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Research article

Some generalized fixed point results via a τ-distance and applications

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Abstract: The aim of this manuscript is to present some new fixed point results in complete partially order metric spaces and to derive some extended forms of Suzuki and Banach fixed point theorems via a τ-distance by applying some new control functions. Our results are extensions of several existing fixed point theorems in the literature. To show the dominance of the established results, some examples and an application are studied.

Keywords: partially ordered metric space; complete metric space; Banach fixed point theorem; Suzuki fixed point theorem; ω-distance; τ-distance
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1. Introduction and preliminaries

Metric fixed point theory is one of the chief branch in the field of nonlinear analysis. The Banach contraction principle (BCP) is an important and celebrated result of this branch, which was initiated by Banach in 1922. In 1974, Ćirić [1] investigated a quasi-contraction mapping and showed that his main result implies the BCP. Varies researchers, as Ekland [2], Hegedus [3], Ghosh [4] and others generalized the BCP in different directions, either by changing the underlaying spaces, or the contractive condition. In the fixed point theory, one of the most unusual and interesting easily formulated and useful results is due to caristi [5]. Khojasteh et al. [6] introduced the idea that many known fixed point theorems can easily be derived from the Caristi theorem. Karapınar et al. [7] proposed a new fixed point theorem inspired from both Caristi and Banach. Du and Karapinar [8] introduced the concept of Caristi-type cyclic mappings and presented new convergence and best proximity point theorems. Karapinar [9] used lower semi-continuous mappings to generalise Caristi-Kirk's fixed point theorem on partial metric spaces. In [10], Kada et al. studied Caristi result and provided a new generalization by using the ω-distance (for more details, see [11–14]). Gil et al. [15] established a proof for Bianchini and Grandolfi Theorem in the context of quasi-metric spaces via a modified ω-distance. Suzuki [16] initiated the idea of the τ-distance and studied the generalized distance with an existence theorem in metric spaces. On the basis of the τ-distance, Suzuki proved the Caristi result for single-value mappings. In 2003, by generalizing Caristi work, Bae et al. [17] investigated fixed point results in case of weakly contractive multi-valued mappings. By studying the result of Bae et al., Suzuki [18] gave a new generalization of Caristi result. In 2008, Khamsi [19] presented some remarks on Caristi result.

On the other hand, in 2015, Suzuki et al. [20] worked on Caristi result by utilizing the completeness of an υ-generalized metric space in the sense of Branciari. In 2016, Du [21] initiated the idea of an essential distance that generalized the notion of a T-function and provided an upgrade form of Caristi fixed point result. Kozlowski [22] studied Caristi result and provided its proof by applying a pure metric technique. In 2018, Suzuki [23] initiated the idea of $\sum$-completeness in semi-metric spaces and extended Caristi result to $\sum$-complete semi-metric spaces. Also, Isik et al. [24] studied a new generalization of Caristi and Banach fixed point theorems.

Now, we recall Caristi fixed point result as follows:

Theorem 1.1. *[5] Assume that $(\mathcal{H}, d)$ is a complete metric space and $\mathcal{T} : \mathcal{H} \to \mathcal{H}$ is a self-mapping. Suppose that*

$$\Psi : \mathcal{H} \to [0, +\infty)$$

is a lower semi-continuous mapping such as, for all $v \in \mathcal{H}$,

$$d(v, \mathcal{T}(v)) + \Psi(\mathcal{T}(v)) \quad \leq \quad \Psi(v).$$

Then there exists a fixed point of $\mathcal{T}$ in $\mathcal{H}$.

Using a new approach, Bae *et al.* [17] upgraded Caristi result as follows:

Theorem 1.2. *Let $(\mathcal{H}, d)$ represent a complete metric space and $A : \mathcal{H} \to [0, +\infty)$ be a lower semi-continuous mapping. Let $\mathcal{T} : \mathcal{H} \to \mathcal{H}$ represent a self-mapping such that, for all $v \in \mathcal{H}$,*

$$d(v, \mathcal{T}(v)) \quad \leq \quad \alpha(A(v) - A(\mathcal{T}(v))),$$

where

$$\alpha = \max\{b(A(v)), b(A(\mathcal{T}(v)))\}$$

and $b : [0, +\infty) \to (0, +\infty)$ is a right upper semi-continuous function. Then $\mathcal{T}$ has a fixed point in $\mathcal{H}$.

Initially, Kada *et al.* [7] launched the idea of the ω-distance as follows:

Definition 1.3. Assume that $(\mathcal{H}, d)$ is a metric space and $\omega : \mathcal{H} \times \mathcal{H} \to [0, +\infty)$ is a function. ω is called the *ω-distance* if, for all $p, q, r \in \mathcal{H}$, the following conditions hold:

(i) $\omega(p, r) \leq \omega(p, q) + \omega(q, r)$;

(ii) $\omega(p, \cdot) : \mathcal{H} \to [0, +\infty)$ is lower semi-continuous;

(iii) If there is $\beta > 0$ such that $\omega(r, p) \leq \beta$ and $\omega(r, q) \leq \beta$ for any $\epsilon > 0$, then $d(p, q) \leq \epsilon$.

In [7], Kada *et al.* upgraded Caristi results by the ω-distance as in the following.

Theorem 1.4. *Let $(\mathcal{H}, d)$ represent a complete metric space and ω be the ω-distance on $\mathcal{H}$. If $\mathcal{T} : \mathcal{H} \to \mathcal{H}$ is a mapping on $\mathcal{H}$ such that, for any $v \in \mathcal{H}$,*

$$A(v) - A(\mathcal{T}(v)) \geq \omega(v, A(v)),$$

where $A : \mathcal{H} \to [0, +\infty)$ is a lower semi-continuous function, then there is $s \in \mathcal{H}$ so that $\mathcal{T}(s) = s$ and $\omega(s, s) = 0$.

In 2001, Suzuki [16] provided the idea of the τ-distance, which is an upgrade form of the ω-distance.

Definition 1.5. Let $\mathcal{H}$ be a metric space endowed with a metric d. A *τ-distance* on $\mathcal{H}$ is a function $Q : \mathcal{H} \times \mathcal{H} \to [0, +\infty)$ ensuring that for all $p, q, r \in \mathcal{H}$,

($\mathbf{Q_1}$) $Q(p, r) \leq Q(p, q) + Q(q, r)$;

($\mathbf{Q_2}$) $\xi(p, 0) = 0$ and $\xi(p, v) \geq v$ for all $p \in \mathcal{H}$ and $v \geq 0$, where ξ is continuous, concave and non-decreasing in its second variable;

($\mathbf{Q_3}$) If $\limsup\{\xi(r_n, Q(r_n, p_m)) : m \geq n\} = 0$ and $\lim_{n \to +\infty} p_n = p$, then $Q(v, p) \leq \liminf_{n \to +\infty} Q(v, p_n)$ for all $v \in \mathcal{H}$;

($\mathbf{Q_4}$) If $\lim_{n \to +\infty} \xi(p_n, v_n) = 0$ and $\limsup\{Q(p_n, q_m) : m \geq n\} = 0$, then $\lim_{n \to +\infty} \xi(q_n, v_n) = 0$;

($\mathbf{Q_5}$) If $\lim_{n \to +\infty} \xi(r_n, Q(r_n, q_n)) = 0$ and $\lim_{n \to +\infty} \xi(r_n, Q(r_n, p_n)) = 0$, then $\lim_{n \to +\infty} d(p_n, q_n) = 0$.

