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Research article

On sequential fractional Caputo (p, q)-integrodifference equations via three-point fractional Riemann-Liouville (p, q)-difference boundary condition

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Abstract: In this paper, we aim to study the problem of a sequential fractional Caputo (p, q)-integrodifference equation with three-point fractional Riemann-Liouville (p, q)-difference boundary condition. We use some properties of (p, q)-integral in this study and employ Banach fixed point theorems and Schauder's fixed point theorems to prove existence results of this problem.

Keywords: fractional (p, q)-integral; fractional (p, q)-difference; three-point boundary value problems; existence
Mathematics Subject Classification: 39A10, 39A13, 39A70

1. Introduction

The q-calculus is one type of quantum calculus that was proposed by Jackson [1, 2] in 1910. This calculus has been employed in several fields of many fields such as engineering, electrical networks, dynamical system, control theory, physical problems, economics, applied sciences and so on [3–14].

Later, the development of quantum calculus based on two parameters p and q was proposed by Chakrabarti and Jagannathan [15]. This calculus is called (p, q)-calculus or post quantum calculus. The extension of studies of (p, q)-calculus including with its applications can be found in [16–29]. For instance, the fundamental theorems of (p, q)- calculus and some (p, q)-Taylor formulas were studied in [18]. In [25], the (p, q)-Melin transform and its applications were studied. The Picard and Gauss-Weierstrass singular integral in (p, q)-calculus were studied in [26]. For the boundary value problems for (p, q)-difference equations were introduced in [27–29].

For fractional quantum calculus, Agarwal [30] and Al-Salam [31] proposed fractional q-calculus, and Díaz and Osler [32] proposed fractional difference calculus. In 2017, Brikshavana and Sitthiwirattham [33] introduced fractional Hahn difference calculus. In 2019, Patanarapeelert and Sitthiwirattham [34] studied fractional symmetric Hahn difference calculus. For the basic theory and applications of fractional calculus, as well as for some recent developments in the field, we refer to [35–41] and the references cited therein.

Recently, Soontharanon and Sitthiwirattham [42] introduced the operators of fractional (p,q)-calculus and its properties. However, there are a few literature on the study of the boundary value problems for fractional (p,q)-difference equation since fractional (p,q)-operators have been defined lately. Now, this calculus was used in the inequalities [43, 44] and the boundary value problems [45–47]. In 2020, the existence results of a fractional (p,q)-integrodifference equation with Robin boundary condition were first introduced [45]. The existence results of solution and positive solution for the boundary value problem of a class of fractional (p,q)-difference equations involving the Riemann–Liouville fractional derivative [46] were studied in 2021. In the same year, the authors investigated the boundary value problem of a class of fractional (p,q)-difference Schrödinger equations [47].

Motivated by the above papers, to enrich the work in this new research area, in this paper, we study the boundary value problem for fractional (p,q)-difference of Caputo type involving function F which depends on fractional (p,q)-integral and fractional (p,q)-difference of Riemann-Liouville type, and the boundary condition is nonlocal. Our problem is a sequential fractional Caputo (p,q)-integrodifference equation with three-point fractional Riemann-Liouville (p,q)-difference boundary conditions of the form

$$
\begin{aligned}
&{}^{C}D^{\alpha}_{p,q}\left[{}^{C}D^{\beta}_{p,q}\left(1+\rho\left(t\right)\right)\right]u(t) = F\left[t,u(t),\left(\Psi^{\gamma}_{p,q}u\right)(t),\left(\Upsilon^{\nu}_{p,q}u\right)(t)\right], \quad t\in I^{T}_{p,q},\\
&u(0)=\phi(u),\\
&u\left(\frac{T}{p}\right) = D^{\theta}_{p,q}g(\eta)u(\eta), \quad \eta\in I^{T}_{p,q}-\left\{0,\frac{T}{p}\right\}
\end{aligned}
\tag{1.1}
$$

