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Research article

New Lyapunov-type inequalities for fractional multi-point boundary value problems involving Hilfer-Katugampola fractional derivative

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Abstract: In this paper, we present new Lyapunov-type inequalities for Hilfer-Katugampola fractional differential equations. We first give some unique properties of the Hilfer-Katugampola fractional derivative, and then by using these new properties we convert the multi-point boundary value problems of Hilfer-Katugampola fractional differential equations into the equivalent integral equations with corresponding Green's functions, respectively. Finally, we make use of the Banach's contraction principle to derive the desired results, and give a series of corollaries to show that the current results extend and enrich the previous results in the literature.

Keywords: Lyapunov-type inequality; Hilfer-Katugampola fractional derivative; multi-point boundary condition
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1. Introduction

Consider the following second-order differential equation with Dirichlet boundary conditions,

$$\begin{cases} x''(t) + q(t)x(t) = 0, \ t \in (a, b), \\ x(a) = x(b) = 0, \end{cases} \tag{1.1}$$

where $q(t) \in C([a, b], \mathbb{R})$. If there exists a nontrivial solution $x(t)$ of the boundary value problem (BVP for short) given in Eq (1.1), then the inequality,

$$\int_a^b |q(s)|ds > \frac{4}{b-a}, \tag{1.2}$$

holds. The above inequality (1.2) is known as Lyapunov inequality, and it was first proved by Lyapunov [1]. The inequality given in Eq (1.2) and its generalizations have been used successfully in various branches of mathematics, such as stability problems, oscillation theory, and eigenvalue bounds for ordinary differential equations, see for instance [2, 3] and the references cited therein.

In recent years, with the successful development of fractional calculus theory, the Lyapunov inequalities have been generalized to fractional BVPs, see [4–24] and the references cited therein. Especially, in 2013, Ferreira [4] firstly proved the following result.

Theorem 1.1. If the fractional BVP

$$\begin{cases} ({}_aD^{\alpha}x)(t) + q(t)x(t) = 0, \ t \in (a, b), \ 1 < \alpha \le 2, \\ x(a) = x(b) = 0, \end{cases}$$

has a nontrivial solution, where ${}_aD^{\alpha}$ is the Riemann-Liouville fractional derivative of order α and $q(t) \in C([a, b], \mathbb{R})$, then,

$$\int_a^b |q(s)|ds > \Gamma(\alpha)\Big(\frac{4}{b-a}\Big)^{\alpha-1}. \tag{1.3}$$

Inequality expressed in Eq (1.3) is called Lyapunov-type inequality, and it is a generalization of the inequality given by Eq (1.2) above in the sense of fractional derivative. Since then, many scholars have been tremendously interested in developing Lyapunov-type inequalities, and based on different definitions of fractional calculus, the inequality (1.2) has been generalized to various forms. Examples include Lyapunov-type inequalities for BVPs involving Caputo fractional derivative, Hilfer fractional derivative, Caputo-Fabrizio fractional derivative, Hadamard fractional derivative, Katugampola fractional derivative, conformable fractional derivative, local fractional derivative, and so on. For more details, we refer the interested reader to the survey [15] for a review of recent developments in these problems.

In recent years, several papers have been published on the study of Lyapunov-type inequalities for fractional differential equations with nonlocal boundary conditions, see for example [16–24]. However, only a few considered similar inequalities for fractional m-point BVPs, see [20–24]. In 2018, Wang et al. [20] derived a Lyapunov-type inequality for fractional differential equation involving Hilfer fractional derivative subject to m-point boundary conditions,

$$\begin{cases} D_{a+}^{\alpha,\beta}x(t) + q(t)x(t) = 0, \ t \in (a, b), \ 1 < \alpha \le 2, \ 0 \le \beta \le 1, \\ x(a) = 0, \quad x(b) = \sum\limits_{i=1}^{m-2} \beta_i x(\xi_i), \end{cases} \tag{1.4}$$

where $D_{a+}^{\alpha,\beta}$ denotes the Hilfer fractional derivative of order α and type β; $a<\xi_1<\xi_2<\cdots<\xi_{m-2}<b$, $\beta_i \ge 0$ $(i=1, 2, \cdots, m-2)$, $0 \le \sum_{i=1}^{m-2} \beta_i(\xi_i - a)^{1-(2-\alpha)(1-\beta)} < (b-a)^{1-(2-\alpha)(1-\beta)}$ and $q(t) \in C([a, b], \mathbb{R})$. By converting the BVP (1.4) into the equivalent integral equation with corresponding Green's function and using norm estimation method, the authors reached the following conclusion.

Theorem 1.2. If there exists a nontrivial continuous solution of the fractional BVP (1.4), then

$$\int_a^b |q(s)|ds \ge \frac{\Gamma(\alpha)}{(b-a)^{\alpha-1}L} \cdot \frac{1}{1 + \sum_{i=1}^{m-2} \beta_i T(b)},$$

where

$$L = \frac{(\alpha-1)^{\alpha-1}(\alpha-1+2\beta-\alpha\beta)^{\alpha-1+2\beta-\alpha\beta}}{(2\alpha-2+2\beta-\alpha\beta)^{2\alpha-2+2\beta-\alpha\beta}},$$

$$T(b) = \frac{(b-a)^{1-(2-\alpha)(1-\beta)}}{(b-a)^{1-(2-\alpha)(1-\beta)} - \sum_{i=1}^{m-2}\beta_i(\xi_i-a)^{1-(2-\alpha)(1-\beta)}}.$$

Later, Aouafi and Adjeroud [21], obtained Lyapunov-type inequality for the fractional differential equation of higher order under m-point boundary conditions

$$\begin{cases} {}_a^C D^\alpha x(t) + q(t)x(t) = 0, \ t \in (a,b), \ 3 < \alpha \le 4, \\ x(a) = x'(a) = x'''(a) = 0, \ x''(b) = \sum\limits_{i=1}^{m-2} \eta_i x(\xi_i), \end{cases} \tag{1.5}$$

where ${}_a^C D^\alpha$ is the Caputo fractional derivative of order α; $a < \eta_i, \xi_i < b$, $i = 1, 2, \cdots, m-2$, with $a < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < b$, $0 < \sum_{i=1}^{m-2} \eta_i(\xi_i - a)^2 < 2$ and $q(t) \in C([a,b], \mathbb{R})$. By converting the BVP (1.5) into the equivalent integral equation with corresponding Green's function and using norm estimation method, the authors obtained the following result.

Theorem 1.3. If there exists a nontrivial continuous solution of the fractional BVP (1.5), then

$$\int_a^b |q(s)|ds \ge \frac{2\Gamma(\alpha-2)}{(b-a)^\alpha}\Big(1 + \frac{(b-a)^2 \sum_{i=1}^{m-2}|\eta_i|}{2 - \sum_{i=1}^{m-2}\eta_i(\xi_i-a)^2}\Big)^{-1}.$$

More recently, in [24], the authors analyzed Lyapunov-type inequality for the fractional BVP involving Caputo-Hadamard fractional derivative supplemented with m-point boundary conditions

$$\begin{cases} {}_H^C D_{a+}^\alpha x(t) + q(t)x(t) = 0, \ 0 < a < t < b, \ 1 < \alpha < 2, \\ x(a) = 0, \quad x(b) = \sum\limits_{i=1}^{m-2} \beta_i x(\xi_i), \end{cases} \tag{1.6}$$

where ${}_H^C D_{a+}^\alpha$ denotes the Caputo-Hadamard fractional derivative of order α; $\beta_i \ge 0$, $a < \xi_i < b$, $(i = 1, 2, \cdots, m-2)$, with $a < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < b$, $0 \le \sum_{i=1}^{m-2}\beta_i < 1$ and $q(t) \in C([a,b], \mathbb{R})$. By converting the BVP (1.6) into the equivalent integral equation with corresponding Green's function and using norm estimation method, the authors given the following result.

Theorem 1.4. If there exists a nontrivial continuous solution of the Caputo-Hadamard fractional BVP (1.6), then

$$\int_a^b |q(s)|ds \ge \frac{a\alpha^\alpha\Gamma(\alpha)}{[(\alpha-1)(\ln b - \ln a)]^{\alpha-1}} \cdot \frac{\ln\frac{b}{a} - \sum_{i=1}^{m-2}\beta_i \ln\frac{\xi_i}{a}}{\ln\frac{b}{a} + \sum_{i=1}^{m-2}\beta_i \ln\frac{b}{\xi_i}}.$$

Notice the diversity of definitions for fractional derivative, and thus it is challenging to know which definition is the most suitable to use in studying fractional differential equations. One way to overcome such a problem is to work with more general fractional operators, see for example [25, 26]. In particular, Oliveira et al. [26] applying the idea of the fractional derivative in the Hilfer sense, proposed a new fractional derivative called Hilfer-Katugampola fractional derivative, which formulation interpolates the well-known fractional derivatives of Hilfer, Katugampola, Hilfer-Hadamard, Riemann-Liouville, Hadamard, Caputo, Caputo-Hadamard, Weyl. Recently, many scholars have been interested in Hilfer-Katugampola fractional derivative and have obtained many exciting and essential results of the existence, uniqueness, and stability of solutions for fractional differential equations using the Hilfer-Katugampola fractional derivative, such as [27, 28].