Suzuki [16] used the idea of the τ-distance. He upgraded the result of Kada *et al.* [7] and generalized the BCP.

Theorem 1.6. *[16] Assume that $(\mathcal{H}, d)$ is a complete metric space and Q represents a τ-distance on $\mathcal{H}$. Let $\mathcal{T} : \mathcal{H} \to \mathcal{H}$ be a self-mapping on $\mathcal{H}$ such that for each $v \in \mathcal{H}$,*

$$Q(v, A(v)) \quad \leq \quad A(v) - A(\mathcal{T}(v)),$$

where $A : \mathcal{H} \to [0, +\infty)$ is a lower semi-continuous function. Then there is $s \in \mathcal{H}$ so that $\mathcal{T}(s) = s$ and $Q(s, s) = 0$.

Theorem 1.7. *[16] Let $\mathcal{H}$ be a complete metric space and $\mathcal{T} : \mathcal{H} \to \mathcal{H}$ represent a self-mapping. Suppose that Q is the τ-distance on $\mathcal{H}$ and $t \in [0, 1)$ such that $q(\mathcal{T}g, \mathcal{T}^2 g) \leq t.q(g, \mathcal{T}g)$ for all $g \in \mathcal{H}$. Suppose that following conditions hold:*

($\mathbf{c_1}$) *If* $\limsup_n\{Q(g_n, g_m) : m > n\} = 0$, $\lim\limits_n Q(g_n, \mathcal{T}g_n) = 0$ *and* $\lim\limits_n Q(g_n, h) = 0$, *then* $\mathcal{T}h = h$;

($\mathbf{c_2}$) *If* $\{g_n\}$ *and* $\{\mathcal{T}g_n\}$ *converge to* h, *then* $\mathcal{T}h = h$;

($\mathbf{c_3}$) $\mathcal{T}$ *is continuous.*

Then there is $g_0 \in \mathcal{H}$ *that ensures* $\mathcal{T}g_0 = g_0$ *and* $Q(g_0, g_0) = 0$.

In 2005, Suzuki [18] generalized the results of Bae *et al.* [17] as follows:

Theorem 1.8. *Suppose* $(\mathcal{H}, d)$ *represents a complete metric space and* $h : \mathcal{H} \to (0, +\infty)$ *is a mapping so that for some* $q > 0$, $\sup\{h(v) : v \in \mathcal{H}, A(v) \leq \inf_{z\in\mathcal{H}} A(z) + q\} < +\infty$, *where* $A : \mathcal{H} \to [0, +\infty)$ *is a lower semi-continuous function. Assume* $\mathcal{T} : \mathcal{H} \to \mathcal{H}$ *is a self-mapping such that for all* $v \in \mathcal{H}$,

$$d(v, \mathcal{T}(v)) \quad \leq \quad h(v)(A(v) - A(\mathcal{T}(v))).$$

Then there is a fixed point of $\mathcal{T}$ *in* $\mathcal{H}$.

By using the concept of the τ-distance, Suzuki [16] generalized Theorem 1.6 and Theorem 1.8.

Theorem 1.9. *Let* $(\mathcal{H}, d)$ *be a complete metric space,* Q *denote the* τ*-distance and* $h : \mathcal{H} \to (0, +\infty)$ *be a map so that for some* $q > 0$,

$$\sup\{h(v) : v \in \mathcal{H}, A(v) \leq \inf_{z\in\mathcal{H}} A(z) + q\} < +\infty,$$

where $A : \mathcal{H} \to (0, +\infty)$ *is a lower semi-continuous function. Let* $\mathcal{T} : \mathcal{H} \to \mathcal{H}$ *be a self-mapping so that for all* $v \in \mathcal{H}$,

$$Q(v, \mathcal{T}(v)) \quad \leq \quad h(v)(A(v) - A(\mathcal{T}(v))).$$

Then there is $a_0 \in \mathcal{H}$ *such that* $\mathcal{T}(a_0) = a_0$ *and* $Q(a_0, a_0) = 0$.

Recently, Isik *et al.* [24] proposed the following novel extension of Banach and Caristi's results.

Theorem 1.10. *Assume that* $(\mathcal{H}, d)$ *is a complete metric space and* $\mathcal{T} : \mathcal{H} \to \mathcal{H}$ *is a self-mapping. Let* W *represent the set of functions* $\Psi : R \to (0, +\infty)$ *fulfilling the following properties:*

(**i**) Ψ *is increasing strictly and continuous;*

(**ii**) *For every sequence* $a_n \subseteq \mathbb{R}^+$, $\lim\limits_{n\to+\infty} a_n = 0$ *if and only if* $\lim\limits_{n\to+\infty} \Psi(a_n) = 1$;

(**iii**) *For all* $a, b \in \mathbb{R}$, $\Psi(a+b) \leq \Psi(a)\Psi(b)$, *where* R *and* R^+ *represent the sets of real and non-negative real numbers respectively.*

If there is a bounded-below and lower semi-continuous map $\beta : \mathcal{H} \to R$ *so that*

$$\Psi(d(\mu, \mathcal{T}\mu)) \quad \leq \quad \frac{\Psi(\beta(\mu))}{\Psi(\beta(\mathcal{T}\mu))} \quad for\ all \mu \in \mathcal{H},$$

then $\mathcal{T}$ *has a fixed point.*

Theorem 1.11. *[24] Let $(\mathcal{H}, d)$ be a complete metric and $\mathcal{T} : \mathcal{H} \to \mathcal{H}$ be a continuous self-mapping. If $F : [0, +\infty) \to [0, +\infty)$ exists so that $\lim_{x\to 0} F(x) = 0$, $F(0) = 0$ and*

$$d(\mathcal{T}(a), \mathcal{T}(b)) \leq F(d(a, b)) - F(d(\mathcal{T}(a), \mathcal{T}(b))) \ for \ all a, b \in \mathcal{H},$$

then there is a fixed point of $\mathcal{T}$.

Karapinar [25] investigated the existence and uniqueness of fixed points of Jaggi type contractions by using a simulation function in the framework of partial metric spaces. Gupta *et al.* [26] established the following fixed point result in partially ordered metric spaces by applying generalized (φ, α, β)-contractive mappings.

Theorem 1.12. *Let $(\mathcal{H}, d, \preceq)$ be a partially ordered complete metric space and $\mathcal{T}$ be a non-decreasing self-mapping such that for all $x, y \in X$,*

$$\varphi(d(\mathcal{T}x, \mathcal{T}y)) \leq \alpha(\lambda(x, y)) - \beta(\lambda(x, y)),$$

with $x \preceq y$, $x \neq y$ with

$$\lambda(x, y) = \max\{\frac{d(x, \mathcal{T}x).d(y, \mathcal{T}y)}{d(x, y)}, d(x, y)\}, \tag{1.1}$$

where φ is an alternating distance function and $\alpha, \beta : [0, +\infty) \to [0, +\infty)$ verifying $\alpha(t) \geq \beta(t)$ for each $t \geq 0$, are continuous functions satisfying the condition: $\Psi(t) > \alpha(t) - \beta(t)$ for each $t > 0$. Also, suppose

(C1) *either $\mathcal{T}$ is continuous, or $(\{x_n\} \in \mathcal{H}$ is non-decreasing so that $x_n \to x$, then $x = \sup\{x_n\})$;*

(C2) *if there exists $x_0 \in \mathcal{H}$ so that $x_0 \preceq \mathcal{T}x_0$.*

Then $\mathcal{T}$ has a fixed point.