where $\alpha,\beta,\gamma,\nu,\theta\in(0,1]$; $\rho, g\in C\left(I^{T}_{p,q},\mathbb{R}^{+}\right)$ and $F\in C\left(I^{T}_{p,q}\times\mathbb{R}^{3},\mathbb{R}\right)$ are given functions; ϕ is given functional; and for $\varphi,\psi\in C(I^{T}_{p,q}\times I^{T}_{p,q},[0,\infty))$, we define operators

$$
\left(\Psi^{\gamma}_{p,q}u\right)(t) := \left(\mathcal{I}^{\gamma}_{p,q}\varphi\, u\right)(t) = \frac{1}{p^{\binom{\gamma}{2}}\Gamma_{p,q}(\gamma)}\int_0^t (t-qs)^{\underline{\gamma-1}}_{p,q}\varphi\left(t,\frac{s}{p^{\gamma-1}}\right)u\left(\frac{s}{p^{\gamma-1}}\right)d_{p,q}s, \tag{1.2}
$$

$$
\left(\Upsilon^{\nu}_{p,q}u\right)(t) := \left(D^{\nu}_{p,q}\psi\, u\right)(t) = \frac{1}{p^{\binom{-\nu}{2}}\Gamma_{p,q}(-\nu)}\int_0^t (t-qs)^{\underline{-\nu-1}}_{p,q}\psi\left(t,\frac{s}{p^{-\nu-1}}\right)u\left(\frac{s}{p^{-\nu-1}}\right)d_{p,q}s. \tag{1.3}
$$

We emphasize that our problem contains fractional (p,q)-difference operators of Riemann-Liouville and Caputo types, which is the new idea and different from the previous works. We aim to show the existence and uniqueness of a solution to the problem (1.1) by using the Banach fixed point theorem. The existence of at least one solution by using the Schauder's fixed point theorem. An example will be provide to illustrate our results.

2. Preliminaries

In this section, we recall some basic definitions, notations and lemmas as follows. For $0 < q < p \leq 1$

$$[k]_q := \begin{cases} \dfrac{1-q^k}{1-q}, & k \in \mathbb{N} \\ 1, & k = 0, \end{cases}$$

$$[k]_{p,q} := \begin{cases} \dfrac{p^k-q^k}{p-q} = p^{k-1}[k]_{\frac{q}{p}}, & k \in \mathbb{N} \\ 1, & k = 0, \end{cases}$$

$$[k]_{p,q}! := \begin{cases} [k]_{p,q}[k-1]_{p,q}\cdots[1]_{p,q} = \displaystyle\prod_{i=1}^{k} \frac{p^i-q^i}{p-q}, & k \in \mathbb{N} \\ 1, & k = 0. \end{cases}$$

The (p,q)-forward jump and the (p,q)-backward jump operators are defined as

$$\sigma^k_{p,q}(t) := \left(\tfrac{q}{p}\right)^k t \quad \text{and} \quad \rho^k_{p,q}(t) := \left(\tfrac{p}{q}\right)^k t, \quad \text{for } k \in \mathbb{N}, \text{ respectively.}$$

The q-analogue of the power function $(a-b)_q^{\underline{n}}$ where $n \in \mathbb{N}_0 := \{0, 1, 2, ...\}$ is given by

$$(a-b)_q^{\underline{0}} := 1, \qquad (a-b)_q^{\underline{n}} := \prod_{i=0}^{n-1}(a-bq^i), \qquad a, b \in \mathbb{R}.$$

The (p,q)-analogue of the power function $(a-b)_{p,q}^{\underline{n}}$ where $n \in \mathbb{N}_0$ is given by

$$(a-b)_{p,q}^{\underline{0}} := 1, \qquad (a-b)_{p,q}^{\underline{n}} := \prod_{k=0}^{n-1}(ap^k-bq^k), \qquad a, b \in \mathbb{R}.$$