Motivated by the earlier papers, this study aims to establish new Lyapunov-type inequalities for fractional BVPs involving Hilfer-Katugampola fractional derivative subject to m-point boundary conditions. In precise terms, we consider here the following BVPs:

$$\begin{cases} {}^{\rho}D_{a+}^{\alpha,\beta}x(t) + q(t)x(t) = 0, \ 0 < a < t < b, \ 1 < \alpha < 2, \ \rho > 0, \\ x(a) = 0, \quad x(b) = \sum\limits_{i=1}^{m-2} \gamma_i x(\eta_i), \end{cases} \tag{1.7}$$

and

$$\begin{cases} {}^{\rho}D_{a+}^{\alpha,\beta}x(t) + q(t)x(t) = 0, \ 0 < a < t < b, \ 1 < \alpha < 2, \ \rho > 0, \\ x(a) = 0, \quad t^{1-\rho}\frac{d}{dt}x(t)|_{t=b} = \sum\limits_{i=1}^{m-2} \sigma_i x(\xi_i), \end{cases} \tag{1.8}$$

where ${}^{\rho}D_{a+}^{\alpha,\beta}$ is Hilfer-Katugampola fractional derivative of order α and type β $(0 \leq \beta \leq 1)$; $q(t) \in C([a,b],\mathbb{R})$; $\gamma_i, \sigma_i \geq 0$, $a < \eta_i, \xi_i < b$, $(i = 1, 2, \cdots, m-2)$, with $a < \eta_1 < \eta_2 < \cdots < \eta_{m-2} < b$, $a < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < b$ and they are subject to the following conditions:

(A_1) $\sum\limits_{i=1}^{m-2} \gamma_i(\eta_i^{\rho}-a^{\rho})^{1-(2-\alpha)(1-\beta)}<(b^{\rho}-a^{\rho})^{1-(2-\alpha)(1-\beta)}$.

(A_2) $\sum\limits_{i=1}^{m-2} \sigma_i(\xi_i^{\rho}-a^{\rho})^{1-(2-\alpha)(1-\beta)}<[1-(2-\alpha)(1-\beta)]\rho(b^{\rho}-a^{\rho})^{-(2-\alpha)(1-\beta)}$.

The main contributions to our results can be summarized as follows:

- We prove some new properties of Hilfer-Katugampola calculus and correct Lemma 2.11 in [28].
- We study the Lyapunov-type inequalities for the m-point fractional BVPs (1.7) and (1.8), which generalize and complement some previous results. Indeed, in the limit case $\beta = 0$ or $\beta = 1$ and $\rho \to 0^+$ or $\rho \to 1$, the conclusions of this paper can be reduced to the results presented in [6, 9, 20, 24].
- In the present work, we derive the Lyapunov-type inequalities for BVPs (1.7) and (1.8) by using Banach's contraction principle, which is quite different from the previous research work.
- To the best of our knowledge, the fractional BVPs involving Hilfer-Katugampola fractional derivatives are rarely studied for Lyapunov-type inequalities. So, the results obtained in this paper are new.

The rest of the paper is organized as follows: In Section 2, we recall some definitions on the fractional integral and derivative, and related basic properties which will be used in the sequel. In Section 3, we prove some new properties of Hilfer-Katugampola fractional calculus. Our main results are given in Section 4. Finally, we summarize our results and specify new directions for the future works in Section 5.

2. Preliminaries

In this section, we recall some definitions and lemmas about fractional integral and fractional derivative which we used in this paper. For $c\in\mathbb{R}$, $p\geq 1$, let $X_c^p(a,b)$ denote the space of all complex-valued Lebesgue measurable functions x on (a,b) with $\|x\|_{X_c^p}<\infty$, where the norm is defined by

$$\|x\|_{X_c^p} = \Big(\int_a^b |t^c x(t)|^p \frac{dt}{t}\Big)^{1/p} < \infty.$$

Definition 2.1. [29, 30] The left-sided Katugampola fractional integral of order $\alpha > 0$ and $\rho > 0$ of $x \in X_c^p(a,b)$ for $0 < a < t < b < \infty$, is defined by

$$({}^\rho I_{a+}^\alpha x)(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^t (t^\rho - s^\rho)^{\alpha-1} s^{\rho-1} x(s)ds, \quad t \in [a,b]. \tag{2.1}$$

Definition 2.2. [29, 30] Let $\alpha>0$, $n=[\alpha]+1$ and $\rho>0$. The left-side Katugampola fractional derivative, associated with the Katugampola fractional integral (2.1), is defined, for $0\leq a<t<b\leq\infty$, by

$$\begin{aligned}({}^\rho D_{a+}^\alpha x)(t) =&\delta_\rho^n({}^\rho I_{a+}^{n-\alpha} x)(t)\\ =&\frac{\rho^{1-n+\alpha}}{\Gamma(n-\alpha)}\Big(t^{1-\rho}\frac{d}{dt}\Big)^n \int_a^t \frac{s^{\rho-1}x(s)}{(t^\rho - s^\rho)^{1-n+\alpha}}ds,\end{aligned}$$

where $\delta_\rho^n = (t^{1-\rho}d/dt)^n$.

Definition 2.3. [26] Let $\alpha > 0$, $n = [\alpha] + 1$ and $\rho > 0$. The left-side Hilfer-Katugampola fractional derivative of order α and type β $(0 \leq \beta \leq 1)$ of a function x is defined by

$$({}^\rho D_{a+}^{\alpha,\beta} x)(t) = \Big({}^\rho I_{a+}^{\beta(n-\alpha)}\Big(t^{1-\rho}\frac{d}{dt}\Big)^n {}^\rho I_{a+}^{(1-\beta)(n-\alpha)} x\Big)(t).$$

Lemma 2.1. [29, 30] Let $\alpha,\beta > 0$, $1 \leq p \leq \infty$, $0 < a < b < \infty$ and $\rho > 0$. Then, for $x \in X_c^p(a,b)$ the semigroup property is valid. That is,

$$({}^\rho I_{a+}^\alpha\, {}^\rho I_{a+}^\beta x)(t) = ({}^\rho I_{a+}^{\alpha+\beta} x)(t).$$

Lemma 2.2. [31] Let $\alpha > 0$, $n = [\alpha] + 1$, $x \in X_c^p(a,b)$ and ${}^\rho I_{a+}^\alpha x \in AC_{\delta_\rho}^n[a,b]$. Then

$$({}^\rho I_{a+}^\alpha\, {}^\rho D_{a+}^\alpha x)(t) = x(t) - \sum_{j=1}^n \frac{(\delta_\rho^{n-j}({}^\rho I_{a+}^{n-\alpha} x))(a)}{\Gamma(\alpha - j + 1)}\Big(\frac{t^\rho - a^\rho}{\rho}\Big)^{\alpha-j},$$

where $AC_{\delta_\rho}^n[a,b]$ is defined by

$$AC_{\delta_\rho}^n[a,b] = \{x : [a,b] \to R | \delta_\rho^{n-1} x \in AC[a,b]\},$$

and $AC[a,b]$ denote the space of all absolutely continuous real valued function on $[a,b]$.

Lemma 2.3. [26, 31] Let $\alpha > 0$, $n = [\alpha] + 1$, $\rho > 0$, $a > 0$, $\xi > 0$ and $\lambda > \alpha - 1$. Then

$$ {}^{\rho}I^{\alpha}_{a+}\Big(\frac{t^{\rho} - a^{\rho}}{\rho}\Big)^{\xi-1} = \frac{\Gamma(\xi)}{\Gamma(\alpha+\xi)}\Big(\frac{t^{\rho} - a^{\rho}}{\rho}\Big)^{\alpha+\xi-1}, \tag{2.2} $$

$$ {}^{\rho}D^{\alpha}_{a+}\Big(\frac{t^{\rho} - a^{\rho}}{\rho}\Big)^{\lambda} = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+1-\alpha)}\Big(\frac{t^{\rho} - a^{\rho}}{\rho}\Big)^{\lambda-\alpha}, \tag{2.3} $$

$$ {}^{\rho}D^{\alpha}_{a+}\Big(\frac{t^{\rho} - a^{\rho}}{\rho}\Big)^{\alpha-j} = 0, \quad j = 1, 2, \cdots, n. \tag{2.4} $$

Lemma 2.4. [28] Let α>0, then the homogeneous differential equation with Hilfer-Katugampola fractional derivative

$$ {}^{\rho}D^{\alpha,\beta}_{a+}x(t) = 0, $$

has a solution

$$ x(t)=c_0\Big(\frac{t^{\rho}-a^{\rho}}{\rho}\Big)^{\gamma-1}+c_1\Big(\frac{t^{\rho}-a^{\rho}}{\rho}\Big)^{\gamma+2\beta-2}+\cdots+c_n\Big(\frac{t^{\rho}-a^{\rho}}{\rho}\Big)^{\gamma+n(2\beta)-(n+1)}. $$

Remark 2.1. By Lemma 2.4, let $0<\alpha\leq 1$, then the homogeneous differential equation with Hilfer-Katugampola fractional derivative

$$ {}^{\rho}D^{\alpha,\beta}_{a+}x(t)=0, $$

has a solution

$$ x(t)=c_0\Big(\frac{t^{\rho} - a^{\rho}}{\rho}\Big)^{\gamma-1}+c_1\Big(\frac{t^{\rho} - a^{\rho}}{\rho}\Big)^{\gamma+2\beta-2}. $$

Remark 2.2. The conclusion of Remark 2.1 is incorrect, which means that Lemma 2.4 is not rigorous. Therefore, it is necessary to correct the conclusion of Lemma 2.4. To this end, we will give some new properties of Hilfer-Katugampola fractional calculus in the following section.