Furthermore, if for all $x, y \in \mathcal{H}$, there is $z \in \mathcal{H}$ comparable to x and y, then there exists a unique fixed point of $\mathcal{T}$.

In this manuscript, we give some fixed point theorems in partially ordered complete metric spaces and derive some extended forms of Suzuki and Banach fixed point theorems via a τ-distance by applying some new control functions. The presented work upgrades some popular results from literature, particularly Theorems 1.9–1.12 in the context of usual and partially ordered metric spaces. For the authenticity of the established results, some remarks, examples and applications are also discussed.

2. Fixed point results

Let W represent family of mappings $\Psi : \mathbb{R} \to (0, +\infty)$ fulfilling properties (i)-(iii) of Theorem 1.10. The first result of this section is as follows:

Theorem 2.1. *Let $(\mathcal{H}, d, \preceq)$ be a partially ordered complete metric space and $\mathcal{T} : \mathcal{H} \to \mathcal{H}$ be non-decreasing such that for all $x, y \in \mathcal{H}$,*

$$\Psi(\varphi(d(\mathcal{T}(x), \mathcal{T}(y)))) \leq \frac{\Psi(\alpha(\lambda(x, y)))}{\Psi(\beta(\lambda(x, y)))} \ with \quad x \succeq y, \ x \neq y, \tag{2.1}$$

where

$$\lambda(x,y) \;=\; \max\left\{\frac{d(x,\mathcal{T}(x))d(y,\mathcal{T}(y))}{d(x,y)}, d(x,y)\right\},$$

where φ *is an altering distance function and* $\alpha,\beta : [0,+\infty) \to [0,+\infty)$ *are continuous satisfying* $\alpha(t) \geq \beta(t)$ *for each* $t \geq 0$ *such that*

$$\Psi(\varphi(t)) \;>\; \frac{\Psi(\alpha(t))}{\Psi(\beta(t))}, \quad \textit{for each } \; t > 0.$$

Further, assume that

- **(i)** *either* $\mathcal{T}$ *is continuous, or (if* $\{x_n\} \in X$ *is non-decreasing such that* $x_n \to x$ *,then* $x = \sup\{x_n\}$*);*
- **(ii)** *there is* $x_0 \in X$ *such that* $x_0 \preceq \mathcal{T}x_0$*.*

Then there is a fixed point of $\mathcal{T}$*. Moreover, if for all* $x,y \in \mathcal{H}$*, there is* $z \in \mathcal{H}$ *comparable to* x *and* y*, then the fixed point of* $\mathcal{T}$ *is unique.*

Proof. Let $\{x_n\}$ be a sequence given as $x_{n+1} = \mathcal{T}x_n$ for $n \geq 0$. If there is n_0 such that $x_{n_0} = x_{n_0+1} = \mathcal{T}x_0$, so x_{n_0} is a fixed point of $\mathcal{T}$. Otherwise, assume that $x_n \neq x_{n+1}$ for all $n \geq 0$ then $d(x_n,x_{n+1}) \neq 0$ for each $n \geq 0$. Since x_{n-1} and x_n are comparable for all $n \geq 1$, we have from (2.1),

$$\Psi(\varphi(d(x_{n+1},x_n))) \;=\; \Psi(\varphi(d(\mathcal{T}x_n,\mathcal{T}x_{n-1})))$$

$$\Psi(\varphi(d(x_{n+1},x_n))) \;\leq\; \frac{\Psi(\alpha(\lambda(x_n,x_{n-1})))}{\Psi(\beta(\lambda(x_n,x_{n-1})))} \tag{2.2}$$

where,

$$\begin{aligned}\lambda(x_n,x_{n-1}) &= \max\left\{\frac{d(x_n,\mathcal{T}x_n)d(x_{n-1},\mathcal{T}x_{n-1})}{d(x_n,x_{n-1})}, d(x_n,x_{n-1})\right\}\\ &= \max\left\{\frac{d(x_n,x_{n+1})d(x_{n-1},x_n)}{d(x_n,x_{n-1})}, d(x_n,x_{n-1})\right\}.\end{aligned}$$

Thus, from (2.2),

$$\Psi(\varphi(d(x_{n+1},x_n))) \;\leq\; \frac{\Psi\Big(\alpha(max\{d(x_n,x_{n+1}),d(x_n,x_{n-1})\})\Big)}{\Psi\Big(\beta(max\{d(x_n,x_{n+1}),d(x_n,x_{n-1})\})\Big)}. \tag{2.3}$$

If for some n, $d(x_n,x_{n+1}) \geq d(x_n,x_{n-1})$, then by (2.3)

$$\Psi(\varphi(d(x_{n+1},x_n))) \;\leq\; \frac{\Psi\Big(\alpha(d(x_n,x_{n+1}))\Big)}{\Psi\Big(\beta(d(x_n,x_{n+1}))\Big)}. \tag{2.4}$$

By assumptions, we find

$$\Psi(\varphi(d(x_{n+1}, x_n))) \quad < \quad \Psi(\varphi(d(x_{n+1}, x_n))),$$

which is a contradiction. Thus, for all $n \geq 1$, we have

$$d(x_n, x_{n+1}) \quad < \quad d(x_n, x_{n-1}).$$

That is, $\{d(x_n, x_{n+1})\}$ is a non negative real, decreasing sequence, so there exists $\delta \geq 0$ such that

$$\lim_{n\to+\infty} d(x_n, x_{n+1}) \quad = \quad \delta.$$

Assume that $\delta > 0$. Since $d(x_n, x_{n+1}) < d(x_n, x_{n-1})$, by 2.3 we have

$$\Psi(\varphi(d(x_{n+1}, x_n))) \quad \leq \quad \frac{\Psi\Big(\alpha(d(x_n, x_{n-1}))\Big)}{\Psi\Big(\beta(d(x_n, x_{n-1}))\Big)}.$$

Taking limit on both sides, we get

$$\Psi(\varphi(\delta)) \quad \leq \quad \frac{\Psi(\alpha(\delta))}{\Psi(\beta(\delta))} < \Psi(\varphi(\delta)).$$

It is a contradiction, i.e., $\delta = 0$. That is,

$$\lim_{n\to+\infty} d(x_n, x_{n+1}) \quad = \quad 0. \tag{2.5}$$

Further, to prove that $\{x_n\}$ is Cauchy, suppose the contrary there are $\varepsilon > 0$ and subsequences $\{x_{n_k}\}$ and $\{x_{m_k}\}$ of $\{x_n\}$ with $n_k > m_k > k, k \geq 1$ satisfying

$$d(x_{m_k}, x_{n_k}) \quad \geq \quad \varepsilon \tag{2.6}$$

and

$$d(x_{m_k}, x_{n_{k-1}}) \quad < \quad \varepsilon. \tag{2.7}$$

For all $k \geq 0$, we have

$$\varepsilon \leq d(x_{m_k}, x_{n_k}) \leq d(x_{m_k}, x_{n_{k-1}}) + d(x_{n_{k-1}}, x_{n_k}) < \varepsilon + d(x_{n_{k-1}}, x_{n_k}). \tag{2.8}$$