Generally, for $\alpha \in \mathbb{R}$,

$$(a-b)_q^{\underline{\alpha}} = a^\alpha \prod_{i=0}^{\infty} \frac{1-\left(\frac{b}{a}\right)q^i}{1-\left(\frac{b}{a}\right)q^{\alpha+i}}, \ a \neq 0.$$

$$(a-b)_{p,q}^{\underline{\alpha}} = p^{\binom{\alpha}{2}}(a-b)_{\frac{q}{p}}^{\underline{\alpha}} = a^\alpha \prod_{i=0}^{\infty} \frac{1}{p^\alpha}\left[\frac{1-\frac{b}{a}\left(\frac{q}{p}\right)^i}{1-\frac{b}{a}\left(\frac{q}{p}\right)^{i+\alpha}}\right], \ a \neq 0,$$

and $a_q^{\underline{\alpha}} = a^\alpha$, $a_{p,q}^{\underline{\alpha}} = \left(\frac{a}{p}\right)^\alpha$ and $(0)_q^{\underline{\alpha}} = (0)_{p,q}^{\underline{\alpha}} = 0$ for $\alpha > 0$.
The (p,q)-gamma and (p,q)-beta functions are defined by

$$\Gamma_{p,q}(x) := \begin{cases} \frac{(p-q)_{p,q}^{\underline{x-1}}}{(p-q)^{x-1}} = \frac{\left(1-\frac{q}{p}\right)_{p,q}^{\underline{x-1}}}{\left(1-\frac{q}{p}\right)^{x-1}}, & x \in \mathbb{R} \setminus \{0, -1, -2, ...\} \\ [x-1]_{p,q}!, & x \in \mathbb{N}, \end{cases}$$

$$B_{p,q}(x,y) := \int_0^1 t^{x-1}(1-qt)_{p,q}^{\underline{y-1}} d_{p,q}t = p^{\frac{1}{2}(y-1)(2x+y-2)} \frac{\Gamma_{p,q}(x)\Gamma_q p, q(y)}{\Gamma_{p,q}(x+y)},$$

respectively.

We next provide the definitions of the (p,q)-difference and (p,q)-integral as follows.

Definition 2.1. *For* $0<q<p\leq 1$ *and* $f:[0,T]\to\mathbb{R}$*, the* (p,q)*-difference of* f *is defined as*

$$D_{p,q}f(t) \quad := \quad \begin{cases} \dfrac{f(pt)-f(qt)}{(p-q)(t)}, & for\ \ t\neq 0 \\ f'(0), & for\ \ t=0 \end{cases}$$

provided that f is differentiable at 0 *and* f *is called* (p,q)*-differentiable on* $I^T_{p,q}$ *if* $D_{p,q}f(t)$ *exists for all* $t\in I^T_{p,q}$ *.*

Definition 2.2. *Let I be any closed interval of* $\mathbb{R}$ *containing* a,b *and* 0*. Assuming that* $f:I\to\mathbb{R}$ *is a given function,* (p,q)*-integral of f from a to b is defined by*

$$\int_a^b f(t)d_{p,q}t := \int_0^b f(t)d_{p,q}t - \int_0^a f(t)d_{p,q}t,$$

where

$$\mathcal{I}_{p,q}f(x) = \int_0^x f(t)d_{p,q}t = (p-q)x\sum_{k=0}^{\infty}\frac{q^k}{p^{k+1}}f\left(\frac{q^k}{p^{k+1}}x\right), \quad x\in I,$$

provided that the series converges at $x=a$ *and* $x=b$*, and* f *is called* (p,q)*-integrable on* $[a,b]$ *if it is* (p,q)*-integrable on* $[a,b]$ *for all* $a,b\in I$*.*

Next, operator $\mathcal{I}^N_{p,q}$ where $N\in\mathbb{N}$ is defined as

$$\mathcal{I}^0_{p,q}f(x) = f(x) \ \text{ and } \ \mathcal{I}^N_{p,q}f(x) = \mathcal{I}_{p,q}\mathcal{I}^{N-1}_{p,q}f(x), N\in\mathbb{N}.$$

The relations between (p,q)-difference and (p,q)-integral are given by

$$D_{p,q}\mathcal{I}_{p,q}f(x) = f(x) \ \text{ and } \ \mathcal{I}_{p,q}D_{p,q}f(x) = f(x)-f(0).$$

The fractioanal (p,q)-integral, fractional (p,q)-difference of Riemann-Liouville type and Caputo type are defined as follows.