3. New properties of Hilfer-Katugampola fractional calculus

In this section, we will present some new properties of Hilfer-Katugampola fractional calculus and give the modified results of Lemma 2.4.

Lemma 3.1. Let $\alpha > 0$, $n = [\alpha]+1$, $\rho > 0$, $c \in \mathbb{R}$ and $1 \leq p < \infty$. If $x \in X^p_c(a, b)$ and ${}^{\rho}I^{(n-\alpha)(1-\beta)}_{a+}x \in AC^n_{\delta_\rho}[a, b]$, then

$$ ({}^{\rho}I^{\alpha}_{a+}{}^{\rho}D^{\alpha,\beta}_{a+}x)(t) = x(t) - \sum_{k=0}^{n-1}\frac{(\delta^k_{\rho}({}^{\rho}I^{(n-\alpha)(1-\beta)}_{a+}x))(a)}{\Gamma[k-(n-\alpha)(1-\beta)+1]}\Big(\frac{t^{\rho} - a^{\rho}}{\rho}\Big)^{k-(n-\alpha)(1-\beta)}. $$

Proof. According to the Definitions 2.2, 2.3, and Lemma 2.1, we have

$$ ({}^{\rho}I^{\alpha}_{a+}{}^{\rho}D^{\alpha,\beta}_{a+}x)(t) = ({}^{\rho}I^{\alpha}_{a+}{}^{\rho}I^{\beta(n-\alpha)}_{a+}\delta^{n}_{\rho}{}^{\rho}I^{(1-\beta)(n-\alpha)}_{a+}x)(t) = ({}^{\rho}I^{\alpha+\beta(n-\alpha)}_{a+}{}^{\rho}D^{\alpha+\beta(n-\alpha)}_{a+}x)(t). $$

Let $\nu = \alpha + \beta(n - \alpha)$, then $n - 1 < \nu \leq n$. An argument similar to the one used in Lemma 2.2 ([31], Theorem 2.7) shows that

$$ ({}^{\rho}I^{\nu}_{a+}{}^{\rho}D^{\nu}_{a+}x)(t) = \delta_{\rho}\Big[{}^{\rho}I^{1}_{a+}x(t) - \sum_{j=1}^{n}\frac{(\delta^{n-j}_{\rho}({}^{\rho}I^{n-\nu}_{a+}x))(a)}{\Gamma(\alpha+2-j)}\Big(\frac{t^{\rho} - a^{\rho}}{\rho}\Big)^{\nu-j+1}\Big]. $$

Let $k = n - j$, then we can rewrite

$$({}^\rho I^\nu_{a+} {}^\rho D^\nu_{a+} x)(t) = \delta_\rho\Big[{}^\rho I^1_{a+} x(t) - \sum_{k=0}^{n-1} \frac{(\delta^k_\rho({}^\rho I^{n-\nu}_{a+} x))(a)}{\Gamma(\nu - n + k + 2)}\Big(\frac{t^\rho - a^\rho}{\rho}\Big)^{\nu-n+k+1}\Big].$$

Therefore, by using the Lemma 2.2, we finally have

$$({}^\rho I^\nu_{a+} {}^\rho D^\nu_{a+} x)(t) = x(t) - \sum_{k=0}^{n-1} \frac{(\delta^k_\rho({}^\rho I^{n-\nu}_{a+} x))(a)}{\Gamma(\nu - n + k + 1)}\Big(\frac{t^\rho - a^\rho}{\rho}\Big)^{\nu-n+k},$$

which completes the proof. □

As a direct consequence of Lemma 3.1, we will have

Corollary 3.1. Let $\alpha > 0,\ n = [\alpha] + 1,\ \rho > 0,\ 0 \leq \beta \leq 1$, then the homogeneous fractional differential equation

$${}^\rho D^{\alpha,\beta}_{a+} x(t) = 0,$$

has a general solution of the form

$$x(t) = c_0\Big(\frac{t^\rho - a^\rho}{\rho}\Big)^{-(n-\alpha)(1-\beta)} + c_1\Big(\frac{t^\rho - a^\rho}{\rho}\Big)^{1-(n-\alpha)(1-\beta)} + \cdots + c_{n-1}\Big(\frac{t^\rho - a^\rho}{\rho}\Big)^{n-1-(n-\alpha)(1-\beta)},$$

where $c_j \in \mathbb{R}$ $(j=0, 1, \cdots, n-1)$ are arbitrary constants.

Lemma 3.2. Let $\alpha > 0$, $n = [\alpha] + 1$, $0 \leq \beta \leq 1$, $\rho > 0$, $a > 0$ and $\lambda > \alpha - 1$, then

$${}^\rho D^{\alpha,\beta}_{a+}\Big(\frac{t^\rho - a^\rho}{\rho}\Big)^{\lambda+\beta(n-\alpha)} = \frac{\Gamma[\lambda + 1 + \beta(n - \alpha)]}{\Gamma[\lambda + 1 - \alpha + \beta(n - \alpha)]}\Big(\frac{t^\rho - a^\rho}{\rho}\Big)^{\beta(n-\alpha)+\lambda-\alpha}, \tag{3.1}$$

in particular,

$${}^\rho D^{\alpha,\beta}_{a+}\Big(\frac{t^\rho - a^\rho}{\rho}\Big)^{\lambda-j+\beta(n-\alpha)} = 0, \quad j = 1, 2, \cdots, n. \tag{3.2}$$

Proof. From the definition of Hilfer-Katugampola fractional derivative, we have

$$\begin{aligned} {}^\rho D^{\alpha,\beta}_{a+}\Big(\frac{t^\rho - a^\rho}{\rho}\Big)^{\lambda+\beta(n-\alpha)} &= {}^\rho I^{\beta(n-\alpha)}_{a+} \delta^n_\rho {}^\rho I^{(1-\beta)(n-\alpha)}_{a+}\Big(\frac{t^\rho - a^\rho}{\rho}\Big)^{\lambda+\beta(n-\alpha)} \\ &= {}^\rho I^{\beta(n-\alpha)}_{a+} {}^\rho D^{\alpha+\beta(n-\alpha)}_{a+}\Big(\frac{t^\rho - a^\rho}{\rho}\Big)^{\lambda+\beta(n-\alpha)}, \end{aligned} \tag{3.3}$$

and

$${}^\rho D^{\alpha,\beta}_{a+}\Big(\frac{t^\rho - a^\rho}{\rho}\Big)^{\alpha-j+\beta(n-\alpha)} = {}^\rho I^{\beta(n-\alpha)}_{a+} {}^\rho D^{\alpha+\beta(n-\alpha)}_{a+}\Big(\frac{t^\rho - a^\rho}{\rho}\Big)^{\alpha+\beta(n-\alpha)-j}. \tag{3.4}$$

On the one hand, we obtain from the Eqs (2.2), (2.3) and (3.3) that

$$\begin{aligned} {}^\rho D^{\alpha,\beta}_{a+}\Big(\frac{t^\rho - a^\rho}{\rho}\Big)^{\lambda+\beta(n-\alpha)} &= \frac{\Gamma[\lambda + 1 + \beta(n - \alpha)]}{\Gamma(\lambda + 1 - \alpha)}\ {}^\rho I^{\beta(n-\alpha)}_{a+}\Big(\frac{t^\rho - a^\rho}{\rho}\Big)^{\lambda-\alpha} \\ &= \frac{\Gamma[\lambda + 1 + \beta(n - \alpha)]}{\Gamma[\lambda + 1 - \alpha + \beta(n - \alpha)]}\Big(\frac{t^\rho - a^\rho}{\rho}\Big)^{\lambda+\beta(n-\alpha)-\alpha}. \end{aligned}$$

On the other hand, by using the Eqs (2.4) and (3.4), we obtain (3.2) immediately. The proof is completed. □

4. Main results

4.1. Green's functions of BVPs (1.7) and (1.8)

In this subsection we discuss the Green's functions of problems (1.7) and (1.8), and present some of their properties.

