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Research article

Remarks on topological spaces on $\mathbb{Z}^n$ which are related to the Khalimsky n-dimensional space

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Abstract: The present paper intensively studies various properties of certain topologies on the set of integers $\mathbb{Z}$ (resp. $\mathbb{Z}^n$) which are either homeomorphic or not homeomorphic to the typical Khalimsky line topology (resp. n-dimensional Khalimsky topology). This finding plays a crucial role in addressing some problems which remain open in the field of digital topology.

Keywords: Khalimsky topology; $T_{\frac{1}{2}}$-separation axiom; Alexandroff topology; quasi-discrete; digital topology
Mathematics Subject Classification: 54A05, 54J05, 54F05, 54C08, 54F65, 68U05

1. Introduction

A recent paper [8] developed infinitely many types of topological structures on $\mathbb{Z}^n$ which need not be homeomorphic to the n-dimensional Khalimsky topological space. In this paper, since we will often use the term "Khalimsky", hereafter, we are willing to use the notation "K-" for brevity instead of the "Khalimsky" if there is no danger of ambiguity. Besides, we also take the notations $\mathbb{N}$, $\mathbb{Z}_0$, $\mathbb{Z}_1$, and $\mathbb{Z}^n$ to indicate the sets of natural numbers, even integers, odd integers, and the n-fold of Cartesian product of the set of integers $\mathbb{Z}$, respectively. In addition, the notation " :=" will be used to introduce a new term. As usual we denote by $X^\sharp$ the cardinal number of the given set X.

After formulating infinitely many types of subbases on $\mathbb{Z}$, the paper [8] proposed various kinds of topologies generated by the given subbases. Furthermore, it proved that many of the obtained topologies are not homeomorphic to the K-topological line topology (or 1-dimensional K-topology). Motivated by this finding, we can address some issues which remain open in the fields of digital topology and digital geometry. Meanwhile, the paper [8] contains some misprinted parts which should be improved (see Remark 5.6). Hence the present paper makes them fixed and widely extends various properties of the obtained results in [8] from the viewpoint of digital topology and further, it investigates some properties of $(\mathbb{Z}, T_{-k})$ or $(\mathbb{Z}^n, (T_k)^n)$. Owing to this improvement, for $k \in \mathbb{Z}$, we can in turn confirm some utilities of the topological spaces $(\mathbb{Z}, T_k)$, $(\mathbb{Z}^2, (T_k \times T_k))$, $(\mathbb{Z}^n, (T_k)^n)$, and so forth (see Sections 3–6 in the present paper) where $T_k := T_{S_k}$ is the topology generated by S_k (see (2.1)) as a subbase. For each $k \in \mathbb{Z}$, given $(\mathbb{Z}, T_k)$ (for details, see Section 2 in the present paper), in relation to the above work, we now focus on the following issues.
(•1) Characterization of the closures of singletons of $(\mathbb{Z}, T_k)$ and $(\mathbb{Z}, T_{-k})$.
(•2) Investigation of the numbers of components of $(\mathbb{Z}, T_k)$ and $(\mathbb{Z}, T_{-k})$.
(•3) Determination of the number k leading to a homeomorphism between $(\mathbb{Z}, T_k)$ and $(\mathbb{Z}, T_{-k})$.
(•4) Finding a necessary and sufficient condition supporting connectedness of $(\mathbb{Z}, T_k)$.

Indeed, the present paper aims at investigating various properties of $(\mathbb{Z}^n, (T_k)^n), n \in \mathbb{N}, k \in \mathbb{Z} \setminus \{0\}$. Besides, the paper deals with some topics such as connectedness, homeomorphisms of Alexandroff (topological) structures [1, 2] and so on. In particular, this approach plays an important role in the fields of fixed point theory, digital topological rough set theory, digital geometry, and so forth [9, 10]. Thus, the study of the recently-established topologies on $\mathbb{Z}^n$ in [8] which are different from the well-known topologies on $\mathbb{Z}^n$ can activate some studies of pure and applied sciences including computer science.

The rest of the paper is organized as follows: Section 2 refers to some notions relating to various structures of Alexandroff spaces. Section 3 investigates certain structures of the closures of singletons of $(\mathbb{Z}, T_k), k \in \mathbb{Z}$. Section 4 proves non-connectedness of the topological spaces $(\mathbb{Z}, T_k), k \in \mathbb{Z} \setminus \{-1, 1\}$. We say that a topological space X satisfies the separation axiom $T_{\frac{1}{2}}$ [3, 4, 14] if every singleton of X is either an open or a closed set. Owing to the topological structure of $T_k, k \neq 0$, satisfying the $T_{\frac{1}{2}}$-separation axiom and the closures of singletons of $\mathbb{Z}$, we can study various structures of $(\mathbb{Z}, T_k)$ and $(\mathbb{Z}^2, T_k \times T_k)$ more efficiently. Finally, it proves that for $k \in \mathbb{Z} \setminus \{0\}$ the topological space $(\mathbb{Z}, T_k)$ has k components. Section 5 proves that $(\mathbb{Z}, T_i)$ is not homeomorphic to $(\mathbb{Z}, T_j)$ if $i \neq j, i, j \in \mathbb{N} \cup \{0\}$. Furthermore, $(\mathbb{Z}, T_k)$ is homeomorphic to $(\mathbb{Z}, T_{-k})$ if $k \in \mathbb{N}$. Section 6 concludes the paper and refers to further works.

2. Preliminaries

In this section, we refer to several concepts which are used in this paper. We say that a topological space (X, T) is an Alexandroff (topological) space [1, 2] if every point $x \in X$ has the smallest (or minimal) open neighborhood in (X, T). It turns out that the n-dimensional Khalimsky topological space is an Alexandroff space [10–13]. For details, let us now recall basic notions related to the K-topological structure on $\mathbb{Z}^n$. The Khalimsky line topology on $\mathbb{Z}$, denoted by $(\mathbb{Z}, \kappa)$, is induced by the set $\{[2n-1, 2n+1]_{\mathbb{Z}} \mid n \in \mathbb{Z}\}$ as a subbase [11], where for $a, b \in \mathbb{Z}$, $[a, b]_{\mathbb{Z}} := \{x \in \mathbb{Z} \mid a \leq x \leq b\}$. It turns out that $(\mathbb{Z}, \kappa)$ places between the semi-$T_{\frac{1}{2}}$ and T_1-separation axiom [3]. The product topology on $\mathbb{Z}^n$

induced by $(\mathbb{Z}, \kappa)$ is called the Khalimsky product topology on $\mathbb{Z}^n$ (or the n-dimensional K-topological space or the Khalimsky nD space), denoted by $(\mathbb{Z}^n, \kappa^n)$ and further, various properties of $(\mathbb{Z}^n, \kappa^n)$ have been investigated [3, 10–13].

