Definition 2.3 (iii)) if \mathcal{T} preserves edges of G , i.e.,

$$(x, y) \in E(G) \Rightarrow (\mathcal{T}x, \mathcal{T}y) \in E(G),$$

and T non-increases weights of edges of G in the following way:

$$(x, y) \in E(G) \Rightarrow \|\mathcal{T}x - \mathcal{T}y\| \leq \|x - y\|.$$

If x and y are vertices in a graph G , then a path in G from x to y of length N ($N \in \mathbb{N} \cup \{0\}$) is a sequence $\{x_i\}_{i=0}^N$ of $N + 1$ vertices such that $x_0 = x$, $x_N = y$ and $(x_i, x_{i+1}) \in E(G)$ for $i = 0, 1, \dots, N - 1$. A graph G is connected if there is a path between any two vertices. A directed graph $G = (V(G), E(G))$ is said to be transitive if, for any $x, y, z \in V(G)$ such that (x, y) and (y, z) are in $E(G)$, we have $(x, z) \in E(G)$. We denote G^{-1} the conversion of a graph G and

$$E(G^{-1}) = \{(x, y) \in X \times X : (y, x) \in E(G)\}.$$

Let $x_0 \in V(G)$ and \mathcal{A} a subset of $V(G)$. We say that \mathcal{A} is dominated by x_0 if $(x_0, x) \in E(G)$ for all $x \in \mathcal{A}$. \mathcal{A} dominates x_0 if for each $x \in \mathcal{A}$, $(x, x_0) \in E(G)$.

In this paper, we use \rightarrow and \rightharpoonup to denote the strong convergence and weak convergence, respectively.

A mapping $\mathcal{T} : C \rightarrow C$ is said to be G -demiclosed at 0 if, for any sequence $\{x_n\}$ in C such that $(x_n, x_{n+1}) \in E(G)$, $x_n \rightharpoonup x$ and $\mathcal{T}x_n \rightarrow 0$ imply $\mathcal{T}x = 0$.

Let X be a Banach space with dimension $X \geq 2$. The modulus of X is the function $\delta_X : (0, 2] \rightarrow [0, 1]$ defined by

$$\delta_X = \inf \left\{ 1 - \left\| \frac{1}{2}(x + y) \right\| : \|x\| = 1, \epsilon = \|x - y\| \right\}.$$

Banach space \mathcal{X} is uniformly convex if and only if $\delta_{\mathcal{X}}(\epsilon) > 0$ for all $\epsilon \in (0, 2]$.

Recall that a Banach space \mathcal{X} is said to satisfy Opial's condition [21] if $x_n \rightharpoonup x$ and $x \neq y$ implying that

$$\limsup_{n \rightarrow \infty} \|x_n - x\| < \limsup_{n \rightarrow \infty} \|x_n - y\|.$$

Let C be a nonempty closed convex subset of a real uniformly convex Banach space \mathcal{X} . Recall that the mappings \mathcal{T}_i ($i = 1, 2, 3$) on C are said to satisfy condition (C) [19] if there exists a nondecreasing function $f : [0, \infty) \rightarrow [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r > 0$ such that for all $x \in C$,

$$\max\{\|x - \mathcal{T}_1 x\|, \|x - \mathcal{T}_2 x\|, \|x - \mathcal{T}_3 x\|\} \geq f(d(x, \mathcal{F})),$$

where $\mathcal{F} = \mathcal{F}(\mathcal{T}_1) \cap \mathcal{F}(\mathcal{T}_2) \cap \mathcal{F}(\mathcal{T}_3)$, $\mathcal{F}(\mathcal{T}_i)$ ($i = 1, 2, 3$) are the sets of fixed points of \mathcal{T}_i and $d(x, \mathcal{F}) = \inf\{\|x - q\| : q \in \mathcal{F}\}$.

Let C be a subset of a metric space (\mathcal{X}, d) . A mapping $\mathcal{T} : C \rightarrow C$ is semi-compact [22] if for a sequence $\{x_n\}$ in C with $\lim_{n \rightarrow \infty} d(x_n, \mathcal{T}x_n) = 0$, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \rightarrow p \in C$.

Let C be a nonempty subset of a normed space \mathcal{X} and let $G = (V(G), E(G))$ be a directed graph such that $V(G) = C$. Then, C is said to have *Property G* (see [19]) if for each sequence in C converging weakly to $x \in C$ and $(x_n, x_{n+1}) \in E(G)$, there is a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $(x_{n_j}, x) \in E(G)$ for all $j \in \mathbb{N}$.

Remark 1. If G is transitive, then *Property G* is equivalent to the property: if $\{x_n\}$ is a sequence in C with $(x_n, x_{n+1}) \in E(G)$ such that for any subsequence $\{x_{n_j}\}$ of the sequence $\{x_n\}$ converging weakly to x in \mathcal{X} , then $(x_n, x) \in E(G)$ for all $n \in \mathbb{N}$.

Definition 1. ([23], Definition 3.1) Let \mathcal{X} be a vector space and \mathcal{D} be a nonempty subset of $\mathcal{X} \times \mathcal{X}$. Then \mathcal{D} is said to be coordinate-convex if for all $(p, u), (p, v), (u, p), (v, p) \in \mathcal{D}$ and for all $t \in [0, 1]$, we have

$$t(p, u) + (1 - t)(p, v) \in \mathcal{D} \text{ and } t(u, p) + (1 - t)(v, p) \in \mathcal{D}.$$

Remark 2. If \mathcal{D} is convex in $\mathcal{X} \times \mathcal{X}$, then \mathcal{D} is coordinate-convex in $\mathcal{X} \times \mathcal{X}$.

In the sequel, the following lemmas are needed to prove our main results.

Lemma 1. ([24]) Let \mathcal{X} be a uniformly convex Banach space, and $\{\alpha_n\}$ a sequence in $[\delta, 1 - \delta]$ for some $\delta \in (0, 1)$. Suppose that sequences $\{x_n\}$ and $\{y_n\}$ in \mathcal{X} are such that $\limsup_{n \rightarrow \infty} \|x_n\| \leq c$, $\limsup_{n \rightarrow \infty} \|y_n\| \leq c$ and $\limsup_{n \rightarrow \infty} \|\alpha_n x_n + (1 - \alpha_n)y_n\| = c$ for some $c \geq 0$. Then $\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0$.

Lemma 2. ([25]) Let \mathcal{X} be a Banach space that satisfies Opial's condition and let $\{x_n\}$ be a sequence in \mathcal{X} . Let $u, v \in \mathcal{X}$ be such that $\lim_{n \rightarrow \infty} \|x_n - u\|$ and $\lim_{n \rightarrow \infty} \|x_n - v\|$ exist. If $\{x_{n_j}\}$ and $\{x_{n_k}\}$ are subsequences of $\{x_n\}$ that converge weakly to u and v , respectively, then $u = v$.

The following lemmas show some useful *Property G* of C on a uniformly convex Banach space.

Lemma 3. ([19]) Suppose that \mathcal{X} is a Banach space having Opial's condition, C has *Property G* and let $\mathcal{T} : C \rightarrow C$ be a G -nonexpansive mapping. Then, $I - \mathcal{T}$ is G -demiclosed at 0, i.e., if $x_n \rightharpoonup x$ and $x_n - \mathcal{T}x_n \rightarrow 0$, then $x \in \mathcal{F}(T)$, where $\mathcal{F}(T)$ is the set of fixed points of \mathcal{T} .

Lemma 4. ([26]) Let C be a nonempty closed convex subset of a uniformly convex Banach space \mathcal{X} and suppose that C has *Property G*. Let \mathcal{T} be a G -nonexpansive mapping on C . Then $I - \mathcal{T}$ is G -demiclosed at 0.

The following useful result is due to [27].

Lemma 5. ([27]) Let $\{x_n\}$ be a bounded sequence in a reflexive Banach space \mathcal{X} . If for any weakly convergent subsequence $\{x_{n_j}\}$ of $\{x_n\}$, both $\{x_{n_j}\}$ and $\{x_{n_j+1}\}$ converge weakly to the same point in \mathcal{X} , then the sequence $\{x_n\}$ is weakly convergent.

3. Main results

Throughout the section, we let C be a nonempty closed convex subset of a Banach space \mathcal{X} endowed with a directed graph G such that $V(G) = C$ and $E(G)$ is coordinate-convex. We also suppose that the graph G is transitive. The mappings \mathcal{T}_i ($i = 1, 2, 3$) are G -nonexpansive from C to C with $\mathcal{F} = \mathcal{F}(\mathcal{T}_1) \cap \mathcal{F}(\mathcal{T}_2) \cap \mathcal{F}(\mathcal{T}_3)$ nonempty. For an arbitrary $x_0 \in C$, defined the sequence $\{x_n\}$ by

$$\begin{aligned} z_n &= (1 - \gamma_n)x_n + \gamma_n\mathcal{T}_3x_n, \\ y_n &= (1 - \beta_n)\mathcal{T}_3x_n + \beta_n\mathcal{T}_2z_n, \\ x_{n+1} &= (1 - \alpha_n)\mathcal{T}_1y_n + \alpha_n\mathcal{T}_2z_n, \quad n \geq 0, \end{aligned} \quad (3.1)$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are appropriate real sequences in $[0, 1]$.

Proposition 1. Let $q_0 \in \mathcal{F}$ be such that $(x_0, q_0), (q_0, x_0)$ are in $E(G)$. Then $(x_n, q_0), (z_n, q_0), (y_n, q_0), (q_0, x_n), (q_0, z_n), (q_0, y_n), (x_n, y_n), (x_n, z_n)$ and (x_n, x_{n+1}) are in $E(G)$.

Proof. We proceed by induction. Since \mathcal{T}_3 is edge-preserving and $(x_0, q_0) \in E(G)$, we have $(\mathcal{T}_3x_0, q_0) \in E(G)$. Note that

$$(z_0, q_0) = ((1 - \gamma_0)x_0 + \gamma_0\mathcal{T}_3x_0, q_0) = (1 - \gamma_0)(x_0, q_0) + \gamma_0(\mathcal{T}_3x_0, q_0).$$

Using $(x_0, q_0), (\mathcal{T}_3x_0, q_0) \in E(G)$ and the coordinate-convexity of $E(G)$, we have $(z_0, q_0) \in E(G)$. Again, by edge-preserving of \mathcal{T}_2 and $(z_0, q_0) \in E(G)$, we have $(\mathcal{T}_2z_0, q_0) \in E(G)$. We also have

$$(y_0, q_0) = ((1 - \beta_0)\mathcal{T}_3x_0 + \beta_0\mathcal{T}_2z_0, q_0) = (1 - \beta_0)(\mathcal{T}_3x_0, q_0) + \beta_0(\mathcal{T}_2z_0, q_0).$$

Since $(\mathcal{T}_3x_0, q_0), (\mathcal{T}_2z_0, q_0) \in E(G)$ by the coordinate-convexity of $E(G)$, we have $(y_0, q_0) \in E(G)$. Then, since \mathcal{T}_1 is edge-preserving, we get $(\mathcal{T}_1y_0, q_0) \in E(G)$. Note that

$$(x_1, q_0) = ((1 - \alpha_0)\mathcal{T}_1y_0 + \alpha_0\mathcal{T}_2z_0, q_0) = (1 - \alpha_0)(\mathcal{T}_1y_0, q_0) + \alpha_0(\mathcal{T}_2z_0, q_0).$$

By the coordinate-convexity of $E(G)$ and $(\mathcal{T}_2z_0, q_0), (\mathcal{T}_1y_0, q_0) \in E(G)$, we get $(x_1, q_0) \in E(G)$. Thus, by edge-preserving of \mathcal{T}_3 , $(\mathcal{T}_3x_1, q_0) \in E(G)$. Again, by the coordinate-convexity of $E(G)$ and $(\mathcal{T}_3x_1, q_0), (x_1, q_0) \in E(G)$, we have

$$(z_1, q_0) = ((1 - \gamma_1)x_1 + \gamma_1\mathcal{T}_3x_1, q_0) = (1 - \gamma_1)(x_1, q_0) + \gamma_1(\mathcal{T}_3x_1, q_0) \in E(G).$$

Then, since \mathcal{T}_2 is edge-preserving, $(\mathcal{T}_2z_1, q_0) \in E(G)$. By the coordinate-convexity of $E(G)$ and $(\mathcal{T}_2z_1, q_0), (\mathcal{T}_3x_1, q_0) \in E(G)$, we get

$$(y_1, q_0) = ((1 - \beta_1)\mathcal{T}_3x_1 + \beta_1\mathcal{T}_2z_1, q_0) = (1 - \beta_1)(\mathcal{T}_3x_1, q_0) + \beta_1(\mathcal{T}_2z_1, q_0) \in E(G),$$

and hence, $(\mathcal{T}_1y_1, q_0) \in E(G)$.

