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Research article

Extinction behavior for a parabolic p-Laplacian equation with gradient source and singular potential

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Abstract: We concern with the extinction behavior of the solution for a parabolic p-Laplacian equation with gradient source and singular potential. By energy estimate approach, Hardy-Littlewood-Sobolev inequality, a series of ordinary differential inequalities, and super-solution and sub-solution methods, we obtain the conditions on the occurrence of the extinction phenomenon of the weak solution.

Keywords: extinction; parabolic p-Laplacian equation; gradient source; singular potential
Mathematics Subject Classification: 35K20, 35K55

1. Introduction

We are interested in the extinction properties of the solutions to a p-Laplacian equation with gradient source and singular potential

$$\begin{cases} |x|^{-s}\, u_t = \operatorname{div}(|\nabla u|^{p-2}\nabla u) + \lambda\, |\nabla u|^q\,, & (x,t) \in \Omega \times (0,+\infty), \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,+\infty), \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N\,(N \geq 2)$ with smooth boundary $\partial\Omega$, $1 < p < 2$, $s > 0$, $\lambda > 0$, $q > 0$, $|x| = \sqrt{x_1^2 + \cdots + x_N^2}$ for $x = (x_1, \cdots, x_N) \in \Omega$, $u_0\,(x)$ is a non-negative and bounded function with $u_0 \in W_0^{1,p}(\Omega)$.

Problem (1.1) is encountered in many natural phenomena and physical contexts, such as the compressible fluid flows in a homogeneous isotropic rigid porous medium, the physical theory of growth and roughening of surfaces (see for instance [6, 17, 19] and the references therein, where a more detailed physical background can be found). From a physical point of view, $\operatorname{div}(|\nabla u|^{p-2}\nabla u)$ with

$p \in (1, 2)$ is called a fast diffusion term, which may cause the extinction phenomenon of the solution; $\lambda |\nabla u|^q$ with $\lambda > 0$ is called a gradient source term, which may prevent the extinction phenomenon.

In the last twenty years, there has been a great deal of literature on the parabolic problems with gradient reaction terms (see [4, 5, 8, 9, 11, 13–15, 22, 23]). In particular, Zhang and Li [21] considered problem (1.1) with $s = 0$ and $p > 2$. They gave the conditions on the occurrence of the gradient blow-up phenomenon (i.e., there is a $T > 0$ such that $\sup_{\Omega\times[0,T)} |u| < \infty$ and $\lim_{t\to T^-} \|\nabla u\|_{L^\infty} = +\infty$). More precisely, they pointed out that, when $q > p$, the gradient blow-up phenomenon will occur for suitably large initial data, while the solution exists globally in $W^{1,\infty}$ norm for appropriately small initial data. When $q \le p$, they claimed that all solutions are global in $W^{1,\infty}$ norm.

Later, Zhang [20] gave the gradient blow-up rate in one dimensional case. Mu and Liu [7] dealt with the extinction behavior of the solution to problem (1.1) with $s = 0$ and $p \in (1, 2)$. They concluded that if $p - 1 < q < \frac{p}{2}$, the solution will vanish in finite time for appropriately small initial data, while if $q < p - 1$, the solution will not vanish in finite time for appropriately large λ. When $q = p - 1$, the size of the parameter λ plays a crucial role in the occurrence of the extinction phenomenon.

As far as we know, there is no result for the case $s \neq 0$, especially the extinction results for $s > 0$ and $p \in (1, 2)$. For these reasons, we consider the extinction behavior of the solution to problem (1.1) under the assumptions $s > 0$ and $p \in (1, 2)$. It is worth pointing out the solution of problem (1.1) is global in L^∞ norm. Our main attention will be focused on the roles that the singular potential $|x|^{-s}$, the competition between the fast diffusion term and the gradient source term play. In different ranges of gradient reaction exponents, we give the complete classification of the L^∞ norm global solutions including extinction and non-extinction cases. Our main results are the following three theorems.

