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Research article

Upper paired domination in graphs

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Abstract: A set $PD \subseteq V(G)$ in a graph G is a paired dominating set if every vertex $v \notin PD$ is adjacent to a vertex in PD and the subgraph induced by PD contains a perfect matching. A paired dominating set PD of G is minimal if there is no proper subset $PD' \subset PD$ which is a paired dominating set of G. A minimal paired dominating set of maximum cardinality is called an upper paired dominating set, denoted by $\Gamma_{pr}(G)$-set. Denote by $Upper\text{-}PDS$ the problem of computing a $\Gamma_{pr}(G)$-set for a given graph G. Michael et al. showed the APX-completeness of $Upper\text{-}PDS$ for bipartite graphs with $\Delta = 4$ [11]. In this paper, we show that $Upper\text{-}PDS$ is APX-complete for bipartite graphs with $\Delta = 3$.

Keywords: upper paired domination; APX-completeness
Mathematics Subject Classification: 05C69, 68Q15

1. Introduction

Throughout this paper, all graphs considered are finite, undirected, loopless and without multiple edges. We refer the reader to [3, 18] for terminology and notation in graph theory.

Let $G = (V,E)$ be a graph of order n with *vertex set* $V(G)$ and *edge set* $E(G)$. The *open neighborhood* of a vertex v in G is $N_G(v) = N(v) = \{u \in V(G)|uv \in E(G)\}$, and the *closed neighborhood* of v is $N_G[v] = N[v] = N(v) \cup \{v\}$. The *degree* of a vertex v in the graph G is $d_G(v) = d(v) = |N(v)|$. Let $\delta(G) = \delta$ and $\Delta(G) = \Delta$ denote the *minimum* and *maximum degree* of a graph G, respectively. Denote by $G[H]$ the induced subgraph of G induced by H with $H \subset V(G)$. A vertex v in G is a *leaf* if $d(v) = 1$. A vertex u is a *support vertex* if u has a leaf neighbor. Denote $L(u) = \{v|uv \in E(G), d(v) = 1\}$.

A subset $M \subseteq E(G)$ is called a *matching* in G if no two elements are adjacent in G. A vertex v is said to be M-saturated if some edges of M are incident with v, otherwise, v is M-unsaturated. If every vertex of G is M-saturated, the matching M is perfect. M is a maximum matching if G has no matching

M' with $|M'| > |M|$. Let R be a subgraph of G, M be a matching of G, $v \in V(R)$, $uv \in M$. We say the vertex v is R^I (resp. R^O) if $u \in V(R)$ (resp. $u \notin V(R)$).

A set VC of vertices in a graph G is a *vertex cover* of G if all the edges are touched by the vertices in VC. A vertex cover VC of G is minimal if no proper subset of it is a vertex cover of G. A minimal vertex cover of maximum cardinality is called a VC-set. In 2001, Mishra et al. [13] denote by *MAX-MIN-VC* the problem of finding a VC-set of G. Bazgan et al. [2] showed that *MAX-MIN-VC* is APX-complete for cubic graphs.

A set $PD \subseteq V(G)$ in a graph G is a *paired dominating set* if every vertex $v \notin PD$ is adjacent to a vertex in PD and the subgraph induced by PD contains a perfect matching. Paired domination was proposed in 1996 [9] and was studied for example in [4–6, 12, 16, 17]. A paired dominating set PD of G is minimal if there is no proper subset $PD' \subset PD$ which is a paired dominating set of G. A minimal paired dominating set with maximum cardinality is called a $\Gamma_{pr}(G)$-set. The upper paired domination number of G is the cardinality of a $\Gamma_{pr}(G)$-set of G. Denote by *Upper-PDS* the problem of finding a $\Gamma_{pr}(G)$-set of G. Upper paired domination was introduced by Dorbec et al. in [7]. They investigated the relationship between the upper total domination and upper paired domination numbers of a graph. Later, they established bounds on upper paired domination number for connected claw-free graphs [10]. Denote $Pr(v) = \{u | u \notin PD, N(u) \cap PD = \{v\}, uv \in E(G)\}$, where PD is a minimal paired dominating set of G.

Recently, Michael et al. showed that *Upper-PDS* is NP-hard for split graphs and bipartite graphs, and APX-completeness of *Upper-PDS* for bipartite graphs with $\Delta = 4$ in [11]. In order to improve the results in [11], we show that *Upper-PDS* is APX-complete for bipartite graphs with $\Delta = 3$.

2. APX-completeness

The class APX is the set of NP-optimization problems that allow polynomial-time approximation algorithms with approximation ratio bounded by a constant.

First, we recall the notation of L-reduction [1, 15]. Given two NP-optimization problems H and G and polynomial time transformation f from instances of H to instances of G, we say that f is an L-reduction if there are positive constants α and β such that for every instance x of H:
(i) $opt_G(f(x)) \leq \alpha opt_H(x)$;
(ii) for every feasible solution y of $f(x)$ with objective value $m_G(f(x), y) = a$, we can find a solution y' of x with $m_H(x, y') = b$ in polynomial time such that $|opt_H(x) - b| \leq \beta |opt_G(f(x)) - a|$.

To show that a problem $P \in APX$ is APX-complete, it's enough to show that there is an L-reduction from some APX-complete problems to P.

Denote by *MAX-MIN-VC* the problem of finding a maximum minimal vertex cover of G. Note that, Minimum Domination problem is APX-complete even for bipartite graphs with maximum degree 3 [14], and Minimum Independent Domination problem [8] is the complement problem of *MAX-MIN-VC* in a graph G. We can obtain an L-reduction from Minimum Domination problem to Minimum Independent Domination problem by replacing every edge uv with a path $P_{uv} = uabcv$ with $\alpha = 7$, $\beta = 1$. It's clear that Minimum Independent Domination problem and *MAX-MIN-VC* are in APX. Thus, Minimum Independent Domination problem is APX-complete even for bipartite graphs with maximum degree 3, so is *MAX-MIN-VC* (by Theorem 7 in [2]).

In this section, we show *Upper-PDS* for bipartite graphs with maximum degree 3 is APX-complete by providing an L-reduction f from *MAX-MIN-VC* for bipartite graphs with maximum degree 3.

We formalize the optimization problems as follows.

MAX-MIN-VC

Instance: A graph $G = (V,E)$ with maximum degree 3.
Solution: A maximum minimal vertex cover of G, VC.
Measure: Cardinality of VC.