Next, we assume that $(x_k, q_0) \in E(G)$. Since \mathcal{T}_3 is edge-preserving, we get $(\mathcal{T}_3 x_k, q_0) \in E(G)$. Thus

$$(z_k, q_0) = ((1 - \gamma_k)x_k + \gamma_k \mathcal{T}_3 x_k, q_0) = (1 - \gamma_k)(x_k, q_0) + \gamma_k(\mathcal{T}_3 x_k, q_0) \in E(G),$$

since $E(G)$ is coordinate-convex. Hence, by edge-preserving of \mathcal{T}_2 and $(z_k, q_0) \in E(G)$, we have $(\mathcal{T}_2 z_k, q_0) \in E(G)$. By the coordinate-convexity of $E(G)$ and $(\mathcal{T}_3 x_k, q_0), (\mathcal{T}_2 z_k, q_0) \in E(G)$, we get

$$(y_k, q_0) = ((1 - \beta_k)\mathcal{T}_3 x_k + \beta_k \mathcal{T}_2 z_k, q_0) = (1 - \beta_k)(\mathcal{T}_3 x_k, q_0) + \beta_k(\mathcal{T}_2 z_k, q_0) \in E(G).$$

Also, since \mathcal{T}_1 is edge-preserving and $(y_k, q_0) \in E(G)$, we get $(\mathcal{T}_1 y_k, q_0) \in E(G)$. By the coordinate-convexity of $E(G)$ and $(\mathcal{T}_2 z_k, q_0), (\mathcal{T}_1 y_k, q_0) \in E(G)$, we obtain

$$(x_{k+1}, q_0) = (1 - \alpha_k)(\mathcal{T}_1 y_k + \alpha_k \mathcal{T}_2 z_k, q_0) = (1 - \alpha_k)(\mathcal{T}_1 y_k, q_0) + \alpha_k(\mathcal{T}_2 z_k, q_0) \in E(G).$$

Hence, by edge-preserving of \mathcal{T}_3 , we get $(\mathcal{T}_3 x_{k+1}, q_0) \in E(G)$, and so $(z_{k+1}, q_0) = ((1 - \gamma_{k+1})x_{k+1} + \gamma_{k+1} \mathcal{T}_3 x_{k+1}, q_0) = (1 - \gamma_{k+1})(x_{k+1}, q_0) + \gamma_{k+1}(\mathcal{T}_3 x_{k+1}, q_0) \in E(G)$, since $E(G)$ is coordinate-convex. Again, by edge-preserving of \mathcal{T}_2 , we obtain $(\mathcal{T}_2 z_{k+1}, q_0) \in E(G)$. By the coordinate-convexity of $E(G)$ and $(\mathcal{T}_3 x_{k+1}, q_0), (\mathcal{T}_2 z_{k+1}, q_0) \in E(G)$, we obtain $(y_{k+1}, q_0) = ((1 - \beta_{k+1})\mathcal{T}_3 x_{k+1} + \beta_{k+1} \mathcal{T}_2 z_{k+1}, q_0) = (1 - \beta_{k+1})(\mathcal{T}_3 x_{k+1}, q_0) + \beta_{k+1}(\mathcal{T}_2 z_{k+1}, q_0) \in E(G)$. Therefore, $(x_n, q_0), (z_n, q_0), (y_n, q_0) \in E(G)$ for all $n \geq 1$.

Since \mathcal{T}_3 is edge-preserving and $(q_0, x_0) \in E(G)$, we have $(q_0, \mathcal{T}_3 x_0) \in E(G)$, and so $(q_0, z_0) = (q_0, (1 - \gamma_0)x_0 + \gamma_0 \mathcal{T}_3 x_0) = (1 - \gamma_0)(q_0, x_0) + \gamma_0(q_0, \mathcal{T}_3 x_0) \in E(G)$. Again, since \mathcal{T}_2 is edge-preserving and $(q_0, z_0) \in E(G)$, we have $(q_0, \mathcal{T}_2 z_0) \in E(G)$. By the coordinate-convexity of $E(G)$ and $(q_0, \mathcal{T}_3 x_0), (q_0, \mathcal{T}_2 z_0) \in E(G)$, so $(q_0, y_0) = (q_0, (1 - \beta_0)\mathcal{T}_3 x_0 + \beta_0 \mathcal{T}_2 z_0) = (1 - \beta_0)(q_0, \mathcal{T}_3 x_0) + \beta_0(q_0, \mathcal{T}_2 z_0) \in E(G)$.

Using a similar argument, we can show that $(q_0, x_n), (q_0, z_n), (q_0, y_n) \in E(G)$ under the assumption that $(q_0, x_0), (q_0, z_0), (q_0, y_0) \in E(G)$. By transitivity of G , we get $(x_n, y_n), (x_n, z_n), (x_n, x_{n+1}) \in E(G)$. This completes the proof. \square

Lemma 6. Let \mathcal{X} be a uniformly convex Banach space. Suppose that $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are real sequences in $[\delta, 1 - \delta]$ for some $\delta \in (0, 1)$ and $(x_0, q_0), (q_0, x_0) \in E(G)$ for arbitrary $x_0 \in \mathcal{C}$ and $q_0 \in \mathcal{F}$. Then

- (i) $\lim_{n \rightarrow \infty} \|x_n - q_0\|$ exists;
- (ii) $\lim_{n \rightarrow \infty} \|x_n - \mathcal{T}_1 x_n\| = \lim_{n \rightarrow \infty} \|x_n - \mathcal{T}_2 x_n\| = \lim_{n \rightarrow \infty} \|x_n - \mathcal{T}_3 x_n\| = 0$.

Proof. (i) Let $q_0 \in \mathcal{F}$. By Proposition 1, we have $(x_n, q_0), (y_n, q_0)$ and $(z_n, q_0) \in E(G)$. Then, by G -nonexpansiveness of $\mathcal{T}_i (i = 1, 2, 3)$ and using (3.1), we have

$$\begin{aligned} \|z_n - q_0\| &= \|(1 - \gamma_n)x_n + \gamma_n \mathcal{T}_3 x_n - q_0\| \\ &= \|(1 - \gamma_n)(x_n - q_0) + \gamma_n(\mathcal{T}_3 x_n - q_0)\| \\ &\leq (1 - \gamma_n)\|x_n - q_0\| + \gamma_n\|\mathcal{T}_3 x_n - q_0\| \\ &\leq (1 - \gamma_n)\|x_n - q_0\| + \gamma_n\|x_n - q_0\| \\ &= \|x_n - q_0\|, \end{aligned} \tag{3.2}$$

$$\|y_n - q_0\| = \|(1 - \beta_n)\mathcal{T}_3 x_n + \beta_n \mathcal{T}_2 z_n - q_0\|$$

$$\begin{aligned}
&= \|(1 - \beta_n)(\mathcal{T}_3 x_n - q_0) + \beta_n(\mathcal{T}_2 z_n - q_0)\| \\
&\leq (1 - \beta_n)\|\mathcal{T}_3 x_n - q_0\| + \beta_n\|\mathcal{T}_2 z_n - q_0\| \\
&\leq (1 - \beta_n)\|x_n - q_0\| + \beta_n\|z_n - q_0\| \\
&\leq (1 - \beta_n)\|x_n - q_0\| + \beta_n\|x_n - q_0\| \\
&= \|x_n - q_0\|,
\end{aligned} \tag{3.3}$$

and so

$$\begin{aligned}
\|x_{n+1} - q_0\| &= \|(1 - \alpha_n)\mathcal{T}_1 y_n + \alpha_n \mathcal{T}_2 z_n - q_0\| \\
&= \|(1 - \alpha_n)(\mathcal{T}_1 y_n - q_0) + \alpha_n(\mathcal{T}_2 z_n - q_0)\| \\
&\leq (1 - \alpha_n)\|\mathcal{T}_1 y_n - q_0\| + \alpha_n\|\mathcal{T}_2 z_n - q_0\| \\
&\leq (1 - \alpha_n)\|y_n - q_0\| + \alpha_n\|z_n - q_0\| \\
&\leq (1 - \alpha_n)\|x_n - q_0\| + \alpha_n\|x_n - q_0\| \\
&\leq \|x_n - q_0\|.
\end{aligned} \tag{3.4}$$

As $\{\|x_n - q_0\| : n \in \mathbb{N}\}$ is decreasing sequence which is obviously bounded below. Therefore $\lim_{n \rightarrow \infty} \|x_n - q_0\|$ exists. In particular, the sequence $\{x_n\}$ is bounded.

(ii) Assume that $\lim_{n \rightarrow \infty} \|x_n - q_0\| = c$. If $c = 0$, then by G -nonexpansiveness of \mathcal{T}_i ($i = 1, 2, 3$), we get

$$\begin{aligned}
\|x_n - \mathcal{T}_i x_n\| &\leq \|x_n - q_0\| + \|q_0 - \mathcal{T}_i x_n\| \\
&\leq \|x_n - q_0\| + \|q_0 - x_n\|.
\end{aligned}$$

Therefore, the result follows. Suppose that $c > 0$. Taking the \limsup on both sides in the inequality (3.3), we obtain

$$\limsup_{n \rightarrow \infty} \|y_n - q_0\| \leq \limsup_{n \rightarrow \infty} \|x_n - q_0\| = c. \tag{3.5}$$

In addition, by G -nonexpansiveness of \mathcal{T}_1 , we have $\|\mathcal{T}_1 y_n - q_0\| \leq \|y_n - q_0\|$, taking the \limsup on both sides in this inequality and using (3.5), we obtain

$$\limsup_{n \rightarrow \infty} \|\mathcal{T}_1 y_n - q_0\| \leq c. \tag{3.6}$$

Taking the \limsup on both sides in the inequality (3.2), we obtain

$$\limsup_{n \rightarrow \infty} \|z_n - q_0\| \leq \limsup_{n \rightarrow \infty} \|x_n - q_0\| = c. \tag{3.7}$$

In addition, by G -nonexpansiveness of \mathcal{T}_2 , we have $\|\mathcal{T}_2 z_n - q_0\| \leq \|z_n - q_0\|$, taking the \limsup on both sides in this inequality and using (3.7), we obtain

$$\limsup_{n \rightarrow \infty} \|\mathcal{T}_2 z_n - q_0\| \leq c. \tag{3.8}$$

Since $\lim_{n \rightarrow \infty} \|x_{n+1} - q_0\| = c$. Letting $n \rightarrow \infty$ in the inequality (3.4), we have

$$\lim_{n \rightarrow \infty} \|(1 - \alpha_n)(\mathcal{T}_1 y_n - q_0) + \alpha_n(\mathcal{T}_2 z_n - q_0)\| = c. \quad (3.9)$$

By using (3.6), (3.8) and (3.9) and Lemma 1, we have

$$\lim_{n \rightarrow \infty} \|\mathcal{T}_2 z_n - \mathcal{T}_1 y_n\| = 0. \quad (3.10)$$

Moreover, we see that

$$\begin{aligned} \|x_{n+1} - q_0\| &= \|(1 - \alpha_n)\mathcal{T}_1 y_n + \alpha_n \mathcal{T}_2 z_n - q_0\| \\ &\leq (1 - \alpha_n)\|\mathcal{T}_1 y_n - q_0\| + \alpha_n\|\mathcal{T}_2 z_n - q_0\| \\ &\leq (1 - \alpha_n)(\|\mathcal{T}_1 y_n - \mathcal{T}_2 z_n\| + \|\mathcal{T}_2 z_n - q_0\|) + \alpha_n(\|\mathcal{T}_2 z_n - q_0\|) \\ &= (1 - \alpha_n)\|\mathcal{T}_1 y_n - \mathcal{T}_2 z_n\| + (1 - \alpha_n)\|\mathcal{T}_2 z_n - q_0\| + \alpha_n(\|\mathcal{T}_2 z_n - q_0\|) \\ &= (1 - \alpha_n)\|\mathcal{T}_1 y_n - \mathcal{T}_2 z_n\| + \|\mathcal{T}_2 z_n - q_0\|. \end{aligned} \quad (3.11)$$

Using (3.10) and (3.11), we have

$$\liminf_{n \rightarrow \infty} \|\mathcal{T}_2 z_n - q_0\| \geq c, \quad (3.12)$$

and so by (3.8) and (3.12), we have

$$\lim_{n \rightarrow \infty} \|\mathcal{T}_2 z_n - q_0\| = c. \quad (3.13)$$

On the other hand,

$$\begin{aligned} \|\mathcal{T}_2 z_n - q_0\| &\leq \|\mathcal{T}_2 z_n - \mathcal{T}_1 y_n\| + \|\mathcal{T}_1 y_n - q_0\| \\ &\leq \|\mathcal{T}_2 z_n - \mathcal{T}_1 y_n\| + \|y_n - q_0\|, \end{aligned}$$

and this yields to

$$\liminf_{n \rightarrow \infty} \|y_n - q_0\| \geq c. \quad (3.14)$$

From (3.5) and (3.14),

$$\lim_{n \rightarrow \infty} \|y_n - q_0\| = c. \quad (3.15)$$

In addition, by G-noneapansives of \mathcal{T}_3 we have $\|\mathcal{T}_3 x_n - q_0\| \leq \|x_n - q_0\|$, taking the lim sup on both sides in this inequality, we obtain

$$\limsup_{n \rightarrow \infty} \|\mathcal{T}_3 x_n - q_0\| \leq c. \quad (3.16)$$

By (3.3) and (3.15), we have

$$\lim_{n \rightarrow \infty} \|(1 - \beta_n)(\mathcal{T}_3 x_n - q_0) + \beta_n(\mathcal{T}_2 z_n - q_0)\| = c. \quad (3.17)$$

Using (3.8), (3.16) and (3.17) and Lemma 1,

$$\lim_{n \rightarrow \infty} \|\mathcal{T}_2 z_n - \mathcal{T}_3 x_n\| = 0. \quad (3.18)$$

Moreover,

$$\begin{aligned} \|x_{n+1} - q_0\| &= \|(1 - \alpha_n)\mathcal{T}_1 y_n + \alpha_n \mathcal{T}_2 z_n - q_0\| \\ &\leq (1 - \alpha_n)\|\mathcal{T}_1 y_n - q_0\| + \alpha_n \|\mathcal{T}_2 z_n - q_0\| \\ &\leq (1 - \alpha_n)\|\mathcal{T}_1 y_n - q_0\| + \alpha_n (\|\mathcal{T}_2 z_n - \mathcal{T}_1 y_n\| + \|\mathcal{T}_1 y_n - q_0\|) \\ &= (1 - \alpha_n)\|\mathcal{T}_1 y_n - q_0\| + \alpha_n \|\mathcal{T}_2 z_n - \mathcal{T}_1 y_n\| + \alpha_n \|\mathcal{T}_1 y_n - q_0\| \\ &= \|\mathcal{T}_1 y_n - q_0\| + \alpha_n \|\mathcal{T}_2 z_n - \mathcal{T}_1 y_n\|. \end{aligned} \quad (3.19)$$