Lemma 4.1. Assume that (A_1) holds. Then, for $x(t) \in C[a,b]$ is a solution of the BVP (1.7) if and only if $x(t)$ satisfies the integral equation

$$x(t) = \int_a^b G(t,s)q(s)x(s)ds + Q(t)\sum_{i=1}^{m-2}\gamma_i \int_a^b G(\eta_i,s)q(s)x(s)ds, \quad t \in [a,b], \tag{4.1}$$

where $Q(t)$ is defined by

$$Q(t) = \frac{(t^\rho - a^\rho)^{1-(2-\alpha)(1-\beta)}}{(b^\rho - a^\rho)^{1-(2-\alpha)(1-\beta)} - \sum_{i=1}^{m-2}\gamma_i(\eta_i^\rho - a^\rho)^{1-(2-\alpha)(1-\beta)}}, \quad t \in [a,b],$$

and $G(t,s)$ is the Green's function given by

$$G(t,s) = \frac{\rho^{1-\alpha}s^{\rho-1}}{\Gamma(\alpha)(b^\rho - a^\rho)^{1-(2-\alpha)(1-\beta)}}\begin{cases} h_1(t,s), & a \le s \le t \le b, \\ h_2(t,s), & a \le t \le s \le b, \end{cases}$$

with

$$\begin{aligned} h_1(t,s) &= (t^\rho - a^\rho)^{1-(2-\alpha)(1-\beta)}(b^\rho - s^\rho)^{\alpha-1} - (b^\rho - a^\rho)^{1-(2-\alpha)(1-\beta)}(t^\rho - s^\rho)^{\alpha-1}, \\ h_2(t,s) &= (t^\rho - a^\rho)^{1-(2-\alpha)(1-\beta)}(b^\rho - s^\rho)^{\alpha-1}. \end{aligned}$$

Proof. Using Lemma 3.1, the fractional differential equation in (1.7) can be transformed into an equivalent integral equation

$$x(t) = -{}^\rho I_{a+}^\alpha q(t)x(t) + c_0\Big(\frac{t^\rho - a^\rho}{\rho}\Big)^{-(2-\alpha)(1-\beta)} + c_1\Big(\frac{t^\rho - a^\rho}{\rho}\Big)^{1-(2-\alpha)(1-\beta)},$$

where $c_0, c_1 \in \mathbb{R}$. From the first boundary condition $x(a) = 0$, we get $c_0 = 0$, then

$$x(t) = -{}^\rho I_{a+}^\alpha q(t)x(t) + c_1\Big(\frac{t^\rho - a^\rho}{\rho}\Big)^{1-(2-\alpha)(1-\beta)}. \tag{4.2}$$

The second boundary condition $x(b) = \sum_{i=1}^{m-2}\gamma_i x(\eta_i)$ yields

$$x(b) = -{}^\rho I_{a+}^\alpha q(t)x(t)|_{t=b} + c_1\Big(\frac{b^\rho - a^\rho}{\rho}\Big)^{1-(2-\alpha)(1-\beta)} = \sum_{i=1}^{m-2}\gamma_i x(\eta_i),$$

from which we obtain

$$c_1 = \Big(\frac{b^\rho - a^\rho}{\rho}\Big)^{-1+(2-\alpha)(1-\beta)}\Big(\sum_{i=1}^{m-2}\gamma_i x(\eta_i) + {}^\rho I_{a+}^\alpha q(t)x(t)|_{t=b}\Big).$$

Substituting the value of c_1 into (4.2), we have

$$\begin{aligned} x(t) &= -{}^{\rho}I^{\alpha}_{a^+}q(t)x(t) + \Big(\frac{t^{\rho}-a^{\rho}}{b^{\rho}-a^{\rho}}\Big)^{1-(2-\alpha)(1-\beta)}\Big(\sum_{i=1}^{m-2}\gamma_i x(\eta_i) + {}^{\rho}I^{\alpha}_{a^+}q(t)x(t)|_{t=b}\Big) \\ &= \int_a^b G(t,s)q(s)x(s)ds + \sum_{i=1}^{m-2}\gamma_i x(\eta_i)\Big(\frac{t^{\rho}-a^{\rho}}{b^{\rho}-a^{\rho}}\Big)^{1-(2-\alpha)(1-\beta)}. \end{aligned} \tag{4.3}$$

Then, we deduce

$$\sum_{i=1}^{m-2}\gamma_i x(\eta_i) = \sum_{i=1}^{m-2}\gamma_i\int_a^b G(\eta_i,s)q(s)x(s)ds + \sum_{i=1}^{m-2}\gamma_i x(\eta_i)\sum_{i=1}^{m-2}\gamma_i\Big(\frac{\eta_i^{\rho}-a^{\rho}}{b^{\rho}-a^{\rho}}\Big)^{1-(2-\alpha)(1-\beta)},$$

which gives

$$\sum_{i=1}^{m-2}\gamma_i x(\eta_i) = \frac{\sum_{i=1}^{m-2}\gamma_i\int_a^b G(\eta_i,s)q(s)x(s)ds(b^{\rho}-a^{\rho})^{1-(2-\alpha)(1-\beta)}}{(b^{\rho}-a^{\rho})^{1-(2-\alpha)(1-\beta)} - \sum_{i=1}^{m-2}\gamma_i(\eta_i^{\rho}-a^{\rho})^{1-(2-\alpha)(1-\beta)}}. \tag{4.4}$$

Using Eq (4.4) in Eq (4.3), we obtain the solution (4.1). The converse follows by direct computation. The proof is completed. □

Lemma 4.2. Assume that (A_2) holds. Then, for $x(t) \in C[a,b]$ is a solution of the BVP (1.8) if and only if $x(t)$ satisfies the integral equation

$$x(t) = \int_a^b K(t,s)q(s)x(s)ds + R(t)\sum_{i=1}^{m-2}\sigma_i\int_a^b K(\xi_i,s)q(s)x(s)ds, \quad t\in[a,b], \tag{4.5}$$

where $R(t)$ is defined by

$$R(t) = \frac{(t^{\rho}-a^{\rho})^{1-(2-\alpha)(1-\beta)}}{[1-(2-\alpha)(1-\beta)]\rho(b^{\rho}-a^{\rho})^{-(2-\alpha)(1-\beta)} - \sum\limits_{i=1}^{m-2}\sigma_i(\xi_i{}^{\rho}-a^{\rho})^{1-(2-\alpha)(1-\beta)}}, \quad t\in[a,b],$$

and $K(t,s)$ is the Green's function defined by

$$K(t,s) = \frac{(b^{\rho}-s^{\rho})^{\alpha-2}\rho^{1-\alpha}s^{\rho-1}}{[1-(2-\alpha)(1-\beta)]\Gamma(\alpha)}\begin{cases} k_1(t,s), & a\le s\le t\le b, \\ k_2(t,s), & a\le t\le s\le b, \end{cases}$$

with

$$\begin{aligned} k_1(t,s) &= (\alpha-1)(b^{\rho}-a^{\rho})^{(2-\alpha)(1-\beta)}(t^{\rho}-a^{\rho})^{1-(2-\alpha)(1-\beta)} - [1-(2-\alpha)(1-\beta)]\frac{(t^{\rho}-s^{\rho})^{\alpha-1}}{(b^{\rho}-s^{\rho})^{\alpha-2}}, \\ k_2(t,s) &= (\alpha-1)(b^{\rho}-a^{\rho})^{(2-\alpha)(1-\beta)}(t^{\rho}-a^{\rho})^{1-(2-\alpha)(1-\beta)}. \end{aligned}$$

Proof. As argued in Lemma 4.1, the solutions of fractional differential equation in (1.8) can be written as

$$x(t) = -{}^{\rho}I^{\alpha}_{a+}q(t)x(t) + c_0\Big(\frac{t^{\rho}-a^{\rho}}{\rho}\Big)^{-(2-\alpha)(1-\beta)} + c_1\Big(\frac{t^{\rho}-a^{\rho}}{\rho}\Big)^{1-(2-\alpha)(1-\beta)},$$

where $c_0, c_1 \in \mathbb{R}$. Using the first boundary condition $x(a) = 0$, we find that $c_0 = 0$, which gives

$$x(t) = -{}^{\rho}I^{\alpha}_{a+}q(t)x(t) + c_1\Big(\frac{t^{\rho} - a^{\rho}}{\rho}\Big)^{1-(2-\alpha)(1-\beta)}. \tag{4.6}$$

Differentiating the equality (4.6) with respect to t, and then multiplying the both sides of the equation by $t^{1-\rho}$, we get

$$t^{1-\rho}\frac{d}{dt}x(t) = -{}^{\rho}I^{\alpha-1}_{a+}q(t)x(t) + \frac{c_1[1-(2-\alpha)(1-\beta)](t^{\rho}-a^{\rho})^{-(2-\alpha)(1-\beta)}}{\rho^{-(2-\alpha)(1-\beta)}},$$

which, together with the boundary condition $t^{1-\rho}\frac{d}{dt}x(t)|_{t=b} = \sum\limits_{i=1}^{m-2}\sigma_i x(\xi_i)$, yields

$$c_1 = \frac{\rho^{-(2-\alpha)(1-\beta)}}{[1-(2-\alpha)(1-\beta)](b^{\rho}-a^{\rho})^{-(2-\alpha)(1-\beta)}}\Big[\sum_{i=1}^{m-2}\sigma_i x(\xi_i) + {}^{\rho}I^{\alpha-1}_{a+}q(t)x(t)|_{t=b}\Big].$$

Substituting the value of c_1 into (4.6), we obtain the solution

$$\begin{aligned} x(t) &= -{}^{\rho}I^{\alpha}_{a+}q(t)x(t) + \frac{\Big[\sum_{i=1}^{m-2}\sigma_i x(\xi_i)+{}^{\rho}I^{\alpha-1}_{a+}q(t)x(t)|_{t=b}\Big](t^{\rho}-a^{\rho})^{1-(2-\alpha)(1-\beta)}}{[1-(2-\alpha)(1-\beta)]\rho(b^{\rho}-a^{\rho})^{-(2-\alpha)(1-\beta)}} \\ &= \int_a^b K(t,s)q(s)x(s)ds + \frac{(t^{\rho}-a^{\rho})^{1-(2-\alpha)(1-\beta)}}{[1-(2-\alpha)(1-\beta)]\rho(b^{\rho}-a^{\rho})^{-(2-\alpha)(1-\beta)}}\sum_{i=1}^{m-2}\sigma_i x(\xi_i), \end{aligned} \tag{4.7}$$

it follows that

$$\sum_{i=1}^{m-2}\sigma_i x(\xi_i) = \sum_{i=1}^{m-2}\sigma_i\int_a^b K(\xi_i,s)q(s)x(s)ds + \sum_{i=1}^{m-2}\frac{\sigma_i(\xi_i^{\rho}-a^{\rho})^{1-(2-\alpha)(1-\beta)}\sum_{i=1}^{m-2}\sigma_i x(\xi_i)}{[1-(2-\alpha)(1-\beta)]\rho(b^{\rho}-a^{\rho})^{-(2-\alpha)(1-\beta)}}. \tag{4.8}$$