As usual, we denote the cardinality of an (infinite) denumerable set with $\aleph_0$. Besides, we will often use the following notations in this paper: $[m, +\infty)_{\mathbb{Z}} := \{x \in \mathbb{Z} \,|\, m \leq x\}$ and $(-\infty, n]_{\mathbb{Z}} := \{x \in \mathbb{Z} \,|\, x \leq n\}$. Given a universal set U, for a subset $A \subset U$, as usual we use the notation A^c to indicate the complement of A in U. A topology T is called quasi-discrete [15] (or clopen or pseudo-discrete [16]) if every open set in T is closed. In view of the K-topological structure of $(\mathbb{Z}^n, \kappa^n)$, it is clear that any infinite subset of $(\mathbb{Z}^n, \kappa^n)$ is not compact in $(\mathbb{Z}^n, \kappa^n)$ and due to the connectedness of $(\mathbb{Z}, \kappa)$ [12], $(\mathbb{Z}^n, \kappa^n)$ is clearly connected. Based on this approach, a map $h : X \to Y$ is called a K-homeomorphism if h is a K-continuous bijection and further, $h^{-1} : Y \to X$ is K-continuous, where for two K-topological spaces $X := (X, \kappa_X^{n_1})$ and $Y := (Y, \kappa_Y^{n_2})$, we say that a map $f : X \to Y$ is K-continuous if for every point $x \in X$, $f(SN_K(x)) \subset SN_K(f(x))$, and $SN_K(x)$ (resp. $SN_K(f(x))$ means the smallest open set of x (resp. $f(x)$) in X (resp. Y).

In the paper [8], many types of subbases, say $S_k, k \in \mathbb{Z}$ (see (2.1) in Section 2 of the present paper), were introduced to establish various types of topologies on $\mathbb{Z}$. Then, it intensively explored some topological features of $T_k, k \in \mathbb{Z}$, with respect to the separation axioms and an Alexandroff space structure. In view of each element $[2n-1, 2n+1]_{\mathbb{Z}}$ of the subbase of the K-line topology, the topology on $\mathbb{Z}$ generated by the set $\{[2n, 2n+2]_{\mathbb{Z}} \,|\, n \in \mathbb{Z}\}$ as a subbase is indeed homeomorphic to $(\mathbb{Z}, \kappa)$. Thus, in digital topology it is natural to consider other types of elements which are not consecutive, such as for a given number $k \in \mathbb{Z}$, $\{\{2n, 2n+1, 2n+2k+1\} \,|\, n \in \mathbb{Z}\}$, $\{\{2n, 2n+1, 2n+2k\} \,|\, n \in \mathbb{Z}\}$ and so forth. Indeed, this establishment facilitates various studies in digital topology and computer science related to writing parallel algorithms on subsets of $\mathbb{Z}^n$.

Let us now recall various properties of the topologies generated by certain subbases $S_k, k \in \mathbb{Z}$ [8]. For details, given a number $k \in \mathbb{Z}$, assume the set

$$S_k := \{S_{k,n} \,|\, S_{k,n} := \{2n, 2n+1, 2n+2k+1\},\ n \in \mathbb{Z}\}. \tag{2.1}$$

Consider a topology on $\mathbb{Z}$ generated by S_k as a subbase, denoted by $T_k := T_{S_k}$. Then, it turns out that $(\mathbb{Z}, T_k)$ is an Alexandroff space [8], $k \in \mathbb{Z}$. Hereinafter, with $(\mathbb{Z}, T_k)$, given a point $x \in \mathbb{Z}$, we denote by

$$SN_k(x) \text{ the smallest open neighborhood of } x. \tag{2.2}$$

For instance, consider the topology $T_{-1} := T_{S_{-1}}$ generated by the set $S_{-1} = \{\{2n-1, 2n, 2n+1\} \,|\, n \in \mathbb{Z}\}$ as a subbase [8]. Then, for any $n \in \mathbb{Z}$, we respectively have the smallest open neighborhood of $2n$ and $2n+1$ in $(\mathbb{Z}, T_{-1})$ denoted by

$$SN_{-1}(2n) = [2n-1, 2n+1]_{\mathbb{Z}} \text{ and } SN_{-1}(2n+1) = \{2n+1\}, \tag{2.3}$$

from which T_{-1} is proved to be an Alexandroff topology.

Next, consider the topology $T_0 := T_{S_0}$ generated by the set $S_0 = \{\{2n, 2n+1\} \,|\, n \in \mathbb{Z}\}$ as a subbase [8]. Then, for any $n \in \mathbb{Z}$ we respectively obtain the smallest open neighborhood of $2n$ and $2n+1$ in $(\mathbb{Z}, T_0)$ denoted by

$$SN_0(2n) = SN_0(2n+1) = \{2n, 2n+1\}, \tag{2.4}$$

which implies that T_0 is also an Alexandroff topology.

Finally, assume the topology $T_1 := T_{S_1}$ generated by the set $S_1 = \{\{2n, 2n+1, 2n+3\} \mid n \in \mathbb{Z}\}$ as a subbase [8]. Then, for any $n \in \mathbb{Z}$ we respectively have the smallest open neighborhood of $2n$ and $2n+1$ in $(\mathbb{Z}, T_1)$ denoted by

$$SN_1(2n) = \{2n, 2n+1, 2n+3\} \text{ and } SN_1(2n+1) = \{2n+1\}, \tag{2.5}$$

which means that T_1 is also an Alexandroff topology.

Based on the structures of (2.3)–(2.5), for distinct numbers $k_1, k_2 \in \mathbb{Z}$, we obtain the following [8]:

(1) $S_{k_1} \cap S_{k_2} = \emptyset$.

(2) $B_{S_{k_1}} \neq B_{S_{k_2}}$, where $B_{S_{k_i}}$ means the base generated by the subbase $S_{k_i}, i \in \{1, 2\}$.

(3) $(\mathbb{Z}, T_0)$ is not a Kolmogorov space.

(4) $(\mathbb{Z}, T_k)$ is a $T_{\frac{1}{2}}$-space, i.e., T_k satisfies the separation axiom $T_{\frac{1}{2}}$, $k \in \mathbb{Z} \setminus \{0\}$.

3. Characterization of the closures of singletons in $(\mathbb{Z}, T_k), k \in \mathbb{Z}$

In this section, we intensively characterize the topological spaces $(\mathbb{Z}, T_k), k \in \mathbb{Z}$. One important thing is that there are infinitely many types of topologies on $\mathbb{Z}$ generated by the given subbases S_k in (2.1) according to the number $k \in \mathbb{Z}$. Furthermore, they are related to a quasi-discrete, a K-topological, an Alexandroff topological structure and so forth. It turns out that $(\mathbb{Z}, T_0)$ is a quasi-discrete (not discrete) topological space and further, $(\mathbb{Z}, T_0)$ is not connected [8]. Meanwhile, $(\mathbb{Z}, T_{-1})$ is the K-topological line (see Theorems 1 and 2 of [8]).

Furthermore, for $i \neq j$, $i, j \in \mathbb{Z}$, we find that $T_i \neq T_j$ [8] and further, in the case $i \neq j, i, j \in \mathbb{N}$, it turns out that T_i is not homeomorphic to T_j either (see Theorem 2(4) of [8] and Corollaries 4.4 and 4.8 and also Remark 5.6 in the present paper).

With $(\mathbb{Z}, T_k), k \in \mathbb{Z}$, since the closure of a singleton plays a significant role in studying $(\mathbb{Z}, T_k), k \in \mathbb{Z}$, let us now intensively investigate certain structures of the closures of singletons of $\mathbb{Z}$. For our purposes, in $(\mathbb{Z}, T_k), k \in \mathbb{Z}$, for a subset $A \subset \mathbb{Z}$ we denote by $Cl_k(A)$ the closure of A.