Using (3.10) and (3.19), we have

$$\liminf_{n \rightarrow \infty} \|\mathcal{T}_1 y_n - q_0\| \geq c. \quad (3.20)$$

and so by (3.6) and (3.20), we get

$$\lim_{n \rightarrow \infty} \|\mathcal{T}_1 y_n - q_0\| = c. \quad (3.21)$$

On the other hand,

$$\begin{aligned} \|\mathcal{T}_1 y_n - q_0\| &\leq \|\mathcal{T}_1 y_n - \mathcal{T}_2 z_n\| + \|\mathcal{T}_2 z_n - q_0\| \\ &\leq \|\mathcal{T}_1 y_n - \mathcal{T}_2 z_n\| + \|z_n - q_0\|, \end{aligned}$$

and this yields to

$$\liminf_{n \rightarrow \infty} \|z_n - q_0\| \geq c. \quad (3.22)$$

From (3.7) and (3.22), we get

$$\lim_{n \rightarrow \infty} \|z_n - q_0\| = c. \quad (3.23)$$

From (3.2) and (3.23), we have

$$\lim_{n \rightarrow \infty} \|(1 - \gamma_n)(x_n - q_0) + \gamma_n(\mathcal{T}_3 x_n - q_0)\| = c. \quad (3.24)$$

By using (3.16), (3.24) and Lemma 1,

$$\lim_{n \rightarrow \infty} \|\mathcal{T}_3 x_n - x_n\| = 0. \quad (3.25)$$

Thus, it follows from (3.25) that

$$\begin{aligned} \|z_n - x_n\| &= \|(1 - \gamma_n)x_n + \gamma_n\mathcal{T}_3 x_n - x_n\| \\ &\leq \gamma_n \|\mathcal{T}_3 x_n - x_n\| \\ &\rightarrow 0 \text{ (as } n \rightarrow \infty\text{)}. \end{aligned} \quad (3.26)$$

Using (3.18), (3.25) and (3.26) together with G -nonexpansiveness of \mathcal{T}_2 ,

$$\begin{aligned} \|\mathcal{T}_2 x_n - x_n\| &= \|\mathcal{T}_2 x_n - \mathcal{T}_2 z_n + \mathcal{T}_2 z_n - x_n\| \\ &\leq \|x_n - z_n\| + \|\mathcal{T}_2 z_n - \mathcal{T}_3 x_n\| + \|\mathcal{T}_3 x_n - x_n\| \\ &\rightarrow 0 \text{ (as } n \rightarrow \infty\text{)}. \end{aligned} \quad (3.27)$$

From (3.18) and (3.25), we have

$$\begin{aligned} \|y_n - x_n\| &= \|(1 - \beta_n)(\mathcal{T}_3 x_n - x_n) + \beta_n(\mathcal{T}_2 z_n - x_n)\| \\ &\leq (1 - \beta_n)\|\mathcal{T}_3 x_n - x_n\| + \beta_n\|\mathcal{T}_2 z_n - x_n\| \\ &\leq (1 - \beta_n)\|\mathcal{T}_3 x_n - x_n\| + \beta_n(\|\mathcal{T}_2 z_n - \mathcal{T}_3 x_n\| + \|\mathcal{T}_3 x_n - x_n\|) \\ &= (1 - \beta_n)\|\mathcal{T}_3 x_n - x_n\| + \beta_n\|\mathcal{T}_2 z_n - \mathcal{T}_3 x_n\| + \beta_n\|\mathcal{T}_3 x_n - x_n\| \\ &= \|\mathcal{T}_3 x_n - x_n\| + \beta_n\|\mathcal{T}_2 z_n - \mathcal{T}_3 x_n\| \\ &\rightarrow 0 \text{ (as } n \rightarrow \infty\text{)}. \end{aligned} \quad (3.28)$$

Using (3.10), (3.18), (3.25) and (3.28) together with G -nonexpansiveness of \mathcal{T}_1 , we have

$$\begin{aligned} \|\mathcal{T}_1 x_n - x_n\| &\leq \|\mathcal{T}_1 x_n - \mathcal{T}_1 y_n\| + \|\mathcal{T}_1 y_n - x_n\| \\ &\leq \|x_n - y_n\| + \|\mathcal{T}_1 y_n - \mathcal{T}_2 z_n\| + \|\mathcal{T}_2 z_n - \mathcal{T}_3 x_n\| + \|\mathcal{T}_3 x_n - x_n\| \\ &\rightarrow 0 \text{ (as } n \rightarrow \infty\text{)}. \end{aligned}$$

Therefore, we conclude that $\lim_{n \rightarrow \infty} \|x_n - \mathcal{T}_1 x_n\| = \lim_{n \rightarrow \infty} \|x_n - \mathcal{T}_2 x_n\| = \lim_{n \rightarrow \infty} \|x_n - \mathcal{T}_3 x_n\| = 0$. This completes the proof. \square

We now prove the weak convergence of the sequence generated by the new modified three-step iteration method (3.1) for three G -nonexpansive mappings in a uniformly convex Banach space satisfying Opial's condition.

Theorem 1. Let \mathcal{X} be a uniformly convex Banach space which satisfies Opial's condition and C has Property G . Suppose that $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are real sequences in $[\delta, 1 - \delta]$ for some $\delta \in (0, 1)$. If $(x_0, q_0), (q_0, x_0) \in E(G)$ for arbitrary $x_0 \in C$ and $q_0 \in \mathcal{F}$, then $\{x_n\}$ converges weakly to a common fixed point of $\mathcal{T}_1, \mathcal{T}_2$ and \mathcal{T}_3 .

Proof. Let $q_0 \in \mathcal{F}$ be such that $(x_0, q_0), (q_0, x_0) \in E(G)$. From Lemma 6 (i), $\lim_{n \rightarrow \infty} \|x_n - q_0\|$ exists, thus $\{x_n\}$ is bounded. It follows from Lemma 6 (ii) that $\lim_{n \rightarrow \infty} \|x_n - \mathcal{T}_1 x_n\| = \lim_{n \rightarrow \infty} \|x_n - \mathcal{T}_2 x_n\| = \lim_{n \rightarrow \infty} \|x_n - \mathcal{T}_3 x_n\| = 0$. Since \mathcal{X} is uniformly convex and $\{x_n\}$ is bounded, we may assume that $x_n \rightarrow u$ as $n \rightarrow \infty$, without loss of generality. By Lemma 3, we have $u \in F$. Suppose that subsequences $\{x_{n_k}\}$ and $\{x_{n_j}\}$ of $\{x_n\}$ converge weakly to u and v , respectively. By Lemma 6 (ii), we obtain that $\|x_{n_k} - \mathcal{T}_i x_{n_k}\| \rightarrow 0$ and $\|x_{n_j} - \mathcal{T}_i x_{n_j}\| \rightarrow 0$ as $k, j \rightarrow \infty$. Using Lemma 3, we have $u, v \in \mathcal{F}$. By Lemma 6 (i), $\lim_{n \rightarrow \infty} \|x_n - u\|$ and $\lim_{n \rightarrow \infty} \|x_n - v\|$ exist. It follows from Lemma 3 that $u = v$. Therefore, $\{x_n\}$ converges weakly to a common fixed point of $\mathcal{T}_1, \mathcal{T}_2$ and \mathcal{T}_3 . \square

It is worth noting that the Opial's condition has remained crucial in proving weak convergence theorems. However, each l^p ($1 \leq p < \infty$) satisfies the Opial's condition, while L^p do not have the property unless $p = 2$.

Next, we deal with the weak convergence of the sequence $\{x_n\}$ generated by (3.1) for two G -nonexpansive mappings without assuming the Opial's condition in a uniformly convex Banach space with a directed graph.

Theorem 2. Let \mathcal{X} be a uniformly convex Banach space. Suppose that C has Property G , $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are real sequences in $[\delta, 1 - \delta]$ for some $\delta \in (0, 1)$, \mathcal{F} is dominated by x_0 and \mathcal{F} dominates x_0 . If $(x_0, q_0), (q_0, x_0) \in E(G)$ for arbitrary $x_0 \in C$ and $q_0 \in \mathcal{F}$, then $\{x_n\}$ converges weakly to a common fixed point of $\mathcal{T}_1, \mathcal{T}_2$ and \mathcal{T}_3 .

Proof. Let $q_0 \in \mathcal{F}$ be such that $(x_0, q_0), (q_0, x_0)$ are in $E(G)$. From Lemma 6 (i), $\lim_{n \rightarrow \infty} \|x_n - q_0\|$ exists, so $\{x_n\}$ is bounded in C . Since C is nonempty closed convex subset of a uniformly convex Banach space \mathcal{X} , it is weakly compact and hence there exists a subsequence $\{x_{n_j}\}$ of the sequence $\{x_n\}$ such that $\{x_{n_j}\}$ converges weakly to some point $p \in C$. By Lemma 6 (ii) we obtain that

$$\lim_{j \rightarrow \infty} \|x_{n_j} - \mathcal{T}_1 x_{n_j}\| = \lim_{j \rightarrow \infty} \|x_{n_j} - \mathcal{T}_2 x_{n_j}\| = \lim_{j \rightarrow \infty} \|x_{n_j} - \mathcal{T}_3 x_{n_j}\| = 0. \quad (3.29)$$

In addition, $\|y_n - z_n\| \leq \|y_n - x_n\| + \|x_n - z_n\|$. Using (3.26) and (3.28), we have

$$\lim_{n \rightarrow \infty} \|y_n - z_n\| = 0. \quad (3.30)$$

From (3.26) and G -nonexpansiveness of \mathcal{T}_2 ,

$$\|\mathcal{T}_2 z_n - z_n\| \leq \|\mathcal{T}_2 z_n - \mathcal{T}_2 x_n\| + \|\mathcal{T}_2 x_n - x_n\| + \|x_n - z_n\|$$

$$\begin{aligned} &\leq \|z_n - x_n\| + \|\mathcal{T}_2 x_n - x_n\| + \|x_n - z_n\| \\ &\rightarrow 0 \text{ (as } n \rightarrow \infty). \end{aligned} \quad (3.31)$$

Thus, it follows from (3.10), (3.30) and (3.31) that

$$\begin{aligned} \|\mathcal{T}_1 y_n - y_n\| &\leq \|\mathcal{T}_1 y_n - \mathcal{T}_2 z_n\| + \|\mathcal{T}_2 z_n - z_n\| + \|y_n - z_n\| \\ &\rightarrow 0 \text{ (as } n \rightarrow \infty). \end{aligned} \quad (3.32)$$

Using Lemma 4, we have $I - \mathcal{T}_1$, $I - \mathcal{T}_2$ and $I - \mathcal{T}_3$ are G -demiclosed at 0 so that $p \in \mathcal{F}$. To complete the proof it suffices to show that $\{x_n\}$ converges weakly to p . To this end we need to show that $\{x_n\}$ satisfies the hypothesis of Lemma 5. Let $\{x_{n_j}\}$ be a subsequence of $\{x_n\}$ which converges weakly to some $q \in \mathcal{C}$. By similar arguments as above q is in \mathcal{F} . Now for each $j \geq 1$, using (3.1), we have

$$x_{n_{j+1}} = (1 - \alpha_{n_j})\mathcal{T}_1 y_{n_j} + \alpha_{n_j}\mathcal{T}_2 z_{n_j}. \quad (3.33)$$

It follows from (3.29) that

$$\mathcal{T}_3 x_{n_j} = (\mathcal{T}_3 x_{n_j} - x_{n_j}) + x_{n_j} \rightharpoonup q. \quad (3.34)$$

Now from (3.1) and (3.34),

$$z_{n_j} = (1 - \gamma_{n_j})x_{n_j} + \gamma_{n_j}\mathcal{T}_3 x_{n_j} \rightharpoonup q. \quad (3.35)$$

Using (3.31) and (3.35), we have

$$\mathcal{T}_2 z_{n_j} = (\mathcal{T}_2 z_{n_j} - z_{n_j}) + z_{n_j} \rightharpoonup q. \quad (3.36)$$

Now from (3.1), (3.34) and (3.36),

$$y_{n_j} = (1 - \beta_{n_j})\mathcal{T}_3 x_{n_j} + \beta_{n_j}\mathcal{T}_2 z_{n_j} \rightharpoonup q. \quad (3.37)$$

Also from (3.32) and (3.37),

$$\mathcal{T}_1 y_{n_j} = (\mathcal{T}_1 y_{n_j} - y_{n_j}) + y_{n_j} \rightharpoonup q. \quad (3.38)$$

It follows from (3.33), (3.36) and (3.38) that

$$x_{n_{j+1}} \rightharpoonup q.$$

Therefore, the sequence $\{x_n\}$ satisfies the hypothesis of Lemma 5 which in turn implies that $\{x_n\}$ weakly converges to q so that $p = q$. This completes the proof. \square

The strong convergence of the sequence generated by the new modified three-step iteration method (3.1) for three G -nonexpansive mappings in a uniformly convex Banach space with a directed graph is discussed in the rest of this section.

Theorem 3. Let X be a uniformly convex Banach space. Suppose that $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are real sequences in $[\delta, 1 - \delta]$ for some $\delta \in (0, 1)$, $\mathcal{T}_i (i = 1, 2, 3)$ satisfy condition (C), \mathcal{F} is dominated by x_0 and \mathcal{F} dominates x_0 . Then $\{x_n\}$ converges strongly to a common fixed point of $\mathcal{T}_1, \mathcal{T}_2$ and \mathcal{T}_3 .