Upper-PDS

Instance: A graph $G = (V,E)$ with maximum degree 3.
Solution: A maximum minimal paired-dominating set PD.
Measure: Cardinality of PD.

Lemma 1. *[11] Upper-PDS can be approximated with a factor of* 2Δ *for graphs without isolated vertices and with maximum degree* Δ.

Therefore, *Upper-PDS* is in APX.

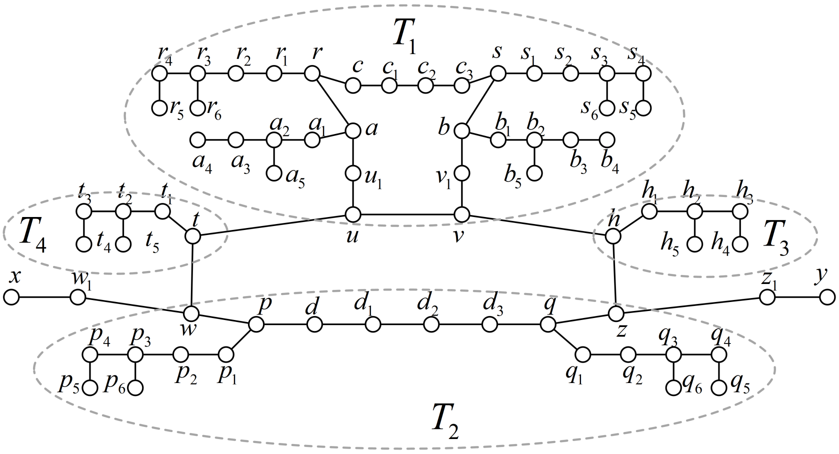

Figure 1. The graph H_{xy}.

Let $G = (V,E)$ be a bipartite graph with $|E| = m$, $\Delta(G) = 3$.

For each edge $xy \in E(G)$, let H_{xy} be the graph which is shown in Figure 1. Let $T_1 = \{\ a,...,a_5,\ b,...,b_5,\ r,...,r_6,\ s,...,s_6,\ c,..,c_3,\ u,u_1,v,v_1\}$, $T_2 = \{p,...,p_6,\ q,...,q_6,\ d,...,d_3,w,z\}$, $T_3 = \{h,..,h_5\}$, $T_4 = \{t,...,t_5\}$, $V(H_{xy}) = V(T_1) \cup V(T_2) \cup V(T_3) \cup V(T_4) \cup \{w_1,z_1,x,y\}$, $|V(H_{xy})| = 70$.

Construct G' by replacing each edge $xy \in E(G)$ with the graph H_{xy}.

It's clear, $\Delta(G') = 3$ and G' is a bipartite graph.

Let $S_p = \{p,p_1,...,p_6\}$, $S_q = \{q,q_1,...,q_6\}$, $S_a = \{a,a_1,...,a_5\}$, $S_b = \{b,b_1,...,b_5\}$, $S_r = \{r,r_1,...,r_6\}$, $S_s = \{s,s_1,...,s_6\}$, $S_c = \{c,c_1,c_2,c_3\}$, $S_d = \{d,d_1,d_2,d_3\}$.

Let $xy = e \in E(G)$, $H'_e = H'_{xy} = H_{xy} - \{x,y\}$, $|V(H'_{xy})| = 68$.

Let PD be a paired dominating set of G', $uv \in E(G)$. We say H'_{uv} is $[I,O]$ if u is H^I_{uv} and v is H^O_{uv}, or if v is H^I_{uv} and u is H^O_{uv}. We say H'_{uv} is $[I,0]$ if u is H^I_{uv} and $v \notin PD$, or if $u \notin PD$ and v is H^I_{uv}. Analogously, H'_{uv} could be $[0,0]$ ($[I,I]$ or $[O,O]$ or $[O,0]$).

Note that

$$\begin{aligned}|T_1 \cap PD| =& |S_a \cap PD| + |S_b \cap PD| + |S_c \cap PD| \\ &+ |S_r \cap PD| + |S_s \cap PD| + |\{u,u_1,v,v_1\} \cap PD|,\end{aligned} \tag{2.1}$$

$$|T_2 \cap PD| = |S_p \cap PD| + |S_q \cap PD| + |S_d \cap PD| + |\{w,z\} \cap PD|, \tag{2.2}$$

$$\begin{aligned}|V(H'_{xy}) \cap PD| =& |T_1 \cap PD| + |T_2 \cap PD| + |T_3 \cap PD| + |T_4 \cap PD| \\ &+ |\{w_1,z_1\} \cap PD|.\end{aligned} \tag{2.3}$$

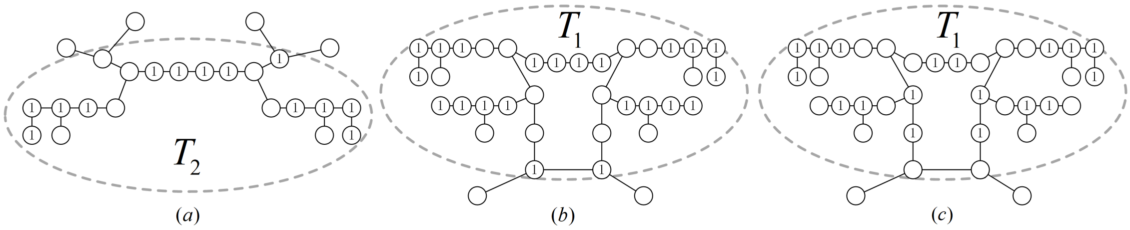

Figure 2. (a) $|T_2 \cap PD| = 13$, (b) $|T_1 \cap PD| = 22$, (c) $|T_1 \cap PD| = 18$.

The following lemma is immediate.