Theorem 3.1. *With* $(\mathbb{Z}, T_k), k \in \mathbb{N} \cup \{0\}$,

$$\begin{Bmatrix} Cl_0(\{2n+1\}) = \{2n, 2n+1\} = Cl_0(\{2n\}), \text{ and} \\ Cl_k(\{2n+1\}) = \{2n-2k, 2n, 2n+1\}, \ Cl_k(\{2n\}) = \{2n\} \text{ if } k \in \mathbb{N}. \end{Bmatrix}$$

Proof: Before starting the proof, we had better remind that for any $k \in \mathbb{N}$ and any even $x \in \mathbb{Z}$, $Cl_k(x) = \{x\}$. Therefore, in the proof we can consider only odd integers. According to the structure of $(\mathbb{Z}, T_k), k \in \mathbb{Z}$, we have the following cases.

(Case 1) In the case of $k = 0$, owing to the quasi-discrete structure of $(\mathbb{Z}, T_0)$ with $SN_0(2n) = \{2n, 2n+1\} = SN_0(2n+1)$ (see (2.4)) [8], the proof is completed.

(Case 2) In the case of $k \in \mathbb{N}$, according to the number k we have the following:

(Case 2-1) In the case of $k = 1$, based on $(\mathbb{Z}, T_1)$, for each $2n+1$, to find the closure of the singleton $\{2n+1\}$, we look for the largest open set that does not contain the element $2n+1$. In fact, we find that it is equal to $\mathbb{Z} \setminus \{2n-2, 2n, 2n+1\}$. To be specific, the set (see Figure 1(1))

$$\bigcup_{m \in 2\mathbb{Z}, m \leq 2n-4} \{m, m+1, m+3\} \cup [2n+2, +\infty)_{\mathbb{Z}}$$

is the maximal open set excluding the set $\{2n-2, 2n, 2n+1\}$. Thus we obtain

$$Cl_1(\{2n+1\}) = \{2n-2, 2n, 2n+1\}.$$

(Case 2-2) In the case of $k = 2$, based on $(\mathbb{Z}, T_2)$, we obviously have that the set (see Figure 1(2))

$$\bigcup_{m\in 2\mathbb{Z}, m\leq 2n-6} \{m, m+1, m+5\} \cup \{2n-2, 2n-1, 2n+3\} \cup [2n+2, +\infty)_{\mathbb{Z}}$$

is the maximal open set excluding the set $\{2n-4, 2n, 2n+1\}$. Thus we have

$$Cl_2(\{2n+1\}) = \{2n-4, 2n, 2n+1\}.$$

(Case 2-3) In the case of $k = 3$, based on $(\mathbb{Z}, T_3)$, it is clear that the set (see Figure 1(3))

$$\left\{\begin{array}{l} \bigcup_{m\in 2\mathbb{Z}, m\leq 2n-8} \{m, m+1, m+7\} \cup \{2n-4, 2n-3, 2n+3\} \cup \\ \{2n-2, 2n-1, 2n+5\} \cup [2n+2, +\infty)_{\mathbb{Z}} \end{array}\right\}$$

is the maximal open set excluding the set $\{2n-6, 2n, 2n+1\}$. Thus we have

$$Cl_3(\{2n+1\}) = \{2n-6, 2n, 2n+1\}.$$

Using a method similar to this approach, in general, we obtain the following:

(Case l) In the case of $k = l \geq 4$, for each $2n+1 \in \mathbb{Z}$

$$\left\{\begin{array}{l} \bigcup_{m\in 2\mathbb{Z}, m\leq 2n-2l-2} \{m, m+1, m+2l+1\} \\ \cup \{2n-2l+2, 2n-2l+3, 2n+3\} \cup \cdots \\ \cup \{2n-2, 2n-1, 2n+2l-1\} \cup [2n+2, +\infty)_{\mathbb{Z}}. \end{array}\right\} \tag{3.1}$$

is the maximal open set excluding the set $\{2n-2l, 2n, 2n+1\}$. Finally, we obtain

$$Cl_l(\{2n+1\}) = \{2n-2l, 2n, 2n+1\}.$$

Thus, it is clear that the complement of $\{2n-2k, 2n, 2n+1\}$ in $(\mathbb{Z}, T_k)$ is the largest open set that does not contain the element $2n+1$.

Next, since the singleton $\{2n\}$ is a closed set, it is clear that $Cl_k(\{2n\}) = \{2n\}$. □

Example 3.1. *(1) As an example guaranteeing (Case 2-1), we have* $Cl_1(\{3\}) = \{0, 2, 3\}$ *(see Figure 1(1)) because*

$$\bigcup_{m\in 2\mathbb{Z}, m\leq -2} \{m, m+1, m+3\} \cup [4, +\infty)_{\mathbb{Z}}$$

is the maximal open set excluding the set $\{0, 2, 3\}$.

(2) To support the proof of (Case 2-2), we have $Cl_2(\{3\}) = \{-2, 2, 3\}$ *(see Figure 1(2)) because*

$$\bigcup_{m\in 2\mathbb{Z}, m\leq -4} \{m, m+1, m+5\} \cup \{0, 1, 5\} \cup [4, +\infty)_{\mathbb{Z}}$$

is the maximal open set excluding the set $\{-2, 2, 3\}$.
(3) To guarantee (Case 2-3), we have $Cl_3(\{7\}) = \{0, 6, 7\}$ *because*

$$\bigcup_{m\in 2\mathbb{Z}, m\leq -2} \{m, m+1, m+7\} \cup \{2, 3, 9\} \cup \{4, 5, 11\} \cup [8, +\infty)_{\mathbb{Z}}$$

is the maximal open set excluding the set $\{0, 6, 7\}$.

So far, we have studied some structures of singletons of the topological space $(\mathbb{Z}, T_k)$. Let us examine certain structure of closures of the singletons in $(\mathbb{Z}, T_{-k})$.

Theorem 3.2. *With* $(\mathbb{Z}, T_{-k}), k \in \mathbb{N} \setminus \{1\}$, *we obtain*

$$Cl_{-k}(\{2n+1\}) = \{2n, 2n+1, 2n+2k\},\ \ Cl_{-k}(\{2n\}) = \{2n\}.$$

Proof: Using a method used in the proof of Theorem 3.1 (see (3.1)), for each $2n+1 \in \mathbb{Z}$, we obtain

$$Cl_{-k}(\{2n+1\}) = \{2n, 2n+1, 2n+2k\}$$

because the complement of $\{2n, 2n+1, 2n+2k\}$ in $\mathbb{Z}$ is the maximal open set in $(\mathbb{Z}, T_{-k})$ excluding the set $\{2n, 2n+1, 2n+2k\}$.

Next, since the singleton $\{2n\}$ is a closed set in $(\mathbb{Z}, T_{-k})$, it is clear that $Cl_{-k}(\{2n\}) = \{2n\}$. □

To support the proof of Theorem 3.2, we can consider the following example. In $(\mathbb{Z}, T_{-2})$, we obtain $Cl_{-2}(\{3\}) = \{2, 3, 6\}$ (see Figure 1(4)).

