Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 7(1): 1333–1345.
DOI: 10.3934/math.2022079
Received: 25 August 2021
Accepted: 14 October 2021
Published: 25 October 2021

Research article

On a generalization of fractional Langevin equation with boundary conditions

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Abstract: In this work, we consider a generalization of the nonlinear Langevin equation of fractional orders with boundary value conditions. The existence and uniqueness of solutions are studied by using the results of the fixed point theory. Moreover, the previous results of fractional Langevin equations are a special case of our problem.

Keywords: nonlinear Langevin equation; fractional order; existence of solution
Mathematics Subject Classification: 26A33, 34A08, 34A12

1. Introduction

The Langevin equation has been applied to describe phenomena that have stochastic properties. First, Langevin equation [14] used for a particle submerged in a fluid with Brownian motion. Langevin equation has various versions which proposed in different fields. For instance, Pineda and Stamatakis [19] investigated the stochastic modelling of surface reactions by using reflected chemical Langevin equations. In [15], based on Langevin equation, molecular dynamics simulation was used in the study of structural, thermal properties of matter in different phases. Moreover, the modified Langevin equation was applied as a macroscopic stochastic nonlinear model of many geophysical processes in [3]. Mendoza-Méndez et al. [18] proposed a dynamic equivalence between the longtime dynamic properties of atomic and colloidal liquids by a more formal fundamental derivation of the generalized Langevin equation for a tracer particle in an atomic liquid.

The study of existence results of equations with applications in engineering industries and modeling different processes is gaining much importance and attention. Recently, Fallah and Mehrdoust studied [5] the existence and uniqueness of the solution to the stochastic differential equation of the double Heston model which is defined by two independent variance processes with non-Lipschitz diffusion. The existence of global mild solutions was obtained for the incompressible nematic liquid crystal flow in the whole space in [22]. Marasi et al. [17] studied the existence and uniqueness of positive solutions for nonlinear fractional differential equation boundary value problems by using new fixed point results. For some recent works on existence results, see [7–12, 16, 21].

According to the applications of Langevin equation, on the existence theory of solutions for this class of equations is an important subject. Ahmad and Nieto [1] discussed the existence of solutions of nonlinear Langevin equation involving two fractional orders with Dirichlet boundary conditions as follows:

$$\begin{aligned} &{}^{c}D_{0+}^{\beta}\left({}^{c}D_{0+}^{\alpha}+\lambda\right)x(t)=f\left(t,x(t)\right),\ t\in(0,1),\\ &x(0)=\gamma_1,\ x(1)=\gamma_2,\ \ 0<\alpha,\beta\leqslant 1,\end{aligned}$$

where ${}^{c}D^{\alpha}$ is the Caputo fractional derivative of order α, $f:[0,1]\times\mathbb{X}\to\mathbb{X}$ is a continuous function, $\mathbb{X}$ is a Banach space, λ is a real number and $\gamma_1,\gamma_2\in\mathbb{X}$. Also, they introduce a q-fractional variant of nonlinear Langevin equation of different orders with q-fractional antiperiodic boundary conditions in [2].

Fazli and J. Nieto [6] investigated the existence and uniqueness of solutions for nonlinear Langevin equation of fractional orders with anti-periodic boundary conditions as:

$$\begin{aligned} &{}^{c}D_{0+}^{\beta}\left({}^{c}D_{0+}^{\alpha}+\lambda\right)x(t)=f\left(t,x(t)\right),\ t\in(0,1),\ 0<\alpha\leqslant 1,\ 1<\beta\leqslant 2,\\ &x(0)+x(1)=0,\ \ {}^{c}D_{0+}^{\alpha}x(0)+\ {}^{c}D_{0+}^{\alpha}x(1)=0,\ \ \mathcal{D}_{0+}^{2\alpha}x(0)+\mathcal{D}_{0+}^{2\alpha}x(1)=0,\end{aligned}$$

where the function $f:[0,1]\times\mathbb{R}\to\mathbb{R}$ is continuous and λ is a real number and $\mathcal{D}_{0+}^{2\alpha}$ is the sequential fractional derivative. Recently, the existence and uniqueness of initial value problems for nonlinear Langevin equation involving three fractional orders was studied in [4]

This manuscript is concerned to study the existence and uniqueness of solutions a generalization of the nonlinear Langevin equation of different fractional orders with four-point boundary conditions provided as:

$$ {}^{c}D_{0+}^{\beta}\left({}^{c}D_{0+}^{\alpha}+\gamma\right)x(t)=f\left(t,x(t),x'(t)\right),\ t\in(0,1), \tag{1.1}$$

$$ x(0)=x(1)=x'(0)=x'(1)=0,\ \ 0<\alpha\leqslant 1,\ 2<\beta\leqslant 3, \tag{1.2}$$

where ${}^{c}D^{\alpha}$ is the Caputo fractional derivative of order α, $f:[0,1]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is a continuous function and γ is a real number. Under suitable assumptions on part of nonlinear and by application of the fixed point theory, we obtain the existence results of the considered problem.

The paper is scheduled as follows: In Section 2 are recalled some necessary preliminaries from fractional calculus. Next, we obtain the Green functions corresponding to the problem. In section 3 is devoted to the existence and uniqueness of solutions for boundary value problem (1.1) and (1.2). Also, an example to illustrate our results is given. Finally, we propose some conclusions.

2. Preliminaries

In this section, we present some definitions and lemmas which are needed for our results.

Definition 2.1. *[13, 20] The Riemann-Liouville fractional integral of order $\rho>0$ of a function $g:(0,\infty)\to\mathbb{R}$ is defined by*

$$I_{0^+}^{\rho}g(\omega)=\frac{1}{\Gamma(\rho)}\int_0^{\omega}(\omega-s)^{\rho-1}g(s)ds,$$

provided the right-hand side is pointwise defined on $(0, \infty)$.