Lemma 2. *Let PD be a minimal paired dominating set of G, M be a perfect matching of $G[PD]$. If v,u are support vertices, $uv \in E(G)$, $x \in L(v)$, $y \in L(u)$, then $|\{x,y\} \cap PD| \le 1$.*

Lemma 3. *Let PD be a minimal paired dominating set of G', M be a perfect matching of $G'[PD]$. For each H_{xy}, we have*
(a) $|S_c \cap PD| = 4$ if and only if $r,s \notin PD$, $Pr(c_3) \neq \emptyset$ or $Pr(c) \neq \emptyset$.
(b) $|S_d \cap PD| = 4$ if and only if $p,q \notin PD$, $Pr(d_3) \neq \emptyset$ or $Pr(d) \neq \emptyset$.
(c) $|T_3 \cap PD| \le 4$ with equality if and only if (i) or (ii) holds,
(i) $Pr(h_1) \neq \emptyset$ if $h \notin PD$,
(ii) $N(h) \cap PD = \{h_1\}$ or $Pr(h) \neq \emptyset$ if $h \in PD$.
And if h is $G[T_3]^O$, $|T_3 \cap PD| = 3$.
(d) $|T_4 \cap PD| \le 4$ with equality if and only if (i) or (ii) holds,
(i) $Pr(t_1) \neq \emptyset$ if $t \notin PD$,
(ii) $N(t) \cap PD = \{t_1\}$ or $Pr(t) \neq \emptyset$ if $t \in PD$.
And if t is $G[T_4]^O$, $|T_4 \cap PD| = 3$.
(e) $|S_a \cap PD| \le 4$ with equality if and only if (i) or (ii) holds,
(i) $Pr(a_1) \neq \emptyset$ if $a \notin PD$,
(ii) $N(a) \cap PD = \{a_1\}$ or $Pr(a) \neq \emptyset$ if $a \in PD$.
And if a is $G[S_a]^O$, $|S_a \cap PD| = 3$.
(f) $|S_b \cap PD| \le 4$ with equality if and only if (i) or (ii) holds,
(i) $Pr(b_1) \neq \emptyset$ if $b \notin PD$,

(ii) $N(b) \cap PD = \{b_1\}$ *or* $Pr(b) \neq \emptyset$ *if* $b \in PD$.

And if b *is* $G[S_b]^O$, $|S_b \cap PD| = 3$.

(g) $3 \leq |S_r \cap PD| \leq 4$. *And if* $r \in PD$ *and* r *is* $G[S_r]^O$, $|S_r \cap PD| = 3$.

(h) $3 \leq |S_s \cap PD| \leq 4$. *And if* $s \in PD$ *and* s *is* $G[S_r]^O$, $|S_s \cap PD| = 3$.

(i) $3 \leq |S_p \cap PD| \leq 4$. *And if* $p \in PD$ *and* p *is* $G[S_p]^O$, $|S_p \cap PD| = 3$.

(j) $3 \leq |S_q \cap PD| \leq 4$. *And if* $q \in PD$ *and* q *is* $G[S_q]^O$, $|S_q \cap PD| = 3$.

(k) $|T_2 \cap PD| \leq 13$ *with equality if and only if* $|\{w, z\} \cap PD| = 1$.

(l) $|T_1 \cap PD| \leq 22$ *with equality if and only if* $\{u, v\} \subseteq PD$.

(m) If $\{u, v\} \cap PD = \emptyset$, $|T_1 \cap PD| \leq 18$.

Proof. (a) W.l.o.g. we consider $r \in PD$. If $rc \in M$, $|S_c \cap PD| \neq 4$. Otherwise, $c_1c_2, c_3s \in M$ and let $PD' = PD \setminus \{c_1, c_2\}$, $M' = M \setminus \{c_1c_2\}$. Then, PD' is a paired dominating set and PD is not a minimal paired dominating set, a contradiction. If $rc \notin M$, $|S_c \cap PD| \neq 4$. Otherwise, $cc_1, c_2c_3 \in M$ and let $PD' = PD \setminus \{c, c_1\}$, $M' = M \setminus \{cc_1\}$. Then, PD' is a paired dominating set and PD is not a minimal paired dominating set, a contradiction.

If $Pr(c_3) = \emptyset$ and $Pr(c) = \emptyset$, let $PD' = PD \setminus \{c, c_3\}$ and $M' = M \setminus \{cc_1, c_2c_3\} \cup \{c_1c_2\}$. Then, PD' is a paired dominating set and PD is not a minimal paired dominating set, a contradiction.

(b) The proof is analogous to that of (a), and the proof is omitted.

(c) Clearly, $|T_3 \cap PD| \leq 4$. If $|T_3 \cap PD| = 4$, $\{h_1, h_2, h_3, h_4\} \subseteq PD$ or $\{h, h_1, h_2, h_3\} \subseteq PD$. If $\{h_1, h_2, h_3, h_4\} \subseteq PD$, $Pr(h_1) \neq \emptyset$. Otherwise, let $PD' = PD \setminus \{h_4, h_1\}$, $M' = M \setminus \{h_3h_4, h_1h_2\} \cup \{h_2h_3\}$. Then, PD' is a paired dominating set and PD is not a minimal paired dominating set, a contradiction. If $\{h, h_1, h_2, h_3\} \subseteq PD$, $N(h) \cap PD = \{h_1\}$ or $Pr(h) \neq \emptyset$. Otherwise, let $PD' = PD \setminus \{h, h_1\}$, $M' = M \setminus \{hh_1\}$. Then, PD' is a paired dominating set and PD is not a minimal paired dominating set, a contradiction.

If h is $G[T_3]^O$, and since $|T_3 \setminus \{h\} \cap PD|$ is even, we have $|T_3 \cap PD| = 3$.

(d)–(f) We obtain the conclusions with a similar proof of (c).

(g) Clearly, $3 \leq |S_r \cap PD| \neq 6$. If $|S_r \cap PD| = 5$, we obtain $S_r \cap PD = \{r, r_1, r_2, r_3, r_4\}$ or $S_r \cap PD = \{r, r_2, r_3, r_4, r_5\}$ by Lemma 2. Therefore, PD is not a minimal paired dominating set, a contradiction. Thus, $3 \leq |S_r \cap PD| \leq 4$.

If r is $G[S_r]^O$, and since $|S_r \setminus \{r\} \cap PD|$ is even, we have $|S_r \cap PD| = 3$.

(k) Since $|S_d \cap PD| \leq 4$, $|T_2 \cap PD| \leq 14$ by (i)–(j) and Eq (2.1).

If $|\{w, z\} \cap PD| = 0$, $|T_2 \cap PD| \leq 12$.

If $|\{w, z\} \cap PD| = 1$, $|T_2 \cap PD| \leq 13$.

Then we consider $|\{w, z\} \cap PD| = 2$. If w is $G[T_2]^I$ or z is $G[T_2]^I$, we may assume w is $G[T_2]^I$. We obtain $wp \in M$, $|S_p \cap PD| = 3$ by (i), $|S_d \cap PD| \leq 3$ by (b). Therefore, $|T_2 \cap PD| \leq 12$ by Eq (2.1). If w, z are $G[T_2]^O$, $|S_d \cap PD| \leq 3$ by (b). Since $|T_2 \cap PD|$ is even, $|T_2 \cap PD| \leq 12$ by Eq (2.1).