Taking $\lim_{k\to+\infty}$ and using Eqs (2.8) and (2.7),

$$\lim_{k\to+\infty} d(x_{m_k}, x_{n_k}) \quad = \quad \varepsilon. \tag{2.9}$$

Using triangle inequality, we get

$$\begin{aligned} \Psi(d(x_{m_k}, x_{n_k})) \quad &\leq \quad \Psi(d(x_{m_k}, x_{m_{k-1}}).\Psi(d(x_{m_{k-1}}, x_{m_{k-2}}).\Psi(d(x_{m_{k-2}}, x_{n_k}) \\ \Psi(d(x_{m_{k-1}}, x_{n_{k-1}}) \quad &\leq \quad \Psi(d(x_{m_{k-1}}, x_{m_k}).\Psi(d(x_{m_k}, x_{n_k}).\Psi(d(x_{n_k}, x_{n_{k-1}}). \end{aligned}$$

Taking $k \to +\infty$ in above inequalities and using Eqs (2.5) and (2.9), we have

$$\lim_{k\to+\infty} d(x_{m_{k-1}}, x_{n_{k-1}}) = \varepsilon. \tag{2.10}$$

As $n_k > m_k$, and $x_{n_{k-1}}$ and $x_{m_{k-1}}$ are comparable, then from Eq (2.1),

$$\Psi\Big(\varphi(d(x_{n_k}, x_{m_k}))\Big) = \Psi\Big(\varphi(d(\mathcal{T}x_{n_{k-1}}, \mathcal{T}x_{m_{k-1}}))\Big) \leq \frac{\Psi\Big(\alpha(\lambda(x_{n_{k-1}}, x_{m_{k-1}}))\Big)}{\Psi\Big(\beta(\lambda(x_{n_{k-1}}, x_{m_{k-1}}))\Big)}, \tag{2.11}$$

where

$$\begin{aligned}\lambda(x_{n_{k-1}}, x_{m_{k-1}}) &= \max\left\{\frac{d(x_{n_{k-1}}, \mathcal{T}x_{n_{k-1}}).d(x_{m_{k-1}}, \mathcal{T}x_{m_{k-1}})}{d(x_{n_{k-1}}, x_{m_{k-1}})}, d(x_{n_{k-1}}, x_{m_{k-1}})\right\}\\ &= \max\left\{\frac{d(x_{n_{k-1}}, x_{n_k}).d(x_{m_{k-1}}, x_{m_k})}{d(x_{n_k}, x_{m_{k-1}})}, d(x_{n_{k-1}}, x_{m_{k-1}})\right\}.\end{aligned}$$

Taking $k \to +\infty$ in the above inequality and using (2.5), (2.9) and (2.10), we have

$$\lim_{k\to+\infty} \lambda(x_{n_{k-1}}, x_{m_{k-1}}) = \max\{0, \varepsilon\} = \varepsilon. \tag{2.12}$$

Taking $k \to +\infty$ in (2.11) and using (2.12), we have

$$\Psi(\varphi(\varepsilon)) \leq \frac{\Psi(\alpha(\varepsilon))}{\Psi(\beta(\varepsilon))} < \Psi(\varphi(\varepsilon)).$$

It is a contradiction. Therefore, $\{x_n\}$ is Cauchy. So, there exists $z \in \mathcal{H}$ such that

$$\lim_{n\to+\infty} x_n = z. \tag{2.13}$$

Next, we prove that $\mathcal{T}(z) = z$.
Case:1 $\mathcal{T}$ is continuous. We have

$$\begin{aligned} z &= \lim_{n\to+\infty} x_{n+1}\\ &= \lim_{n\to+\infty} \mathcal{T}x_n\\ &= \mathcal{T}(\lim_{n\to+\infty} x_n)\\ &= \mathcal{T}z.\end{aligned}$$

Case:2 Let $\{x_n\} \in \mathcal{H}$ be a non-decreasing sequence such that $x_n \to z$, then $z = \sup\{x_n\}$. In particular, $x_n \preceq z$ for all n. Since $\mathcal{T}$ is non-decreasing, we have $\mathcal{T}x_n \preceq \mathcal{T}z$ for all n, that is, $x_{n+1} \preceq \mathcal{T}z$ for all n. Moreover, as $x_n \preceq x_{n+1} \preceq \mathcal{T}z$ for all n and $z = \sup\{x_n\}$, hence $z \leq \mathcal{T}z$. Now, define $\{y_n\}$ by $y_0 = z$, $y_{n+1} = \mathcal{T}y_n$, $n = 0, 1, 2, \cdots$. Since $y_0 \preceq \mathcal{T}y_0$, therefore $\{y_n\}$ is non-decreasing and thus $z = \sup\{x_n\}$. Hence for all n, we have

$$x_n < z = y_0 \leq \mathcal{T}z \leq y_n \leq y.$$

Assume that $z \neq y$, hence from (2.2),

$$\Psi(\varphi(d(y_{n+1}, x_{n+1}))) = \Psi(\varphi(d(\mathcal{T} y_n, \mathcal{T} x_n))) \leq \frac{\Psi(\alpha(\lambda(y_n, x_n)))}{\Psi(\beta(\lambda(y_n, x_n)))}, \tag{2.14}$$

where,

$$\begin{aligned} \lambda(y_n, x_n) &= \max\left\{\frac{d(y_n, \mathcal{T} y_n).d(x_n, \mathcal{T} x_n)}{d(y_n, x_n)}, d(y_n, x_n)\right\} \\ &= \max\left\{\frac{d(y_n, y_{n+1}).d(x_n, x_{n+1})}{d(y_n, x_n)}, d(y_n, x_n)\right\}. \end{aligned}$$

Taking $n \to +\infty$, in the above inequality, we obtain

$$\lim_{n\to+\infty} \lambda(y_n, x_n) = max\{0, d(y, z)\} = d(y, z). \tag{2.15}$$

Applying $\lim_{n\to\infty}$ in (2.14) and using (2.15), we get

$$\Psi(\varphi(d(y, z))) \leq \frac{\Psi(\alpha(d(y, z)))}{\Psi(\beta(d(y, z)))}.$$

Thus, $d(y, z) = 0$, therefore $y = z$. Thus, we have $z \leq \mathcal{T} z \leq z$. Hence, $\mathcal{T} z = z$.