Solving Eq (4.8), we get

$$\sum_{i=1}^{m-2}\sigma_i x(\xi_i) = \frac{[1-(2-\alpha)(1-\beta)]\rho(b^{\rho}-a^{\rho})^{-(2-\alpha)(1-\beta)}\sum_{i=1}^{m-2}\sigma_i\int_a^b K(\xi_i,s)q(s)x(s)ds}{[1-(2-\alpha)(1-\beta)]\rho(b^{\rho}-a^{\rho})^{-(2-\alpha)(1-\beta)} - \sum_{i=1}^{m-2}\sigma_i(\xi_i^{\rho}-a^{\rho})^{1-(2-\alpha)(1-\beta)}}. \tag{4.9}$$

By substituting (4.9) into (4.7), we obtain (4.5). Conversely, by direct computation, it can be established that (4.5) satisfies the problem (1.8). This completes the proof. □

Lemma 4.3. [5] If $1 < \upsilon < 2$, then

$$\frac{2-\upsilon}{(\upsilon-1)^{\frac{\upsilon-1}{\upsilon-2}}} \le \frac{(\upsilon-1)^{\upsilon-1}}{\upsilon^{\upsilon}}.$$

Lemma 4.4. The Green's functions $G(t,s)$ and $K(t,s)$ given by Lemmas 4.1 and 4.2, respectively, satisfy the following properties:

(i) $G(t,s)$ and $K(t,s)$ are two continuous functions in $[a,b]\times[a,b]$;

(ii) $|G(t,s)| \leq \dfrac{(\alpha-1)^{\alpha-1}[\alpha-1+\beta(2-\alpha)]^{\alpha-1+\beta(2-\alpha)}}{[2(\alpha-1)+\beta(2-\alpha)]^{2(\alpha-1)+\beta(2-\alpha)}\Gamma(\alpha)}\rho^{1-\alpha}s^{\rho-1}(b^\rho-a^\rho)^{\alpha-1}$ for any $(t,s)\in[a,b]\times[a,b]$;

(iii) $|K(t,s)| \leq \dfrac{(b^\rho-s^\rho)^{\alpha-2}\rho^{1-\alpha}s^{\rho-1}}{[1-(2-\alpha)(1-\beta)]\Gamma(\alpha)}(b^\rho-a^\rho)\max\{\beta(2-\alpha),\alpha-1\}$ for any $(t,s)\in[a,b]\times[a,b]$.

Proof. Clearly, (i) is true. Let's now prove that properties (ii) and (iii). Firstly, we show that (ii) holds. In fact, by the expression for the function $h_2(t,s)$, we can easily obtain that

$$0 \leq h_2(t,s) \leq h_2(s,s), \quad (t,s) \in [a,b]\times[a,b].$$

Now, we turn our attention to the function $h_1(t,s)$. Differentiating $h_1(t,s)$ with respect to s for every fixed $t \in [a,b]$, we get

$$\frac{\partial h_1(t,s)}{\partial s}=(\alpha-1)\rho s^{\rho-1}(b^\rho-a^\rho)^{1-(2-\alpha)(1-\beta)}(t^\rho-s^\rho)^{\alpha-2}\Big[1-\Big(\frac{t^\rho-s^\rho}{b^\rho-s^\rho}\Big)^{2-\alpha}\Big(\frac{t^\rho-a^\rho}{b^\rho-a^\rho}\Big)^{1-(2-\alpha)(1-\beta)}\Big]\geq 0.$$

This means that $h_1(t,s)$ is increasing with respect to $s \in [a,t]$ for any fixed $t \in [a,b]$. Hence, we have

$$h_1(t,a) \leq h_1(t,s) \leq h_1(t,t).$$

Note that

$$\begin{aligned} h_1(t,a) &=(t^\rho - a^\rho)^{1-(2-\alpha)(1-\beta)}(b^\rho - a^\rho)^{\alpha-1} - (b^\rho - a^\rho)^{1-(2-\alpha)(1-\beta)}(t^\rho - a^\rho)^{\alpha-1} \\ &=(t^\rho - a^\rho)^{1-(2-\alpha)(1-\beta)}(b^\rho - a^\rho)^{\alpha-1}\Big[1-\Big(\frac{b^\rho - a^\rho}{t^\rho - a^\rho}\Big)^{2\beta-\alpha\beta}\Big] \leq 0, \end{aligned}$$

we obtain

$$|h_1(t,s)| \leq \max\Big\{\max_{t\in[a,b]} h_1(t,t),\ \max_{t\in[a,b]} -h_1(t,a)\Big\}.$$

Denote

$$g_1(t) = h_1(t,t) = (t^\rho - a^\rho)^{1-(2-\alpha)(1-\beta)}(b^\rho - t^\rho)^{\alpha-1}, \quad t \in [a,b].$$

Then, differentiating $g_1(t)$ on (a,b), we get

$$\begin{aligned} g_1'(t) =&\rho t^{\rho-1}(t^\rho - a^\rho)^{-(2-\alpha)(1-\beta)}(b^\rho - t^\rho)^{\alpha-2} \\ &\times\{[1-(2-\alpha)(1-\beta)](b^\rho - t^\rho) - (\alpha-1)(t^\rho - a^\rho)\}. \end{aligned}$$

Observe that $g_1'(t) = 0$ has a unique zero on (a,b), attained at the point

$$\begin{aligned} t = t^* =&\Big\{a^\rho + \frac{[\beta(2-\alpha)+(\alpha-1)](b^\rho - a^\rho)}{2(\alpha-1)+\beta(2-\alpha)}\Big\}^{\frac{1}{\rho}} \\ =&\Big\{b^\rho - \frac{(\alpha-1)(b^\rho - a^\rho)}{2(\alpha-1)+\beta(2-\alpha)}\Big\}^{\frac{1}{\rho}}. \end{aligned}$$

Since $g_1(a) = g_1(b) = 0$ and $g_1(t) > 0$ on (a,b), we conclude that $g_1(t)$ reaches the maximum at $t = t^*$, that is,

$$\begin{aligned} \max_{t\in[a,b]} g_1(t) =&g_1(t^*)=\Big\{\frac{[\beta(2-\alpha)+(\alpha-1)](b^\rho-a^\rho)}{2(\alpha-1)+\beta(2-\alpha)}\Big\}^{1-(2-\alpha)(1-\beta)}\Big[\frac{(\alpha-1)(b^\rho-a^\rho)}{2(\alpha-1)+\beta(2-\alpha)}\Big]^{\alpha-1} \\ =&\frac{(2\beta-\alpha\beta+\alpha-1)^{2\beta-\alpha\beta+\alpha-1}(b^\rho-a^\rho)^{2\beta+2\alpha-\alpha\beta-2}(\alpha-1)^{\alpha-1}}{(2\alpha-2+2\beta-\alpha\beta)^{2\alpha-2+2\beta-\alpha\beta}}. \end{aligned}$$

Denote

$$\begin{aligned}g_2(t)=&-h_1(t,a)=(b^\rho-a^\rho)^{1-(2-\alpha)(1-\beta)}(t^\rho-a^\rho)^{\alpha-1}-(t^\rho-a^\rho)^{1-(2-\alpha)(1-\beta)}(b^\rho-a^\rho)^{\alpha-1}\\ =&(b^\rho-a^\rho)^{1-(2-\alpha)(1-\beta)}(t^\rho-a^\rho)^{\alpha-1}\Big[1-\Big(\frac{t^\rho-a^\rho}{b^\rho-a^\rho}\Big)^{\beta(2-\alpha)}\Big],\ t\in[a,b].\end{aligned}$$

We claim that

$$\max_{t\in[a,b]} g_2(t) \le \max_{t\in[a,b]} g_1(t). \tag{4.10}$$

In fact, if $\beta(2-\alpha)=0$, then (4.10) holds obviously. If $\beta(2-\alpha)\neq 0$, differentiating $g_2(t)$ on (a,b), we have

$$g_2'(t)=\rho t^{\rho-1}(b^\rho-a^\rho)^{\alpha-1}(t^\rho-a^\rho)^{\alpha-2}[(\alpha-1)(b^\rho-a^\rho)^{\beta(2-\alpha)}-(\alpha+2\beta-\alpha\beta-1)(t^\rho-a^\rho)^{\beta(2-\alpha)}].$$

Observe that $g_2'(t)=0$ has a unique zero on (a,b), attained at the point

$$t=\tilde{t}=\Big[a^\rho+\Big(\frac{\alpha-1}{\alpha+2\beta-\alpha\beta-1}\Big)^{\frac{1}{\beta(2-\alpha)}}(b^\rho-a^\rho)\Big]^{\frac{1}{\rho}}.$$