Thus, $|T_2 \cap PD| \leq 13$ with equality if and only if $|\{w, z\} \cap PD| = 1$, see Figure 2 (a).

(h)–(j) Using similar arguments of (g), the conclusions follow.

(l)–(m) We discuss the following cases.

Case 1. $|\{u, v\} \cap PD| = 2$.

In this case, we have $|\{u_1, v_1\} \cap PD| \geq 1$, $|S_a \cap PD| \geq 3$ and $|S_c \cap PD| \geq 3$, otherwise, $|T_1 \cap PD| \leq 22$ by (e)–(h) and Eq (2.2).

W.l.o.g. we assume $u_1 \in PD$.

First, we assume that $|S_a \cap PD| = 4$. We obtain $aa_1, uu_1 \in M$, $\{r, c, r_1\} \cap PD = \emptyset$ by (e). Thus, $|S_c \cap PD| \leq 3$. Then, we consider $|S_c \cap PD| = 3$, that is, $c_3s \in M$. By (h), we have $|S_s \cap PD| = 3$. Therefore, $|T_1 \cap PD| \leq 22$ by Eq (2.2).

Then, we consider $|S_a \cap PD| = 3$. Therefore, $v_1 \in PD$, otherwise, $|T_1 \cap PD| \leq 22$ by Eq (2.2). We have $|S_b \cap PD| = 4$, otherwise, $|T_1 \cap PD| \leq 22$ by Eq (2.2). By (f), $\{s, s_1, c_3\} \cap PD = \emptyset$. Thus, $|S_c \cap PD| \leq 3$. Therefore, $|T_1 \cap PD| \leq 22$ by Eq (2.2), see Figure 2 (b).

Case 2. $|\{u, v\} \cap PD| = 1$.

W.l.o.g. we assume $u \in PD$.

We have $|\{u_1, v_1\} \cap PD| \geq 1$ and $|S_a \cap PD| \geq 3$, otherwise, $|T_1 \cap PD| \leq 21$ by Eq (2.2).

Case 2.1 $u_1 \in PD$.

If $|S_a \cap PD| = 4$, $\{r, r_1, c\} \cap PD = \emptyset$ by (e). Then $|S_c \cap PD| \leq 3$. If $|S_c \cap PD| = 3$, we have $c_3s \in M$, $|S_s \cap PD| \leq 3$ by (h). Therefore, $|T_1 \cap PD| \leq 21$ by Eq (2.2). If $|S_c \cap PD| = 2$, $|T_1 \cap PD| \leq 21$ by Eq (2.2).

Now we consider $|S_a \cap PD| = 3$. If $v_1 \notin PD$, $|T_1 \cap PD| \leq 21$ by Eq (2.2). Thus, $v_1 \in PD$, that is, $v_1b \in M$. Therefore, $|S_b \cap PD| = 3$ by (f), $|T_1 \cap PD| \leq 21$ by Eq (2.2).

Case 2.2 $u_1 \notin PD$.

If $v_1 \in PD$, $v_1b \in M$. Therefore, $|S_b \cap PD| = 3$ by (f), $|T_1 \cap PD| \leq 21$ by Eq (2.2). Thus, $v_1 \notin PD$, and $|T_1 \cap PD| \leq 21$ by Eq (2.2).

Case 3. $|\{u, v\} \cap PD| = 0$.

In this case, $|T_1 \cap PD|$ is even.

Case 3.1 $|\{u_1, v_1\} \cap PD| \geq 1$.

W.l.o.g. we assume $u_1 \in PD$. Then $u_1a \in M$, $|S_a \cap PD| = 3$ by (e). If $v_1 \notin PD$, $b \in PD$. By (a), $|S_c \cap PD| \leq 3$. Therefore, $|T_1 \cap PD| \leq 19$ by Eq (2.2). If $v_1 \in PD$, we obtain $v_1b \in M$, $|S_b \cap PD| = 3$ by (f). $|S_c \cap PD| \leq 3$ by (a). Therefore, $|T_1 \cap PD| \leq 19$ by Eq (2.2).

Case 3.2 $|\{u_1, v_1\} \cap PD| = 0$.

In this case, $a, b \in PD$. By (a), $|S_c \cap PD| \leq 3$. Therefore, $|T_1 \cap PD| \leq 19$ by Eq (2.2).

Note that $|T_1 \cap PD|$ is even, so $|T_1 \cap PD| \leq 18$, see Figure 2 (c).

Thus, (l) and (m) hold. □

Lemma 4. *Let PD be a minimal paired dominating set of G'.*
(a) $|V(H'_{xy}) \cap PD| \leq 43$.
(b) If $\{xw_1, yz_1\} \subset M$, $|V(H'_{xy}) \cap PD| \leq 42$.
(c) If $\{xw_1, yz_1\} \cap M = \emptyset$ *and* $\{w_1, z_1\} \subseteq PD$, $|V(H'_{xy}) \cap PD| \leq 42$.
(d) If $xw_1 \notin M(G)$, $w_1 \in PD$ *and* $y \notin PD$, *then* $|V(H'_{xy}) \cap PD| \leq 42$.

Proof. (a) By Lemma 3 and Eq (2.3),

$$
\begin{aligned}
&|V(H'_{xy}) \cap PD| \\
=&|T_1 \cap PD| + |T_2 \cap PD| + |T_3 \cap PD| + |T_4 \cap PD| + |\{w_1, z_1\} \cap PD| \\
\leq& 22 + 13 + 4 + 4 + 2 = 45.
\end{aligned}
$$

We consider that $\{w_1, z_1\} \cap PD \neq \emptyset$, $|T_4 \cap PD| \geq 3$ and $|T_3 \cap PD| \geq 3$, otherwise, $|V(H'_{xy}) \cap PD| \leq 43$. Then, w.l.o.g. we assume that $w_1 \in PD$.

If $|T_4 \cap PD| = 4$, $\{tt_1, t_2t_3\} \subseteq M$ or $\{t_1t_2, t_3t_4\} \subseteq M$.