Let us assume that $x, y \in \mathcal{H}$ are two fixed points of $\mathcal{T}$, so there exists $u \in \mathcal{H}$ which is comparable to x and y. Define the sequence $\{u_n\}$ as $u_0 = u, u_{n+1} = \mathcal{T} u_n$ for all $n = 0, 1, 2, 3, \ldots$. Since u is comparable with x, we can assume that $u \leq x$. Continuing in this way, we can show that $u_n \leq x$ for all n. Suppose that there exists $n_0 \geq 1$ such that $u_{n_0} = x$, then $u_n = \mathcal{T} u_{n-1} = \mathcal{T} x = x$ for all $n \geq n_0 - 1$. This implies that $u_n \to \infty$. If $u_n \neq x$ for all n, then from (2.2) we have

$$\Psi(\varphi(d(x, u_n))) = \Psi(\varphi(d(\mathcal{T} x, \mathcal{T} u_{n-1}))) \leq \frac{\Psi(\alpha(\lambda(x, u_{n-1})))}{\Psi(\beta(\lambda(x, u_{n-1})))}, \tag{2.16}$$

where,

$$\lambda(x, u_n) = \max\left\{\frac{d(x, \mathcal{T} x).d(u_{n-1}, \mathcal{T} u_{n-1})}{d(x, u_{n-1})}, d(x, u_{n-1})\right\} = d(x, u_{n-1}).$$

Thus, from (2.16), we obtain

$$\Psi(\varphi(d(x, u_n))) \leq \frac{\Psi(\alpha(d(x, u_{n-1})))}{\Psi(\beta(d(x, u_{n-1})))} \leq \Psi(\varphi(d(x, u_{n-1}))) \tag{2.17}$$

$$\Psi(\varphi(d(x, u_n))) \leq \Psi(\varphi(d(x, u_{n-1})) \tag{2.18}$$

$$\varphi(d(x, u_n)) \leq \varphi(d(x, u_{n-1}). \tag{2.19}$$

Since φ as an altering distance functions, one has $d(x, u_n) \leq d(x, u_{n-1})$ for all $n \geq 1$ that is, $\{d(x, u_n)\}$ is a decreasing sequence of positive real numbers, therefore there exists $v \geq 0$ such that $d(x, u_n) \to v$. Assume that $v > 0$. By taking $\lim_{n\to\infty}$ of both side of (2.15), we get a contradiction and hence $v = 0$. Thus, in both cases we have $u_n \to x$. Similarly, $u_n \to y$. The limit is unique, therefore $x = y$ and so $\mathcal{T}$ has a unique fixed point. □

The following lemma is useful in the sequel.

Lemma 2.2. *Let $\mathcal{H}$ be a metric space equipped with a τ-distance Q and $\Phi : \mathcal{H} \to (-\infty, +\infty]$ be a bounded below and proper lower semi-continuous map. Define a set*

$$Mg = \{h \in \mathcal{H} : \Psi(\Phi(h)) \cdot \Psi(Q(g,h)) \leq \Psi(\Phi(g))\},$$

where $\Psi \in W$. Let $v \in \mathcal{H}$ and $c \in \mathbb{R}^+$ so that

$$\Phi(v) < +\infty, \quad Mv \neq \emptyset, \quad c \geq \Phi(v) - \inf \Phi(Mv).$$

Then $p : \mathcal{H} \times \mathcal{H} \to R^+$ defined by

$$p(g,h) = \begin{cases} \Phi(g) - \inf \Phi(Mg) & \text{if} \quad g \in Mv \wedge h \in Mg, \\ c + Q(g,h) & \text{if} \quad g \notin Mv \vee h \notin Mg \end{cases}$$

is a τ-distance.

Proof. Suppose that $\xi : \mathcal{H} \times R^+ \to R^+$ is a function fulfilling the properties (Q2)–(Q5) of Definition 1.5. If $h \in Mg$ and $i \in Mh$, then $i \in Mg$ is such that

$$\begin{aligned} \Psi(\Phi(i)) \cdot \Psi(Q(g,i)) &\leq \Psi(\Phi(i)) \cdot \Psi(Q(g,h) + Q(h,i)) \\ &\leq \Psi(\Phi(i)) \cdot \Psi(Q(h,i)) \cdot \Psi(Q(g,h)) \\ &\leq \Psi(\Phi(h)) \cdot \Psi(Q(g,h)) \\ &\leq \Psi(\Phi(g)). \end{aligned}$$

If $g \in Mv$ and $h \in Mg$, then we have

$$\Psi(\Phi(h) \cdot \Psi(Q(g,h)) \leq \Psi(\Phi(g)),$$

$$\Psi(\Phi(h) + Q(g,h)) \leq \Psi(\Phi(g)),$$

$$\Phi(h) + Q(g,h) \leq \Phi(g),$$

$$\begin{aligned} Q(g,h) &\leq \Phi(g) - \Phi(h) \\ &\leq p(g,h) \\ &= \Phi(g) - \inf \Phi(Mg) \\ &\leq \Phi(v) - \inf \Phi(Mv) \\ &\leq c \end{aligned}$$

and hence

$$Q(g,h) \leq p(g,h) \leq c + Q(g,h),$$

for all $g, h \in \mathcal{H}$.

To show that p is a τ-distance, we need to show that the properties $(Q1)$ and $(Q3)$ are satisfied for a function p. Fix $g, h, i \in \mathcal{H}$. If $g \in Mv$, $h \in Mv$ and $h \in Mg$, $i \in Mh$, then $i \in Mg$. Thus, we have

$$p(g, i) = p(g, h) \leq p(g, h) + p(h, i).$$

Also, we have

$$\begin{aligned} p(g, i) &\leq c + Q(g, i) \\ &\leq c + Q(g, h) + Q(h, i) \\ &\leq p(g, h) + p(h, i) \end{aligned}$$

and hence $(Q1)$ is proved.

Now, for the proof of $(Q3)$, suppose $\lim_{n\to+\infty} g_n = g$ and

$$\limsup\{\xi(i_n, p(i_n, g_m)) : m \geq n\} = 0.$$

Fix $t \in \mathcal{H}$. From

$$\limsup\{\xi(i_n, Q(i_n, g_m)) : m \geq n\} = 0,$$

we have

$$Q(t, g) \leq \liminf_{n\to+\infty} Q(t, g_n).$$

If $t \in Mv$, then there exists $\{g_{n_k}\} \subset \{g_n\}$ such that $g_{n_k} \in Mt$ for all $k \in N$. Then we have $g \in Mt$. In fact, we have

$$\begin{aligned} \Psi(\Phi(g)) \cdot \Psi(Q(t, g)) &\leq \Psi(\liminf \Phi(g_n)) \cdot \Psi(\liminf(Q(t, g_n))) \\ &\leq \liminf(\Psi(\Phi(g_n)) \cdot \Psi(Q(t, g_n))) \\ &\leq \liminf(\Psi(\Phi(g_{n_k})) \cdot \Psi(Q(t, g_{n_k}))) \\ &\leq \Psi(\Phi(t)). \end{aligned}$$

Thus,

$$\begin{aligned} p(t, g) &= \Phi(t) - \inf \Phi(Mt) \\ &= \lim_{k\to+\infty} p(t, g_{n_k}) \\ &\leq \liminf_{n\to+\infty} p(t, g_n). \end{aligned}$$

Also, one has

$$\begin{aligned} p(t, g) &\leq c + Q(t, g) \\ &\leq \liminf_{n\to+\infty}(c + Q(t, g_n)) \\ &= \liminf_{n\to+\infty} p(t, g_n). \end{aligned}$$

Hence, p is a τ-distance. □

Proposition 2.3. *Let $\mathcal{H}$ represent a complete metric space having a τ-distance Q. Assume that $\Phi : \mathcal{H} \to (-\infty, +\infty]$ is a bounded-below and proper lower semi-continuous map. Define a set Mg by*

$$Mg = \{h \in \mathcal{H} : \Psi(\Phi(h)) \cdot \Psi(Q(g,h)) \leq \Psi(\Phi(g))\},$$

where $\Psi \in W$. Then, for all $v \in \mathcal{H}$ with $Mv \neq \emptyset$, there exists $g_0 \in Mv$ such that $Mg_0 \subset \{g_0\}$.