Definition 2.2. *[13, 20] The Riemann-Liouville fractional derivative of order* $\rho > 0$ *of a continuous function* $g : (0, \infty) \to \mathbb{R}$ *is defined by*

$$D_{0^+}^{\rho} g(\omega) = \frac{1}{\Gamma(m-\rho)} \left(\frac{d}{d\omega}\right)^m \int_0^{\omega} (\omega - s)^{m-\rho-1} g(s) ds,$$

where $m = [\rho] + 1$, *provided that right-hand side is pointwise defined on* $(0, \infty)$.

Definition 2.3. *[13, 20] For a function g given on the interval* $[0, \infty)$, *the Caputo fractional derivative of order* $\rho > 0$ *of g is defined by*

$$^{c}D_{0^+}^{\rho} g(\omega) = \frac{1}{\Gamma(m-\rho)} \int_0^{\omega} (\omega - s)^{m-\alpha-1} g^{(n)}(s) ds,$$

where $m = [\rho] + 1$.

Lemma 2.4. *[13, 20] Let* $\alpha, \beta \geqslant 0$ *and* $n \in \mathbb{N}$, *then the following relations hold:*

1) $D_{a+}^{\alpha} I_{a+}^{\alpha} f(t) = f(t)$,
2) $D_{a+}^{\beta} I_{a+}^{\alpha} f(t) = D_{a+}^{\beta-\alpha} f(t)$, *(if* $\beta \geqslant \alpha$*)*,
3) $D_{a+}^{\beta} I_{a+}^{\alpha} f(t) = I_{a+}^{\alpha-\beta} f(t)$, *(if* $\alpha \geqslant \beta$*)*,
4) $I_{a+}^{\alpha} (t-a)^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} (t-a)^{\alpha+\beta}$,
5) $D_{a+}^{\alpha} (t-a)^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} (t-a)^{\beta-\alpha}$.

Lemma 2.5. *[13] Let* $\rho > 0$. *Then, the fractional differential equation* $^{c}D^{\rho} u(t) = 0$, *has a general solution as*

$$u(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_{m-1} t^{m-1},$$

for some $a_k \in \mathbb{R}$, $k = 1, \ldots, m-1$, *and* $m = [\rho] + 1$.

Lemma 2.6. *[13] Let* $\rho > 0$. *Then, we have*

$$I_{0^+}^{\rho}\, {}^{c}D_{0^+}^{\rho} u(t) = u(t) + a_0 + a_1 t + \cdots + a_{m-1} t^{m-1},$$

where $a_k \in \mathbb{R}$, $k = 1, \ldots, m-1$, *and* $m = [\rho] + 1$.

Lemma 2.7. *Let* $y \in C[0, 1]$, *a unique solution of boundary value problem for following fractional Langevin equation*

$$\begin{aligned} &{}^{c}D_{0+}^{\beta} \left({}^{c}D_{0+}^{\alpha} + \gamma\right) x(t) = y(t), \ t \in (0, 1), \\ &x(0) = x(1) = x'(0) = x'(1) = 0, \ \ 0 < \alpha \leqslant 1, \ 2 < \beta \leqslant 3, \end{aligned}$$

is given by

$$x(t) = \int_0^1 G(t, s) y(s) ds + \int_0^1 H(t, s) x(s) ds,$$

where

$$G(t,s)=\frac{1}{\Gamma(\alpha+\beta)}\begin{cases}(t-s)^{\alpha+\beta-1}+\left[(\alpha+1)t^{\alpha+2}-(\alpha+2)t^{\alpha+1}\right](1-s)^{\alpha+\beta-1}\\ +(\alpha+\beta-1)\left(t^{\alpha+1}-t^{\alpha+2}\right)(1-s)^{\alpha+\beta-2}, & s\leqslant t,\\ \\ \left[(\alpha+1)t^{\alpha+2}-(\alpha+2)t^{\alpha+1}\right](1-s)^{\alpha+\beta-1}\\ +(\alpha+\beta-1)\left(t^{\alpha+1}-t^{\alpha+2}\right)(1-s)^{\alpha+\beta-2}, & t\leqslant s,\end{cases}$$

and

$$H(t,s)=\frac{1}{\Gamma(\alpha)}\begin{cases}-\gamma(t-s)^{\alpha+\beta-1}+\gamma\left[(\alpha+2)t^{\alpha+1}-(\alpha+1)t^{\alpha+2}\right](1-s)^{\alpha-1}\\ +\gamma(\alpha-1)\left(t^{\alpha+2}-t^{\alpha+1}\right)(1-s)^{\alpha-2}, & s\leqslant t,\\ \\ \gamma\left[(\alpha+2)t^{\alpha+1}-(\alpha+1)t^{\alpha+2}\right](1-s)^{\alpha-1}\\ +\gamma(\alpha-1)\left(t^{\alpha+2}-t^{\alpha+1}\right)(1-s)^{\alpha-2}, & t\leqslant s.\end{cases}$$

Proof. For $2<\beta\leqslant 3$, Lemma 2.6 yields

$${}^{c}D^{\alpha}x(t)=I_{0^+}^{\beta}y(t)-\gamma x(t)+c_0+c_1t+c_2t^2,$$

where $c_0,c_1,c_2\in\mathbb{R}$. Also, for $0<\alpha\leqslant 1$, we have

$$x(t)=I_{0^+}^{\alpha+\beta}y(t)-\gamma I_{0^+}^{\alpha}x(t)+c_0\frac{t^{\alpha}}{\Gamma(\alpha+1)}+c_1\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}+c_2\frac{2t^{\alpha+2}}{\Gamma(\alpha+3)}+c_3. \tag{2.1}$$

By the condition $x(0)=0$, we give $c_3=0$. Differentiation of (2.1) with respect to t produces

$$x'(t)=I_{0^+}^{\alpha+\beta-1}y(t)-\gamma I_{0^+}^{\alpha-1}x(t)+c_0\frac{t^{\alpha-1}}{\Gamma(\alpha)}+c_1\frac{t^{\alpha}}{\Gamma(\alpha+1)}+c_2\frac{2t^{\alpha+1}}{\Gamma(\alpha+2)}. \tag{2.2}$$