If $\{tt_1, t_2t_3\} \subseteq M$, $Pr(t) \neq \emptyset$ or $N(t) \cap PD = \{t_1\}$. If $Pr(t) \neq \emptyset$, $u \in Pr(t)$. By Lemma 3 (m), $|V(H'_{xy}) \cap PD| \leq 43$. If $N(t) \cap PD = \{t_1\}$, we have $u, w_1 \notin PD$, $|T_1 \cap PD| \leq 21$ by Lemma 3 (l). Then we obtain $z \in PD$, otherwise, $|V(H'_{xy}) \cap PD| \leq 43$ by Lemma 3 (k) and Eq (2.3). If $|T_3 \cap PD| = 4$, we have $v \notin PD$, therefore, $|V(H'_{xy}) \cap PD| \leq 43$ by Eq (2.3). If $|T_3 \cap PD| = 3$, $|V(H'_{xy}) \cap PD| \leq 43$ by Eq (2.3).

If $\{t_1t_2, t_3t_4\} \subseteq M$, $\{w, u\} \cap PD = \emptyset$. By Lemma 3 (l), $|T_1 \cap PD| \leq 21$, and $v \in PD$. If $z \notin PD$, $|V(H'_{xy}) \cap PD| \leq 43$ by Lemma 3 (k) and Eq (2.3). If $z \in PD$, $|T_3 \cap PD| \leq 3$ by Lemma 3 (c). Therefore, $|V(H'_{xy}) \cap PD| \leq 43$ by Eq (2.3).

If $|T_4 \cap PD| = 3$, we consider $|T_1 \cap PD| = 22$, and $\{u, v, z_1\} \in PD$. We have $|T_3 \cap PD| \neq 4$ by Lemma 3 (c). Therefore, $|V(H'_{xy}) \cap PD| \leq 43$ by Eq (2.3).

(b)–(d) Since $|V(H'_{xy}) \cap PD| \leq 43$, and, $|V(H'_{xy}) \cap PD|$ is even in those cases, so $|V(H'_{xy}) \cap PD| \leq 42$. □

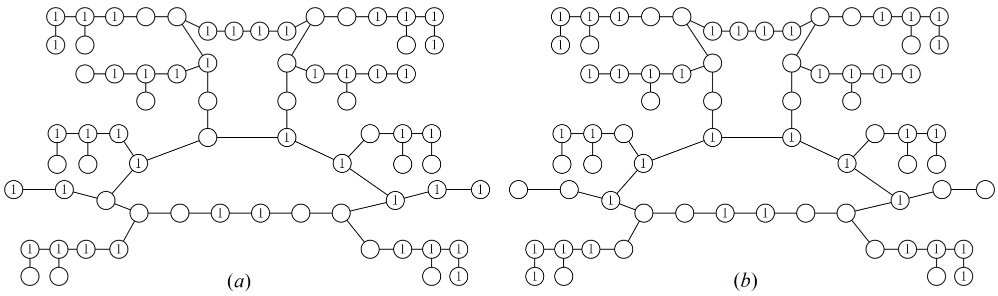

Figure 3. (a) $|V(H'_{xy}) \cap PD| = 41$, (b) $|V(H'_{xy}) \cap PD| = 40$.

Lemma 5. *Let PD be a minimal paired dominating set of G', M be a perfect matching of $G'[PD]$.*
(a) If $\{x, w_1, y, z_1\} \subset PD$, $xw_1 \in M(G)$ and $yz_1 \notin M$, we have $|V(H'_{xy}) \cap PD| \leq 41$.
(b) If $\{x, y\} \cap PD = \emptyset$, $|V(H'_{xy}) \cap PD| \leq 40$.

Proof. (a) In this case, we have $z \in PD$ and $zz_1 \in M$.

Since $|V(H'_{xy}) \cap PD|$ is odd, it's sufficient to show $|V(H'_{xy}) \cap PD| \leq 42$. We only consider $\{u, v\} \cap PD \neq \emptyset$ by Lemma 3 (m).

Case 1. $|T_4 \cap PD| = 4$.

In this case, we have $\{t_1t_2, t_3t_4\} \subseteq M$ or $\{tt_1, t_2t_3\} \subseteq M$. If $\{t_1t_2, t_3t_4\} \subseteq M$ (or $\{tt_1, t_2t_3\} \subseteq M$), we obtain $u, w \notin PD$, $v \in PD$. Since $z, v \in PD$, $|T_3 \cap PD| \leq 3$ by Lemma 3 (c). If $|T_3 \cap PD| = 2$, $|V(H'_{xy}) \cap PD| \leq 42$ by Eq (2.3). If $|T_3 \cap PD| = 3$, $hv \in M$. Thus, $\{q, q_1, d_3\} \cap PD = \emptyset$, otherwise, let $PD' = PD \setminus \{z, z_1\}$, $M' = M \setminus \{zz_1\}$. Then, PD' is a paired dominating set and PD is not a minimal paired dominating set, a contradiction. Since $zz_1 \in M$, we obtain that $|T_2 \cap PD|$ is odd. So $|T_1 \cap PD| \leq 12$. Therefore, $|V(H'_{xy}) \cap PD| \leq 42$ by Eq (2.3), see Figure 3 (a).

Case 2. $|T_4 \cap PD| = 3$.

If $tw \in M$, $u \notin PD$ or $\{p, p_1, d\} \cap PD = \emptyset$. Otherwise, let $PD' = PD \setminus \{t, w\}$,$M' = M \setminus \{tw\}$. Then, PD' is a paired dominating set and PD is not a minimal paired dominating set, a contradiction. If $\{p, p_1, d\} \cap PD = \emptyset$, $|S_d \cap PD| \leq 3$. W have $|S_d \cap PD| = 3$, $d_3q \in M$, otherwise, $|V(H'_{xy}) \cap PD| \leq 42$ by

Eq (2.3). Thus $|S_q \cap PD| \leq 3$ by Lemma 3 (j) and Eq (2.3), and $|V(H'_{xy}) \cap PD| \leq 42$. If $u \notin PD$, $v \in PD$. Thus, $|T_3 \cap PD| \leq 3$ by Lemma 3 (c) and Eq (2.3), and $|V(H'_{xy}) \cap PD| \leq 42$.

If $tu \in M$, $|T_3 \cap PD| \leq 3$ by Lemma 3 (c). We have $|T_3 \cap PD| = 3$, $hv \in M$, otherwise, $|V(H'_{xy}) \cap PD| \leq 42$ by Eq (2.3). Let $PD' = PD \setminus \{t,h\}$, $M' = M \setminus \{tu, hv\} \cup \{uv\}$. Therefore, PD' is a paired dominating set and PD is not a minimal paired dominating set, a contradiction.