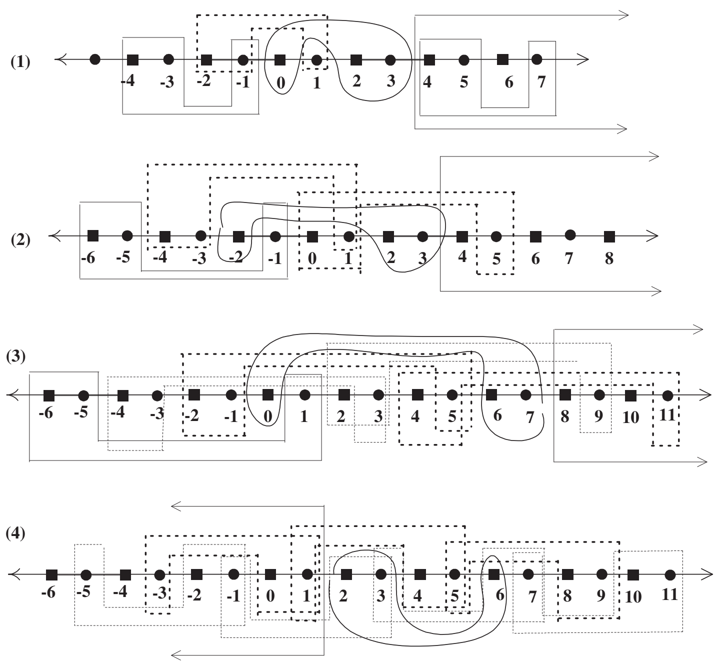

Figure 1. Configuration of the closure of the singleton $\{2n+1\}$ in $(\mathbb{Z}, T_k), k \in \{1, 2, 3\}$. For details, in (1) we obtain $Cl_1(\{3\}) = \{0, 2, 3\}$. In (2) we obtain $Cl_2(\{3\}) = \{-2, 2, 3\}$. In (3) we obtain $Cl_3(\{7\}) = \{0, 6, 7\}$. In (4), in $(\mathbb{Z}, T_{-2})$, $Cl_{-2}(\{3\}) = \{2, 3, 6\}$.

Besides, in $(\mathbb{Z}, T_{-3})$, we obtain $Cl_{-3}(\{3\}) = \{2, 3, 8\}$ (see Figure 2)

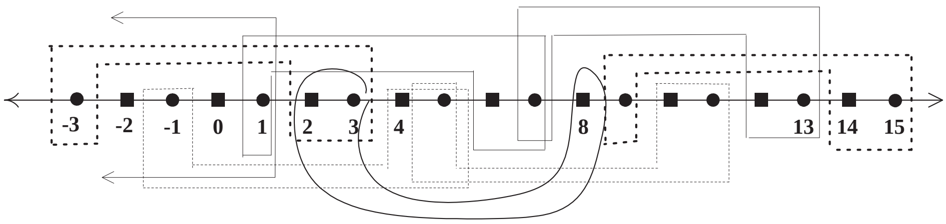

Figure 2. In $(\mathbb{Z}, T_{-3})$, configuration of $Cl_{-3}(\{3\})$.

It is obvious that $(\mathbb{Z}, T_0)$ is not a Kolmogorov space. Let us now examine the other cases.

Remark 3.3. *For each $k \in \mathbb{Z}\setminus\{0\}$, even though $(\mathbb{Z}, T_k)$ is a Kolmogorov space, it is not a Fréchet space, i.e., it does not satisfy the separation axiom T_1.*

Proof: By Theorems 3.1 and 3.2, we observe that not every singleton is closed in $(\mathbb{Z}, T_k)$, which implies that $(\mathbb{Z}, T_k)$ is not a T_1-space. To be specific, for any two points $2n, 2n+1 \in (\mathbb{Z}, T_k)$, $k \in \mathbb{Z}\setminus\{0\}$, we have $2n+1 \in SN_k(2n)$ and $2n \notin SN_k(2n+1) = \{2n+1\}$. □

4. The numbers of components of $(\mathbb{Z}, T_k), k \in \mathbb{Z}$

Let us now calculate the numbers of components of $(\mathbb{Z}, T_k), k \in \mathbb{Z}$. This study is essential for characterizing topological structures of $(\mathbb{Z}, T_k), k \in \mathbb{Z}$.

Lemma 4.1. $(\mathbb{Z}, T_k), k \in \mathbb{Z}$ *is locally connected.*

Proof: For each $k \in \mathbb{Z}$, since $(\mathbb{Z}, T_k)$ is an Alexandroff topological space (see Lemma 1 of [8]), for each point $x \in \mathbb{Z}$, the minimal open neighborhood of x in $(\mathbb{Z}, T_k)$ is connected. □

Hereinafter, for our purposes, given two numbers $i, m \in \mathbb{N}$, we use the notation $m\mathbb{Z} + i := \{mn + i \,|\, n \in Z\}$.

Proposition 4.2. *With the topological space $(\mathbb{Z}, T_k)$ and $k \in \mathbb{N}$, for $i \in [1, k]_{\mathbb{Z}}$, let*

$$C_i := \bigcup_{n \in 2k\mathbb{Z}+2(i-1)} \{n, n+1, n+2k+1\}. \tag{4.1}$$

Then the set $\{C_i \,|\, i \in [1, k]_{\mathbb{Z}}\}$ is a partition of $\mathbb{Z}$ such that C_i is both an open and a closed set in $(\mathbb{Z}, T_k)$, $i \in [1, k]_{\mathbb{Z}}$, and $C_i^{\sharp} = \aleph_0$.

Proof: Depending on the number k of $(\mathbb{Z}, T_k)$, we have the following:

(Case 1) In the case of $k = 1$, we obtain (see Figure 3(1))

$$C_1 := \bigcup_{n \in 2\mathbb{Z}} \{n, n+1, n+3\} = \mathbb{Z}. \tag{4.2}$$

(Case 2) In the case of $k = 2$, we obtain (see Figure 3(2-1) and (2-2))

$$\begin{cases} (1)\, C_1 := \bigcup_{n\in 4\mathbb{Z}} \{n, n+1, n+5\}, \text{ and} \\ (2)\, C_2 := \bigcup_{n\in 4\mathbb{Z}+2} \{n, n+1, n+5\}, \end{cases} \tag{4.3}$$

such that for $i \in [1, 2]_{\mathbb{Z}}$

$$C_1 \cup C_2 = \mathbb{Z}, C_1 \cap C_2 = \emptyset, \text{ and } C_i, C_i^c \in T_2,$$

where C_i^c means the complement of the given set C_i in $\mathbb{Z}$.

(Case 3) In the case of $k = 3$, we have (see Figure 3(3-1)–(3-3))

$$\begin{cases} (1)\, C_1 := \bigcup_{n\in 6\mathbb{Z}} \{n, n+1, n+7\}, \\ (2)\, C_2 := \bigcup_{n\in 6\mathbb{Z}+2} \{n, n+1, n+7\}, \text{ and} \\ (3)\, C_3 := \bigcup_{n\in 6\mathbb{Z}+4} \{n, n+1, n+7\}, \end{cases} \tag{4.4}$$

such that for distinct numbers $i, j \in [1, 3]_{\mathbb{Z}}$

$$C_1 \cup C_2 \cup C_3 = \mathbb{Z}, C_i \cap C_j = \emptyset, \text{ and } C_i, C_i^c \in T_3.$$