The condition $x'(0)=0$, for (2.2), implies that $c_0=0$. Now, by conditions $x(1)=0$ and $x'(1)=0$, we give

$$\frac{1}{\Gamma(\alpha+\beta)}\int_0^1(1-s)^{\alpha+\beta-1}y(s)ds-\frac{\gamma}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}x(s)ds+c_1\frac{1}{\Gamma(\alpha+2)}+c_2\frac{2}{\Gamma(\alpha+3)}=0, \tag{2.3}$$

and

$$\frac{1}{\Gamma(\alpha+\beta-1)}\int_0^1(1-s)^{\alpha+\beta-2}y(s)ds-\frac{\gamma}{\Gamma(\alpha-1)}\int_0^1(1-s)^{\alpha-2}x(s)ds+c_1\frac{1}{\Gamma(\alpha+1)}+c_2\frac{2}{\Gamma(\alpha+2)}=0, \tag{2.4}$$

respectively. By solving the system of (2.3) and (2.4), we obtain

$$\begin{aligned}c_1 \;=\;& \frac{\Gamma(\alpha+2)}{\Gamma(\alpha+\beta-1)}\int_0^1(1-s)^{\alpha+\beta-2}y(s)ds-\frac{\Gamma(\alpha+3)}{\Gamma(\alpha+\beta)}\int_0^1(1-s)^{\alpha+\beta-1}y(s)ds\\ &+\frac{\gamma\Gamma(\alpha+3)}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}x(s)ds-\frac{\gamma\Gamma(\alpha+2)}{\Gamma(\alpha-1)}\int_0^1(1-s)^{\alpha-2}x(s)ds,\end{aligned}$$

and

$$c_2 = \frac{(\alpha+1)\Gamma(\alpha+3)}{2\Gamma(\alpha+\beta)}\int_0^1 (1-s)^{\alpha+\beta-1} y(s)ds - \frac{\Gamma(\alpha+3)}{2\Gamma(\alpha+\beta-1)}\int_0^1 (1-s)^{\alpha+\beta-2} y(s)ds$$
$$+\frac{\gamma\Gamma(\alpha+3)}{2\Gamma(\alpha-1)}\int_0^1 (1-s)^{\alpha-2} x(s)ds - \frac{\gamma(\alpha+1)\Gamma(\alpha+3)}{2\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1} x(s)ds.$$

By substituting the values of c_0, c_1, c_2, c_2 in (2.1), then we give

$$x(t) = \frac{1}{\Gamma(\alpha+\beta)}\int_0^t (t-s)^{\alpha+\beta-1} y(s)ds - \frac{\gamma}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} x(s)ds$$
$$+\left[(\alpha+2)t^{\alpha+1} - (\alpha+1)t^{\alpha+2}\right]\frac{\gamma}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1} x(s)ds + \frac{\gamma\left(t^{\alpha+2}-t^{\alpha+1}\right)}{\Gamma(\alpha-1)}\int_0^1 (1-s)^{\alpha-2} x(s)ds$$
$$+\frac{t^{\alpha+1}-t^{\alpha+2}}{\Gamma(\alpha+\beta-1)}\int_0^1 (1-s)^{\alpha+\beta-2} y(s)ds + \frac{(\alpha+1)t^{\alpha+2}-(\alpha+2)t^{\alpha+1}}{\Gamma(\alpha+\beta)}\int_0^1 (1-s)^{\alpha+\beta-1} y(s)ds$$
$$= \int_0^t \left(\frac{(t-s)^{\alpha+\beta-1} + \left[(\alpha+1)t^{\alpha+2} - (\alpha+2)t^{\alpha+1}\right](1-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\right.$$
$$\left.+\frac{(\alpha+\beta-1)\left(t^{\alpha+1}-t^{\alpha+2}\right)(1-s)^{\alpha+\beta-2}}{\Gamma(\alpha+\beta)}\right) y(s)ds$$
$$+\int_t^1 \frac{\left[(\alpha+1)t^{\alpha+2} - (\alpha+2)t^{\alpha+1}\right](1-s)^{\alpha+\beta-1} + (\alpha+\beta-1)\left(t^{\alpha+1}-t^{\alpha+2}\right)(1-s)^{\alpha+\beta-2}}{\Gamma(\alpha+\beta)} y(s)ds$$
$$+\int_0^t \left(\frac{-\gamma(t-s)^{\alpha+\beta-1} + \gamma\left[(\alpha+2)t^{\alpha+1} - (\alpha+1)t^{\alpha+2}\right](1-s)^{\alpha-1}}{\Gamma(\alpha)}\right.$$
$$\left.+\frac{\gamma(\alpha-1)\left(t^{\alpha+2}-t^{\alpha+1}\right)(1-s)^{\alpha-2}}{\Gamma(\alpha)}\right) x(s)ds$$
$$+\int_t^1 \frac{\gamma\left[(\alpha+2)t^{\alpha+1} - (\alpha+1)t^{\alpha+2}\right](1-s)^{\alpha-1} + \gamma(\alpha-1)\left(t^{\alpha+2}-t^{\alpha+1}\right)(1-s)^{\alpha-2}}{\Gamma(\alpha)} x(s)ds$$
$$= \int_0^1 G(t,s)\, y(s)ds + \int_0^1 H(t,s)\, x(s)ds.$$

□

3. Main results

In this section, we propose the existence results of Langevin differential equation of fractional orders (1.1) and (1.2). Let the Banach space $B = C^1[0,1]$ be equipped with the norm:

$$\|x\| = \max_{t\in[0,1]} |x(t)| + \max_{t\in[0,1]} |x'(t)|. \tag{3.1}$$

To prove the main results, we need the following assumptions:

$A_1)$ $f : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous function.
$A_2)$ There exists a constant $w > 0$ such that

$$|f(t,x,y) - f(t,u,v)| \leq w\left(|x-u| + |y-v|\right),$$

for all $t \in [0,1]$, $x, y, u, v \in \mathbb{R}$.
$A_3)$ There exists a nonnegative function $\sigma \in L[0,1]$ such that

$$|f(t,x,y)| \leq \sigma(t) + a_1|x|^{\tau_1} + a_2|y|^{\tau_2},$$

where $a_1, a_2 \in \mathbb{R}^+$ and $0 < \tau_1, \tau_2 < 1$.