It follows from $g_2(a)=g_2(b)=0$ and $g_2(t)>0$ on (a,b) that $g_2(t)$ has maximum at point $\tilde{t}$, that is,

$$\max_{t\in[a,b]} g_2(t)=g_2(\tilde{t})=\frac{\beta(2-\alpha)}{\alpha+2\beta-\alpha\beta-1}\Big(\frac{\alpha-1}{\alpha+2\beta-\alpha\beta-1}\Big)^{\frac{\alpha-1}{\beta(2-\alpha)}}(b^\rho-a^\rho)^{\alpha-(2-\alpha)(1-\beta)}.$$

We now show that $g_2(\tilde{t})\le g_1(t^*)$. Let $\upsilon=\frac{2\alpha-2+2\beta-\alpha\beta}{\alpha+2\beta-\alpha\beta-1}$, then by Lemma 4.3, we obtain

$$\begin{aligned}g_2(\tilde{t})=&\frac{\beta(2-\alpha)}{\alpha+2\beta-\alpha\beta-1}\Big(\frac{\alpha-1}{\alpha+2\beta-\alpha\beta-1}\Big)^{\frac{\alpha-1}{\beta(2-\alpha)}}(b^\rho-a^\rho)^{\alpha-(2-\alpha)(1-\beta)}\\ \le&\Big\{\frac{(\alpha-1)^{\alpha-1}[\alpha-1+\beta(2-\alpha)]^{\alpha-1+\beta(2-\alpha)}}{[2(\alpha-1)+\beta(2-\alpha)]^{2(\alpha-1)+\beta(2-\alpha)}}\Big\}^{\frac{1}{\alpha-1+\beta(2-\alpha)}}(b^\rho-a^\rho)^{\alpha-(2-\alpha)(1-\beta)}\\ \le&\frac{(\alpha-1)^{\alpha-1}[\alpha-1+\beta(2-\alpha)]^{\alpha-1+\beta(2-\alpha)}}{[2(\alpha-1)+\beta(2-\alpha)]^{2(\alpha-1)+\beta(2-\alpha)}}(b^\rho-a^\rho)^{\alpha-(2-\alpha)(1-\beta)}\\ =&g_1(t^*),\end{aligned}$$

which completes the proof of the claim. So we have

$$\begin{aligned}|h_1(t,s)|\le&\max\Big\{\max_{t\in[a,b]} h_1(t,t),\ \max_{t\in[a,b]} -h_1(t,a)\Big\}\\ =&\max\Big\{\max_{t\in[a,b]} g_1(t), \max_{t\in[a,b]} g_2(t)\Big\}\\ =&\max_{t\in[a,b]} g_1(t)=\frac{(\alpha-1)^{\alpha-1}[\alpha-1+\beta(2-\alpha)]^{\alpha-1+\beta(2-\alpha)}}{[2(\alpha-1)+\beta(2-\alpha)]^{2(\alpha-1)+\beta(2-\alpha)}}(b^\rho-a^\rho)^{\alpha-(2-\alpha)(1-\beta)}.\end{aligned}$$

Thus we have derived that

$$\begin{aligned}|G(t,s)|\le&\frac{\rho^{1-\alpha}s^{\rho-1}}{\Gamma(\alpha)}(b^\rho-a^\rho)^{(2-\alpha)(1-\beta)-1}\max_{t\in[a,b]} h_1(t,t)\\ =&\frac{(\alpha-1)^{\alpha-1}[\alpha-1+\beta(2-\alpha)]^{\alpha-1+\beta(2-\alpha)}}{[2(\alpha-1)+\beta(2-\alpha)]^{2(\alpha-1)+\beta(2-\alpha)}}\cdot\frac{\rho^{1-\alpha}s^{\rho-1}(b^\rho-a^\rho)^{\alpha-1}}{\Gamma(\alpha)}.\end{aligned}$$

Therefore, (ii) is true. Finally, we have to prove that (iii) is also holds. In fact, for any $(t,s) \in [a,b]\times[a,b]$, it is easy to see that

$$0 \leq k_2(t,s) \leq k_2(s,s) = k_1(s,s).$$

We now consider the function $k_1(t,s)$. Differentiating it with respect to t, we get

$$\begin{aligned}\frac{\partial k_1(t,s)}{\partial t} =&(\alpha-1)[1-(2-\alpha)(1-\beta)]\rho t^{\rho-1}(b^\rho-a^\rho)^{(2-\alpha)(1-\beta)}(t^\rho-a^\rho)^{-(2-\alpha)(1-\beta)}\\ &-(\alpha-1)[1-(2-\alpha)(1-\beta)]\rho t^{\rho-1}\frac{(t^\rho-s^\rho)^{\alpha-2}}{(b^\rho-s^\rho)^{\alpha-2}}\\ =&(\alpha-1)[1-(2-\alpha)(1-\beta)]\rho t^{\rho-1}\Big[-\Big(\frac{b^\rho-s^\rho}{t^\rho-s^\rho}\Big)^{2-\alpha}+\Big(\frac{b^\rho-a^\rho}{t^\rho-a^\rho}\Big)^{(2-\alpha)(1-\beta)}\Big]\leq 0.\end{aligned}$$

This means that for fixed $s \in [a,b]$, $k_1(t,s)$ is a decreasing function of $t \in [s,b]$, it follows

$$k_1(b,s) \leq k_1(t,s) \leq k_1(s,s) = k_2(s,s).$$

Thus we have

$$|k_1(t,s)| \leq \max\Big\{\max_{s\in[a,b]}|k_1(b,s)|, \max_{s\in[a,b]}|k_1(s,s)|\Big\}. \tag{4.11}$$

Note that

$$\begin{aligned}k_1(s,s) =&(\alpha-1)(b^\rho-a^\rho)^{(2-\alpha)(1-\beta)}(s^\rho-a^\rho)^{1-(2-\alpha)(1-\beta)}\\ \leq& k_1(b,b)=(\alpha-1)(b^\rho-a^\rho).\end{aligned} \tag{4.12}$$
$$k_1(b,s)=(\alpha-1)(b^\rho-a^\rho)-[1-(2-\alpha)(1-\beta)](b^\rho-s^\rho).$$

It can easily be seen that $k_1(b,s)$ is an increasing function with respect to $s \in [a,b]$. Thus,

$$k_1(b,a) \leq k_1(b,s) \leq k_1(b,b).$$

Since

$$\begin{aligned}k_1(b,a) &= -\beta(2-\alpha)(b^\rho-a^\rho) \leq 0,\\ k_1(b,b) &=(\alpha-1)(b^\rho-a^\rho) > 0,\end{aligned}$$

then we have

$$|k_1(b,s)| \leq \max\{k_1(b,b), -k_1(b,a)\} = (b^\rho-a^\rho)\max\{\beta(2-\alpha), \alpha-1\}. \tag{4.13}$$

Combined with (4.11)–(4.13), we get

$$|k_1(t,s)| \leq (b^\rho-a^\rho)\max\{\beta(2-\alpha), \alpha-1\}.$$

Thus we are led to the conclusion that

$$|K(t,s)| \leq \frac{(b^\rho-s^\rho)^{\alpha-2}\rho^{1-\alpha}s^{\rho-1}}{[1-(2-\alpha)(1-\beta)]\Gamma(\alpha)}(b^\rho-a^\rho)\max\{\beta(2-\alpha), \alpha-1\}.$$

The proof of the Lemma 4.4 is now completed. □

4.2. Lyapunov-type inequality for BVP (1.7)

In this subsection we present the Lyapunov-type inequality for problem (1.7). To show this, we define $X = C[a, b]$ be the Banach space endowed with norm $\|x\|_\infty = \max_{t\in[a,b]}|x(t)|$.

Theorem 4.1. Suppose that (A_1) holds. If the BVP (1.7) has a nontrivial continuous solution $x(t) \in X$, where $q(t)$ is a real and continuous function in $[a, b]$, then

$$\int_a^b |q(s)|ds \geq \frac{[2(\alpha-1)+\beta(2-\alpha)]^{2(\alpha-1)+\beta(2-\alpha)}\Gamma(\alpha)\rho^{\alpha-1}}{\Delta_1[1+Q(b)\sum_{i=1}^{m-2}\gamma_i]\max\{a^{\rho-1}, b^{\rho-1}\}}, \tag{4.14}$$

where

$$\Delta_1 := (\alpha-1)^{\alpha-1}[\alpha-1+\beta(2-\alpha)]^{\alpha-1+\beta(2-\alpha)}(b^\rho - a^\rho)^{\alpha-1}.$$

Proof. By Lemma 4.1, we define the linear operator $T : X \to X$ as follow:

$$Tx(t)=\int_a^b G(t,s)q(s)x(s)ds+Q(t)\sum_{i=1}^{m-2}\gamma_i\int_a^b G(\eta_i,s)q(s)x(s)ds,\ \ x(t)\in C[a,b],\ \ t\in[a,b].$$