Proof. By Lemma 6 (i), $\lim_{n \rightarrow \infty} \|x_n - q\|$ exists and so $\lim_{n \rightarrow \infty} d(x_n, \mathcal{F})$ exists for any $q \in \mathcal{F}$. Also by Lemma 6 (ii), $\lim_{n \rightarrow \infty} \|x_n - \mathcal{T}_1 x_n\| = \lim_{n \rightarrow \infty} \|x_n - \mathcal{T}_2 x_n\| = \lim_{n \rightarrow \infty} \|x_n - \mathcal{T}_3 x_n\| = 0$. It follows from condition (C) that $\lim_{n \rightarrow \infty} f(d(x_n, \mathcal{F})) = 0$. Since $f : [0, \infty) \rightarrow [0, \infty)$ is a nondecreasing function satisfying $f(0) = 0$, $f(r) > 0$ for all $r \in (0, \infty)$, we obtain that $\lim_{n \rightarrow \infty} d(x_n, \mathcal{F}) = 0$. Hence, we can find a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ and a sequence $\{u_j\} \subset \mathcal{F}$ such that $\|x_{n_j} - u_j\| \leq \frac{1}{2^j}$. Put $n_{j+1} = n_j + k$ for some $k \geq 1$. Then

$$\|x_{n_{j+1}} - u_j\| \leq \|x_{n_j+k-1} - u_j\| \leq \|x_{n_j} - u_j\| \leq \frac{1}{2^j}.$$

We obtain that $\|u_{j+1} - u_j\| \leq \frac{3}{2^{j+1}}$, thus $\{u_j\}$ is a Cauchy sequence. We assume that $u_j \rightarrow q_0 \in C$ as $j \rightarrow \infty$. Since \mathcal{F} is closed, we get $q_0 \in \mathcal{F}$. Therefore, we have $x_{n_j} \rightarrow q_0 \in C$ as $j \rightarrow \infty$. Since $\lim_{n \rightarrow \infty} \|x_n - q_0\|$ exists, we obtain $x_n \rightarrow q_0$. This completes the proof. \square

Theorem 4. Let X be a uniformly convex Banach space. Suppose that C has *Property G*, $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are real sequences in $[\delta, 1 - \delta]$ for some $\delta \in (0, 1)$, \mathcal{F} is dominated by x_0 and \mathcal{F} dominates x_0 . If one of $\mathcal{T}_i (i = 1, 2, 3)$ is semi-compact, then $\{x_n\}$ converges strongly to a common fixed point of $\mathcal{T}_1, \mathcal{T}_2$ and \mathcal{T}_3 .

Proof. It follows from Lemma 6 that $\{x_n\}$ is bounded and $\lim_{n \rightarrow \infty} \|x_n - \mathcal{T}_1 x_n\| = \lim_{n \rightarrow \infty} \|x_n - \mathcal{T}_2 x_n\| = \lim_{n \rightarrow \infty} \|x_n - \mathcal{T}_3 x_n\| = 0$. Since one of $\mathcal{T}_1, \mathcal{T}_2$ and \mathcal{T}_3 is semi-compact, then there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \rightarrow q \in C$ as $j \rightarrow \infty$. Since C has *Property G* and transitivity of graph G , we obtain $(x_{n_j}, q) \in E(G)$. Notice that, for each $i \in \{1, 2, 3\}$, $\lim_{j \rightarrow \infty} \|x_{n_j} - \mathcal{T}_i x_{n_j}\| = 0$. Then

$$\begin{aligned} \|q - \mathcal{T}_i q\| &\leq \|q - x_{n_j}\| + \|x_{n_j} - \mathcal{T}_i x_{n_j}\| + \|\mathcal{T}_i x_{n_j} - \mathcal{T}_i q\| \\ &\leq \|q - x_{n_j}\| + \|x_{n_j} - \mathcal{T}_i x_{n_j}\| + \|x_{n_j} - q\| \\ &\rightarrow 0 \text{ (as } j \rightarrow \infty\text{)}. \end{aligned}$$

Hence $q \in \mathcal{F}$. Thus $\lim_{n \rightarrow \infty} d(x_n, \mathcal{F})$ exists by Theorem 3. We note that $d(x_{n_j}, \mathcal{F}) \leq d(x_{n_j}, q) \rightarrow 0$ as $j \rightarrow \infty$, hence $\lim_{n \rightarrow \infty} d(x_n, \mathcal{F}) = 0$. It follows, as in the proof of Theorem 3, that $\{x_n\}$ converges strongly to a common fixed point of $\mathcal{T}_1, \mathcal{T}_2$ and \mathcal{T}_3 . This completes the proof. \square

4. Numerical examples

For supporting our main theorem, the approximate solutions of common fixed-point problems for a class of G -nonexpansive mapping on the closed interval are discussed. The following definition will be useful in this context.

Definition 2. [28] Suppose $\{\zeta_n\}$ is a sequence that converges to ζ , with $\zeta_n \neq \zeta$ for all n . If positive constants ν and ψ exist with

$$\lim_{n \rightarrow \infty} \frac{|\zeta_{n+1} - \zeta|}{|\zeta_n - \zeta|^\psi} = \nu,$$

then $\{\zeta_n\}$ converges to ζ of order ψ , with asymptotic error constant ν . If $\psi = 1$ (and $\nu < 1$), the sequence is linearly convergent.

In 2011, Phuengrattana and Suantai [29] showed that the Ishikawa iteration converges faster than the Mann iteration for a class of continuous functions on the closed interval in a real line. In order to study the order of convergence of a real sequence $\{\zeta_n\}$ converging to ζ , we usually use the well-known terminology in numerical analysis, see [28], for example.

Example 1. Let $X = \mathbb{R}$, $C = [0, 2]$ and $G = (V(G), E(G))$ be a directed graph defined by $V(G) = C$ and $(x, y) \in E(G)$ if and only if $0.50 \leq x \neq y \leq 1.70$ or $x = y \in C$. Then $E(G)$ is coordinate-convex and $\{(x, x) : x \in V(G)\} \subset E(G)$. Define mappings $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3 : C \rightarrow C$ where

$$\mathcal{T}_1 x = \frac{2}{3} \arcsin(x - 1) + 1, \quad \mathcal{T}_2 x = \frac{1}{3} \tan(x - 1) + 1, \quad \mathcal{T}_3 x = \sqrt{x},$$

for any $x \in C$. It is easy to show that $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3$ are G -nonexpansive but $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3$ are not nonexpansive because

$$|\mathcal{T}_1 x - \mathcal{T}_1 y| > 0.50 = |x - y|,$$

$$|\mathcal{T}_2 u - \mathcal{T}_2 v| > 0.07 = |u - v|,$$

and

$$|\mathcal{T}_3 p - \mathcal{T}_3 q| > 0.45 = |p - q|,$$

when $x = 1.95, y = 1.45, u = 0.08, v = 0.01, p = 0.5$ and $q = 0.05$.

Let

$$\alpha_n = \frac{n+1}{5n+3}, \beta_n = \frac{n+2}{\sqrt{8n+5}} \text{ and } \gamma_n = \frac{n+4}{10n+7}. \quad (4.1)$$

Let $\{x_n\}$ be a sequence generated by (3.1) and $\{y_n\}, \{z_n\}$ be sequences generated by the three-step Noor iteration (1.1) and SP-iteration, respectively. Example 1 shows the convergence behavior of these three comparative methods and $x = 1$ is a common fixed point of $\mathcal{T}_1, \mathcal{T}_2$ and \mathcal{T}_3 . We choose $z_1 = y_1 = w_1 = x_1 = 1.65$ and set the relative error $|\zeta_n - x|/|x| < 1.00e - 08$ as stopping criterion where $\{\zeta_n\}$ be all of comparative sequences. The results of three comparative algorithms with the permutation of the operators $\mathcal{T}_1, \mathcal{T}_2$ and \mathcal{T}_3 are presented. However, the permutation of the group of these three operators gives 27 case studies that we need to consider. But we are interested in the absence of duplicate operator being applied to the proposed algorithm (3.1). That is the following 6 cases of three comparative algorithms consists of the proposed algorithm, three-step Noor iteration (1.1) and SP-iteration (1.2) are demonstrated and discussed.

Case I. Three comparative algorithms with the operator order $\mathcal{T}_1 - \mathcal{T}_2 - \mathcal{T}_3$.

Case II. Three comparative algorithms with the operator order $\mathcal{T}_1 - \mathcal{T}_3 - \mathcal{T}_2$.

Case III. Three comparative algorithms with the operator order $\mathcal{T}_2 - \mathcal{T}_1 - \mathcal{T}_3$.

Case IV. Three comparative algorithms with the operator order $\mathcal{T}_2 - \mathcal{T}_3 - \mathcal{T}_1$.

Case V. Three comparative algorithms with the operator order $\mathcal{T}_3-\mathcal{T}_1-\mathcal{T}_2$.

Case VI. Three comparative algorithms with the operator order $\mathcal{T}_3-\mathcal{T}_2-\mathcal{T}_1$.

All numerical experiments for common fixed point solution by using three comparative methods with the permutation of the operators $\mathcal{T}_1, \mathcal{T}_2$ and \mathcal{T}_3 are shown in Figures 1–6. Each figure contains three graphs showing the behavior of numerical solution sequences, relative error sequences and asymptotic error sequences for three comparative methods, respectively. The first two graphs of each figure shows that all sequences generated by these three comparative methods converge to a common fixed point solution $x = 1$ and the relative errors of these three comparative methods are also decreased to zero when the number of iteration increased. The remaining graph of each figure shows the tendency of the asymptotic error constant σ for sequence $\{\zeta_n\}$ results from the formula $|\zeta_{n+1} - 1| / |\zeta_n - 1|$ of three-step Noor, SP and proposed methods.

It can be seen from the remaining graph of each figure that all methods are linearly convergent. This message is being made more confident by using Definition 2. Since, the smaller of asymptotic error constant gives us the faster convergence of the considering sequence then it can be concluded from the remaining graph that the proposed method converge faster than three-step Noor and SP methods.

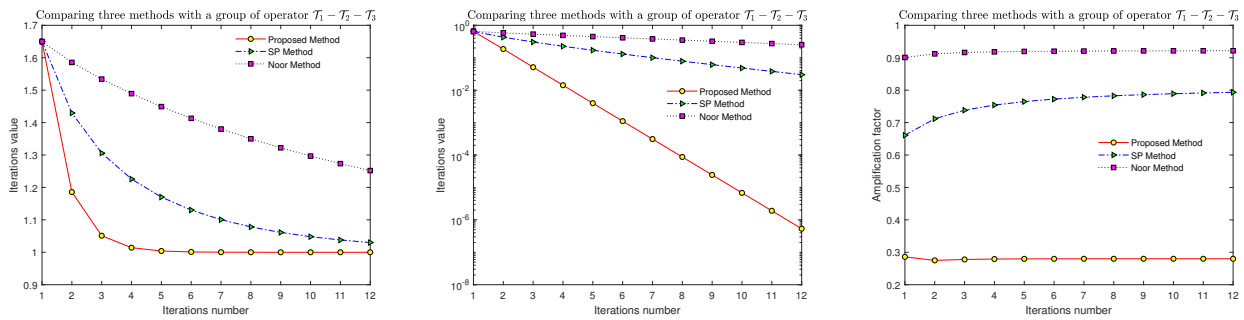


Figure 1. Numerical solution, relative error and asymptotic error constant of three comparative methods for case I.

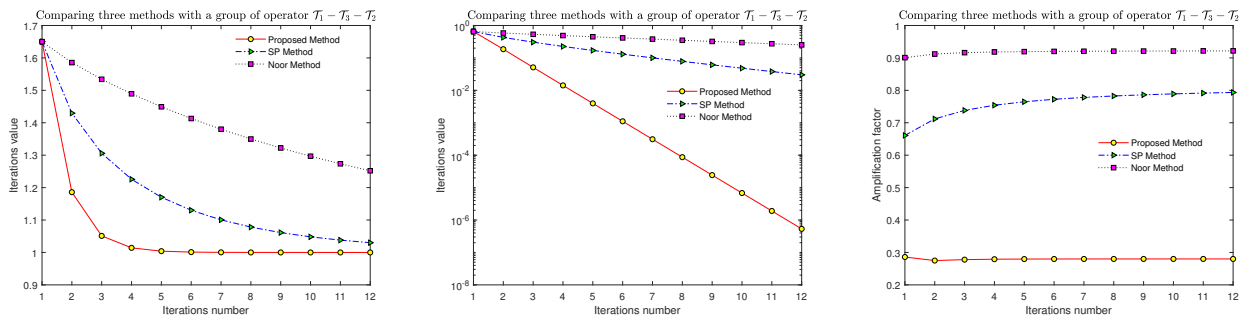


Figure 2. Numerical solution, relative error and asymptotic error constant of three comparative methods for case II.

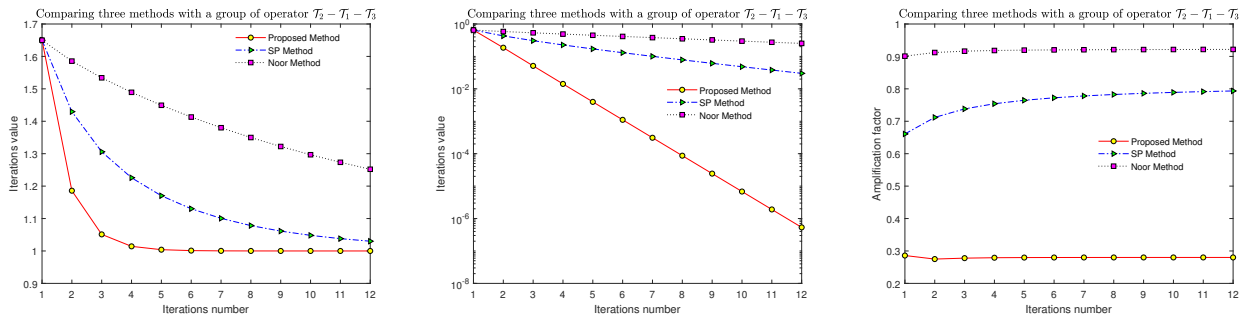


Figure 3. Numerical solution, relative error and asymptotic error constant of three comparative methods for case III.