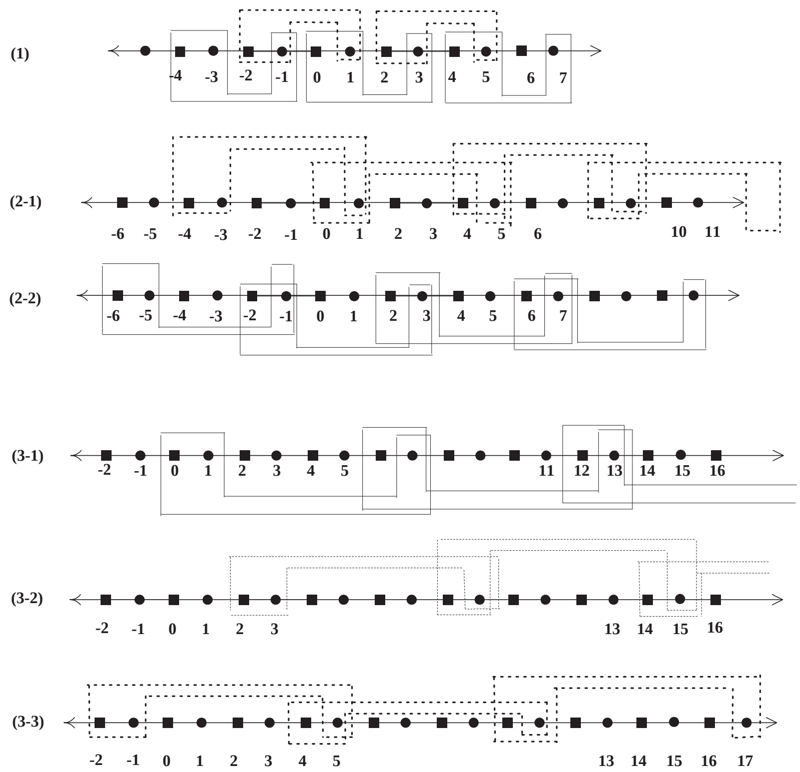

Figure 3. (1) In $(\mathbb{Z}, T_1)$, the only one component is C_1; (2) In $(\mathbb{Z}, T_2)$, the only two components are C_1 (see (2-1)) and C_2 (see (2-2)); (3) In $(\mathbb{Z}, T_3)$, the only three components are C_1 (see (3-1)), C_2 (see (3-2)), and C_3 (see (3-3)).

In general, using a method similar to the above approach, we obtain the following:

(Case m) In the case of $k = m$, we have

$$\left\{\begin{array}{l}(1)\, C_1 := \bigcup\limits_{n\in 2m\mathbb{Z}} \{n, n+1, n+2m+1\},\\ (2)\, C_2 := \bigcup\limits_{n\in 2m\mathbb{Z}+2} \{n, n+1, n+2m+1\},\\ (3)\, C_3 := \bigcup\limits_{n\in 2m\mathbb{Z}+4} \{n, n+1, n+2m+1\},\\ \cdots\\ \cdots\\ (m)\, C_m := \bigcup\limits_{n\in 2m\mathbb{Z}+2(m-1)} \{n, n+1, n+2m+1\},\end{array}\right\} \tag{4.5}$$

such that for distinct numbers $i, j \in [1, k]_{\mathbb{Z}}$

$$\bigcup_{i\in[1,k]_{\mathbb{Z}}} C_i = \mathbb{Z}, C_i \cap C_j = \emptyset, \text{ and } C_i, C_i^c \in T_k.$$

In view of (4.5), in $(\mathbb{Z}, T_k)$, for distinct numbers $i, j \in [1, k]_{\mathbb{Z}}$, we obtain

$$\mathbb{Z} = \bigcup_{i\in[1,k]_{\mathbb{Z}}} C_i \text{ such that } C_i \cap C_j = \emptyset, i \neq j. \tag{4.6}$$

Furthermore, owing to the structure C_i, $i \in [1, k]_{\mathbb{Z}}$ which is a union of the smallest open sets, each C_i is clearly open set in $(\mathbb{Z}, T_k)$. Finally, by (4.6), we conclude that each C_i has the property

$$C_i = \mathbb{Z} \setminus \bigcup_{j\in[1,k]_{\mathbb{Z}}\setminus\{i\}} C_j.$$

Hence each set $C_i, i \in [1, k]_{\mathbb{Z}}$, is also a closed set in $(\mathbb{Z}, T_k)$, which completes the proof. □

In view of Lemma 4.1, since $(\mathbb{Z}, T_k)$, $k \in \mathbb{N}$, is locally connected, we can confirm that each component C_i in (4.1) is both an open and a closed set in $(\mathbb{Z}, T_k)$, $k \in \mathbb{N}$.

In view of Proposition 4.2, we obtain the following:

Theorem 4.3. *(1)* $(\mathbb{Z}, T_0)$ *has* $\aleph_0$ *components.*

(2) For the topological spaces $(\mathbb{Z}, T_k)$ *and each* $k \in \mathbb{N}$, $(\mathbb{Z}, T_k)$ *has* k *components.*

Proof: (1) Owing the quasi-discrete structure of $(\mathbb{Z}, T_0)$, we observe that for any $n \in \mathbb{Z}$

$$\left\{\begin{array}{l}(1)\, SN_0(2n) = \{2n, 2n+1\} = SN_0(2n+1), \text{ and}\\ (2)\, Cl_0(2n) = \{2n, 2n+1\} = Cl_0(2n+1).\end{array}\right\}$$

Thus, after letting $C_n := \{2n, 2n+1\}, n \in \mathbb{Z}$, we obtain that $\{C_n \,|\, n \in \mathbb{Z}\}$ is a partition of $\mathbb{Z}$ consisting of the open sets and components C_n, $n \in \mathbb{Z}$, in $(\mathbb{Z}, T_0)$. To be specific,

$$\mathbb{Z} = \bigcup_{n\in\mathbb{Z}} C_n, C_n \in T_0 \text{ and further, } C_i \cap C_j = \emptyset, i \neq j, i, j \in \mathbb{Z}.$$

(2) In view of Proposition 4.2, for each $k \in \mathbb{N}$ and $i \in [1, k]_{\mathbb{Z}}$, it suffices to prove that each C_i of (4.1) is a component in $(\mathbb{Z}, T_k)$. To be specific, for each $n \in 2k\mathbb{Z} + 2(i-1)$ of (4.1), consider the subset $\{n, n+1, n+2k+1\}$ of C_i of (4.1). Then the set $\{n, n+1, n+2k+1\}$ is a certain $S_{k,n} \in S_k$ (see (2.1)), and it is one of the smallest open set containing the element n in the Alexandroff topological space $(\mathbb{Z}, T_k)$. Hence $\{n, n+1, n+2k+1\}$ is connected in $(\mathbb{Z}, T_k)$. Then, take another smallest open set which is a neighbor of $\{n, n+1, n+2k+1\}$ which is also the smallest open set in $(\mathbb{Z}, T_k)$, so that it is also connected in $(\mathbb{Z}, T_k)$. Furthermore, the intersection of these two smallest open sets is not an empty set. Thus the union of these two smallest open sets is connected in $(\mathbb{Z}, T_k)$. Using this method, we clearly see that the set C_i of (4.1) is connected in $(\mathbb{Z}, T_k)$ and further, it is a component of the element $n \in 2k\mathbb{Z} + 2(i-1)$. Besides, as proved in Proposition 4.2, for $i \neq j, i, j \in [1, k]_{\mathbb{Z}}$, since

$$C_i \cap C_j = \emptyset \text{ and } \bigcup_{i \in [1,k]_{\mathbb{Z}}} C_i = \mathbb{Z},$$

which implies that for each $k \in \mathbb{N}$, $(\mathbb{Z}, T_k)$ has k components.