Proof. Fix $v \in \mathcal{H}$ with $Mv \neq \emptyset$ and select $v_1 \in Mv$ with $\Phi(v_1) < +\infty$. If $Mv_1 = \emptyset$, then there is nothing to prove. Let us suppose that $Mv_1 \neq \emptyset$ and $Mg \cap (\mathcal{H} \setminus \{g\}) \neq \emptyset$ for all $g \in Mv_1$. Fix $v_2 \in Mv_1$. From $\Phi(h) \leq \Phi(g)$ for all $g \in \mathcal{H}$ and $h \in Mg$, we define the following function $S : \mathcal{H} \to \mathcal{H}$ as follows: For all $g \in Mv_1$, Sg satisfies

$$Sg \in Mg, \quad Sg \neq g, \quad \Phi(Sg) \leq \frac{S(g) + \inf S(Mg)}{2}.$$

For all $g \notin Mv_1$, define $Sg = v_2 \neq g$. Also, we define a mapping $p : \mathcal{H} \times \mathcal{H} \to \mathbb{R}^+$ by

$$p(g,h) = \begin{cases} \Phi(g) - \inf \Phi(Mg) & \text{if } \ g \in Mv_1 \wedge h \in Mg, \\ 2(\Phi(v_1) - \inf \Phi(Mv_1)) + 1 + Q(g,h) & \text{if } \ g \notin Mv_1 \vee h \notin Mg. \end{cases}$$

Thus, from Lemma 2.2, p is a τ-distance. $h \in Mg$ together with $i \in Mh$ imply that $i \in Mg$. Thus, $Sg \in Mv_1$ and $MSg \subset Mg$ for each $g \in Mv_1$. If $g \in Mv_1$, then we have

$$\begin{aligned} p(Sg, S^2 g) &= \Phi(Sg) - \inf \Phi(MSg) \leq \Phi(Sg) - \inf \Phi(Mg) \\ &\leq \frac{\Phi(g) - \inf \Phi(Mg)}{2} = \frac{p(g, Sg)}{2}. \end{aligned}$$

If $g \notin Mv_1$, then we have

$$\begin{aligned} p(Sg, S^2 g) &= p(v_2, Sv_2) = \Phi(v_2) - \inf \Phi(Mv_2) \\ &\leq \Phi(v_1) - \inf \Phi(Mv_1) \\ &\leq \frac{p(g, v_2)}{2} = \frac{p(g, Sg)}{2}. \end{aligned}$$

Now, we use Theorem 1.7, that is, suppose that

$$\lim_{n \to +\infty} \sup\{p(g_n, g_m) : m > n\} = 0, \quad \lim_{n \to +\infty} p(g_n, h) = 0.$$

By the definition of p, let us suppose $g_n \in Mv_1$ and $h \in Mg_n$ for all $n \in \mathbb{N}$. Then $h \in Mv_1$ and so $Sh \in Mh \subset Mg_n$. We have $\lim_{n \to +\infty} p(g_n, Sh) = \lim_{n \to +\infty} p(g_n, h) = 0$ and so $Sh = h$, which is a contradiction because by the definition of S, $Sh \neq h$. Hence, there exists $g_0 \in Mv_1 \subset Mv$ such that $Mg_0 \subset \{g_0\}$. □

Theorem 2.4. *Let $\mathcal{H}$ represent a complete metric space endowed with a τ-distance Q. Let $\mathcal{T} : \mathcal{H} \to \mathcal{H}$ be a self-mapping and $\Phi : \mathcal{H} \to (-\infty, +\infty]$ be a bounded-below and proper lower semi-continuous function. Assume that $g_0 \in \mathcal{H}$ such that*

$$\Psi(\Phi(\mathcal{T}(g))) \cdot \Psi(Q(g, \mathcal{T}(g))) \leq \Psi(\Phi(g)),$$

for all $g \in \mathcal{H}$. Then $\mathcal{T}(g_0) = g_0$ and $Q(g_0, g_0) = 0$.

Proof. Define a set Mg by

$$Mg = \{h \in \mathcal{H} : \Psi(\Phi(\mathcal{T}(g))) \cdot \Psi(Q(g, \mathcal{T}(g)) \leq \Psi(\Phi(g)))\},$$

where $\Psi \in W$. By Proposition 2.3, there is $g_0 \in \mathcal{H}$ so that $Mg_0 \subset \{g_0\}$. Since $\mathcal{T}(g_0) \in Mg_0$, we have $\mathcal{T}(g_0) = g_0$ and $\Phi(g_0) < +\infty$. Indeed, if $\Phi(g_0) = +\infty$, then $\mathcal{H} = Mg_0 \subset \{g_0\}$, which is a contradiction. So $\Phi(g_0) < +\infty$ and

$$\begin{aligned}
\Psi(\Phi(g_0)) \cdot \Psi(Q(g_0, g_0)) &= \Psi(\Phi(\mathcal{T}(g_0))) \cdot \Psi(Q(g_0, \mathcal{T}(g_0))) \\
&\leq \Psi(\Phi(g_0)) \\
\Psi(\Phi(g_0)) \cdot \Psi(Q(g_0, g_0)) &\leq \Psi(\Phi(g_0)) \\
\Phi(g_0) + Q(g_0, g_0) &\leq \Phi(g_0) \\
Q(g_0, g_0) &= 0.
\end{aligned}$$

Hence, the conclusion follows. □

Here, we provide a new form of the Banach type fixed point result using a τ-distance.

Theorem 2.5. *Let $\mathcal{H}$ represent a complete metric space with a metric d, $\mathcal{T} : \mathcal{H} \to \mathcal{H}$ be a continuous self-mapping and Q be a τ-distance. Suppose there exists a function $F : [0, +\infty) \to [0, +\infty)$ so that $\lim_{x\to 0} F(x) = 0$, $F(0) = 0$. Also, if the following conditions hold:*

$$\lim_{n\to+\infty} \sup\{Q(a_n, a_m) : m > n\} = 0, \quad \lim_{n\to+\infty} Q(a_n, \mathcal{T}(a_n)) = 0, \quad \lim_{n\to+\infty} Q(a_n, b) = 0,$$

then $\mathcal{T}(b) = b$ and

$$Q(\mathcal{T}(a), \mathcal{T}(b)) \leq F(Q(a, b)) - F(Q(\mathcal{T}(a), \mathcal{T}(b)))$$

for all $a, b \in \mathcal{H}$, then there is a fixed point of $\mathcal{T}$.