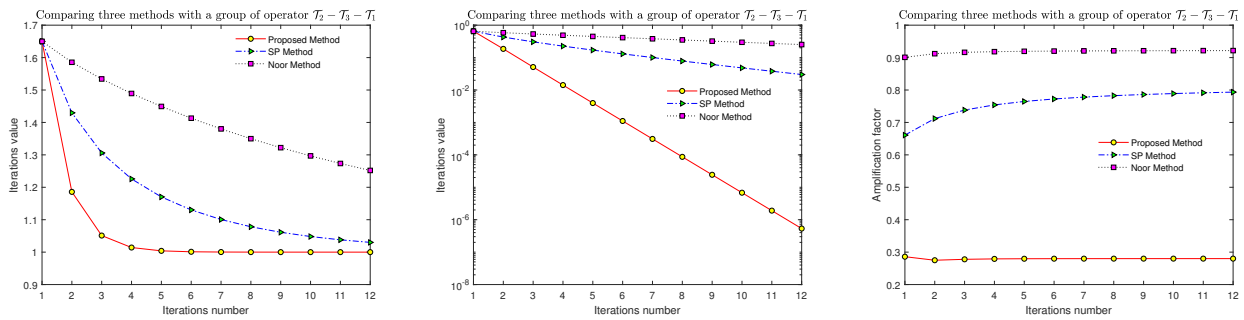


Figure 4. Numerical solution, relative error and asymptotic error constant of three comparative methods for case IV.

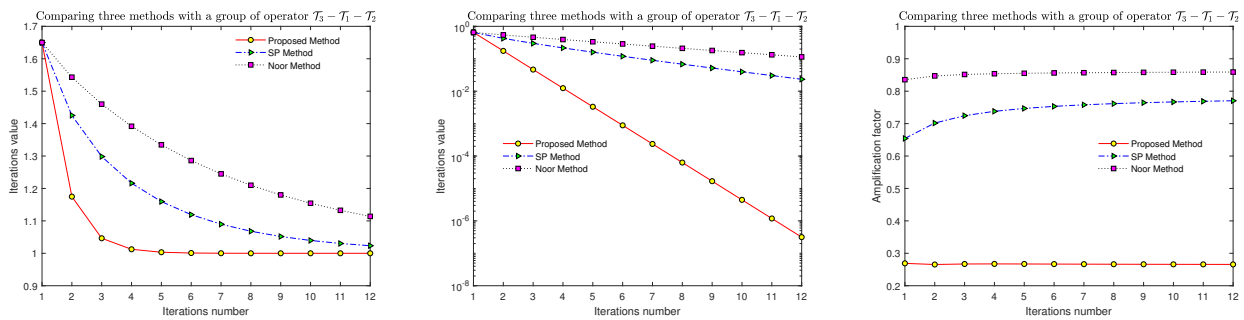


Figure 5. Numerical solution, relative error and asymptotic error constant of three comparative methods for case V.

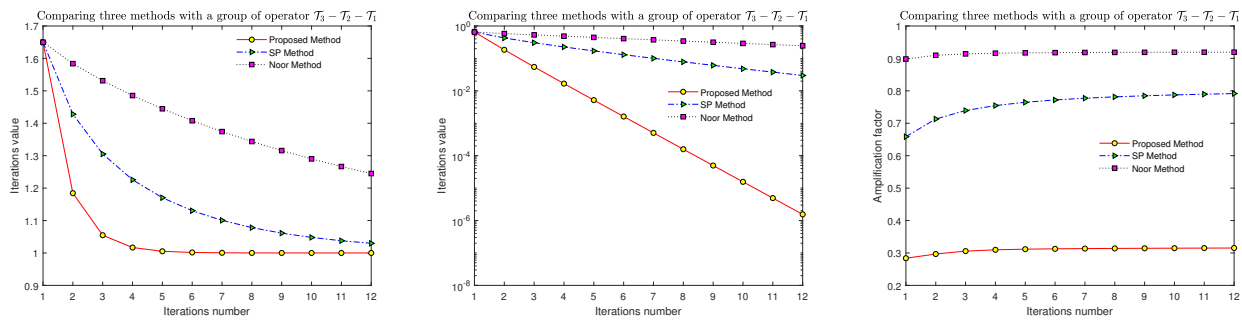


Figure 6. Numerical solution, relative error and asymptotic error constant of three comparative methods for case VI.

Thus, it can be concluded from Figures 1–6 that the proposed method with the absence of duplicate operator being applied to the proposed algorithm give us faster convergence compare with three-step Noor and SP methods. And the fastest case occurs when the proposed method with the operator order $\mathcal{T}_3 - \mathcal{T}_1 - \mathcal{T}_2$ is used.

5. Simulated results for image deblurring and signal recovering problems

Now, we apply our proposed algorithms to solve certain image deblurring and signal recovering problems where all codes were written in Matlab and run on laptop Intel core i7, 16.00 GB RAM, windows 10 (64-bit).

The minimization problem of the sum of two functions is to find a solution of

$$\min_{x \in \mathbb{R}^n} \{F(x) := f(x) + h(x)\}, \tag{5.1}$$

where $h : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{\infty\}$ is proper convex and lower semicontinuous function, and $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is convex differentiable function with gradient ∇f being L -Lipschitz constant for some $L > 0$. The solution of (5.1) can be characterized by using Fermats rule, Theorem 16.3 of Bauschke and Combettes [30] as follows:

$$x^* \text{ is a minimizer of } (f + h) \text{ if and only if } 0 \in \partial h(x^*) + \nabla f(x^*),$$

where ∂h is the subdifferential of h and ∇f is the gradient of f . The subdifferential of h at x^* , denoted by $\partial h(x^*)$, is defined by

$$\partial h(x^*) := \{u : h(x) - h(x^*) \geq \langle u, x - x^* \rangle \text{ for all } x\}.$$

It is also well-known that the solution of (5.1) is characterized by the following fixed point problem:

$$x^* \text{ is a minimizer of } (f + h) \text{ if and only if } x^* = prox_{\mu h}(I - \mu \nabla f)(x^*),$$

where $c > 0$, $prox_h$ is the proximity operator of h defined by $prox_h := argmin\{h(y) + \frac{1}{2} \|x - y\|_2^2\}$, see [31] for more details. It is also known that $prox_{\mu h}(I - \mu \nabla f)$ is a nonexpansive mapping when $\mu \in (0, \frac{2}{L})$.

5.1. Application to image deblurring problems

Let \mathcal{B} is the degraded image of the true image \mathcal{X} in the matrix form of \tilde{m} rows and \tilde{n} columns ($\mathcal{B}, \mathcal{X} \in \mathbb{R}^{\tilde{m} \times \tilde{n}}$). The key to obtaining the image restoration model is to rearrange the elements of the images \mathcal{B} and \mathcal{X} into the column vectors by stacking the columns of these images into two long vectors b and x where $b = \text{vec}(\mathcal{B})$ and $x = \text{vec}(\mathcal{X})$, both of length $n = \tilde{m}\tilde{n}$. The image restoration problem can be modelled in one dimensional vector by the following linear equation system:

$$b = Mx, \quad (5.2)$$

where $x \in \mathbb{R}^n$ is an original image, $b \in \mathbb{R}^n$ is the observed image, $M \in \mathbb{R}^{n \times n}$ is the blurring matrix and $n = \tilde{m}\tilde{n}$. In order to solve problem (5.2), we aim to approximate the original image, vector b , which is known as the following least squares (LS) problem:

$$\min_x \frac{1}{2} \|b - Mx\|_2^2, \quad (5.3)$$

where $\|\cdot\|_2$ is defined by $\|x\|_2 = \sqrt{\sum_{i=1}^n |x_i|^2}$. We can apply the minimization problem of the sum of two functions (5.1) to the LS-problem (5.3) by setting $h(x) = 0$ and $f(x) = \frac{1}{2} \|b - Mx\|_2^2$ where $\nabla f(x) = M^T(Mx - b)$. Thus, LS-problem can be solved with our method (3.1) by setting

$$\mathcal{T}x = x - \mu M^T(Mx - b), \quad \mu \in \left(0, \frac{2}{\|M^T M\|_2}\right).$$

And, the following proposed method is obtained in finding the solution of the image deblurring problem:

$$\begin{aligned} z_n &= (1 - \gamma_n)x_n + \gamma_n(x_n - \mu M^T(Mx_n - b)), \\ y_n &= (1 - \beta_n)(x_n - \mu M^T(Mx_n - b)) + \beta_n(z_n - \mu M^T(Mz_n - b)), \\ x_{n+1} &= (1 - \alpha_n)(y_n - \mu M^T(My_n - b)) + \alpha_n(z_n - \mu M^T(Mz_n - b)), \end{aligned} \quad (5.4)$$

where $\mu \in \left(0, \frac{2}{\|M^T M\|_2}\right)$ and $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[\delta, 1 - \delta]$ for all $n \in \mathbb{N}$ and for some δ in $(0, 1)$. The algorithm (5.4) is used in solving the image deblurring problem (5.2) with the default parameter (4.1) and $\mu = 1/\|M^T M\|_2$.

The goal on image deblurring problem is to find the original image from the observed image without knowing which one is the blurring matrix. However, the blurring matrix M must be known in applying algorithm (5.4). Now, we present the new idea in solving the image deblurring problem when the observed image b_1, b_2, \dots, b_N can be restored by using the blurring matrices M_1, M_2, \dots, M_N , respectively in which

$$b_i = M_i x, \quad i = 1, 2, \dots, N. \quad (5.5)$$

That is, the original image x is a common solution of the N -determined linear Eq (5.5). Now, Let us consider the following LS-problems:

$$\min_{x \in \mathbb{R}^n} \frac{1}{2} \|M_1 x - b_1\|_2^2, \min_{x \in \mathbb{R}^n} \frac{1}{2} \|M_2 x - b_2\|_2^2, \dots, \min_{x \in \mathbb{R}^n} \frac{1}{2} \|M_N x - b_N\|_2^2, \quad (5.6)$$

where x is the common solution of problem (5.6). In applying the proposed algorithm (5.4) in finding the original image x , Only a group of the chosen three blurring matrices M_j, M_k, M_l from known blurring matrices M_1, M_2, \dots, M_N . And, we can apply the minimization problem of the sum of two functions with our method (3.1) for solving the LS-problems by setting

$$\mathcal{T}_i x = x - \mu_i \nabla f_i(x) c$$

with $h_i(x) = 0$ and $f_i(x) = \frac{1}{2} \|b_i - M_i x\|_2^2$ where $\nabla f_i(x) = M_i^T (M_i x - b_i)$. Now, we presented the proposed algorithm with $M_j - M_k - M_l$:

$$\begin{aligned} z_n &= (1 - \gamma_n)x_n + \gamma_n (x_n - \mu_j M_j^T (M_j x_n - b_j)), \\ y_n &= (1 - \beta_n) (x_n - \mu_j M_j^T (M_j x_n - b_j)) + \beta_n (z_n - \mu_k M_k^T (M_k z_n - b_k)), \\ x_{n+1} &= (1 - \alpha_n) (y_n - \mu_l M_l^T (M_l y_n - b_l)) + \alpha_n (z_n - \mu_k M_k^T (M_k z_n - b_k)), \end{aligned} \quad (5.7)$$

with the default parameter (4.1), $\mu_j = 1/\|M_j^T M_j\|_2, \mu_k = 1/\|M_k^T M_k\|_2, \mu_l = 1/\|M_l^T M_l\|_2$ and called it as the proposed algorithm with $M_j - M_k - M_l$. The implemented algorithm (5.7) is proposed in solving the image deblurring problem by using every three blurring matrices from N known blurring matrices with the default parameter (4.1). The original RGB format for color image shown in Figure 7 is used to demonstrate the practicability of the proposed algorithm. The relative errors with the stopping criterion of the proposed algorithms are defined as $\|x_n - x\|_\infty / \|x\|_\infty < 10^{-3}$. The performance of the comparing algorithms at x_n on image deblurring process is measured quantitatively by the means of the peak signal-to-noise ratio (PSNR), which is defined by

$$\text{PSNR}(x_n) = 20 \log_{10} \left(\frac{255^2}{\text{MSE}} \right),$$

where $\text{MSE} = \|x_n - x\|_2^2$. Three different types of the original RGB image degraded by the blurring matrices M_1, M_2 and M_3 are shown in Figure 8. These are used to test the implemented algorithm.



Figure 7. The original RGB image with matrix size $248 \times 356 \times 3$.



Figure 8. The original RGB image degraded by blurred matrices M_1 , M_2 and M_3 respectively.

Next, we present restoration of images that have been corrupted by the following blur types:

Type I. Gaussian blur of filter size 9×9 with standard deviation $\sigma = 4$ (The original image has been degraded by the blurring matrix M_1).

Type II. Out of focus blur (Disk) with radius $r = 6$ (The original image has been degraded by the blurring matrix M_2).

Type III. Motion blur specifying with motion length of 21 pixels ($len = 21$) and motion orientation 11° ($\theta = 11$) (The original image has been degraded by the blurring matrix M_3).

Since, the using image X and three different types of the blurring image B (See on Figure 8) are represented in the red-green-blue component. Then, we denote X_r, X_g, X_b and B_r, B_g, B_b as the gray-scale images that constitute the red-green-blue channels of the using image X and the blurring image B respectively. Thus, we define the column vector x and b from color image X and B and both of length $n = 3\tilde{m}\tilde{n}$. After that, we apply the proposed algorithms in getting the common solution of the image deblurring problem with these three blurring matrices. And, we are also compare with three-step Noor iteration (1.1) and SP-iteration (1.2).

The permutation of the group of these three blurring matrices gives 27 case studies that we need to consider. First, we are interested in the simple three cases in which all three blurring matrices are the same. Second, we are also interested in the case of the absence of duplicate blurring matrices. That is the following 9 cases of three comparative algorithms consists of the proposed algorithm (5.7), three-step Noor iteration (1.1) and SP-iteration (1.2) are demonstrated and discussed.

Case I. Three comparative algorithms with $M_1-M_1-M_1$.

Case II. Three comparative algorithms with $M_1-M_2-M_3$.