Definition 2.3. *For* $\alpha>0$, $0<q<p\leq 1$ *and* f *defined on* $I^T_{p,q}$*, the fractional* (p,q)*-integral is defined by*

$$\begin{aligned} \mathcal{I}^{\alpha}_{p,q}f(t) \quad &:= \quad \frac{1}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)}\int_0^t (t-qs)^{\underline{\alpha-1}}_{p,q} f\left(\frac{s}{p^{\alpha-1}}\right)d_{p,q}s \\ &= \quad \frac{(p-q)t}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)}\sum_{k=0}^{\infty}\frac{q^k}{p^{k+1}}\left(t-\left(\frac{q}{p}\right)^{k+1}t\right)^{\underline{\alpha-1}}_{p,q} f\left(\frac{q^k}{p^{k+\alpha}}t\right), \end{aligned}$$

and $(\mathcal{I}^0_{p,q}f)(t) = f(t)$*.*

Definition 2.4. *For $\alpha > 0,\ 0 < q < p \leq 1$ and f defined on $I^T_{p,q}$, the fractional (p,q)-difference operator of Riemann-Liouville type of order α is defined by*

$$\begin{aligned} D^{\alpha}_{p,q}f(t) &:= D^N_{p,q}\mathcal{I}^{N-\alpha}_{p,q}f(t) \\ &= \frac{1}{p^{\binom{-\alpha}{2}}\Gamma_{p,q}(-\alpha)}\int_0^t (t-qs)^{\underline{-\alpha-1}}_{p,q} f\left(\frac{s}{p^{-\alpha-1}}\right) d_{p,q}s, \end{aligned}$$

and $D^0_{p,q}f(t) = f(t)$, where $N-1 < \alpha < N,\ N \in \mathbb{N}$.

Definition 2.5. *For $\alpha > 0,\ 0 < q < p \leq 1$ and f defined on $I^T_{p,q}$, the fractional (p,q)-difference operator of Caputo type of order α is defined by*

$$\begin{aligned} {}^C D^{\alpha}_{p,q}f(t) &:= \mathcal{I}^{N-\alpha}_{p,q}D^N_{p,q}f(t) \\ &= \frac{1}{p^{\binom{N-\alpha}{2}}\Gamma_{p,q}(N-\alpha)}\int_0^t (t-qs)^{\underline{N-\alpha-1}}_{p,q} D^N_{p,q}f\left(\frac{s}{p^{N-\alpha-1}}\right) d_{q,\omega}s, \end{aligned}$$

and ${}^C D^0_{p,q}f(t) = f(t)$, where $N-1 < \alpha < N,\ N \in \mathbb{N}$.

Next, we introduce lemmas that are used in the main results.

Lemma 2.1. *[42] Let $\alpha \in (N-1, N),\ N \in \mathbb{N},\ 0 < q < p \leq 1$ and $f : I^T_{p,q} \to \mathbb{R}$. Then,*

$$\mathcal{I}^{\alpha}_{p,q}\,{}^C D^{\alpha}_{p,q}f(t) = f(t) + C_1 t^{N-1} + C_2 t^{N-2} + ... + C_N,$$

for some $C_i \in \mathbb{R}, i = 1, 2, ..., N$.