For the sake of convenience, we define the following constants:

$$\mathcal{K}_1 = \left\{\tfrac{4(\alpha+1)+2\beta}{\Gamma(\alpha+\beta+1)}\right\}, \mathcal{K}_2 = \left\{\tfrac{2\alpha+4}{\Gamma(\alpha+\beta)} + \tfrac{2(\alpha+1)(\alpha+2)}{\Gamma(\alpha+\beta+1)}\right\}, \mathcal{K} = \mathcal{K}_1 + \mathcal{K}_2,$$
$$\mathcal{L}_1 = \left\{\tfrac{|\gamma|4(\alpha+1)}{\Gamma(\alpha+1)}\right\}, \mathcal{L}_2 = \left\{\tfrac{|\gamma|(2\alpha+4)}{\Gamma(\alpha)} + \tfrac{2|\gamma|(\alpha+1)(\alpha+2)}{\Gamma(\alpha+1)}\right\}, \mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2.$$

Theorem 3.1. *Under the hypotheses* (A_1) *and* (A_3)*, the fractional Langevin Eqs (1.1) and (1.2) has a solution.*

Proof. We define the operator $\mathcal{Z} : B \to B$ as follow:

$$(\mathcal{Z}x)(t) = \int_0^1 G(t,s) f\left(s, x(s), x'(s)\right) ds + \int_0^1 H(t,s)\, x(s) ds. \tag{3.2}$$

Lemma 2.7 implies that the fixed points of the operator $\mathcal{Z}$ in (3.2) are the same solutions of the boundary value problem (1.1) and (1.2). We consider a ball $U_r = \{x \in B, \|x\| \leq r\}$ so that $\max\left\{4\mathcal{K}\|\sigma\|, (4a_1\mathcal{K})^{1/1-\tau_1}, (4a_2\mathcal{K})^{1/1-\tau_2}, 4r\mathcal{L}\right\} \leq r$. For any $x \in U_r$ and by (A_3), we show that $\mathcal{Z}U_r \subset U_r$, then

$$\begin{aligned}
|(\mathcal{Z}x)(t)| &\leq \frac{1}{\Gamma(\alpha+\beta)} \int_0^t (t-s)^{\alpha+\beta-1} \left|f\left(s,x(s),x'(s)\right)\right| ds \\
&+ \frac{t^{\alpha+1} + t^{\alpha+2}}{\Gamma(\alpha+\beta-1)} \int_0^1 (1-s)^{\alpha+\beta-2} \left|f\left(s,x(s),x'(s)\right)\right| ds \\
&+ \frac{(\alpha+1)t^{\alpha+2} + (\alpha+2)t^{\alpha+1}}{\Gamma(\alpha+\beta)} \int_0^1 (1-s)^{\alpha+\beta-1} \left|f\left(s,x(s),x'(s)\right)\right| ds + \frac{|\gamma|}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |x(s)| ds \\
&+ \left[(\alpha+2)t^{\alpha+1} + (\alpha+1)t^{\alpha+2}\right] \frac{|\gamma|}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} |x(s)| ds \\
&+ \frac{|\gamma|\left(t^{\alpha+2} + t^{\alpha+1}\right)}{\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} |x(s)| ds \\
&\leq \frac{1}{\Gamma(\alpha+\beta)} \int_0^t (t-s)^{\alpha+\beta-1} \left(\sigma(s) + a_1|x(s)|^{\tau_1} + a_2|x'(s)|^{\tau_2}\right) ds \\
&+ \frac{2}{\Gamma(\alpha+\beta-1)} \int_0^1 (1-s)^{\alpha+\beta-2} \left(\sigma(s) + a_1|x(s)|^{\tau_1} + a_2|x'(s)|^{\tau_2}\right) ds \\
&+ \frac{2\alpha+3}{\Gamma(\alpha+\beta)} \int_0^1 (1-s)^{\alpha+\beta-1} \left(\sigma(s) + a_1|x(s)|^{\tau_1} + a_2|x'(s)|^{\tau_2}\right) ds
\end{aligned}$$

$$
\begin{aligned}
&+\frac{|\gamma|\,\|x\|}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\,ds+\frac{(2\alpha+3)\,|\gamma|\,\|x\|}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\,ds+\frac{2\,|\gamma|\,\|x\|}{\Gamma(\alpha-1)}\int_0^1 (1-s)^{\alpha-2}\,ds\\
&\le (\|\sigma\|+a_1r^{\tau_1}+a_2r^{\tau_2})\left\{\frac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}+\frac{2}{\Gamma(\alpha+\beta)}+\frac{2\alpha+3}{\Gamma(\alpha+\beta+1)}\right\}\\
&+\|x\|\left\{\frac{|\gamma|\,t^{\alpha}}{\Gamma(\alpha+1)}+\frac{(2\alpha+3)\,|\gamma|}{\Gamma(\alpha+1)}+\frac{2\,|\gamma|}{\Gamma(\alpha)}\right\}\\
&\le (\|\sigma\|+a_1r^{\tau_1}+a_2r^{\tau_2})\left\{\frac{4\,(\alpha+1)+2\beta}{\Gamma(\alpha+\beta+1)}\right\}+r\left\{\frac{|\gamma|\,4\,(\alpha+1)}{\Gamma(\alpha+1)}\right\}\\
&=(\|\sigma\|+a_1r^{\tau_1}+a_2r^{\tau_2})\,\mathcal{K}_1+r\mathcal{L}_1, \qquad (3.3)
\end{aligned}
$$