For instance, let us consider the case $(\mathbb{Z}, T_3)$ as mentioned in (4.4). Then the set

$$\{C_1, C_2, C_3\}$$

is a partition of $\mathbb{Z}$ and each $C_i, i \in [1, 3]_{\mathbb{Z}}$, is a component in $(\mathbb{Z}, T_3)$. To be specific, based on (4.1), for $i \in [1, 3]_{\mathbb{Z}}$, we have

$$C_i := \bigcup_{n \in 6\mathbb{Z} + 2(i-1)} \{n, n+1, n+7\},$$

where

$$\begin{Bmatrix} C_1 = \cdots \cup \{0, 1, 7\} \cup \{6, 7, 13\} \cup \{12, 13, 19\} \cup \cdots \\ C_2 = \cdots \cup \{2, 3, 9\} \cup \{8, 9, 15\} \cup \{14, 15, 21\} \cup \cdots \\ C_3 = \cdots \cup \{4, 5, 11\} \cup \{10, 11, 17\} \cup \{16, 17, 23\} \cup \cdots . \end{Bmatrix}$$

Then, it is clear that C_1 is a component of the element 0, C_2 is a component of the element 2, and C_3 is that of the element 4. Besides, for $i \neq j, i, j \in [1, 3]_{\mathbb{Z}}$, we obviously have

$$C_1 \cup C_2 \cup C_3 = \mathbb{Z} \text{ and } C_i \cap C_j = \emptyset. \square$$

Corollary 4.4. *For distinct numbers $i, j \in \mathbb{N} \cup \{0\}$, $(\mathbb{Z}, T_i)$ is not homeomorphic to $(\mathbb{Z}, T_j)$.*

Proof: By Theorem 4.3, since each $(\mathbb{Z}, T_i), i \in \mathbb{N}$, has i components, in the case of $i \neq j, i, j \in \mathbb{N}$, it is clear that T_i is not homeomorphic to T_j. Besides, since $(\mathbb{Z}, T_0)$ has $\aleph_0$ components, it is clear that T_0 is not homeomorphic to $T_i, i \in \mathbb{N}$. $\square$

By Proposition 4.2 and Theorem 4.3, we obtain the following:

Corollary 4.5. *(1) $(\mathbb{Z}, T_1)$ is connected.*

(2) $(\mathbb{Z}, T_k)$ is not connected if $k \in (\mathbb{N} \setminus \{1\}) \cup \{0\}$.

With $(\mathbb{Z}, T_{-k})$, depending on the number $k \in \mathbb{N}$, let us now investigate the number of components of $(\mathbb{Z}, T_{-k})$, as follows:

Proposition 4.6. *With the topological space* $(\mathbb{Z}, T_{-k})$, *given* $k \in \mathbb{N}$, *for* $i \in [1, k]_{\mathbb{Z}}$, *let*

$$D_i := \bigcup_{n \in 2k\mathbb{Z}+2(i-1)} \{n - 2k + 1, n, n + 1\}. \tag{4.7}$$

Then the set $\{D_i \,|\, i \in [1, k]_{\mathbb{Z}}\}$ *is a partition of* $\mathbb{Z}$ *such that* D_i *is both an open and a closed set in* $(\mathbb{Z}, T_{-k})$, *and* $D_i^{\sharp} = \aleph_0$.

Proof: Depending on the number $k \in \mathbb{N}$ of $(\mathbb{Z}, T_{-k})$, using a method similar to the construction of C_i in Proposition 4.2, we have the following (see the cases in Figure 4 related to T_{-2}):
In general, with the topological space $(\mathbb{Z}, T_{-k})$, for each $k \in \mathbb{N}$ and $i \in [1, k]_{\mathbb{Z}}$, we have

$$\left\{\begin{array}{l} (1)\, D_1 := \bigcup_{n \in 2k\mathbb{Z}} \{n - 2k + 1, n, n + 1\}, \\ (2)\, D_2 := \bigcup_{n \in 2k\mathbb{Z}+2} \{n - 2k + 1, n, n + 1\}, \\ (3)\, D_3 := \bigcup_{n \in 2k\mathbb{Z}+4} \{n - 2k + 1, n, n + 1\}, \\ \dots \\ (k)\, D_k := \bigcup_{n \in 2k\mathbb{Z}+2(k-1)} \{n - 2k + 1, n, n + 1\}. \end{array}\right\} \tag{4.8}$$

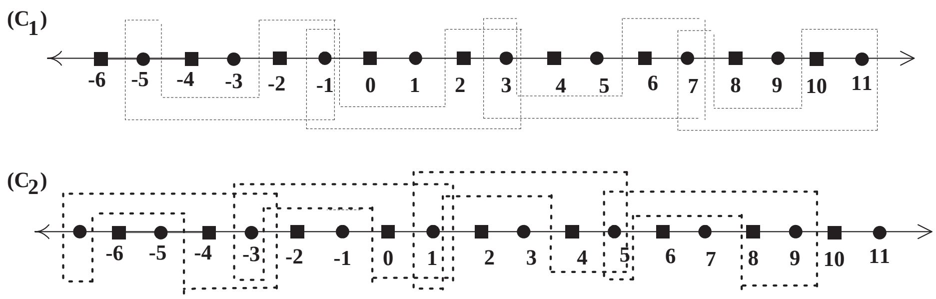

Figure 4. Two components of the topological space $(\mathbb{Z}, T_{-2})$.

In view of (4.8), in $(\mathbb{Z}, T_{-k})$ for distinct numbers $i \in [1, k]_{\mathbb{Z}}$, we obtain

$$\mathbb{Z} = \bigcup_{i \in [1,k]_{\mathbb{Z}}} D_i \text{ such that } D_i \cap D_j = \emptyset. \tag{4.9}$$

Hence, we obtain a partition of $\mathbb{Z}$, as follows:

$$\{D_i \,|\, i \in [1, k]_{\mathbb{Z}}\}.$$

Furthermore, owing to the structure D_i which is a union of the smallest open sets

$$SN_{-k}(n) = \{n - 2k + 1, n, n + 1\} \in S_{-k} \subset T_{-k},$$

each D_i is clearly an open set in $(\mathbb{Z}, T_{-k})$ (see (4.8)). Besides, in view of (4.9), each D_i has the property

$$D_i = \mathbb{Z} \setminus \bigcup_{j \in [1,k]_{\mathbb{Z}} \setminus \{i\}} D_j.$$

Hence each set $D_i, i \in [1, k]_{\mathbb{Z}}$ is also a closed set in $(\mathbb{Z}, T_{-k})$. □

Besides, by Lemma 4.1, since $(\mathbb{Z}, T_{-k})$ is locally connected, we can confirm that D_i is both an open and a closed set in T_{-k}.

Besides, Proposition 4.6 leads to the following:

Corollary 4.7. *For each $k \in \mathbb{N}$, the topological space, $(\mathbb{Z}, T_{-k})$ has k components.*

Proof: By Proposition 4.6 and using a method similar to the proof of Theorem 4.3, it turns out that for each $k \in \mathbb{N}$, $(\mathbb{Z}, T_{-k})$ has k components. □

Corollary 4.8. *For distinct numbers $i, j \in \mathbb{N} \cup \{0\}$, $(\mathbb{Z}, T_{-i})$ is not homeomorphic to $(\mathbb{Z}, T_{-j})$.*

Proof: The proof is identical to that of Corollary 4.4. □

By Corollary 4.7, we obtain the following:

Corollary 4.9. *$(\mathbb{Z}, T_{-k})$ is not connected if $k \in \mathbb{N} \setminus \{1\}$.*

Some further studies of the structures of $(\mathbb{Z}, T_{-i}), i \in \mathbb{Z}$, will intensively be done in Section 5 (see Proposition 5.1).