Theorem 1.1. *Assume that* $0 < p - 1 < q < \frac{p}{2} < 1$, $0 \le s < p$ *and* $u_0(x)$ *is appropriately small such that (3.10) holds. Then the nonnegative weak solution of problem (1.1) vanishes in finite time.*

Theorem 1.2. *Assume that* $0 < q < p - 1 < 1$. *Then for some suitable* Ω, *problem (1.1) at least admits a non-extinction solution.*

Theorem 1.3. *Assume that* $0 < p - 1 = q < 1$. *Then the extinction phenomenon will occur for appropriately small* λ, *while problem (1.1) at least exists a non-extinction solution for some suitable* Ω.

The rest of this article is organized as follows. In section 2, we give the definition of the weak solution of problem (1.1) and collect some useful auxiliary lemmas. The last section is mainly focused on the conditions on the occurrence of the extinction phenomenon of the solution. By using Hardy-Littlewood-Sobolev inequality, the integral norm estimate method and some ordinary differential inequalities, the proofs of the extinction results will be given. Based on super-solution and sub-solution methods, the proofs of the non-extinction results will also be given in section 3.

2. Preliminaries

Since $p \in (1, 2)$, problem (1.1) is singular at the point $x \in \Omega$ such that $\nabla u = 0$. Firstly, we introduce the definition of the weak solution of (1.1) as follows.

Definition 2.1. *For some* $T > 0$, *a function* $u(x, t)$ *defined in* $\overline{\Omega} \times [0, T)$ *is called a weak sub- (super-) solution of problem (1.1) if it satisfies the following assumptions*

- $u \in C\left(\overline{\Omega} \times [0,T)\right) \cap L^{\max\{p,q\}}\left(0,T; W_0^{1,\max\{p,q\}}(\Omega)\right)$, $|x|^{-s} u_t \in L^2(\Omega \times (0,T))$.
- *For any* $0 \le \phi \in C\left(\overline{\Omega} \times [0,T]\right) \cap L^p\left(0,T; W_0^{1,p}(\Omega)\right)$, *one has*

$$\int_0^T \int_\Omega \left(|x|^{-s} u_t \phi \mathrm{d}x + |\nabla u|^{p-2} \nabla u \cdot \nabla \phi\right) \mathrm{d}x\mathrm{d}t \le (\ge) \int_0^T \int_\Omega \lambda |\nabla u|^q \phi \mathrm{d}x\mathrm{d}t. \tag{2.1}$$

- $u(x,t) \le (\ge) 0$ *for* $(x,t) \in \partial\Omega \times (0,T)$.
- $u(x,0) \le (\ge) u_0(x)$ *for* $x \in \Omega$.

A function $u(x,t)$ *is a weak solution of problem (1.1) if it is both a sub-solution and a super-solution of problem (1.1).*

The local existence of the weak solution to problem (1.1) can be obtained by using the standard regularization method and approximation process, the reader may refer to [2,22,23] for more details.

Our goal is to find the conditions on the occurrence of the extinction singularity of the solution to problem (1.1). To this aim, we need the following lemmas.

Lemma 2.1. *(see [1, 3]) Suppose* $N \ge 2$, $1 < \mu < N$, $0 \le \vartheta \le \mu$ *and* $\sigma = \frac{\mu(N-\vartheta)}{N-\mu}$. *Then, there is a positive constant* $\kappa_1 = \kappa_1(\mu, \vartheta, N)$ *such that*

$$\int_\Omega \frac{|u(x)|^\sigma}{|x|^\vartheta} \mathrm{d}x \le \kappa_1 \left(\int_\Omega |\nabla u|^\mu \mathrm{d}x\right)^{\frac{N-\vartheta}{N-\mu}}, \tag{2.2}$$

holds for any $u \in W_0^{1,\mu}(\Omega)$, *where* $\Omega \subset \mathbb{R}^N$ *is a bounded domain.*

Since Ω is a bounded domain in $\mathbb{R}^N$, then there is a ball $B(0,R) \subset \mathbb{R}^N$ centered at 0 with radius

$$R = \sup_{x \in \Omega} \sqrt{x_1^2 + \cdots + x_N^2} < +\infty, \tag{2.3}$$

such that $\Omega \subseteq B(0,R)$.