Case III. Three comparative algorithms with $M_1-M_3-M_2$.

Case IV. Three comparative algorithms with $M_2-M_2-M_2$.

Case V. Three comparative algorithms with $M_2-M_1-M_3$.

Case VI. Three comparative algorithms with $M_2-M_3-M_1$.

Case VII. Three comparative algorithms with $M_3-M_3-M_3$.

Case VIII. Three comparative algorithms with $M_3-M_1-M_2$.

Case IX. Three comparative algorithms with $M_3-M_2-M_1$.

Figures 9–17 show the comparative plots behavior of Cauchy error, relative error and PSNR quality of the reconstructed RGB image with 9 cases.

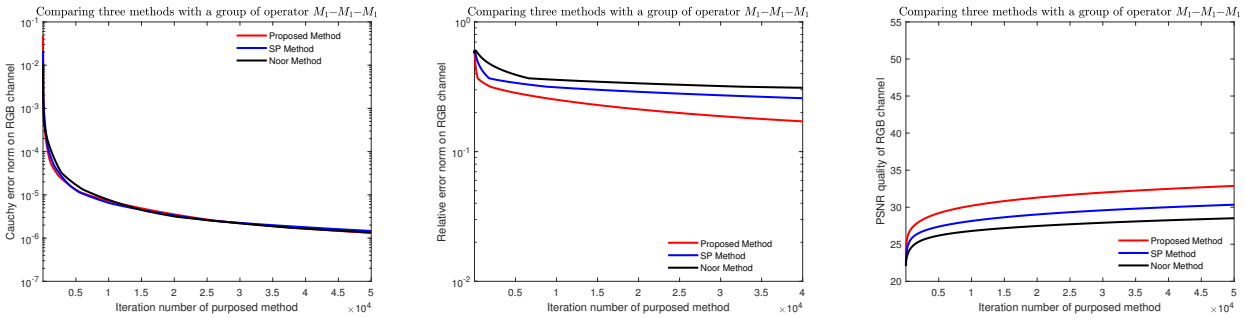


Figure 9. The Cauchy norm, relative norm and PSNR quality plots of three comparative algorithms with case I.

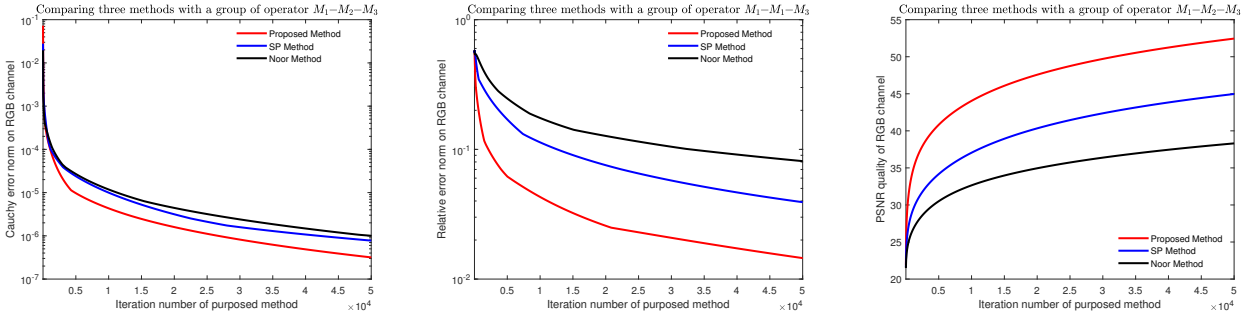


Figure 10. The Cauchy norm, relative norm and PSNR quality plots of three comparative algorithms with case II.

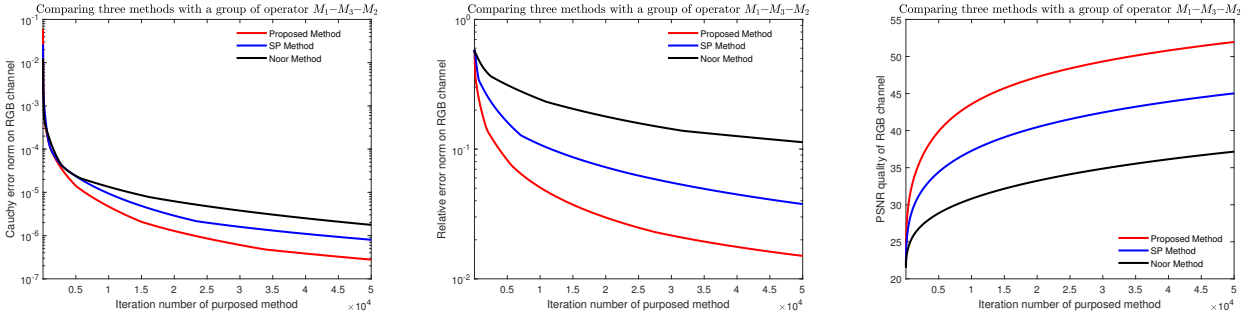


Figure 11. The Cauchy norm, relative norm and PSNR quality plots of three comparative algorithms with case III.

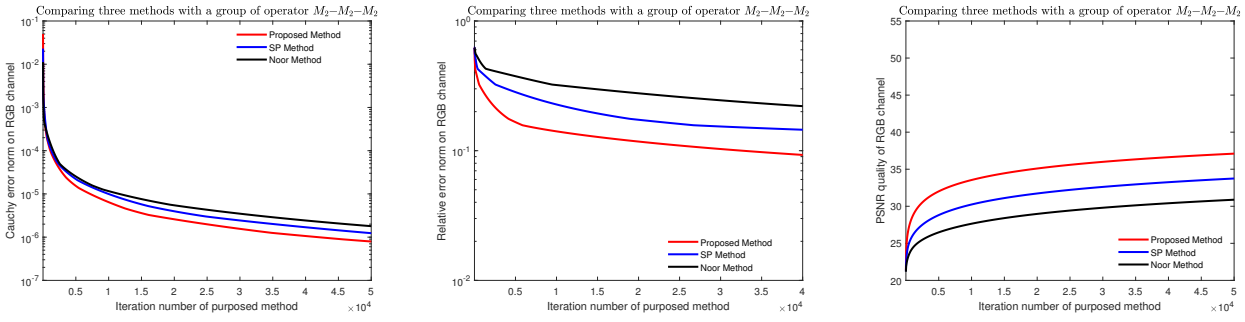


Figure 12. The Cauchy norm, relative norm and PSNR quality plots of three comparative algorithms with case IV.

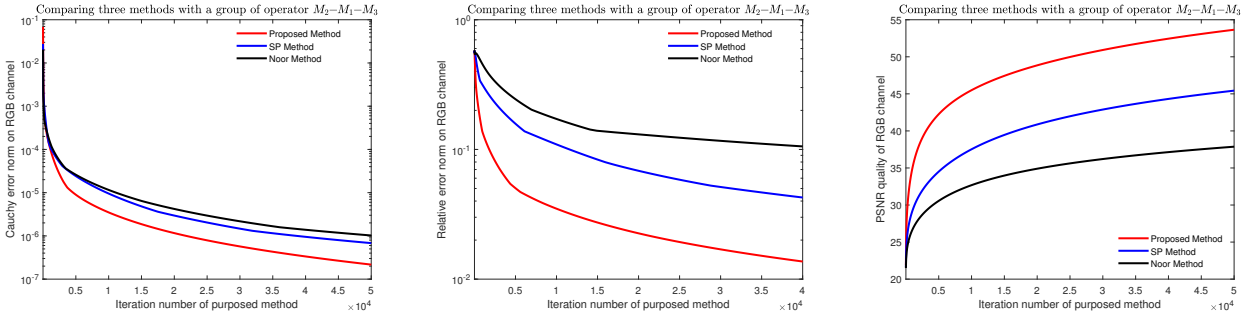


Figure 13. The Cauchy norm, relative norm and PSNR quality plots of three comparative algorithms with case V.

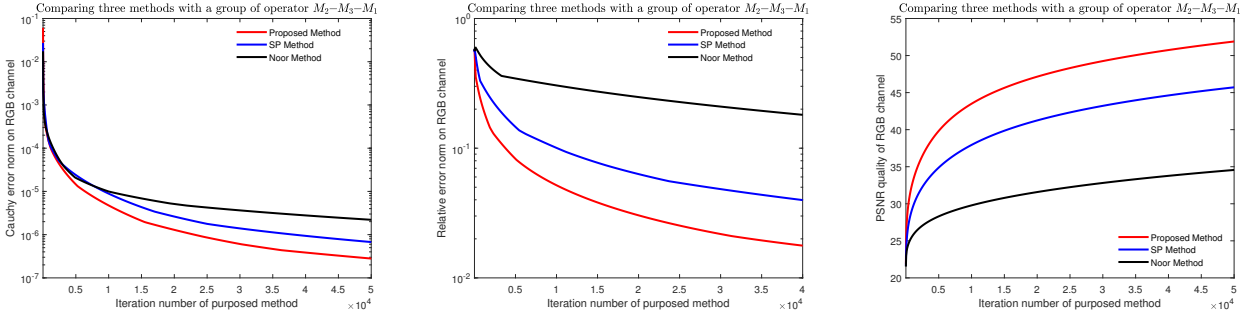


Figure 14. The Cauchy norm, relative norm and PSNR quality plots of three comparative algorithms with case VI.

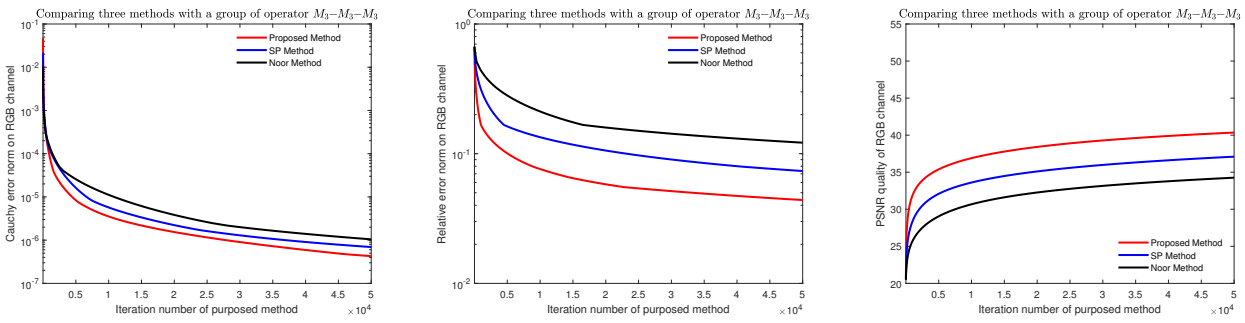


Figure 15. The Cauchy norm, relative norm and PSNR quality plots of three comparative algorithms with case VII.

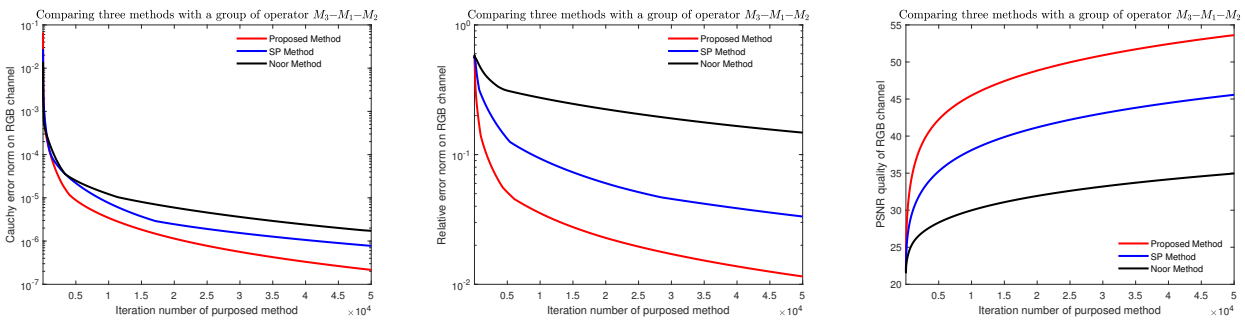


Figure 16. The Cauchy norm, relative norm and PSNR quality plots of three comparative algorithms with case VIII.

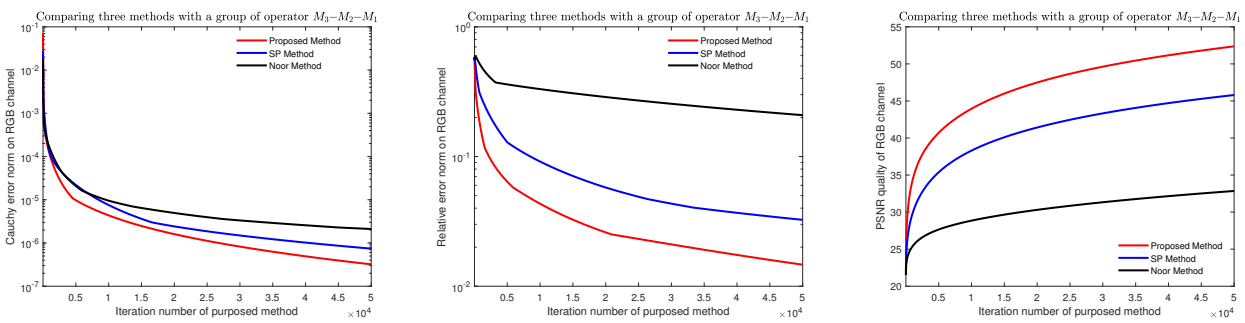


Figure 17. The Cauchy norm, relative norm and PSNR quality plots of three comparative algorithms with case IX.