Corollary 4.10. *Each T_k is connected if and only if $k \in \{-1, 1\}$.*

Proof: By Theorem 4.3(2) and Corollary 4.7, the proof is completed. □

5. Homeomorphisms between $(\mathbb{Z}, T_{-k})$ and $(\mathbb{Z}, T_k), k \in \mathbb{N}$

Comparing the numbers of components of $(\mathbb{Z}, T_k), k \in \mathbb{N} \cup \{0\}$, it turns out that T_i is not homeomorphic to T_j if $i \neq j$ and $i, j \in \mathbb{N} \cup \{0\}$ (see Corollary 4.4). Furthermore, T_{-i} is proved not to be homeomorphic to T_{-j} if $i \neq j$ and $i, j \in \mathbb{N} \cup \{0\}$ (see Corollary 4.8). Let us now investigate various topological properties of $(\mathbb{Z}, T_{-k}), k \in \mathbb{N}$ and further, examine if $(\mathbb{Z}, T_{-k})$ is homeomorphic to $(\mathbb{Z}, T_k), k \in \mathbb{N}$.

Proposition 5.1. *$(\mathbb{Z}, T_{-k})$ is homeomorphic to $(\mathbb{Z}, T_k), k \in \mathbb{N}$.*

Before proving this assertion, note that there are infinitely many homeomorphisms between $(\mathbb{Z}, T_{-k})$ and $(\mathbb{Z}, T_k)$.

Proof: Let us consider the map $h : (\mathbb{Z}, T_{-k}) \to (\mathbb{Z}, T_k), k \in \mathbb{N}$, defined by

$$h(x) = \begin{Bmatrix} x, x \in \mathbb{Z}_0, \text{ and} \\ x + 2k, x \in \mathbb{Z}_1. \end{Bmatrix} \tag{5.1}$$

Then, it is clear that h is bijective. Next, we now prove that each of the maps h and h^{-1} is continuous. (Case 1) Note that for $x \in \mathbb{Z}_0$, we have

$$h(x) = h(2n) = 2n$$

and

$$SN_{-k}(2n) = \{2n - 2k + 1, 2n, 2n + 1\} \text{ and } SN_k(2n) = \{2n, 2n + 1, 2n + 2k + 1\}.$$

(Case 2) Note that for $x \in \mathbb{Z}_1$, we have

$$h(x) = h(2n + 1) = 2n + 2k + 1$$

and

$$SN_{-k}(2n + 1) = \{2n + 1\} \text{ and } SN_k(2n + 2k + 1) = \{2n + 2k + 1\}.$$

Owing to these (Case 1) and (Case 2), we see that the map h is continuous. Finally, through a similar process, we obtain that h^{-1} is continuous. Hence h is a homeomorphism, which completes the proof.

In addition, based on the components $D_i, C_i, i \in [1, k]_{\mathbb{Z}}$ in (4.1) and (4.7), it is clear that for each $D_i, i \in [1, k]_{\mathbb{Z}}$, we can observe that the restriction of h to D_i, say $h|_{D_i} : D_i \to C_i$, is also a homeomorphism. Hence we also have the homeomorphism $h : (\mathbb{Z}, T_{-k}) \to (\mathbb{Z}, T_k)$ as an extension of the given $h|_{D_i}, i \in [1, k]_{\mathbb{Z}}$. For instance, in Figure 6, we may assume the map $F := h$ and $F|_{D_i} := h|_{D_i} := f_i, i \in \{1, 2\}$. □

Example 5.1. *(1)* $(\mathbb{Z}, T_{-1})$ *is homeomorphic to* $(\mathbb{Z}, T_1)$.
(2) $(\mathbb{Z}, T_{-2})$ *is homeomorphic to* $(\mathbb{Z}, T_2)$.

To support these homeomorphisms $h : (\mathbb{Z}, T_{-k}) \to (\mathbb{Z}, T_k), k \in \{1, 2\}$, we consider the two cases, as follows:

(Case 1) Assume the case $k = 1$. Then we take the following map h (see Figure 5).

$$h(x) = \begin{Bmatrix} x, x \in \mathbb{Z}_0, \text{ and} \\ x + 2, x \in \mathbb{Z}_1. \end{Bmatrix}$$

Then, it is clear that the map h is a homeomorphism.

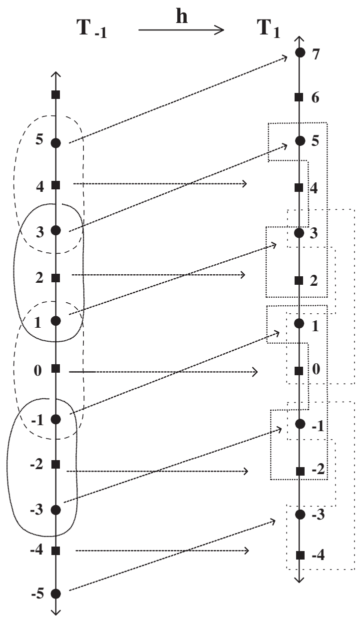

Figure 5. Configuration of a certain homeomorphism between $(\mathbb{Z}, T_{-1})$ and $(\mathbb{Z}, T_1)$.

(Case 2) Assume the case $k = 2$. Then we take the following map h (see Figure 6).

$$h(x) = \begin{cases} x, x \in \mathbb{Z}_0, \text{ and} \\ x + 4, x \in \mathbb{Z}_1. \end{cases} \tag{5.2}$$

Then, it is clear that the map h is a homeomorphism.

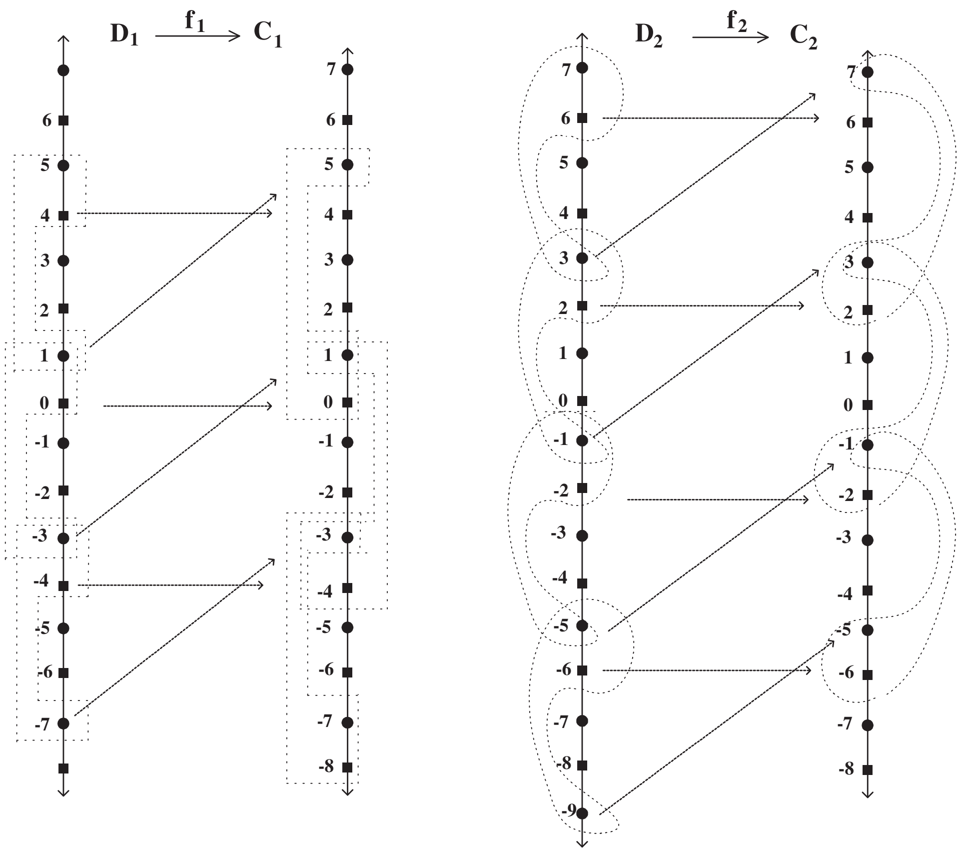

Figure 6. A homeomorphism between $(\mathbb{Z}, T_{-2})$ and $(\mathbb{Z}, T_2)$ formulated by the two homeomorphisms f_1 and f_2. Naively, $F : (\mathbb{Z}, T_{-2}) \to (\mathbb{Z}, T_2)$ can be considered as a union of the two homeomorphisms f_1 and f_2, i.e., $F := f_1 \cup f_2$.