Lemma 2.2. *(see [3, 10]) Suppose* $N > s$ *and* $\Omega \subset \mathbb{R}^N$ *is a bounded domain. Then, one has*

$$\begin{aligned} \int_\Omega |x|^{-s} \mathrm{d}x \le \int_{B(0,R)} |x|^{-s} \mathrm{d}x &= \int_0^R \left[\int_{\partial B(0,r)} |x|^{-s} \mathrm{d}S(x)\right] \mathrm{d}r \\ &= \omega_N \int_0^R r^{-s} r^{N-1} \mathrm{d}r = \frac{\omega_N}{N-s} R^{N-s} \overset{def}{=} \kappa_2 < +\infty, \end{aligned} \tag{2.4}$$

where

$$\omega_N = \frac{N\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}+1\right)},$$

denotes the surface area of the unit sphere $\partial B(0,1)$ *and* Γ *is the usual Gamma function.*

Lemma 2.3. *(see [12]) Suppose* $0 < \theta < \eta \leq 1$. *Let* $g(t)$ *be a solution of the ordinary differential inequality*

$$\begin{cases} \frac{\mathrm{d}g}{\mathrm{d}t} + \gamma_1 g^{\theta} \leq \gamma_2 g^{\eta}, & t > 0, \\ g(0) = g_0 > 0, \end{cases}$$

where $\gamma_1 > 0$ *and* $0 < \gamma_2 < \frac{1}{2}\gamma_1 g_0^{\theta-\eta}$. *Then, there are two positive constants* σ_1, σ_2 *such that, for* $t \geq 0$,

$$0 \leq g(t) \leq \sigma_1 e^{-\sigma_2 t}.$$

3. Proofs of the main results

In this section, we will give the proofs of the main results and the conditions on the occurrence of the extinction phenomenon of the solution $u(x,t)$.

Proof of Theorem 1.1. Taking

$$l > \max\left\{\frac{N(2-p)+s(p-1)-p}{p-s}, 0\right\}, \tag{3.1}$$

multiplying (1.1) by u^l and integrating by parts over Ω yield

$$\frac{1}{l+1}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |x|^{-s} u^{l+1}\mathrm{d}x + \frac{lp^p}{(l+p-1)^p}\int_\Omega \left|\nabla u^{\frac{l+p-1}{p}}\right|^p \mathrm{d}x = \lambda\int_\Omega u^l |\nabla u|^q \,\mathrm{d}x. \tag{3.2}$$

Recalling that $p-1 < q < \frac{p}{2}$ and using Young's inequality lead to

$$\begin{aligned}\int_\Omega u^l |\nabla u|^q \,\mathrm{d}x =& \frac{p^q}{(l+p-1)^q}\int_\Omega u^{\frac{pl-ql+q}{p}}\left|\nabla u^{\frac{l+p-1}{p}}\right|^q \mathrm{d}x \\ \leq& \frac{\epsilon p^q}{(l+p-1)^q}\int_\Omega \left|\nabla u^{\frac{l+p-1}{p}}\right|^p \mathrm{d}x + \frac{C(\epsilon)p^q}{(l+p-1)^q}\int_\Omega u^{\frac{l(p-q)+q}{p-q}}\mathrm{d}x,\end{aligned} \tag{3.3}$$

where

$$\epsilon \in \left(0, \frac{lp^{p-q}}{\lambda(l+p-1)^{p-q}}\right).$$

Substituting (3.3) into (3.2) tells us that

$$\frac{1}{l+1}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |x|^{-s} u^{l+1}\mathrm{d}x + C_1\int_\Omega |\nabla u^{\frac{l+p-1}{p}}|^p \mathrm{d}x \leq C_2\int_\Omega u^{\frac{l(p-q)+q}{p-q}}\mathrm{d}x, \tag{3.4}$$

where

$$C_1 = \frac{lp^p}{(l+p-1)^p} - \frac{\lambda\epsilon p^q}{(l+p-1)^q} \text{ and } C_2 = \frac{\lambda C(\epsilon)p^q}{(l+p-1)^q}.$$