Lemma 2.2. *[42] Let $0 < q < p \leq 1$ and $f : I^T_{p,q} \to \mathbb{R}$ be continuous at 0. Then,*

$$\int_0^x \int_0^s f(\tau)\, d_{p,q}\tau\, d_{p,q}s = \int_0^{\frac{x}{p}} \int_{pq\tau}^x f(\tau)\, d_{p,q}s\, d_{p,q}\tau.$$

Lemma 2.3. *[42] Let $\alpha, \beta > 0,\ 0 < q < p \leq 1$. Then,*

$$\begin{aligned} (a) &\quad \int_0^t (t-qs)^{\underline{\alpha-1}}_{p,q} s^{\beta}\, d_{p,q}s = t^{\alpha+\beta}B_{p,q}(\beta+1, \alpha), \\ (b) &\quad \int_0^t \int_0^x (t-qx)^{\underline{\alpha-1}}_{p,q}(x-qs)^{\underline{\beta-1}}_{p,q}\, d_{p,q}s\, d_{p,q}x = \frac{B_{p,q}(\beta+1,\alpha)}{[\beta]_{p,q}} t^{\alpha+\beta}. \end{aligned}$$

Lemma 2.4. *[45] Let $\alpha, \beta > 0,\ 0 < q < p \leq 1$ and $n \in \mathbb{Z}$. Then,*

$$\begin{aligned} (a) &\quad \int_0^t (t-qs)^{\underline{\alpha-1}}_{p,q}\, d_{p,q}s = p^{\binom{\alpha}{2}} \frac{\Gamma_{p,q}(\alpha)}{\Gamma_{p,q}(\alpha+1)}\, t^{\alpha}, \\ (b) &\quad \int_0^t \int_0^{\frac{x}{p^{-\beta-1}}} (t-qx)^{\underline{-\beta-1}}_{p,q} \left(\frac{x}{p^{-\beta-1}} - qs\right)^{\underline{\alpha-1}}_{p,q} d_{p,q}s\, d_{p,q}x = p^{\binom{\alpha}{2}+\binom{-\beta}{2}} \frac{\Gamma_{p,q}(\alpha)}{\Gamma_{p,q}(\alpha+1)}\, t^{\alpha+\beta}, \\ (c) &\quad \int_0^t (t-qs)^{\underline{-\beta-1}}_{p,q} \left(\frac{s}{p^{-\beta-1}}\right)^{\alpha-n} d_{p,q}s = p^{\binom{-\beta}{2}} \frac{\Gamma_{p,q}(\alpha-n+1)\Gamma_{p,q}(-\beta)}{\Gamma_{p,q}(\alpha-\beta-n+1)}\, t^{\alpha-\beta-n}. \end{aligned}$$

We employ above lemmas to get the new results as follows.

Lemma 2.5. *Let* $\alpha,\beta,\theta > 0$ *and* $0 < q < p \le 1$. *Then,*

$$\int_0^t \int_0^{\frac{y}{p^{-\theta-1}}} \int_0^{\frac{x}{p^{\beta-1}}} (t-qy)_{p,q}^{\underline{-\theta-1}} \left(\tfrac{y}{p^{-\theta-1}} - qx\right)^{\underline{-\beta-1}} \left(\tfrac{x}{p^{\beta-1}} - qs\right)_{p,q}^{\underline{\alpha-1}} d_{p,q}s\, d_{p,q}x\, d_{p,q}y = p^{\binom{\alpha}{2}+\binom{\beta}{2}+\binom{-\theta}{2}} \frac{\Gamma_{p,q}(\alpha)\Gamma_{p,q}(\beta)\Gamma_{p,q}(-\theta)}{\Gamma_{p,q}(\alpha+\beta-\theta+1)} t^{\alpha+\beta-\theta}.$$