Then $x(t)\in X$ is a solution of BVP (1.7) if and only if $x(t)$ is a fixed point of the operator T on X. Using Lemma 4.4 (ii), for any $x_1, x_2\in X$ with $t\in[a, b]$, we have

$$\begin{aligned}
|Tx_1(t)-Tx_2(t)|&\leq\int_a^b |G(t,s)q(s)||x_1(s)-x_2(s)|ds+Q(t)\sum_{i=1}^{m-2}\gamma_i\int_a^b|G(\eta_i,s)q(s)||x_1(s)-x_2(s)|ds\\
&\leq\Big[\int_a^b|G(t,s)q(s)|ds+Q(t)\sum_{i=1}^{m-2}\gamma_i\int_a^b|G(\eta_i,s)q(s)|ds\Big]\|x_1-x_2\|_\infty\\
&\leq\frac{\Delta_1\rho^{1-\alpha}s^{\rho-1}}{[2(\alpha-1)+\beta(2-\alpha)]^{2(\alpha-1)+\beta(2-\alpha)}\Gamma(\alpha)}\Big[1+Q(b)\sum_{i=1}^{m-2}\gamma_i\Big]\int_a^b|q(s)|ds\|x_1-x_2\|_\infty\\
&\leq\frac{\Delta_1\rho^{1-\alpha}\max\{a^{\rho-1},b^{\rho-1}\}}{[2(\alpha-1)+\beta(2-\alpha)]^{2(\alpha-1)+\beta(2-\alpha)}\Gamma(\alpha)}\Big[1+Q(b)\sum_{i=1}^{m-2}\gamma_i\Big]\int_a^b|q(s)|ds\|x_1-x_2\|_\infty.
\end{aligned}$$

Combining this with the Banach's contraction principle, it follows that $x(t)\in X$ is a nontrivial solution of BVP (1.7) iff the inequality given in Eq (4.14) holds. Otherwise, (1.7) has a uniqueness solution $x(t)\equiv 0$. Thus, Theorem 4.1 is proved. □

Notice that the fractional derivative ${}^\rho D_{a+}^{\alpha,\beta}$ is an interpolator of the following fractional derivatives: Caputo-Hadamard (β=1, $\rho\to 0^+$), Katugampola (β=0), Hadamard (β=0, $\rho\to 0^+$), Hilfer ($\rho\to 1$). As special cases of Theorem 4.1, we have the following corollaries:

Corollary 4.1. Consider the following Caputo-Hadamard fractional m-point BVP

$$\begin{cases} {}_H^C D_{a+}^\alpha x(t) + q(t)x(t) = 0,\ 0 < a < t < b,\ 1 < \alpha < 2, \\ x(a) = 0, \quad x(b) = \sum\limits_{i=1}^{m-2}\gamma_i x(\eta_i), \end{cases} \tag{4.15}$$

where $q(t) \in C([a,b],\mathbb{R})$, ${}_{H}^{C}D_{a+}^{\alpha}$ denotes the Caputo-Hadamard fractional derivative of order α, and γ_i, η_i are defined as (1.7). If (4.15) has a nontrivial continuous solution, then

$$\int_a^b |q(s)|ds \geq \frac{a\alpha^{\alpha}\Gamma(\alpha)}{[(\alpha-1)(\ln b - \ln a)]^{\alpha-1}} \cdot \frac{\ln\frac{b}{a} - \sum_{i=1}^{m-2}\gamma_i \ln\frac{\eta_i}{a}}{\ln\frac{b}{a} + \sum_{i=1}^{m-2}\gamma_i \ln\frac{b}{\eta_i}}. \tag{4.16}$$

Proof. If we put β=1 and let $\rho \to 0^+$ in the right-hand side of inequality (4.14), we have

$$\begin{aligned}
&\lim_{\beta=1,\rho\to 0^+} \frac{[2(\alpha-1)+\beta(2-\alpha)]^{2(\alpha-1)+\beta(2-\alpha)}\Gamma(\alpha)\rho^{\alpha-1}}{\Delta_1[1+Q(b)\sum_{i=1}^{m-2}\gamma_i]\max\{a^{\rho-1},b^{\rho-1}\}} \\
&= \frac{a\alpha^{\alpha}\Gamma(\alpha)}{(\alpha-1)^{\alpha-1}} \lim_{\rho\to 0^+} \frac{\rho^{\alpha-1}}{(b^{\rho}-a^{\rho})^{\alpha-1}} \cdot \lim_{\rho\to 0^+} \frac{(b^{\rho}-a^{\rho})-\sum_{i=1}^{m-2}\gamma_i(\eta_i^{\rho}-a^{\rho})}{(b^{\rho}-a^{\rho})(1+\sum_{i=1}^{m-2}\gamma_i)-\sum_{i=1}^{m-2}\gamma_i(\eta_i^{\rho}-a^{\rho})} \\
&\underline{\underline{\text{L'Hospital's rule}}} \frac{a\alpha^{\alpha}\Gamma(\alpha)}{[(\alpha-1)(\ln b - \ln a)]^{\alpha-1}} \cdot \frac{\ln\frac{b}{a} - \sum_{i=1}^{m-2}\gamma_i \ln\frac{\eta_i}{a}}{\ln\frac{b}{a} + \sum_{i=1}^{m-2}\gamma_i \ln\frac{b}{\eta_i}}.
\end{aligned}$$

Therefore, we obtain form (4.14) that (4.16) holds. Obviously, our results matches the results of Theorem 3.7 in [24]. □

Corollary 4.2. Consider the following Katugampola fractional Dirichlet problem

$$\begin{cases} {}^{\rho}D_{a+}^{\alpha}x(t) + q(t)x(t) = 0, \ 0 < a < t < b, \ 1 < \alpha < 2, \\ x(a) = 0, \quad x(b) = 0, \end{cases} \tag{4.17}$$

where $q(t) \in C([a,b],\mathbb{R})$, ${}^{\rho}D_{a+}^{\alpha}$ denotes the Katugampola fractional derivative of order α. If (4.17) has a nontrivial continuous solution, then

$$\int_a^b |q(s)|ds \geq \frac{\Gamma(\alpha)}{\max\{a^{\rho-1},b^{\rho-1}\}}\Big(\frac{4\rho}{b^{\rho}-a^{\rho}}\Big)^{\alpha-1}. \tag{4.18}$$

Proof. Apply Theorem 4.1 for $\beta = 0, \gamma_i = 0$, we derive (4.18) immediately. Clearly, our results matches the results of Theorem 5 in [6]. □

Corollary 4.3. Consider the following Hadamard fractional Dirichlet problem

$$\begin{cases} {}^{H}D_{a+}^{\alpha}x(t) + q(t)x(t) = 0, \ 0 < a < t < b, \ 1 < \alpha < 2, \\ x(a) = 0, \quad x(b) = 0, \end{cases} \tag{4.19}$$

where $q(t) \in C([a,b],\mathbb{R})$, ${}^{\rho}D_{a+}^{\alpha}$ denotes the Hadamard fractional derivative of order α. If (4.19) has a nontrivial continuous solution, then

$$\int_a^b |q(s)|ds \geq 4^{(\alpha-1)}a\Gamma(\alpha)\Big(\ln\frac{b}{a}\Big)^{1-\alpha}. \tag{4.20}$$

Proof. If we take β=0, γ_i=0 and let $\rho \to 0^+$ in the right-hand side of inequality (4.14), we obtain

$$
\begin{aligned}
&\lim_{\beta=0,\rho\to 0^+} \frac{[2(\alpha-1)+\beta(2-\alpha)]^{2(\alpha-1)+\beta(2-\alpha)}\Gamma(\alpha)\rho^{\alpha-1}}{\Delta_1[1+Q(b)\sum_{i=1}^{m-2}\gamma_i]\max\{a^{\rho-1},b^{\rho-1}\}}\\
&=4^{(\alpha-1)}a\Gamma(\alpha)\lim_{\rho\to 0^+}\frac{\rho^{\alpha-1}}{(b^\rho-a^\rho)^{\alpha-1}}\\
&\underline{\underline{\text{L'Hospital's rule}}}\Big(\ln\frac{b}{a}\Big)^{1-\alpha}4^{(\alpha-1)}a\Gamma(\alpha).
\end{aligned}
$$

So we conclude from (4.14) that inequality (4.20) is valid. Evidently, our results matches the results of Theorem 2 in [9]. □

Corollary 4.4. Consider the following Hilfer fractional m-point BVP

$$
\begin{cases}
D_{a+}^{\alpha,\beta}x(t)+q(t)x(t)=0,\ 0<a<t<b,\ 1<\alpha<2,\\
x(a)=0,\quad x(b)=\sum_{i=1}^{m-2}\gamma_i x(\eta_i),
\end{cases}
\tag{4.21}
$$

where $q(t)\in C([a,b],\mathbb{R})$, $D_{a+}^{\alpha,\beta}$ denotes the Hilfer fractional derivative of order α and type β ($0\leq\beta\leq 1$), and γ_i, η_i are defined as (1.15). If (4.21) has a nontrivial continuous solution, then

$$
\int_a^b |q(s)|ds \geq \frac{\Gamma(\alpha)}{\tilde{\Delta}_1}\cdot\frac{1}{1+\sum_{i=1}^{m-2}\gamma_i\tilde{Q}(b)},
\tag{4.22}
$$

where

$$
\begin{aligned}
\tilde{\Delta}_1 &= \lim_{\rho\to 1}\frac{[2(\alpha-1)+\beta(2-\alpha)]^{2(\alpha-1)+\beta(2-\alpha)}}{\Delta_1} = \frac{[2(\alpha-1)+\beta(2-\alpha)]^{2(\alpha-1)+\beta(2-\alpha)}}{(\alpha-1)^{\alpha-1}[\alpha-1+\beta(2-\alpha)]^{\alpha-1+\beta(2-\alpha)}(b-a)^{\alpha-1}},\\
\tilde{Q}(b) &= \lim_{\rho\to 1}Q(b) = \frac{(b-a)^{1-(2-\alpha)(1-\beta)}}{(b-a)^{1-(2-\alpha)(1-\beta)}-\sum_{i=1}^{m-2}\gamma_i(\eta_i-a)^{1-(2-\alpha)(1-\beta)}}.
\end{aligned}
$$