Case 3. $|T_4 \cap PD| = 2$.

Now we only consider $|T_1 \cap PD| = 22$, and $\{u,v\} \subset PD$. By Lemma 3 (c) and Eq (2.3), $|V(H'_{xy}) \cap PD| \leq 42$.

(b) Since $|V(H'_{xy}) \cap PD|$ is even, it's sufficient to show $|V(H'_{xy}) \cap PD| \leq 41$.

Case 1. $|\{z_1, w_1\} \cap PD| = 0$.

We obtain $\{z,w\} \subseteq PD$, $|T_2 \cap PD| \leq 12$ by Lemma 3 (k). If $|T_4 \cap PD| \leq 3$, $|V(H'_{xy}) \cap PD| \leq 41$ by Eq (2.3), see Figure 3 (b). If $|T_4 \cap PD| = 4$, $t \in PD$ and $Pr(t) \neq \emptyset$ by Lemma 3(d). So, $\{u, v, u_1\} \cap PD = \emptyset$. By Lemma 3 (m) and Eq (2.3), $|V(H'_{xy}) \cap PD| \leq 40$.

Case 2. $|\{z_1, w_1\} \cap PD| = 1$.

W.l.o.g. we assume $w_1 \in PD$. Thus, $ww_1 \in M$, $z \in PD$, $|T_2 \cap PD| \leq 12$ by Lemma 3 (k). If $|T_4 \cap PD| = 2$, $|V(H'_{xy}) \cap PD| \leq 41$ by Eq (2.3). If $|T_4 \cap PD| = 4$, we obtain $Pr(t) = \{u\}$ for $t \in PD$, $\{u, u_1, v\} \cap PD = \emptyset$. By Lemma 3 (m) and Eq (2.3), $|V(H'_{xy}) \cap PD| \leq 40$. If $|T_4 \cap PD| = 3$, $tu \in M$. And $v \in PD$, otherwise $|T_1 \cap PD| \leq 21$ by Lemma 3 (l), $|V(H'_{xy}) \cap PD| \leq 41$ by Eq (2.3). Thus, $|T_4 \cap PD| \neq 4$ by Lemma 3 (d). Afterwards, $|V(H'_{xy}) \cap PD| \leq 41$ by Eq (2.3).

Case 3. $|\{z_1, w_1\} \cap PD| = 2$.

Thus, $ww_1 \in M$, $zz_1 \in M$, $|T_2 \cap PD| \leq 12$ by Lemma 3 (k).

If $|T_4 \cap PD| = 4$, $t \in PD$ and $\{u, u_1, v\} \cap PD = \emptyset$. By Lemma 3 (m) and Eq (2.3), we have $|V(H'_{xy}) \cap PD| \leq 40$.

If $|T_4 \cap PD| = 3$, we have $tu \in M$, and $|T_3 \cap PD| \leq 3$ by Lemma 3 (c). If $|T_3 \cap PD| = 2$, $|V(H'_{xy}) \cap PD| \leq 41$ by Eq (2.3). If $|T_3 \cap PD| = 3$, $hv \in M$. Let $PD' = PD \setminus \{t,h\}$, $M' = M \setminus \{tu, hv\} \cup \{uv\}$. Therefore, PD' is a paired dominating set and PD is not a minimal paired dominating set, a contradiction.

If $|T_4 \cap PD| = 2$, we only consider $|T_1 \cap PD| = 22$. Thus, $u, v \in PD$. By Lemma 3 (c), $|T_3 \cap PD| \leq 3$. Therefore, $|V(H'_{xy}) \cap PD| \leq 41$ by Eq (2.3). □

Corollary 6. *Let PD be a minimal paired dominating set of G'. If $|V(H'_{uv}) \cap PD| = 43$ if and only if $|\{u,v\} \cap PD| = 1$, and, u or v is H^I_{uv}.*

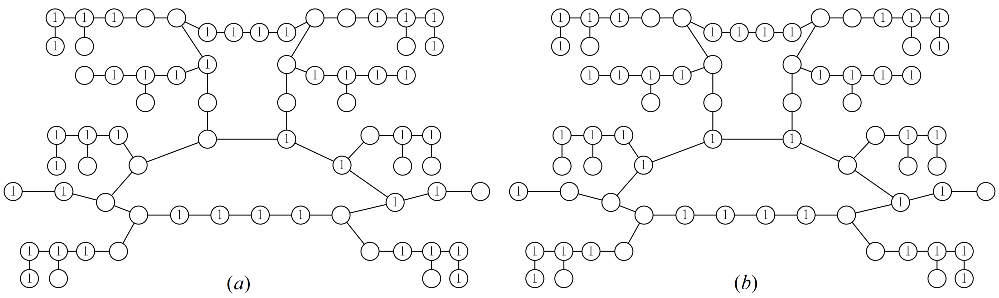

Figure 4. (a) $|V(H'_{xy}) \cap PD| = 43$, (b) $|V(H'_{xy}) \cap PD| = 42$.

Lemma 7. *If VC_1 is a minimal vertex cover of G, there exists a minimal paired dominating set PD_1 of G' with $|PD_1| = 42m + 2|VC|$.*

Proof. A minimal paired dominating set PD_1 can be constructed by the following manner:

For each vertex $x \in VC_1$, we have $|N(x) \cap VC_1| < d(x) \leq 3$. So there exists at least one edge xx_1 with $x_1 \notin VC_1$ in G, and maybe exist edges xx_2 or xx_3.

Therefore, for the edge xx_1, put i into PD' for $i \in \{x, w_1, p_2, p_3, p_4, p_5, d, d_1, d_2, d_3, q_2, q_3, q_4, q_5, z, z_1, h, h_2, h_3, v, b_1, b_2, b_3, b_4, s_2, s_3, s_4, s_5, c, c_1, c_2, c_3, r_2, r_3, r_4, r_5, a, a_1, a_2, a_3, t_1, t_2, t_3, t_4\}$. Put j into M for $j \in \{xw_1, p_5p_4, p_3p_2, dd_1, d_2d_3, q_2q_3, q_4q_5, zz_1, hv, h_2h_3, b_1b_2, b_3b_4, s_2s_3, s_4s_5, cc_1, c_2c_3, r_2r_3, r_4r_5, aa_1, a_2a_3, t_1t_2, t_3t_4\}$. See Figure 4 (a).