It is remarkable that, the Cauchy error and relative error plots on each case of three comparative algorithms decrease as the iteration number increases. Since, the Cauchy and relative errors plot show the validity and confirm the convergence of the presented methods then, it can guarantee that all presented methods on Figures 9–17 converge to common solution of deblurring problem (5.7). Based on the PSNR plots of each case, all restored image using these three comparative algorithms in solving the deblurring problem get the quality improvements when the iteration number increases. It

can be conclude from the comparative plots behavior on Figures 9–17 that the proposed method is more efficiency than three-step Noor and SP methods. Moreover, the PSNR quality of the observed image is much better when the proposed method with $M_j-M_k-M_l$ and $j \neq k \neq l$ is used for solving deblurring problem compare with the proposed methods with $M_1-M_1-M_1$, $M_2-M_2-M_2$ and $M_3-M_3-M_3$. And, the best case in recovering the observed image occurs when the proposed methods with $M_2-M_1-M_3$ and $M_3-M_1-M_2$ are used. Figure 18 demonstrates the crop of reconstructed RGB image presented in 10,000th iteration by using the proposed algorithms in getting the common solution of the deblurring problem with operators $M_1-M_1-M_1$ and $M_2-M_2-M_2$, $M_3-M_3-M_3$ and $M_3-M_1-M_2$ respectively.

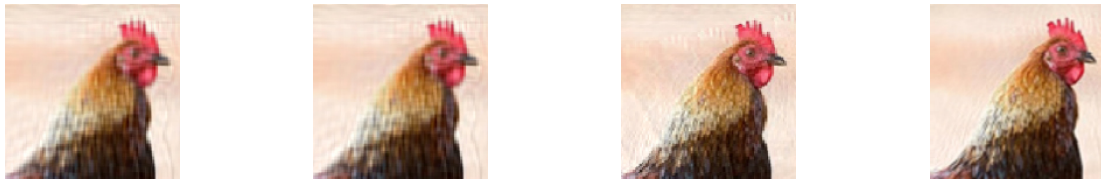


Figure 18. The reconstructed images being used the proposed algorithms with cases I, IV, VI and VIII , respectively present in 10,000th iterations.

It can be seen from these figures that the quality of restored image by using the proposed algorithms with operators $M_3-M_2-M_1$ get the smooth quality of the using degraded image.

Next, we give some numerical examples to the signal recovery.

5.2. Application to signal recovering problems

In signal processing, compressed sensing can be modeled as the following undetermined linear equation:

$$y = Ax + v,$$

where $x \in \mathbb{R}^n$ is an original signal with n components to be recovered, $v, y \in \mathbb{R}^m$ are noise and the observed signal with noisy for m components respectively and $A \in \mathbb{R}^{m \times n}$ is a degraded matrix. Finding the solutions of previous undetermined linear equation can be seen as solving the LASSO problem:

$$\min_{x \in \mathbb{R}^N} \frac{1}{2} \|y - Ax\|_2^2 + \lambda \|x\|_1, \quad (5.8)$$

where $\lambda > 0$. The new idea in solving the signal recovering problem is presented when the observed signal y_1, y_2, \dots, y_N can be recovered by using the known degraded matrices A_1, A_2, \dots, A_N in which

$$y_i = A_i x + v_i, \quad i = 1, 2, \dots, N. \quad (5.9)$$

where a true signal x is common solution of problem (5.9).

Let us consider the following LASSO problems:

$$\min_{x \in \mathbb{R}^N} \frac{1}{2} \|A_1 x - y_1\|_2^2 + \lambda_1 \|x\|_1,$$

$$\begin{aligned} \min_{x \in \mathbb{R}^N} \frac{1}{2} \|A_2 x - y_2\|_2^2 + \lambda_2 \|x\|_1, \\ \vdots \\ \min_{x \in \mathbb{R}^N} \frac{1}{2} \|A_N x - y_N\|_2^2 + \lambda_N \|x\|_1, \end{aligned} \quad (5.10)$$

where a true signal x is common solution of problem (5.10). We can apply the minimization problem of the sum of two functions with our method (3.1) for solving the LASSO problems (5.10) by setting

$$\mathcal{T}_i x = \text{prox}_{\mu_i h_i}(x - \mu_i \nabla f_i(x)), \quad \mu_i \in \left(0, \frac{2}{\|A_i^T A_i\|_2}\right), \quad i = 1, 2, \dots, N,$$

with $h_i(x) = \lambda_i \|x\|_1$ and $f_i(x) = \frac{1}{2} \|y_i - A_i x\|_2^2$ where $\nabla f_i(x) = A_i^T (A_i x - y_i)$. Now, we obtain the following proposed method to find the common solution of LASSO problems (5.10) for a group of three blurring matrices A_j, A_k, A_l chosen from N blurring matrices A_1, A_2, \dots, A_N :

$$\begin{aligned} z_n &= (1 - \gamma_n)x_n + \gamma_n \text{prox}_{\lambda_g}(x_n - \mu_j A_j^T (A_j x_n - y_j)), \\ y_n &= (1 - \beta_n) \text{prox}_{\lambda_g}(x_n - \mu_j A_j^T (A_j x_n - y_j)) + \beta_n \text{prox}_{\lambda_g}(z_n - \mu_k A_k^T (A_k z_n - y_k)), \\ x_{n+1} &= (1 - \alpha_n) \text{prox}_{\lambda_g}(y_n - \mu_l A_l^T (A_l y_n - y_l)) + \alpha_n \text{prox}_{\lambda_g}(z_n - \mu_k A_k^T (A_k z_n - y_k)), \end{aligned} \quad (5.11)$$

with the default parameter (4.1), $\mu_j = 1/\|A_j^T A_j\|_2$, $\mu_k = 1/\|A_k^T A_k\|_2$, $\mu_l = 1/\|A_l^T A_l\|_2$ and called it as the proposed algorithm with A_j - A_k - A_l . The implemented algorithm (5.11) is proposed in solving the signal recovering problem by using every three blurring matrices from N blurring matrices on Eq (5.9) with the default parameter (4.1). And, we are also compare the proposed algorithm (5.11) with three-step Noor iteration (1.1) and SP-iteration (1.2). The following 9 cases of three comparative algorithms consists of the proposed algorithm (5.11), three-step Noor iteration (1.1) and SP-iteration (1.2) are demonstrated and discussed.

Case I. Three comparative algorithms with A_1 - A_1 - A_1 .

Case II. Three comparative algorithms with A_1 - A_2 - A_3 .

Case III. Three comparative algorithms with A_1 - A_3 - A_2 .

Case IV. Three comparative algorithms with A_2 - A_2 - A_2 .

Case V. Three comparative algorithms with A_2 - A_1 - A_3 .

Case VI. Three comparative algorithms with A_2 - A_3 - A_1 .

Case VII. Three comparative algorithms with A_3 - A_3 - A_3 .

Case VIII. Three comparative algorithms with A_3 - A_1 - A_2 .

Case IX. Three comparative algorithms with A_3 - A_2 - A_1 .

Next, some experiments are provided to illustrate the convergence and the effectiveness of the proposed algorithm (5.11). The original signal x with $n = 1024$ generated by the uniform distribution in the interval $[-2, 2]$ with 70 nonzero elements is used to create the observation signal $y_i = A_i x + v_i, i = 1, 2, 3$ with $m = 512$ (See on Figure 19).

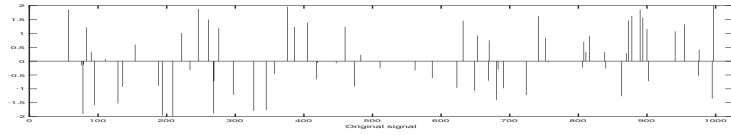


Figure 19. Original Signal (x) with $m = 70$.

The observation signal $y_i = A_i x + n_i$ shows on Figure 20.

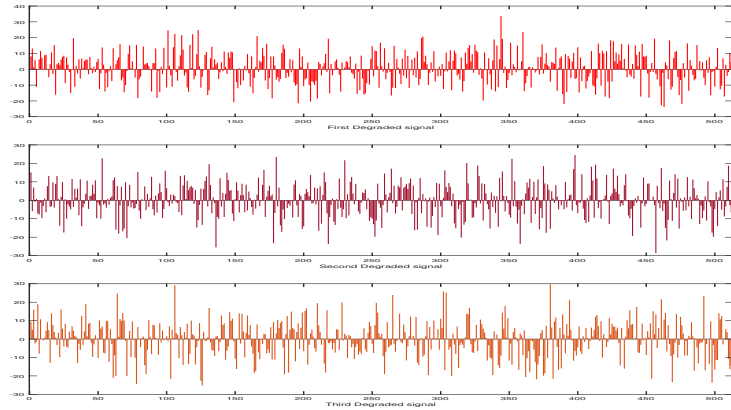


Figure 20. Degraded Signals $y_1, y_2,$ and $y_3,$ respectively.

The process is started when the signal initial data x_0 with $n = 1024$ is picked randomly (See on Figure 21).

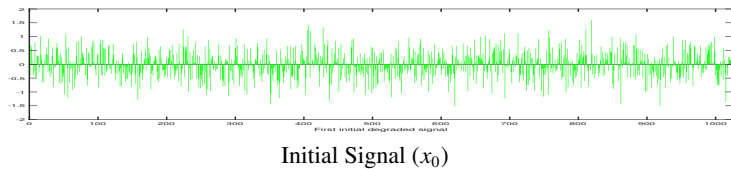


Figure 21. Initial Signals x_0 .

The matrices A_i that generated by the normal distribution with mean zero and variance one and the white Gaussian noise v_i for all $i = 1, 2, 3$ (See on Figure 22).

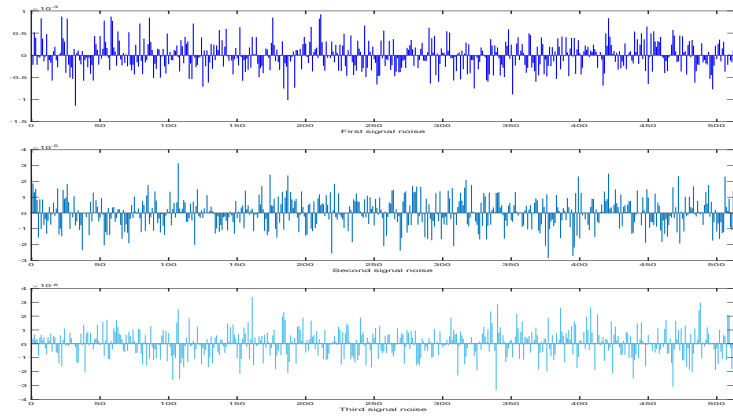


Figure 22. Noise Signals v_1 , v_2 , and v_3 respectively.

The Cauchy error and the relative signal error are measured by using max-norm $\|x_n - x_{n-1}\|_\infty$ and $\|x_n - x\|_\infty / \|x\|_\infty$, respectively. The performance of the comparing method at n^{th} iteration is measured quantitatively by the means of the the signal-to-noise ratio (SNR), which is defined by

$$SNR(x_n) = 20 \log_{10} \left(\frac{\|x_n\|_2}{\|x_n - x\|_2} \right),$$

where x_n is the recovered signal at n^{th} iteration by using the proposed method. The Cauchy error plots and relative error plots on each case of three comparative methods on Figures 23–31 guarantee that all presented methods converge to common solution of signal recovering problem. Based on the SNR plots on each case of three comparative methods, all recovering signal using the proposed method, three-step Noor method and Sridarat method witness quality improvement when the iteration number increases. It can be conclude from the comparative plots behavior on Figures 23–31 that the proposed method is more efficiency than three-step Noor and SP methods.

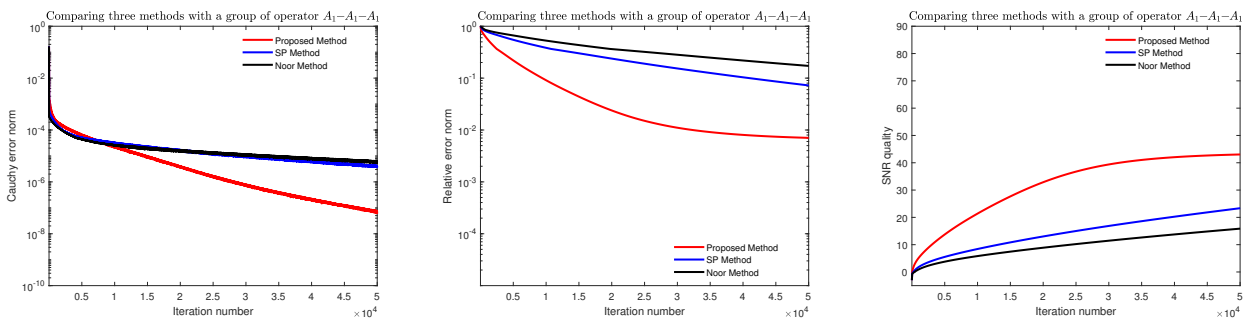


Figure 23. The Cauchy norm, relative norm plots and SNR quality plots of three comparative algorithms with case I.

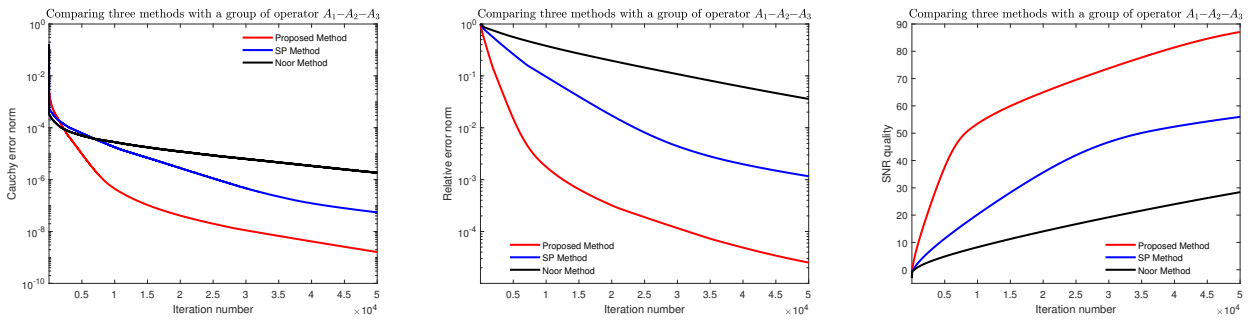


Figure 24. The Cauchy norm, relative norm plots and SNR quality plots of three comparative algorithms with case II.