Proof. Taking $\rho\to 1$ in the right-hand side of inequality (4.14), it follows

$$
\begin{aligned}
&\lim_{\rho\to 1}\frac{[2(\alpha-1)+\beta(2-\alpha)]^{2(\alpha-1)+\beta(2-\alpha)}\Gamma(\alpha)\rho^{\alpha-1}}{\Delta_1[1+\sum_{i=1}^{m-2}\gamma_i Q(b)]\max\{a^{\rho-1},b^{\rho-1}\}}\\
&=\frac{\Gamma(\alpha)}{\tilde{\Delta}_1}\cdot\frac{1}{1+\sum_{i=1}^{m-2}\gamma_i\tilde{Q}(b)}.
\end{aligned}
$$

Then, by Theorem 4.1, we derive (4.22) from (4.14). Apparently, for $a>0$, our results matches the results of Theorem 3.1 in [20]. □

4.3. Lyapunov-type inequality for BVP (1.8)

In this subsection we will prove a Lyapunov-type inequality for problem (1.8). To state our result, we set $E=C[a,b]$ be the Banach space endowed with norm $\|x\|_\infty=\max_{t\in[a,b]}|x(t)|$.

Theorem 4.2. Suppose that (A_2) holds. If the BVP (1.8) has a nontrivial continuous solution $x(t)\in E$, where $q(t)$ is a real and continuous function in $[a,b]$, then

$$\int_a^b (b^\rho - s^\rho)^{\alpha-2}|q(s)|ds \geq \frac{[1-(2-\alpha)(1-\beta)]\rho^{\alpha-1}\Gamma(\alpha)}{\Delta_2[1+R(b)\sum_{i=1}^{m-2}\sigma_i]}, \tag{4.23}$$

where

$$\Delta_2 := (b^\rho - a^\rho)\max\{\beta(2-\alpha), \alpha-1\}\max\{a^{\rho-1}, b^{\rho-1}\}.$$

Proof. By Lemma 4.2, we define the linear operator $\tilde{T}: E \to E$ as follow:

$$\tilde{T}x(t)=\int_a^b K(t,s)q(s)x(s)ds+R(t)\sum_{i=1}^{m-2}\sigma_i\int_a^b K(\xi_i,s)q(s)x(s)ds,\ \ x(t)\in C[a,b],\ \ t\in[a,b],$$

thus $x(t)\in E$ is a solution of BVP (1.8) iff $x(t)$ is a fixed point of the operator $\tilde{T}$ on E. Applying Lemma 4.4 (iii), for any $x_1, x_2\in E$ with $t\in[a,b]$, we get

$$\begin{aligned}
&|\tilde{T}x_1(t)-\tilde{T}x_2(t)|\\
\leq&\int_a^b |K(t,s)q(s)||x_1(s)-x_2(s)|ds+R(t)\sum_{i=1}^{m-2}\sigma_i\int_a^b |K(\xi_i,s)q(s)||x_1(s)-x_2(s)|ds\\
\leq&\Big[\int_a^b |K(t,s)q(s)|ds+R(t)\sum_{i=1}^{m-2}\sigma_i\int_a^b |K(\xi_i,s)q(s)|ds\Big]\|x_1-x_2\|_\infty\\
\leq&\frac{(b^\rho-a^\rho)\max\{\beta(2-\alpha),\alpha-1\}\rho^{1-\alpha}s^{\rho-1}}{[1-(2-\alpha)(1-\beta)]\Gamma(\alpha)}\Big[1+R(b)\sum\nolimits_{i=1}^{m-2}\sigma_i\Big]\int_a^b (b^\rho-s^\rho)^{\alpha-2}|q(s)|ds\|x_1-x_2\|_\infty\\
\leq&\frac{\Delta_2\rho^{1-\alpha}}{[1-(2-\alpha)(1-\beta)]\Gamma(\alpha)}\Big[1+R(b)\sum\nolimits_{i=1}^{m-2}\sigma_i\Big]\int_a^b (b^\rho-s^\rho)^{\alpha-2}|q(s)|ds\|x_1-x_2\|_\infty.
\end{aligned}$$

Thus, combined with the Banach's contraction principle, we deduce that $x(t)\in E$ is a nontrivial solution of BVP (1.8) iff the inequality expressed in Eq (4.23) holds. Otherwise, (1.8) has a uniqueness solution $x(t)\equiv 0$. Therefore, we finish the proof of Theorem 4.2. □

As special cases of Theorem 4.2, we have the following corollaries:

Corollary 4.5. Consider the following Katugampola fractional m-point BVP

$$\begin{cases} {}^\rho D_{a+}^\alpha x(t) + q(t)x(t) = 0,\ 0 < a < t < b,\ 1 < \alpha < 2, \\ x(a) = 0, \quad t^{1-\rho}\frac{d}{dt}x(t)|_{t=b} = \sum\limits_{i=1}^{m-2}\sigma_i x(\xi_i), \end{cases} \tag{4.24}$$

where $q(t) \in C([a,b],\mathbb{R})$, ${}^\rho D_{a+}^\alpha$ denotes the Katugampola fractional derivative of order α, and σ_i, ξ_i are defined as (1.8). If (4.24) has a nontrivial continuous solution, then

$$\int_a^b (b^\rho - s^\rho)^{\alpha-2}|q(s)|ds \geq \frac{\rho^{\alpha-1}\Gamma(\alpha)}{\tilde{\Delta}_2[1+\tilde{R}(b)\sum_{i=1}^{m-2}\sigma_i]}, \tag{4.25}$$

where

$$\tilde{\Delta}_2 := (b^\rho - a^\rho)\max\{a^\rho, b^\rho\},$$

$$\tilde{R}(b) := \frac{(b^\rho - a^\rho)^{\alpha-1}}{(\alpha-1)\rho(b^\rho - a^\rho)^{\alpha-2} - \sum_{i=1}^{m-2}\sigma_i(\xi_i^\rho - a^\rho)^{\alpha-1}}.$$

Proof. Substituting the value of $\beta = 0$ in (4.23), we get (4.25) immediately. □

Corollary 4.6. Consider the following Hilfer fractional m-point BVP

$$\begin{cases} D_{a+}^{\alpha,\beta}x(t) + q(t)x(t) = 0,\ 0 < a < t < b,\ 1 < \alpha < 2, \\ x(a) = 0, \quad x'(b) = \sum\limits_{i=1}^{m-2}\sigma_i x(\xi_i), \end{cases} \tag{4.26}$$

where $q(t) \in C([a,b],\mathbb{R})$, $D_{a+}^{\alpha,\beta}$ denotes the Hilfer fractional derivative of order α and type β $(0\leq\beta\leq1)$, and σ_i, ξ_i are defined as (1.8). If (4.26) has a nontrivial continuous solution, then

$$\int_a^b (b-s)^{\alpha-2}|q(s)|ds \geq \frac{[1-(2-\alpha)(1-\beta)]\Gamma(\alpha)}{\hat{\Delta}_2[1+\hat{R}(b)\sum_{i=1}^{m-2}\sigma_i]}, \tag{4.27}$$

where

$$\hat{\Delta}_2 = \lim_{\rho\to1}\Delta_2 = (b-a)\max\{\beta(2-\alpha), \alpha-1\},$$

$$\hat{R}(b) = \lim_{\rho\to1}R(b) = \frac{(b-a)^{1-(2-\alpha)(1-\beta)}}{[1-(2-\alpha)(1-\beta)](b-a)^{-(2-\alpha)(1-\beta)} - \sum_{i=1}^{m-2}\sigma_i(\xi_i - a)^{1-(2-\alpha)(1-\beta)}}.$$

Proof. Taking $\rho\to1$ in the both sides of inequality (4.23) gives the desired result (4.27). □

5. Conclusions

In the last decades, the study of Lyapunov-type inequalities of fractional BVPs has received significant attention from researchers. This increasing interest is motivated by essential applications of the Lyapunov inequality and the development of the fractional calculus theory. In this study, we obtained Lyapunov-type inequalities for fractional m-point BVPs in the frame of Hilfer-Katugampola fractional derivative. In addition, we showed some new properties of the Hilfer-Katugampola fractional derivative, which play a crucial role in the study of BVPs (1.7) and (1.8). Differing from previous work, we established that new Lyapunov-type inequalities are based on a more general fractional derivative, especially in the limit case $\beta = 0$ or $\beta = 1$ and $\rho \to 0^+$ or $\rho \to 1$, our results can be reduced to some known results in the literature. Finally, we point out that there is still some work to be done in the future, such as: discussing the Lyapunov-type inequalities for a nonlinear fractional hybrid boundary value problems involving Hilfer-Katugampola fractional derivative; studying Lyapunov-type inequalities for Hilfer-Katugampola fractional p-Laplacian equations, considering the Lyapunov-type inequalities for fractional Langevin equations, and so on.

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Conflict of interest

The authors declare that they have no competing interests.

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