Setting

$$a = \frac{N-p}{\theta(N-s)},$$

with $\theta = \frac{l+p-1}{l+1} \in (0,1)$, noticing that $1 < p < 2 \le N$ and $0 \le s < p$, then it follows from (3.1) that $a \in (0,1)$. Hölder's inequality, Lemmas 2.1 and 2.2 give us that

$$\begin{aligned}
\int_\Omega |x|^{-s} u^{l+1} \mathrm{d}x &= \int_\Omega |x|^{-s(a+1-a)} \left(u^{\frac{l+p-1}{p}}\right)^{\frac{p}{\theta}} \mathrm{d}x \\
&\le \left(\int_\Omega \left[|x|^{-sa} \left(u^{\frac{l+p-1}{p}}\right)^{\frac{p}{\theta}}\right]^{\frac{1}{a}} \mathrm{d}x\right)^a \left(\int_\Omega |x|^{-s(1-a)\frac{1}{1-a}}\right)^{1-a} \\
&\le \kappa_2^{1-a} \left(\int_\Omega |x|^{-s} \left(u^{\frac{l+p-1}{p}}\right)^{\frac{p(N-s)}{N-p}} \mathrm{d}x\right)^a \\
&\le \kappa_2^{1-a} \kappa_1^a \left(\int_\Omega \left|\nabla u^{\frac{l+p-1}{p}}\right|^p\right)^\theta,
\end{aligned} \tag{3.5}$$

which implies that

$$\int_\Omega \left|\nabla u^{\frac{l+p-1}{p}}\right|^p \mathrm{d}x \ge \left(\kappa_1^{-a}\kappa_2^{a-1}\right)^\theta \left(\int_\Omega |x|^{-s} u^{l+1} \mathrm{d}x\right)^\theta. \tag{3.6}$$

On the other hand, denoting

$$\eta = \frac{l(p-q)+q}{(p-q)(l+1)} \in (0,1),$$

and using Hölder's inequality again, one has

$$\begin{aligned}
\int_\Omega u^{\frac{l(p-q)+q}{p-q}} \mathrm{d}x &\le |\Omega|^{1-\eta} \left(\int_\Omega u^{l+1} \mathrm{d}x\right)^\eta \\
&= |\Omega|^{1-\eta} \left(\int_\Omega |x|^{-s} |x|^s u^{l+1} \mathrm{d}x\right)^\eta \\
&\le |\Omega|^{1-\eta} R^{s\eta} \left(\int_\Omega |x|^{-s} u^{l+1} \mathrm{d}x\right)^\eta.
\end{aligned} \tag{3.7}$$

Denoting

$$g(t) = \int_\Omega |x|^{-s} u^{l+1} \mathrm{d}x,$$

and substituting (3.6) and (3.7) into (3.4) leads to

$$\frac{\mathrm{d}}{\mathrm{d}t} g(t) + C_3 g^\theta(t) \le C_4 g^\eta(t), \tag{3.8}$$

which is equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}t}g(t)+\frac{C_3}{2}g^{\theta}(t)\le g^{\theta}(t)\left(C_4g^{\eta-\theta}(t)-\frac{C_3}{2}\right),\tag{3.9}$$

where

$$C_3=C_1(l+1)\left(\kappa_1^{-a}\kappa_2^{a-1}\right)^{\theta}\text{ and }C_4=C_2(l+1)|\Omega|^{1-\eta}R^{s\eta}.$$

Since $q>p-1$, one can verify that $\theta<\eta$. If $u_0(x)$ is suitably small satisfying

$$g_0=g(0)=\int_{\Omega}|x|^{-s}u_0^{l+1}\mathrm{d}x<\left(\frac{C_3}{2C_4}\right)^{\frac{1}{\eta-\theta}},\tag{3.10}$$

then Lemma 2.3 tells us that there are two positive constants σ_1 and σ_2 such that

$$0\le g(t)\le\sigma_1e^{-\sigma_2t}.$$

Choosing

$$T_0>\max\left\{0,-\frac{1}{\sigma_2}\ln\left(\frac{1}{\sigma_1}\left(\frac{C_3}{2C_4}\right)^{\frac{1}{\eta-\theta}}\right)\right\},$$

then for any $t\ge T_0$, one can obtain that

$$\frac{\mathrm{d}}{\mathrm{d}t}g(t)+\frac{C_3}{2}g^{\theta}(t)\le g^{\theta}(t)\left[C_4\left(\sigma_1e^{-\sigma_2t}\right)^{\eta-\theta}-\frac{C_3}{2}\right]\le 0.\tag{3.11}$$