and

$$
\begin{aligned}
&\left|(\mathcal{Z}x)'(t)\right|\le\frac{1}{\Gamma(\alpha+\beta-1)}\int_0^t (t-s)^{\alpha+\beta-2}\left|f\left(s,x(s),x'(s)\right)\right|ds\\
&+\frac{(\alpha+1)\,t^{\alpha}+(\alpha+2)\,t^{\alpha+1}}{\Gamma(\alpha+\beta-1)}\int_0^1 (1-s)^{\alpha+\beta-2}\left|f\left(s,x(s),x'(s)\right)\right|ds\\
&+\frac{(\alpha+1)(\alpha+2)\left(t^{\alpha+1}+t^{\alpha}\right)}{\Gamma(\alpha+\beta)}\int_0^1 (1-s)^{\alpha+\beta-1}\left|f\left(s,x(s),x'(s)\right)\right|ds\\
&+\frac{|\gamma|}{\Gamma(\alpha-1)}\int_0^t (t-s)^{\alpha-2}\,|x(s)|\,ds+\frac{|\gamma|\,(\alpha+1)(\alpha+2)\left(t^{\alpha}+t^{\alpha+1}\right)}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\,|x(s)|\,ds\\
&+\frac{|\gamma|\left[(\alpha+2)\,t^{\alpha+1}+(\alpha+1)\,t^{\alpha}\right]}{\Gamma(\alpha-1)}\int_0^1 (1-s)^{\alpha-2}\,|x(s)|\,ds\\
&\le (\|\sigma\|+a_1r^{\tau_1}+a_2r^{\tau_2})\left\{\frac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta)}+\frac{2\alpha+3}{\Gamma(\alpha+\beta)}+\frac{2\,(\alpha+1)(\alpha+2)}{\Gamma(\alpha+\beta+1)}\right\}\\
&+\|x\|\left\{\frac{|\gamma|\,t^{\alpha-1}}{\Gamma(\alpha)}+\frac{2\,|\gamma|\,(\alpha+1)(\alpha+2)}{\Gamma(\alpha+1)}+\frac{|\gamma|\,(2\alpha+3)}{\Gamma(\alpha)}\right\}\\
&\le (\|\sigma\|+a_1r^{\tau_1}+a_2r^{\tau_2})\left\{\frac{2\alpha+4}{\Gamma(\alpha+\beta)}+\frac{2\,(\alpha+1)(\alpha+2)}{\Gamma(\alpha+\beta+1)}\right\}\\
&+r\left\{\frac{|\gamma|\,(2\alpha+4)}{\Gamma(\alpha)}+\frac{2\,|\gamma|\,(\alpha+1)(\alpha+2)}{\Gamma(\alpha+1)}\right\}=(\|\sigma\|+a_1r^{\tau_1}+a_2r^{\tau_2})\,\mathcal{K}_2+r\mathcal{L}_2. \qquad (3.4)
\end{aligned}
$$

So by (3.3) and (3.4), we have

$$
\begin{aligned}
\|\mathcal{Z}x\| &= \max|(\mathcal{Z}x)(t)|+\max\left|(\mathcal{Z}x)'(t)\right|\\
&\le (\|\sigma\|+a_1r^{\tau_1}+a_2r^{\tau_2})\,(\mathcal{K}_1+\mathcal{K}_2)+r\,(\mathcal{L}_1+\mathcal{L}_2)\\
&= (\|\sigma\|+a_1r^{\tau_1}+a_2r^{\tau_2})\,\mathcal{K}+r\mathcal{L}\\
&\le \frac{r}{4}+\frac{r}{4}+\frac{r}{4}+\frac{r}{4}=r.
\end{aligned}
$$

Next, we prove that the operator $\mathcal{Z}$ is completely continuous. The functions f, $G(t,s)$ and $H(t,s)$ are continuous, hence the operator $\mathcal{Z}$ is continuous. Let $M=\max_{t\in[0,1],x\in U_r}|f(t,x(t),x'(t))|+1$, for any $x\in U_r$

and $t_1, t_2 \in [0,1]$ such that $t_1 < t_2$, we have

$$\begin{aligned}
&\left|(\mathcal{Z}x)(t_2) - (\mathcal{Z}x)(t_1)\right| \leq \Bigg|\frac{1}{\Gamma(\alpha+\beta)}\int_0^{t_2}(t_2-s)^{\alpha+\beta-1} f(s,x(s),x'(s))\,ds \\
&-\frac{1}{\Gamma(\alpha+\beta)}\int_0^{t_1}(t_1-s)^{\alpha+\beta-1} f(s,x(s),x'(s))\,ds\Bigg| \\
&+\left|\frac{\left(t_2^{\alpha+1}-t_1^{\alpha+1}\right)+\left(t_2^{\alpha+2}-t_1^{\alpha+2}\right)}{\Gamma(\alpha+\beta-1)}\int_0^1(1-s)^{\alpha+\beta-2} f(s,x(s),x'(s))\,ds\right| \\
&+\left|\frac{(\alpha+1)\left(t_2^{\alpha+2}-t_1^{\alpha+2}\right)+(\alpha+2)\left(t_2^{\alpha+1}-t_1^{\alpha+1}\right)}{\Gamma(\alpha+\beta)}\int_0^1(1-s)^{\alpha+\beta-1} f(s,x(s),x'(s))\,ds\right| \\
&+\left|\frac{\gamma}{\Gamma(\alpha)}\int_0^{t_2}(t_2-s)^{\alpha-1}x(s)ds-\frac{\gamma}{\Gamma(\alpha)}\int_0^{t_1}(t_1-s)^{\alpha-1}x(s)ds\right| \\
&+\left|\left[(\alpha+2)\left(t_2^{\alpha+1}-t_1^{\alpha+1}\right)+(\alpha+1)\left(t_2^{\alpha+2}-t_1^{\alpha+2}\right)\right]\frac{\gamma}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}x(s)ds\right| \\
&+\left|\gamma\frac{\left(t_2^{\alpha+1}-t_1^{\alpha+1}\right)+\left(t_2^{\alpha+2}-t_1^{\alpha+2}\right)}{\Gamma(\alpha-1)}\int_0^1(1-s)^{\alpha-2}x(s)ds\right| \\
&\leq \frac{2M}{\Gamma(\alpha+\beta+1)}(t_2-t_1)^{\alpha+\beta}+\frac{M}{\Gamma(\alpha+\beta+1)}\left(t_2^{\alpha+\beta}-t_1^{\alpha+\beta}\right) \\
&+M\frac{\left(t_2^{\alpha+1}-t_1^{\alpha+1}\right)+\left(t_2^{\alpha+2}-t_1^{\alpha+2}\right)}{\Gamma(\alpha+\beta)}+M\frac{(\alpha+1)\left(t_2^{\alpha+2}-t_1^{\alpha+2}\right)+(\alpha+2)\left(t_2^{\alpha+1}-t_1^{\alpha+1}\right)}{\Gamma(\alpha+\beta+1)} \\
&+\frac{2r|\gamma|}{\Gamma(\alpha+1)}(t_2-t_1)^{\alpha}+\frac{r|\gamma|}{\Gamma(\alpha+1)}\left(t_2^{\alpha}-t_1^{\alpha}\right)+\frac{r|\gamma|}{\Gamma(\alpha)}\left[\left(t_2^{\alpha+1}-t_1^{\alpha+1}\right)+\left(t_2^{\alpha+2}-t_1^{\alpha+2}\right)\right] \\
&+\frac{r|\gamma|}{\Gamma(\alpha+1)}\left[(\alpha+2)\left(t_2^{\alpha+1}-t_1^{\alpha+1}\right)+(\alpha+1)\left(t_2^{\alpha+2}-t_1^{\alpha+2}\right)\right].
\end{aligned} \tag{3.5}$$