Proof. Let $\{a_n\} \in \mathcal{H}$ such that $a_{n+1} = \mathcal{T}(a_n)$ for all $n \geq 0$. Observe that

$$\begin{aligned}
0 &< Q(a_n, a_{n+1}) \\
&\leq F(Q(a_{n-1}, a_n)) - F(Q(\mathcal{T}(a_{n-1}), \mathcal{T}(a_n))) \\
&\leq F(Q(a_{n-1}, a_n)) - F(Q(a_n, a_{n+1})),
\end{aligned}$$

$$F(Q(a_n, a_{n+1})) \leq F(Q(a_{n-1}, a_n)).$$

Hence, the sequence $\{F(Q(a_n, a_{n+1}))\}$ is bounded below and non-increasing. So, there is $r \in \mathbb{R}^+$ so that

$$\lim F(Q(a_n, a_{n+1})) = r.$$

Now, for all $n, m \in N$ with $n < m$, we have

$$\begin{aligned}
Q(a_n, a_m) &\leq \sum_{i=n}^{m-1} Q(a_i, a_{i+1}) \\
&\leq \sum_{i=n}^{m-1} (F(Q(a_{i-1}, a_i)) - F(Q(a_i, a_{i+1}))) \\
&= F(Q(a_{n-1}, a_n)) - r,
\end{aligned}$$

$$Q(a_n, a_m) \leq F(Q(a_{n-1}, a_n)) - r.$$

Applying the limit superior in the above inequality, we find that

$$\limsup_{n\to+\infty} \{Q(a_n, a_m) : m > n\} = 0.$$

Thus, $\{a_n\}$ is p-Cauchy, so $\{a_n\}$ is a Cauchy sequence. As $\mathcal{H}$ is complete, there is $a_0 \in \mathcal{H}$ such that $\lim_{n\to+\infty} a_n = a_0$.

Now, we show that $\mathcal{T}(a_0) = a_0$. In fact, from (Q_3),

$$\begin{aligned}
\limsup_{n\to+\infty} (Q(a_n, \mathcal{T}(a_n)) + Q(a_n, a_0)) &\leq \limsup_{n\to+\infty} Q(a_n, a_{n+1}) + \liminf_{n\to+\infty} Q(a_n, a_m) \\
&\leq 2 \limsup_{n\to+\infty} (Q(a_n, a_m) = 0.
\end{aligned}$$

Hence, $\mathcal{T}(a_0) = a_0$. This completes the proof. □

Taking $F(x) = x^2$ in Theorem 2.5, we get the following corollary.

Corollary 2.6. *Let $(\mathcal{H}, d)$ be a complete metric space and Q be a τ-distance. Let $\mathcal{T} : \mathcal{H} \to \mathcal{H}$ be a continuous self-mapping. Suppose if*

$$\limsup_{n\to+\infty} \{Q(a_n, a_m) : m > n\} = 0, \quad \lim_{n\to+\infty} Q(a_n, \mathcal{T}(a_n)) = 0, \quad \lim_{n\to+\infty} Q(a_n, b) = 0,$$

then $\mathcal{T}(b) = b$, and

$$Q(\mathcal{T}(u), \mathcal{T}(v)) \leq [Q(u, v)]^2 - [Q(\mathcal{T}(u), \mathcal{T}(v))]^2,$$

for all $u, v \in \mathcal{H}$. Then there is a fixed point of $\mathcal{T}$.

Put $F(x) = xe^x$ in Theorem 2.5, we get the following corollary.

Corollary 2.7. *Let $(\mathcal{H}, d)$ be a complete metric space and Q be a τ-distance. Let $\mathcal{T} : \mathcal{H} \to \mathcal{H}$ be a continuous self-mapping. Suppose if*

$$\limsup_{n\to+\infty} \{Q(a_n), a_m : m > n\} = 0, \quad \lim_{n\to+\infty} Q(a_n, \mathcal{T}(a_n)) = 0, \quad \lim_{n\to+\infty} Q(a_n, b) = 0,$$

then $\mathcal{T}(b) = b$, and

$$\frac{Q(\mathcal{T}(u), \mathcal{T}(v))(1 + e^{Q(\mathcal{T}(u), \mathcal{T}(v))}}{Q(u, v) e^{Q(u,v))}} \leq 1 \;\; for\ all u, v \in \mathcal{H} \wedge u \neq v.$$

Then there is a fixed point of $\mathcal{T}$.

Remark 2.8. (1) If we put $\Psi(x) = e^x$ in Theorem 2.1 and Theorem 2.4, we get the result of Gupta *et al.* [26] and Suzuki [7].

(2) If we replace the τ-distance Q in Theorem 2.4 by the usual distance d, we get the result of 1.10 of Isik *et al.* [24].

Remark 2.9. The above result (Theorem 2.5) generalizes the Banach Contraction Principle with respect to the τ-distance. By putting $\Psi(y) = \frac{\alpha}{1-\alpha}y$, where $\alpha \in [0, 1)$, we have

$$Q(\mathcal{T}(a), \mathcal{T}(b)) \leq \frac{\alpha}{1-\alpha}Q(a, b) - \frac{\alpha}{1-\alpha}Q(\mathcal{T}(a), \mathcal{T}(b)),$$

$$(1-\alpha)Q(\mathcal{T}(a), \mathcal{T}(b)) + (\alpha)Q(\mathcal{T}(a), \mathcal{T}(b)) \leq \alpha Q(a, b),$$

$$(1-\alpha+\alpha)Q(\mathcal{T}(a), \mathcal{T}(b)) \leq \alpha Q(a, b).$$

Hence,

$$Q(\mathcal{T}(a), \mathcal{T}(b)) \leq \alpha Q(a, b).$$

Example 2.10. Let $\mathcal{H} = \{0, 1\}$ and $d : \mathcal{H} \times \mathcal{H} \to [0, \infty)$ be defined by

$$d(x, y) = \begin{cases} x + y, & \text{if } x \neq y \\ 0, & \text{if } x = y. \end{cases}$$

then clearly $(\mathcal{H}, d)$ is a complete metric space. Let the partial ordered $\preceq$ be defined by: $x \preceq y$ iff $x \leq y$. Now, define the self-mapping $\mathcal{T} : \mathcal{H} \to \mathcal{H}$ as

$$\mathcal{T}(x) = \begin{cases} x - \frac{1}{2}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0. \end{cases}$$

Let us define $\varphi, \alpha, \beta : [0\ \infty) \to [0\ \infty)\ \forall\ u \geq 0$ as $\varphi(u) = u, \alpha(u) = 4u, \beta(u) = 2u$ and taking $\Psi(u) = \frac{1}{e^u}$, clearly the following conditions are satisfied.

(i) $\frac{\Psi(\alpha(u))}{\Psi(\beta(u))} = \frac{1}{e^{2u}} \leq \frac{1}{e^u} = \Psi(\varphi(u))$; for all $u \geq 0$;

(ii) $\Psi(\varphi(u)) - \frac{\Psi(\alpha(u))}{\Psi(\beta(u))} = 0 \quad if\ u = 0$;

(iii) Also $\alpha(u) \geq (\beta(u))$.

Now, without use of generality, assume that $x > y$, then clearly the self-mapping satisfies all conditions of Theorem 2.1. Also, $0 \preceq 0$ and $\mathcal{T} = 0$, therefore 0 is the unique fixed point of $\mathcal{T}$.

Remark 2.11. As $\alpha(u) - \beta(u) = 2u > u = \varphi(u)$, therefore Theorem 1.12 of Gupta *et al.* [26] failed to satisfy the condition $\varphi(u) < \alpha(u) - \beta(u)$. Therefore, Theorem 1.12 of Gupta *et al.* [26] is not applicable to this example. Hance our result is the generalization of Gupta *et al.* [26].