And hence, one has

$$\frac{\mathrm{d}}{\mathrm{d}t}g^{1-\theta}=(1-\theta)g^{-\theta}\frac{\mathrm{d}}{\mathrm{d}t}g\le-\frac{C_3(1-\theta)}{2},\quad t\ge T_0,$$

and

$$0\le g^{1-\theta}(t)\le g_0^{1-\theta}-\frac{C_3(1-\theta)}{2}t,\quad t\ge T_0.$$

Therefore

$$g(t)\equiv 0\quad\text{and}\quad u(x,t)\equiv 0\quad\text{for}\quad t\ge T_1=T_0+\frac{2g_0^{1-\theta}}{(1-\theta)C_3}.$$

The proof of Theorem 1.1 is complete. □

Proof of Theorem 1.2. Denote λ_1 be the first eigenvalue of the following eigenvalue problem

$$\begin{cases}-\mathrm{div}\left(|\nabla\psi|^{p-2}\nabla\psi\right)=\lambda\psi|\psi|^{p-2}, & x\in\Omega,\\ \psi(x)=0, & x\in\partial\Omega,\end{cases}\tag{3.12}$$

and $\psi(x)$ the corresponding eigenfunction. From Lemmas 2.3 and 2.4 in [16], one can know that $\psi(x)$ is positive in Ω and λ_1 can be expressed as

$$\lambda_1=\inf_{\psi\in W_0^{1,p}(\Omega),\psi\neq 0}\frac{\|\nabla\psi\|_p^p}{\|\psi\|_p^p}.$$

Moreover, Theorem 9.2.1 in [18] tells us that $\psi(x) \in W_0^{1,p}(\Omega) \cap C^{1+\beta}\left(\overline{\Omega}\right)$ for some $\beta \in (0,1)$. For the sake of convenience, we normalize $\psi(x)$ in L^∞ norm. Namely, $\max\limits_{x\in\Omega} \psi(x) = 1$. Define a function $f(t)$ as follows

$$f(t) = d^{\frac{1}{p-1-q}}(1-e^{-ct})^{\frac{1}{1-q}},$$

where $d \in (0,1)$, and $c \in \left(0, (p-1-q)\, d^{\frac{q-1}{p-1-q}}\right)$. Then it is easy to show that $f(0) = 0$ and $f(t) \in (0,1)$ for $t > 0$, and we have

$$f'(t) + \tfrac{1}{d} f^{p-1} - f^q < 0. \tag{3.13}$$

Let

$$\upsilon(x,t) = f(t)\,\psi(x).$$

A series of calculations show that, for $0 \le \xi(x,t) \in C\left(\overline{\Omega} \times [0,T]\right) \cap L^p\left(0,T; W_0^{1,p}(\Omega)\right)$,

$$\begin{aligned} I &= \int_0^T \int_\Omega \left(|x|^{-s}\,\upsilon_\tau \xi + |\nabla\upsilon|^{p-2}\,\nabla\upsilon\cdot\nabla\xi - \lambda\,|\nabla\upsilon|^q\,\xi\right) \mathrm{d}x\mathrm{d}\tau \\ &= \int_0^T \int_\Omega \left\{\left[|x|^{-s} f_\tau(\tau)\,\psi(x) - \lambda f^q(\tau)\,|\nabla\psi|^q\right]\xi(x,\tau) + f^{p-1}(\tau)\,|\nabla\psi|^{p-2}\,\nabla\psi\cdot\nabla\xi\right\} \mathrm{d}x\mathrm{d}\tau \\ &< \underbrace{\int_0^T \int_\Omega \left[|x|^{-s}\left(f^q - \frac{1}{d} f^{p-1}\right)\psi(x) + \lambda_1 f^{p-1}(\tau)\,\psi^{p-1}(x) - \lambda f^q(\tau)\,|\nabla\psi|^q\right]\xi(x,\tau)\,\mathrm{d}x\mathrm{d}\tau}_{J}. \end{aligned}$$