Also,

$$\begin{aligned}
&\left|(\mathcal{Z}x)'(t_2) - (\mathcal{Z}x)'(t_1)\right| \leq \Bigg|\frac{1}{\Gamma(\alpha+\beta-1)}\int_0^{t_2}(t_2-s)^{\alpha+\beta-2} f(s,x(s),x'(s))\,ds \\
&-\frac{1}{\Gamma(\alpha+\beta-1)}\int_0^{t_1}(t_1-s)^{\alpha+\beta-2} f(s,x(s),x'(s))\,ds\Bigg| \\
&+\left|\frac{(\alpha+1)\left(t_2^{\alpha}-t_1^{\alpha}\right)+(\alpha+2)\left(t_2^{\alpha+1}-t_1^{\alpha+1}\right)}{\Gamma(\alpha+\beta-1)}\int_0^1(1-s)^{\alpha+\beta-2} f(s,x(s),x'(s))\,ds\right| \\
&+\left|\frac{(\alpha+1)(\alpha+2)\left[\left(t_2^{\alpha+1}-t_1^{\alpha+1}\right)+\left(t_2^{\alpha}-t_1^{\alpha}\right)\right]}{\Gamma(\alpha+\beta)}\int_0^1(1-s)^{\alpha+\beta-1} f(s,x(s),x'(s))\,ds\right| \\
&+\left|\frac{\gamma}{\Gamma(\alpha-1)}\int_0^{t_2}(t_2-s)^{\alpha-2}x(s)ds-\frac{\gamma}{\Gamma(\alpha-1)}\int_0^{t_1}(t_1-s)^{\alpha-2}x(s)ds\right|
\end{aligned}$$

$$+\left|\frac{\gamma(\alpha+1)(\alpha+2)\left[\left(t_2^{\alpha+1}-t_1^{\alpha+1}\right)+\left(t_2^{\alpha}-t_1^{\alpha}\right)\right]}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}x(s)ds\right|$$
$$+\left|\frac{\gamma\left[(\alpha+2)\left(t_2^{\alpha+1}-t_1^{\alpha+1}\right)+(\alpha+1)\left(t_2^{\alpha}-t_1^{\alpha}\right)\right]}{\Gamma(\alpha-1)}\int_0^1(1-s)^{\alpha-2}x(s)ds\right|$$
$$\leq\frac{2M}{\Gamma(\alpha+\beta)}(t_2-t_1)^{\alpha+\beta-1}+\frac{M}{\Gamma(\alpha+\beta)}\left(t_2^{\alpha+\beta-1}-t_1^{\alpha+\beta-1}\right)$$
$$+\frac{M}{\Gamma(\alpha+\beta)}\left[(\alpha+1)(t_2^{\alpha}-t_1^{\alpha})+(\alpha+2)\left(t_2^{\alpha+1}-t_1^{\alpha+1}\right)\right]$$
$$+\frac{M}{\Gamma(\alpha+\beta+1)}(\alpha+1)(\alpha+2)\left[\left(t_2^{\alpha+1}-t_1^{\alpha+1}\right)+(t_2^{\alpha}-t_1^{\alpha})\right]$$
$$+\frac{2r|\gamma|}{\Gamma(\alpha)}(t_2-t_1)^{\alpha-1}+\frac{r|\gamma|}{\Gamma(\alpha)}\left(t_2^{\alpha-1}-t_1^{\alpha-1}\right)+\frac{r|\gamma|(\alpha+1)(\alpha+2)}{\Gamma(\alpha+1)}\left[\left(t_2^{\alpha+1}-t_1^{\alpha+1}\right)+(t_2^{\alpha}-t_1^{\alpha})\right]$$
$$+\frac{r|\gamma|}{\Gamma(\alpha)}\left[(\alpha+2)\left(t_2^{\alpha+1}-t_1^{\alpha+1}\right)+(\alpha+1)(t_2^{\alpha}-t_1^{\alpha})\right]. \tag{3.6}$$

By (3.5) and (3.6), clearly that the functions $(t_2-t_1)^{\alpha+\beta-i}$, $t_2^{\alpha+\beta-i}-t_1^{\alpha+\beta-i}$, $(t_2-t_1)^{\alpha-i}$ $(i=0,1)$ and $t_2^{\alpha+j}-t_1^{\alpha+j}$ $(j=0,1,2)$ are uniformly continuous on $[0,1]$. Then, $\mathcal{Z}(U_r)$ is equicontinuous and the Arzela–Ascoli theorem implies that $\overline{\mathcal{Z}(U_r)}$ is compact, hence the operator $\mathcal{Z}:U_r\to U_r$ is completely continuous. Therefore, by the Schauder fixed-point theorem, we conclude that the problem (1.1) and (1.2) has a solution. □

By applying Banach fixed point theorem, we prove the uniqueness of solution of the problem (1.1) and (1.2).