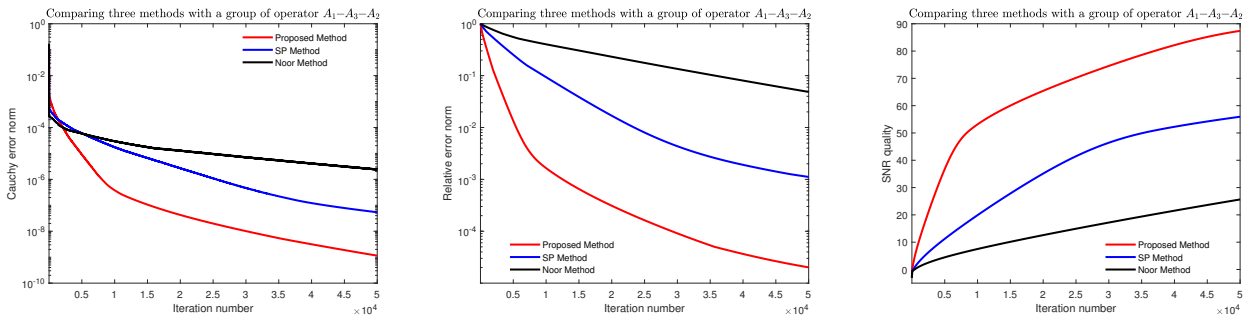


Figure 25. The Cauchy norm, relative norm plots and SNR quality plots of three comparative algorithms with case III.

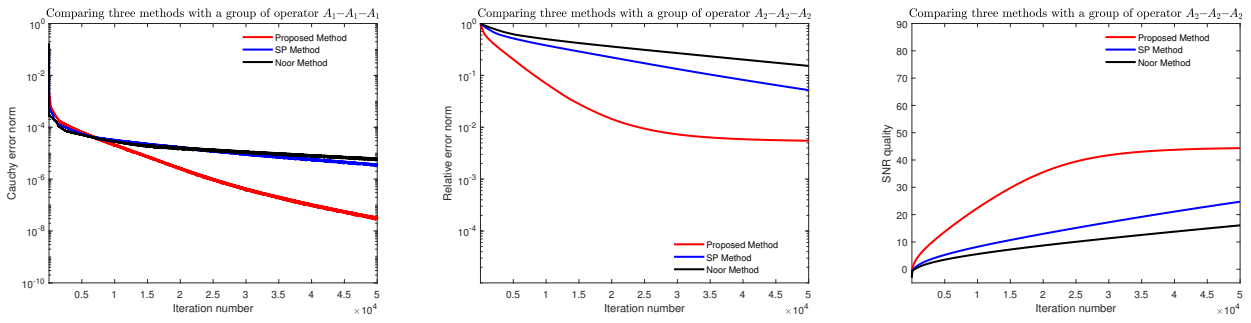


Figure 26. The Cauchy norm, relative norm plots and SNR quality plots of three comparative algorithms with case IV.

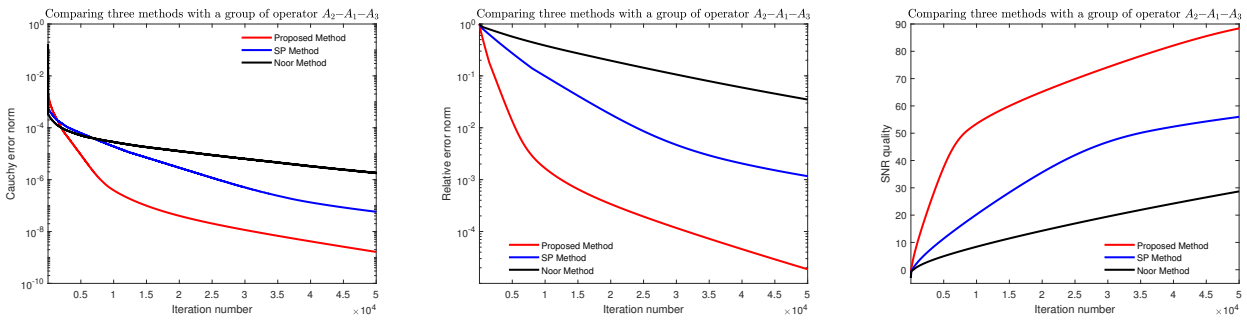


Figure 27. The Cauchy norm, relative norm plots and SNR quality plots of three comparative algorithms with case V.

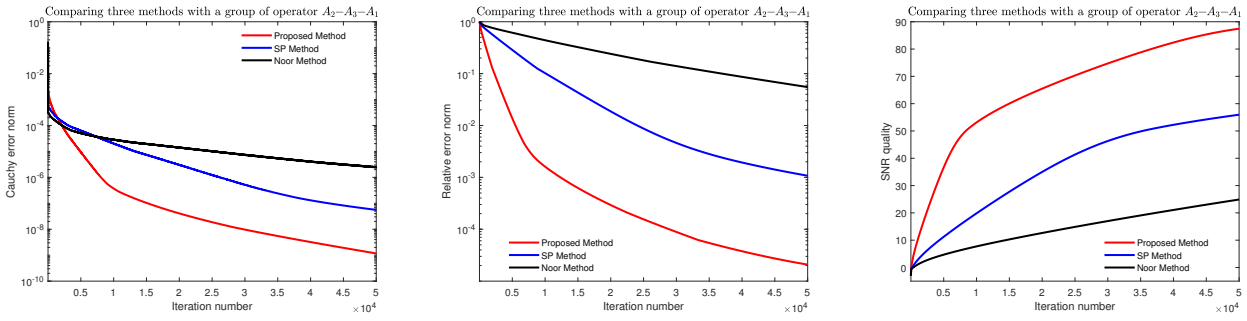


Figure 28. The Cauchy norm, relative norm plots and SNR quality plots of three comparative algorithms with case VI.

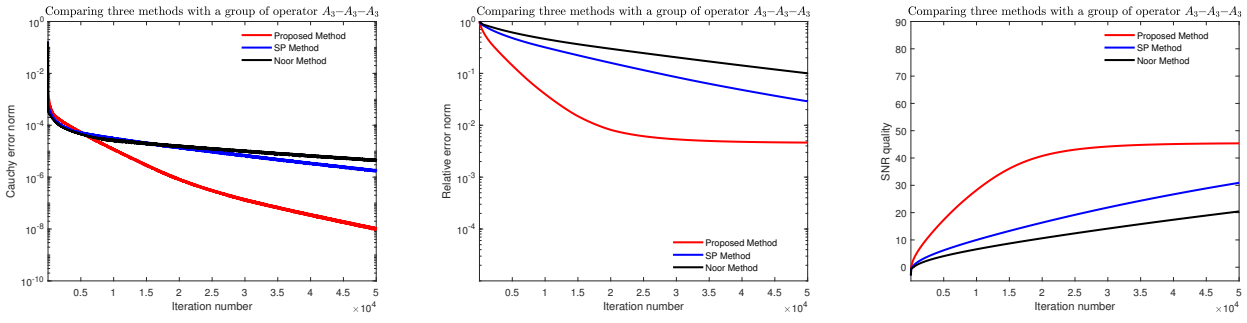


Figure 29. The Cauchy norm, relative norm plots and SNR quality plots of three comparative algorithms with case VII.

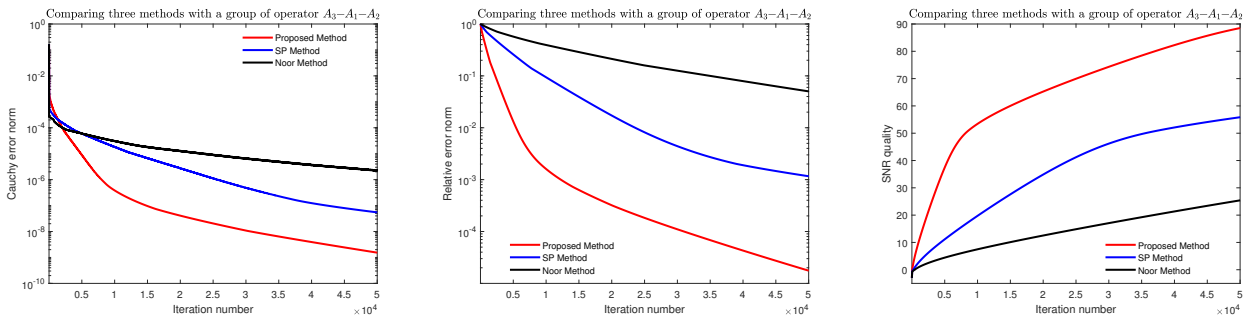


Figure 30. The Cauchy norm, relative norm plots and SNR quality plots of three comparative algorithms with case VIII.

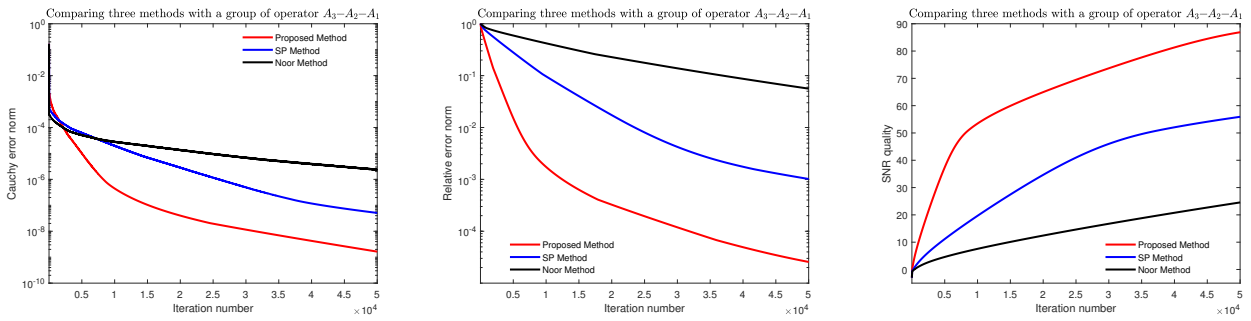


Figure 31. The Cauchy norm, relative norm plots and SNR quality plots of three comparative algorithms with case IX.

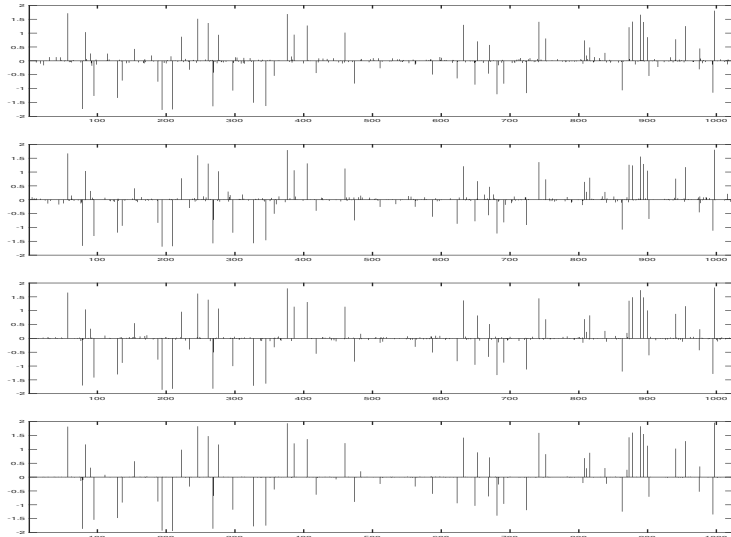


Figure 32. Recovering signals being used the proposed algorithms with cases I, IV, VII and VIII, respectively presented in 10,000th iterations.

Moreover, the SNR quality of the recovering signal is significantly improved when the proposed method with $A_j-A_k-A_l$ and $j \neq k \neq l$ is used compare with the proposed methods with $A_1-A_1-A_1$, $A_2-A_2-A_2$ and $A_3-A_3-A_3$. And, the best case in recovering the observed signal occurred when the proposed method with the operator order $A_2-A_1-A_3$ and $A_3-A_1-A_2$ are used. Figure 32 shows an excellent quality of the restored signal using the proposed algorithm with $A_3-A_1-A_2$.

6. Conclusions

In this article, the efficient modified three-step iteration algorithm is proposed for approximating a common fixed point of three G-nonexpansive mappings on Banach spaces involving a graph. By assuming coordinate-convexity in a uniformly convex Banach space endowed with a directed graph, we have proved strong convergence theorem for above said algorithm and mappings by using condition (C) which is a generalization of condition (A) [32] and a weak convergence theorem by using Opial's condition [21]. Also we have proved weak convergence theorem without using Opial's condition. The conditions for convergence of the method are established by systematic proof. Numerical example illustrating the performance of the suggested algorithm was provided. All numerical experiments for common fixed point solution by using the three-step Noor iteration, SP-iteration and the proposed method with the permutation of three operators are shown in Figures 1–6. Our algorithm was found to be faster than Noor and SP iterations. As applications, we applied our proposed algorithm to solve the image restoration problems with the permutation of the three blurring operators (see Figures 9–17). We also applied our algorithms for solving signal recovery in situations where the type of noise is unknown (see Figures 23–31). We found that our proposed algorithm is flexible and has good quality for common types of blur and noise effects in image deblurring and signal recovering problems. Moreover, we found that when the proposed is used in solving the common solution of image and signal recovering problems, it enhanced the quality range of the recovered image and signal.

Acknowledgments

This research project was supported by the Thailand science research and innovation fund and the University of Phayao (Grant No. FF64-UoE011). The authors acknowledge the partially support provided by Thailand Science Research and Innovation under the Project IRN62W0007. The opinions in the research report belong to the researchers; Thailand Science Research and Innovation do not always have to agree. Also, T.Thianwan was supported by Contract No. PBTSC64019.

Conflict of interest

The authors declare no conflict of interest.

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