3. An application

Alqahtani *et al.* [27] proposed a solution for Volterra type fractional integral equations by using a hybrid type contraction that unifies both nonlinear and linear type inequalities in the context of metric spaces. Rezan *et al.* [28] focused on developing alternative existence and uniqueness criteria for higher-order nonlinear fractional differential equations with integral and multi-point boundary conditions.

Here, we give an existence and uniqueness result of integral equations.
Let $\mathcal{L} : (0,\infty) \to (0,\infty)$ be Lebesgue and local summable. Consider $\mathcal{S} : [0,\infty) \to [0,\infty)$ defined by

$$\mathcal{S}(u) \quad = \quad \int_0^u \mathcal{L}(u)du,$$

where $u > 0$. $\mathcal{S}$ is well-defined, non-decreasing and continuous. Also, $\mathcal{S}(\varepsilon) > 0$ for each $\varepsilon > 0$ and $\mathcal{S}(u) = 0$ if and only if $u = 0$.

Theorem 3.1. *Let $(H, d, \preceq)$ be a complete partially ordered metric space equipped with a metric d. Suppose that $\mathcal{T} : \mathcal{H} \to \mathcal{H}$ is a non-decreasing self-mapping such that for all $x, y \in \mathcal{H}$ with $x \geq y$ and $x \neq y$, we have*

$$\Psi(\varphi(\int_0^{d(\mathcal{T}(x),\mathcal{T}(y))} \mathcal{L}(u)du)) \quad \leq \quad \frac{\Psi(\alpha(\int_0^{\lambda(x,y)} \mathcal{L}(u)du))}{\Psi(\beta(\int_0^{\lambda(x,y)} \mathcal{L}(u)du))}$$

where $\lambda(x, y)$ is defined in Eq (1.1), φ is an altering function, $\mathcal{L}(u)$ is a Lebesgue integrable function and $\alpha, \beta : [0,\infty) \to [0,\infty)$ are continuous functions verifying the conditions $\varphi(v) > \alpha(v) - \beta(v)$ for all $v > 0$ and $\alpha \geq \beta$. Also, assume that:

(i) *Either $\mathcal{T}$ is continuous, or ($\{x_n\}$ is a non-decreasing sequence in $\mathcal{H}$ such that $x_n \to x$, then $x = \sup\{x_n\}$);*

(ii) *There exists $x^* \in \mathcal{H}$ such that $x^* \preceq \mathcal{H}(x^*)$.*

Then $\mathcal{T}$ has a fixed point. If for all $x, y \in \mathcal{H}$, there exists $z \in \mathcal{H}$ which is comparable to x and y, then $\mathcal{H}$ has a unique fixed point.

Proof. Taking $\mathcal{L}(u) = 1$ for all $u \in R$, then the proof can be easily obtained from Theorem 2.1. □

Next, we give the existence result for the following Volterra type integral equation:

$$a(u) = r(u) + \int_0^\Lambda G(u, s)F(s, a(s))ds, \tag{3.1}$$

where $u \in I = [0, \Lambda]$.

Let $I = [0, \Lambda], \mathcal{H} = C(I, R)$ and $Q(a, b) = \sup_{u\in I}[a(u) - b(u)] = ||a - b||$.

Assume that the following conditions are satisfied:

($\mathbf{c_1}$) $F : I \times R \to R$ is continuous;

($\mathbf{c_2}$) $r : I \to R$ is continuous;

($\mathbf{c_3}$) $G : I \times I \to R$ is a measurable and continuous function at $s \in I$ (second variable) and for all $u \in I$;

($\mathbf{c_4}$) $\sup \int_0^\Lambda G(u, s)ds \leq 1$ and $G(u, s) > 0$ for all $u, s \in I$;

($\mathbf{c_5}$) For every $u \in I$ and $a, b \in \mathcal{H}$, with $a \neq b$, $|F(u, a(u)) - F(u, b(u))| \leq |a(u) - b(u)| - 1$.

Theorem 3.2. *If properties* (c_1) – (c_5) *hold, then the integral equation* (3.1) *has a solution in* $\mathcal{H}$.

Proof. Let us define $\mathcal{T} : \mathcal{H} \to \mathcal{H}$ as follows:

$$\mathcal{T}(u) = r(u) + \int_0^\tau G(u, s)F(s, a(s))ds.$$

Now, we have

$$\begin{aligned} \left|\mathcal{T}(a(u)) - \mathcal{T}(b(u))\right| &= \Big|\int_0^\tau G(u, s)(F(s, a(s)) - F(s, b(s)))\Big|ds \\ &\leq \int_0^\tau G(u, s)\Big|F(s, a(s)) - F(s, b(s))\Big|ds \\ &\leq \int_0^\tau G(u, s)(\big|a(u) - b(u)\big| - 1)ds, \end{aligned}$$

$$\begin{aligned} \sup\left|\mathcal{T}(a(u)) - \mathcal{T}(b(u))\right| &\leq \sup\int_0^\tau G(u, s)(\big|a(u) - b(u)\big| - 1)ds \\ &\leq \sup\int_0^\tau G(u, s)(\big\|a(u) - b(u)\big\| - 1)ds \\ &\leq (\big\|a(u) - b(u)\big\| - 1)\sup\int_0^\tau G(u, s)ds, \end{aligned} \tag{3.2}$$

$$\big\|\mathcal{T}(a(u)) - \mathcal{T}(b(u))\big\| \leq \big\|a(u) - b(u)\big\| - 1,$$
$$1 + Q(\mathcal{T}(a(u)), \mathcal{T}(b(u))) \leq Q(a(u), b(u)).$$

This implies that

$$e^{1+Q(\mathcal{T}(a(u)),\mathcal{T}(b(u)))} \leq e^{Q(a(u),b(u))},$$

and so

$$1 + e^{Q(\mathcal{T}(a(u)),\mathcal{T}(b(u)))} \leq e^{1+Q(\mathcal{T}(a(u)),\mathcal{T}(b(u)))} \leq e^{Q(a(u),b(u))}.$$

Therefore, we have

$$1 + e^{Q(\mathcal{T}(a(u)),\mathcal{T}(b(u)))} \leq e^{Q(a(u),b(u))}. \tag{3.3}$$

Thus, from (3.2) and (3.3),

$$Q(\mathcal{T}(a(u)), \mathcal{T}(b(u))(1 + e^{Q(\mathcal{T}(a(u)),\mathcal{T}(b(u)))}) \leq Q(a(u), b(u))e^{Q(a(u),b(u))},$$

$$\frac{Q(\mathcal{T}(a(u)), \mathcal{T}(b(u))(1 + e^{Q(\mathcal{T}(a(u)),\mathcal{T}(b(u)))})}{Q(a(u), b(u))e^{Q(a(u),b(u))}} \leq 1.$$

So, by Corollary 2.7, $\mathcal{T}$ has a fixed point, and hence the integral equation (3.1) has a solution. □

4. Conclusions

In this work, we presented a novel extension of the Banach contraction and Suzuki fixed point theorem by applying some new control functions. The new contraction will be a useful tool for solving integral equations, differential equations, and fractional integro-differential equations that exist. We believe that the multi-valued version is the best option. Researchers can think about the implications of this new contraction. There will be a new multi-valued contraction a useful technique for determining the existence of Volterra-integral inclusions.

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Conflict of interest

All authors declare that they have no competing interest regarding this manuscript.

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