Theorem 3.2. *Let the assumptions* $(A_1$–$A_3)$ *are satisfied, then the boundary value problem (1.1) and (1.2) has a uniqueness solution provided that* $\psi=\psi_1+\psi_2<1$, *where*

$$\psi_1=\left\{\frac{14w}{\Gamma(\alpha+\beta+1)}+\frac{8|\gamma|}{\Gamma(\alpha+1)}\right\},\ \psi_2=\left\{\frac{36w}{\Gamma(\alpha+\beta+1)}+\frac{18|\gamma|}{\Gamma(\alpha+1)}\right\}.$$

Proof. For any $x,y\in B$, $t\in[0,1]$ and by condition (A_2), we give

$$|(\mathcal{Z}x)(t)-(\mathcal{Z}y)(t)|\leq\frac{1}{\Gamma(\alpha+\beta)}\int_0^t(t-s)^{\alpha+\beta-1}\left|f(s,x(s),x'(s))-f(s,y(s),y'(s))\right|ds$$
$$+\frac{(\alpha+1)t^{\alpha+2}+(\alpha+2)t^{\alpha+1}}{\Gamma(\alpha+\beta)}\int_0^1(1-s)^{\alpha+\beta-1}\left|f(s,x(s),x'(s))-f(s,y(s),y'(s))\right|ds$$
$$+\frac{t^{\alpha+1}+t^{\alpha+2}}{\Gamma(\alpha+\beta-1)}\int_0^1(1-s)^{\alpha+\beta-2}\left|f(s,x(s),x'(s))-f(s,y(s),y'(s))\right|ds$$
$$+\frac{|\gamma|}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}|x(s)-y(s)|ds$$
$$+\frac{|\gamma|\left[(\alpha+2)t^{\alpha+1}+(\alpha+1)t^{\alpha+2}\right]}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}|x(s)-y(s)|ds$$

$$+\frac{|\gamma|\left[t^{\alpha+2}+t^{\alpha+1}\right]}{\Gamma(\alpha-1)}\int_0^1(1-s)^{\alpha-2}|x(s)-y(s)|\,ds$$
$$\leq\frac{w\|x-y\|}{\Gamma(\alpha+\beta)}\int_0^t(t-s)^{\alpha+\beta-1}\,ds+\frac{5w\|x-y\|}{\Gamma(\alpha+\beta)}\int_0^1(1-s)^{\alpha+\beta-1}\,ds$$
$$+\frac{2w\|x-y\|}{\Gamma(\alpha+\beta-1)}\int_0^1(1-s)^{\alpha+\beta-2}\,ds+\frac{|\gamma|\|x-y\|}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\,ds$$
$$+\frac{5|\gamma|\|x-y\|}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}\,ds+\frac{2|\gamma|\|x-y\|}{\Gamma(\alpha-1)}\int_0^1(1-s)^{\alpha-2}\,ds$$
$$\leq w\|x-y\|\left\{\frac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}+\frac{5}{\Gamma(\alpha+\beta+1)}+\frac{2}{\Gamma(\alpha+\beta)}\right\}$$
$$+\|x-y\|\left\{\frac{|\gamma|t^{\alpha}}{\Gamma(\alpha+1)}+\frac{5|\gamma|}{\Gamma(\alpha+1)}+\frac{2|\gamma|}{\Gamma(\alpha)}\right\}$$
$$\leq\|x-y\|\left\{\frac{14w}{\Gamma(\alpha+\beta+1)}+\frac{8|\gamma|}{\Gamma(\alpha+1)}\right\}=\psi_1\|x-y\|,\tag{3.7}$$

and

$$\left|(\mathcal{Z}x)'(t)-(\mathcal{T}y)'(t)\right|\leq\frac{1}{\Gamma(\alpha+\beta-1)}\int_0^t(t-s)^{\alpha+\beta-2}\left|f(s,x(s),x'(s))-f(s,y(s),y'(s))\right|ds$$
$$+\frac{(\alpha+1)(\alpha+2)\left(t^{\alpha+1}+t^{\alpha}\right)}{\Gamma(\alpha+\beta)}\int_0^1(1-s)^{\alpha+\beta-1}\left|f(s,x(s),x'(s))-f(s,y(s),y'(s))\right|ds$$
$$+\frac{(\alpha+1)t^{\alpha}+(\alpha+2)t^{\alpha+1}}{\Gamma(\alpha+\beta-1)}\int_0^1(1-s)^{\alpha+\beta-2}\left|f(s,x(s),x'(s))-f(s,y(s),y'(s))\right|ds$$
$$+\frac{|\gamma|}{\Gamma(\alpha-1)}\int_0^t(t-s)^{\alpha-2}|x(s)-y(s)|\,ds$$
$$+\frac{|\gamma|(\alpha+1)(\alpha+2)\left(t^{\alpha}+t^{\alpha+1}\right)}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}|x(s)-y(s)|\,ds$$
$$+\frac{|\gamma|\left[(\alpha+2)t^{\alpha+1}+(\alpha+1)t^{\alpha}\right]}{\Gamma(\alpha-1)}\int_0^1(1-s)^{\alpha-2}|x(s)-y(s)|\,ds$$
$$\leq\frac{w\|x-y\|}{\Gamma(\alpha+\beta-1)}\int_0^t(t-s)^{\alpha+\beta-2}\,ds+\frac{12w\|x-y\|}{\Gamma(\alpha+\beta)}\int_0^1(1-s)^{\alpha+\beta-1}\,ds$$
$$+\frac{5w\|x-y\|}{\Gamma(\alpha+\beta-1)}\int_0^1(1-s)^{\alpha+\beta-2}\,ds+\frac{|\gamma|\|x-y\|}{\Gamma(\alpha-1)}\int_0^t(t-s)^{\alpha-2}\,ds$$
$$+\frac{12|\gamma|\|x-y\|}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}\,ds+\frac{5|\gamma|\|x-y\|}{\Gamma(\alpha-1)}\int_0^1(1-s)^{\alpha-2}\,ds$$
$$\leq w\|x-y\|\left\{\frac{t^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}+\frac{12}{\Gamma(\alpha+\beta+1)}+\frac{5}{\Gamma(\alpha+\beta)}\right\}$$
$$+\|x-y\|\left\{\frac{|\gamma|t^{\alpha-1}}{\Gamma(\alpha)}+\frac{12|\gamma|}{\Gamma(\alpha+1)}+\frac{5|\gamma|}{\Gamma(\alpha)}\right\}$$