For edges xx_2, xx_3, put i into PD' for $i \in \{x, p_2, p_3, p_4, p_5, d, d_1, d_2, d_3, q_2, q_3, q_4, q_5, z, z_1, u, v, h_2, h_3, b_1, b_2, b_3, b_4, s_2, s_3, s_4, s_5, c, c_1, c_2, c_3, r_2, r_3, r_4, r_5, a_1, a_2, a_3, a_4, t_1, t_2, t_3, t_4\}$. Put j into M for $j \in \{p_5p_4, p_3p_2, dd_1, d_2d_3, q_2q_3, q_4q_5, zz_1, h_2h_3, uv, b_1b_2, b_3b_4, s_2s_3, s_4s_5, cc_1, c_2c_3, r_2r_3, r_4r_5, a_1a_2, a_3a_4, tt_1, t_2t_3\}$. See Figure 4 (b).

Let $PD_1 = PD' \cup VC_1$. Since vertex x is M-saturated in PD_1. Therefore, PD_1 is a paired dominating set of G'.

Since $N(w) \cap PD_1 = \{w_1\}$, then $PD_1 \setminus \{w_1\}$ is not a dominating set of G'. So PD_1 is a minimal paired dominating set of G'. And $|PD_1| = |VC_1| + |VC_1| \times 43 + (m - |VC_1|) \times 42$. Therefore, $|PD_1| = 2|VC_1| + 42m$.

□

Let PD be a minimal paired dominating set of G'. Algorithm 1 is to obtain a minimal vertex cover VC of G, and it terminates in polynomial time.

Algorithm 1 CONST-$VC(G', PD)$

Input: A graph G' with a minimal paired dominating set PD
Output: A graph G with a minimal vertex cover VC
$VC = PD$
for every $H_{xy} \subseteq G'$ **do**
  Delete vertices in H'_{xy}
  Add an edge between x and y {obtained the graph G}
  $VC = VC \setminus V(H'_{xy})$
end for
$VC' = VC$
$De = \emptyset$ {Mo is the set of vertex which is removed from VC.}
$In = \emptyset$ {In is the set of vertex which is added into VC.}
$Mo = \emptyset$ {De is the set of vertex which is added into VC at first, then removed from VC.}
while $|N[v] \cap VC| = d(v) + 1$ **do**
  $VC = VC \setminus \{v\}, Mo = Mo \cup \{v\}$
end while
while $uv \in E(G)$ and $u, v \notin VC$ **do**
  $VC = VC \cup \{u\}, In = In \cup \{u\}$
  for $w \in N(u)$ **do**
    if $|N[w] \cap VC| = d(w) + 1$ **then**
      $VC = VC \setminus \{w\}, De = De \cup \{w\}$
    end if
  end for
end while
return VC

Lemma 8. *If PD is a minimal paired dominating set of G' and VC is a minimal vertex cover of G obtained by Algorithm 1,* $|VC| \geq |PD| - 42m - |VC|$.

Proof. Let M be the perfect matching of $G[PD]$, $m_e = V(H'_{xy}) \cap PD$ where $e = xy \in E(G)$, $M_e = \bigcup_{e \in E(G)} m_e$, $Le = V(G) \setminus (Mo \cup In \cup De)$.

In Algorithm 1, we have:

Claim 9. *(a) If v is put into Mo by the while loop (lines 11 to 13) or De (line 18), v will not be put into In later.*

(b) For every vertex $v \in V(G)$, v will be put into Mo (or De or In) at most once.

(c) $Mo \cap De = \emptyset$, $Mo \cap In = \emptyset$.

(d) If $v \in De$, there exists a vertex $w \in N(v) \cap In$.

(e) If vertex $v \in De \cap In$, we have $v \notin VC'$, that is, v is put into In at first and then into De.

(f) If $u, v \in De \cup Mo$, $N(v) \cap N(u) \cap Mo \cap De = \emptyset$.

(g) If $v \in De \setminus In$, there exists a vertex $u \in N(v) \cap (In \setminus De)$, $u \notin VC'$. And $|N(u) \cap De| \leq 2$. What's more, there exists a vertex $w \in N(u) \setminus VC'$. If $w \in In \setminus De$, $|(N(u) \cup N(w)) \cap (De \setminus In)| \leq 3$.

Proof. (a) After v is put into De (or Mo), every $w \in N(v)$ has a neighbor v which does not belong to VC, so w will not be put into De. Therefore, v will not be put into In later.

(b)–(d) By (a), it is immediate.

(e) By (a) and (c), it is immediate.

(f) Suppose v is put into $De \cup Mo$. By (a), $w \in N(v)$ will not be put into $De \cup Mo$.

(g) For vertex $v \in De \setminus In$, by (d) and (f), let $u \in N(v) \cap (In \setminus De)$, and $u \notin VC'$, $|N(u) \cap De| \leq 2$. Since $u \in In \setminus De$, there exists a vertex $w \in N(u) \setminus VC'$.
Since $1 \leq |N(u) \cap (De \setminus In)| \leq 2$, $|N(w) \cap (De \setminus In)| \leq 2$. If $w \in In \setminus De$, we may assume u is put into In at first. Then $N(u) \cap (De \setminus In)| \leq 1$, otherwise, w will not be put into In later. Therefore, $|(N(u) \cup N(w)) \cap (De \setminus In)| \leq 3$. □

Thus,

$$|VC| = |PD| - |M_e| - |Mo| - |De| + |In|. \tag{2.4}$$

To show that $|M_e| + |Mo| + |De| - |In| \leq 42m + |VC|$, we use the following strategy.

Discharging procedure:

In the graph G', we set the initial charge of every vertex v to be $s(v) = 1$ for $v \in Mo \cup M_e \cup (De \setminus In)$, $s(v) = -1$ for $v \in In \setminus De$, $s(v) = 0$ otherwise, $s(H'_{uv}) = \sum_{x \in V(H'_{uv})} s(x)$, $s(G') = \sum_{v \in V(G')} s(v)$.

Obviously,

$$\sum_{v \in V(G')} s(v) = |M_e| + |Mo| + |De| - |In|. \tag{2.5}$$

We use the discharging procedure, leading to a final charge s', defined by applying the following rules:

Rule 1: For the vertex $v \in Mo$, v is M-saturated. Therefore, v is H^I_{uv} for u. If u is H^I_{uv}, $s(v)$ transmits 1 charge to $s(u)$. If u is H^O_{uv}, $s(v)$ transmits 1 charge to $s(H'_{uv})$ which is $[I, O]$.