In view of Proposition 5.1, we observe that for $k \in \mathbb{N} \setminus \{1\}$, $(\mathbb{Z}, T_k)$ and $(\mathbb{Z}, T_{-k})$ are free (topological) sum of C_i and D_i, respectively.

By Proposition 5.1 and Corollary 4.4, we obtain the following:

Corollary 5.2. *There are infinitely many topologies $T_{S_k \times S_k}$, $k \in \mathbb{N} \setminus \{1\}$, which are not homeomorphic to the 2-dimensional K-topological plane, i.e., $(\mathbb{Z}^2, \kappa^2)$, where $T_{S_k \times S_k}$ is the topology on $\mathbb{Z}^2$ generating by the set $S_k \times S_k$ as a subbase.*

Using the method given in Corollary 5.2, for the set $\mathbb{Z}^n$, we can also obtain infinitely many topologies generated by certain n-tuple Cartesian products of S_k. Furthermore, each of these topologies need not be homeomorphic to the n-dimensional K-topological space, i.e., $(\mathbb{Z}^n, \kappa^n)$.

After replacing the subbase S_k of (2.1) by the set S'_k (see (5.3)), where

$$S'_k := \{S'_{k,n} \mid S'_{k,n} = \{2n, 2n+1, 2n+2k\}, n \in \mathbb{Z}\}, \tag{5.3}$$

the paper [8] studied some properties of topologies, denoted by $T'_k := T_{S'_k}$ generated by the set S'_k as a subbase. Then, after comparing topologies T'_k and T_k, we can recognize some similarities and differences between S_k of (2.1) and S'_k of (5.3) [8], as follows:

$$\left\{\begin{array}{l} (1)\, S^{\sharp}_{k,n} = (S'_{k,n})^{\sharp} \text{ for } k, n \in \mathbb{Z}, \\ (2)\, S^{\sharp}_{k} = \aleph_0 = (S'_k)^{\sharp} \text{ for } k \in \mathbb{Z}, \text{ and} \\ (3) \text{ the only difference between } S_{k,n} \text{ and } S'_{k,n} \\ \text{ are the two distinct numbers } 2n+2k+1 \text{ and } 2n+2k. \end{array}\right\}$$

Lemma 5.3. *[8] Given a number $k \in \mathbb{Z}$, $(\mathbb{Z}, T_{S'_k})$ is an Alexandroff space.*

Using a method similar to the proof of Propositions 4.2 and 4.6, we obtain the following:

Proposition 5.4. *For each $i \in \mathbb{N}$, $(\mathbb{Z}, T'_i)$ has i components.*

Proof: Using a method similar to the proof of Propositions 4.2 and 4.6, the proof is completed. □

Corollary 5.5. *For the topological spaces $(\mathbb{Z}, T'_i)$, $i \in \mathbb{N} \cup \{0\}$, we obtain the following:*
For distinct numbers $i, j \in \mathbb{N} \cup \{0\}$, $(\mathbb{Z}, T'_i)$ is not homeomorphic to $(\mathbb{Z}, T'_j)$.

Based on the various properties of $(\mathbb{Z}, T_k)$, we obtain the following which are certain corrections of some assertions proposed in [8].

Remark 5.6. *(Correction) (1) In Theorem 2(5) of [8], the set "$\mathbb{Z}$" should be replace by "$\mathbb{N} \cup \{0\}$" (see Corollary 4.4 in the present paper).*

(2) In Corollary 1 of [8], the part " if $i \in \mathbb{Z} \setminus \{0\}$" should be replaced with " if $i \in \{-1, 1\}$" (see Corollary 4.10 in the present paper).

(3) In Corollary 2(5) of [8], the set "$\mathbb{Z}$" should be replaced by "$\mathbb{N} \cup \{0\}$" (see Corollary 5.5 of the present paper).

(4) Remark 2(3) of [8] should be changed into "$(\mathbb{Z}, T'_i)$ is not homeomorphic to $(\mathbb{Z}, T'_j)$ if two distinct $i, j \in \mathbb{N} \cup \{0\}$" (see Corollary 5.5 of the present paper).

6. Further remark and work

Since the advent of the digital spaces derived from certain digital topological spaces facilitated many works from the viewpoints of digital topology, and based on the works in the paper, we can further study the following:
($\star$1) Unlike Remark 2(3) of [8] (see Remark 5.6(4)), we can examine if $(\mathbb{Z}, T'_i)$ and $(\mathbb{Z}, T_i)$ are homeomorphic to each other.
($\star$2) Comparison among T'_i, T_i, T'_{-i}, and T_{-i} with respect to certain digital space structures associated with these topological structures.

($\star$3) Formulation of digital spaces generated by the topological structures T'_i, T_i, T'_{-i}, and T_{-i}.
($\star$4) Comparison among certain adjacencies on $\mathbb{Z}$ induced by the topological structures T'_i, T_i, T'_{-i}, and T_{-i}.
($\star$5) Under what condition do we have the typical k-adjacencies on $\mathbb{Z}^n$ induced by the product topological structures $(T'_i)^n$, $(T_i)^n$, $(T'_{-i})^n$, $(T_{-i})^n$ and their product topologies?
($\star$6) One of the important problems is that we need to examine if two digital spaces induced by two homeomorphic topologies on $\mathbb{Z}^n$ have certain relationships.
($\star$7) What are the benefits of the topologies T'_i, T'_{-i}, T_i, and T_{-i}?
($\star$8) Based on the locally finite covering approximations in [5–7], using the topological structure of $(\mathbb{Z}^2, T_k)$, we can follow the approach to develop a certain locally finite covering approximation space.

7. Conclusions

We have shown infinitely many topologies on the set $\mathbb{Z}$ which are not homeomorphic to the K-line topology. In particular, we proved that $(\mathbb{Z}, T_k)$ is connected if and only if $k \in \{-1, 1\}$. Furthermore, we proved that $(\mathbb{Z}, T_k), k \in \mathbb{N}$, has k-components. This finding plays an important role in writing parallel algorithms to address some complicated problems. Indeed, these newly-studied topological structures facilitate the studies of digital topological spaces related to the fixed point theory, rough set theory, writing parallel algorithms and so on. As a further work we can intensively study $T'_k := T_{S'_k}$ generated by the set S'_k as a subbase. We further study digital topological structures induced by T'_k, T'_{-k}, T_k, T_{-k} and then applied them to applied sciences as well as computer science.

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Conflict of interest

The authors declare no conflict of interest.

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