$$\leq \|x-y\|\left\{\frac{36w}{\Gamma(\alpha+\beta+1)}+\frac{18|\gamma|}{\Gamma(\alpha+1)}\right\}=\psi_2\|x-y\|. \tag{3.8}$$

Using (3.7) and (3.8), we obtain

$$\|\mathcal{Z}x-\mathcal{Z}y\| \leq \psi\|x-y\|,$$

where $\psi<1$. Hence the operator $\mathcal{Z}$ is a contraction operator and the contraction mapping principle implies that the problem (1.1) and (1.2) has a unique solution. □

In the following examples, we investigate the existence and uniqueness of solutions to fractional Langevin equations.

Example 3.3. *Consider the following nonlinear Langevin equation of fractional orders*

$$\begin{cases} {}^cD_{0+}^{\frac{5}{2}}\left({}^cD_{0+}^{\frac{1}{2}}+\frac{1}{5}\right)u(t)=f(t,u(t),u'(t)),\ t\in(0,1), \\ u(0)=u(1)=0, \quad u'(0)=u'(1)=0, \end{cases} \tag{3.9}$$

where $f(t,u(t),u'(t))=\tan^{-1}t+\left(t-\frac{1}{3}\right)^2(u(t))^{\tau_1}+\frac{t}{e}(u'(t))^{\tau_2}$, $0<\tau_1,\tau_2<1$. *Observe that the function* f *is continuous, also*

$$\begin{aligned} |f(t,u(t),u'(t))| &\leq t+(t-\frac{1}{3})^2|u(t)|^{\tau_1}+\frac{t}{e}|u'(t)|^{\tau_2} \\ &\leq 1+\frac{4}{9}|u(t)|^{\tau_1}+\frac{1}{e}|u'(t)|^{\tau_2}. \end{aligned}$$

Thus, the assumptions (A_1) *and* (A_2) *are satisfied and Theorem 3.1 implies that the problem (3.9) has a solution.*

Example 3.4. *Consider the following boundary value problem of fractional orders*

$$\begin{cases} {}^cD_{0+}^{2.8}\left({}^cD_{0+}^{0.8}+0.1\right)u(t)=\frac{u'(t)-\tan^{-1}u(t)}{4(t+2)^2},\ t\in(0,1) \\ u(0)=u(1)=0, \quad u'(0)=u'(1)=0, \end{cases} \tag{3.10}$$

Here, $\beta=2.8$, $\alpha=0.8$, $\gamma=0.1$ *and* $f(t,u(t),u'(t))=\frac{u'(t)-\tan^{-1}u(t)}{4(t+1)^2}$. *Clearly that the function* f *is continuous and*

$$\begin{aligned} |f(t,u(t),u'(t))-f(t,v(t),v'(t))| &\leq \frac{1}{4(t+1)^2}\left(|u'(t)-v'(t)|+\left|\tan^{-1}u(t)-\tan^{-1}v(t)\right|\right) \\ &\leq \frac{1}{16}\left(|u'(t)-v'(t)|+|u(t)-v(t)|\right). \end{aligned}$$

So $w=\frac{1}{16}$, *also,* $\Psi_1=\frac{14w}{\Gamma(\alpha+\beta+1)}+\frac{8|\gamma|}{\Gamma(\alpha+1)}=\frac{14(1/16)}{\Gamma(0.8+2.8+1)}+\frac{8(0.1)}{\Gamma(2.8+1)}\approx 0.2358$ *and* $\Psi_2=\frac{36w}{\Gamma(\alpha+\beta+1)}+\frac{18|\gamma|}{\Gamma(\alpha+1)}=\frac{36(1/16)}{\Gamma(0.8+2.8+1)}+\frac{18(0.1)}{\Gamma(2.8+1)}\approx 0.5516$, *then* $\Psi_1+\Psi_2\approx 0.7874<1$. *Hence the assumptions* (A_1–A_3) *are satisfied. Thus by applying Theorem 3.2, the BVP (3.10) has a unique solution.*

4. Conclusions

We have provided a generalization of the nonlinear Langevin equation of fractional orders with boundary conditions. By considering the Banach space $C^1[0,1]$ be equipped with the norm $\|x\| = \max_{t\in[0,1]}|x(t)| + \max_{t\in[0,1]}|x'(t)|$, under the assumptions (A_1) and (A_3), we have shown the existence of solution for BVP (1.1) and (1.2). Then, the uniqueness of solution of the problem is proved by Banach fixed point theorem. Also, we have proposed an example to illustrate the results. In future researches, we can investigate the Langevin equations with different fractional derivatives and new boundary value conditions. Moreover, the existence results for the stochastic type of Langevin equation has not been studied so far.

Conflict of interest

The authors declare no conflict of interest.

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