Rule 2: For each $s(H'_{uv}) = 43$, by Corollary 6, $s(H'_{uv})$ transmits 1 charge to $u \in VC'$.

Rule 3: For the vertex $v \in De \setminus In$, by Claim 9 (g), there exists a vertex $u \in N(v) \cap (In \setminus De)$, and a vertex $w \in N(u) \setminus VC'$ and $|N(u) \cap De| \leq 2$. If $|N(u) \cap De| = 2$, $s(v)$ transmits 1 charge to $s(u)$ and transmits 1 charge to $s(H'_{uw})$ which is $[0, 0]$. If $|N(u) \cap De| = 1$, $s(v)$ transmits 2 charge to $s(u)$.

After discharging, we have:

Claim 10. *(a)* $s'(v) \leq 0$ *for* $v \in Mo \cup (De \setminus In) \cup (Le \setminus VC) \cup (In \cap De)$.
(b) For each H'_{xy}, $s'(H'_{xy}) \leq 42$.
(c) $s'(v) \leq 1$ *for* $v \in (In \setminus De) \cup (Le \cap VC)$.

Proof. (a) If $v \in Mo$, by Claim 9 (f), v will not receive any charge by Rules 1 and 3. Since $N[v] \cap VC' = N[v]$. By Lemmas 4 and 5, v will not receive any charge by Rule 2. Therefore, $s'(v) = 0$.

If $v \in De \setminus In$, $v \in VC'$. By Claim 9 (f), $N(v) \cap Mo = \emptyset$. Thus, v will not receive any charge by Rules 1 and 3. Since v is H^I_{uv} for u. By Lemmas 4 and 5, if $u \in VC'$, v will not receive any charge by Rule 2. If $u \notin VC'$, v will receive 1 charge at most by Rule 2. Afterwards, by Rule 3, v will transmit 2 charge to others, so $s'(v) \leq 0$.

If $v \in Le \setminus VC$, v will not receive any charge by Rules 1, 2 and 3.

If $v \in In \cap De$, $v \notin VC'$ by Claim 9 (e). Thus, v will not receive any charge by Rules 1 and 2. By Claim 9 (f), $v \in De$, $N(v) \cap De = \emptyset$. Thus, v will not receive any charge by Rule 3.

(b) If H'_{uw} is $[I,I]$ or $[O,O]$ or $[I,0]$ or $[O,0]$, $s(H'_{uw})$ will not receive any charge by Rules 1, 2 and 3. If H'_{uw} is $[0,0]$, $s(H'_{uw})$ will not receive any charge by Rules 1 and 2.

If H'_{uw} is $[0,0]$, by Claim 9 (g), $|(N(u) \cup N(w)) \cap (De \setminus In)| \leq 3$. Thus, $s(H'_{uw})$ will receive 2 charge at most from $s(x)$ where $x \in N(v) \setminus \{w\}$ by Rule 3.

And if $s(H'_{uw}) = 43$, by Corollary 6, there exists a vertex $u \in VC'$ and u is H^I_{uw}. Therefore, $s'(H'_{uw}) = 42$ by Rule 2.

Thus, by Lemmas 4 and 5, $s'(H'_{uw}) \leq 42$.

(c) If $v \in In \setminus De$, $v \notin VC'$, v will receive any charge by Rules 1 and 2. And there exists a vertex $w \in N(v)$ $w \notin VC'$ and $w \notin De \setminus In$. So v will receive 2 charge at most by Rule 3, $s'(v) \leq -1 + 2 = 1$.

If $v \in Le \cap VC$, v will receive any charge by Rule 3. By Lemmas 4, 5 and Corollary 6, H'_{uv} is $[I,0]$ if $s(H'_{uv}) = 43$. Since v can be M-saturated once, v will receive 1 charge at most by Rules 1 and 2. Thus, $s'(v) \leq 0 + 1 = 1$. □

By Claim 10,

$$\begin{aligned}
& |M_e| + |Mo| + |De| - |In| \\
= & \sum_{uv \in E(G)} s(H'_{uv}) + \sum_{v \in Mo} s(v) + \sum_{v \in De \setminus In} s(v) - \sum_{v \in In \setminus De} s(v) \\
= & \sum_{uv \in E(G)} s'(H'_{uv}) + \sum_{v \in Mo} s'(v) + \sum_{v \in De \setminus In} s'(v) + \sum_{v \in In \setminus De} s'(v) \\
+ & \sum_{v \in In \cap De} s'(v) + \sum_{v \in Le \setminus VC} s'(v) \sum_{v \in Le \cap VC} s'(v) \\
\leq & \; 42m + |In \setminus De| + |Le \cap VC| \\
\leq & \; 42m + |VC|.
\end{aligned}$$

Thus, by Eq (2.4),

$$\begin{aligned}
|VC| &= |PD| - |M_e| - |Mo| - |De| + |In| \\
&\geq |PD| - 42m - |VC|.
\end{aligned}$$

□

Let PD^* be a $\Gamma_{pr}(G')$-set of G', and be the **Input** of Algorithm 1. Then we obtain the **Output** VC by Algorithm 1.

Since

$$|VC^*| \geq \frac{m}{\Delta} = \frac{m}{3}.$$

By Lemma 8,

$$\begin{aligned} |VC| &\geq |PD^*| - 42m - |VC| \geq |PD^*| - 42 \times 3|VC| - |VC| \\ |VC| &\geq |PD^*| - 127|VC| \end{aligned}$$

Let VC^* be a VC-set of G. Since $|VC| \leq |VC^*|$,

$$|PD^*| \leq 128|VC| \leq 128|VC^*| \tag{2.6}$$

By Lemma 7, $|PD^*| \geq |PD_1| = 42m + 2|VC^*|$. By Lemma 8,

$$|PD| - |VC| \leq |VC| + 42m \leq |VC^*| + 42m \leq |PD^*| - |VC^*|.$$

Thus,

$$|VC^*| - |VC| \leq |PD^*| - |PD| \tag{2.7}$$

Therefore, by Eq (2.6) and Eq (2.7), f is an L-reduction with $\alpha = 128$, $\beta = 1$.

3. Conclusions

Upper-PDS for bipartite graphs is proved to be APX-complete with maximum degree 4 and still open with maximum degree 3. In this paper, we show that *Upper-PDS* for bipartite graphs with maximum degree 3 is APX-complete by providing an L-reduction f from *MAX-MIN-VC* for bipartite graphs to it.

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Conflict of interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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