If $\overline{\Omega}$ does not contain coordinate origin, then by the Mean-Value Theorem, one can know that there is a point $(x^\star, \tau^\star) \in \Omega \times (0,T)$ such that

$$\begin{aligned} J &= T\,|\Omega|\,\xi(x^\star,\tau^\star) \\ &\quad \times \left[\left|x^\star\right|^{-s}\left(f^q(\tau^\star) - \frac{1}{d} f^{p-1}(\tau^\star)\right)\psi(x^\star) + \lambda_1 f^{p-1}(\tau^\star)\,\psi^{p-1}(x^\star) - \lambda f^q(\tau^\star)\,|\nabla\psi|^q_{x=x^\star}\right] \\ &\le T\,|\Omega|\,\xi(x^\star,\tau^\star)\left[\left|x^\star\right|^{-s}\left(f^q(\tau^\star) - \frac{1}{d} f^{p-1}(\tau^\star)\right)\psi(x^\star) + \lambda_1 f^{p-1}(\tau^\star)\,\psi^{p-1}(x^\star)\right]. \end{aligned} \tag{3.14}$$

Furthermore, suppose Ω is a suitable domain such that the first eigenvalue λ_1 of the eigenvalue problem (3.12) satisfying

$$\lambda_1 \ge \left|x^\star\right|^{-s} \psi^{2-p}(x^\star)\, f^{q-p+1}(\tau^\star).$$

By choosing $d \in \left(0, \min\left(1, \frac{\psi^{2-p}(x^\star)}{2\lambda_1 |x^\star|^s}\right)\right)$, then (3.14) tells us that

$$J \le T\,|\Omega|\,\xi(x^\star,\tau^\star)\,\psi(x^\star)\,f^q(\tau^\star)\left[\left|x^\star\right|^{-s} - \lambda_1 \psi^{p-2}(x^\star)\,f^{p-1-q}(\tau^\star)\right] < 0.$$

and then $I < 0$. So far, by Definition 2.1, one knows that, under some suitable restrictions on Ω, $\upsilon(x,t)$ is a non-extinction weak sub-solution of problem (1.1). On the other hand, one can prove that $\delta(x,t) = \max\{1, \|u_0(x)\|_{L^\infty}\}$ is a non-extinction super-solution of problem (1.1). Then by an iterated process, one can claim that problem (1.1) at least admits a non-extinction weak solution $u(x,t)$ satisfying $\upsilon(x,t) \le u(x,t) \le \delta(x,t)$. The proof of Theorem 1.2 is complete. □

Proof of Theorem 1.3. From (3.9), one has

$$\frac{\mathrm{d}}{\mathrm{d}t} g(t) + C_5 g^{\theta}(t) \le 0, \tag{3.15}$$

where

$$C_5 = C_3 - C_4.$$

If λ is sufficiently small such that C_5 is greater than zero, then (3.15) leads us to the extinction result of the solution for the case $q = p - 1$.

On the other hand, for some suitable constants $\overline{d} \in (0, 1)$ and $\overline{c} \in \left(0, \frac{(p-1)\overline{d}^{\frac{p-3}{p-1}}}{2}\right)$, repeating a similar argument to that in the proof of Theorem 1.2, one can check that

$$\rho(x,t) = \overline{d}^{\frac{2}{p-1}} \left(1 - e^{-\overline{c}t}\right)^{\frac{2}{3-p}} \psi(x),$$

is a non-extinction weak sub-solution of problem (1.1) for some suitable Ω. Meanwhile,

$$\omega(x,t) = \max\{1, \|u_0(x)\|_{L^\infty}\},$$

is a non-extinction super-solution of problem (1.1). Then by an iterated process, one can claim that problem (1.1) at least admits a non-extinction weak solution $u(x,t)$ satisfying $\rho(x,t) \le u(x,t) \le \omega(x,t)$. The proof of Theorem 1.3 is complete.

□

4. Conclusions

In this article, we analyzed the effects of the singular potential and the competition between the fast diffusion term and the gradient source term on the occurrence of the extinction singularity of the weak solution to a p-Laplacian equation. Using integral norm estimate method and constructing appropriate weak sub-solution and super-solution, we obtained the sufficient conditions for the extinction and non-extinction behaviors of the weak solution.

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Conflict of interest

The authors declare no conflict of interest in this article.

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