Proof. Using Lemma 2.3(a) and definition of the (p,q)-beta function, we obtain

$$\begin{aligned}
&\int_0^t \int_0^{\frac{y}{p^{-\theta-1}}} \int_0^{\frac{x}{p^{\beta-1}}} (t-qy)_{p,q}^{\underline{-\theta-1}} \left(\frac{y}{p^{-\theta-1}} - qx\right)^{\underline{-\beta-1}} \left(\frac{x}{p^{\beta-1}} - qs\right)_{p,q}^{\underline{\alpha-1}} d_{p,q}s\, d_{p,q}x\, d_{p,q}y \\
= &\int_0^t \int_0^{\frac{y}{p^{-\theta-1}}} (t-qy)_{p,q}^{\underline{-\theta-1}} \left(\frac{y}{p^{-\theta-1}} - qx\right)_{p,q}^{\underline{\beta-1}} \left[\int_0^{\frac{x}{p^{\beta-1}}} \left(\frac{x}{p^{\beta-1}} - qs\right)_{p,q}^{\underline{\alpha-1}} d_{p,q}s\right] d_{p,q}x\, d_{p,q}y \\
= &\ p^{\binom{\alpha}{2}} \frac{\Gamma_{p,q}(\alpha)}{\Gamma_{p,q}(\alpha+1)} \int_0^t \int_0^{\frac{y}{p^{-\theta-1}}} (t-qy)_{p,q}^{\underline{-\theta-1}} \left(\frac{y}{p^{-\theta-1}} - qx\right)_{p,q}^{\underline{\beta-1}} \left(\frac{x}{p^{\beta-1}}\right)^{\alpha} d_{p,q}x\, d_{p,q}y \\
= &\ p^{\binom{\alpha}{2}+\binom{\beta}{2}+\binom{-\theta}{2}} \frac{\Gamma_{p,q}(\alpha)\Gamma_{p,q}(\beta)\Gamma_{p,q}(-\theta)}{\Gamma_{p,q}(\alpha+\beta-\theta+1)} t^{\alpha+\beta-\theta}.
\end{aligned}$$

The proof is complete. □

The following lemma is a solution of the linear variant of problem (1.1), plays an important role in the upcoming analysis.

Lemma 2.6. *Let* $\boldsymbol{B} \neq 0$, $0 < q < p \le 1$, $\alpha,\beta,\theta \in (0,1]$, $h \in C(I_{p,q}^T, \mathbb{R})$ *and* $\rho, g \in C(I_{p,q}^T, \mathbb{R}^+)$ *be given functions,* $\phi : C(I_{p,q}^T, \mathbb{R}) \to \mathbb{R}$ *be given functional. Then the problem*

$$ {}^{C}D_{p,q}^{\alpha}\left[{}^{C}D_{p,q}^{\beta}\,(1+\rho(t))\right]u(t) = h(t), \quad t \in I_{p,q}^T \tag{2.1}$$

$$u(0) = \phi(u) \tag{2.2}$$

$$u\left(\frac{T}{p}\right) = D_{p,q}^{\theta} g(\eta)u(\eta), \quad \eta \in I_{p,q}^T - \left\{0, \frac{T}{p}\right\} \tag{2.3}$$

has the unique solution:

$$\begin{aligned}
u(t) = &\ \phi(u)\left[\frac{1+\rho(0)}{1+\rho(t)}\right]\left[1 - \frac{\boldsymbol{A}t^{\beta}}{\boldsymbol{B}\Gamma_{p,q}(\beta+1)}\right] + \left[\frac{\mathbb{O}_{\eta}[h] - \mathbb{O}_{T}[h]}{1+\rho(t)}\right]\frac{t^{\beta}}{\boldsymbol{B}\Gamma_{p,q}(\beta+1)} \\
&+ \frac{1}{(1+\rho(t))\, p^{\binom{\alpha}{2}+\binom{\beta}{2}}\Gamma_{p,q}(\alpha)\Gamma_{p,q}(\beta)} \\
&\times \int_0^t \int_0^{\frac{x}{p^{\beta-1}}} (t-qx)_{p,q}^{\underline{\beta-1}} \left(\frac{x}{p^{\beta-1}} - qs\right)_{p,q}^{\underline{\alpha-1}} h\left(\frac{s}{p^{\alpha-1}}\right) d_{p,q}s\, d_{p,q}x,
\end{aligned} \tag{2.4}$$

where the functionals $\mathbb{O}_{\eta}[h]$ *and* $\mathbb{O}_{T}[h]$ *are defined by*

$$\mathbb{O}_{\eta}[h] := \frac{1}{p^{\binom{\alpha}{2}+\binom{\beta}{2}+\binom{-\theta}{2}}\Gamma_{p,q}(\alpha)\Gamma_{p,q}(\beta)\Gamma_{p,q}(-\theta)}$$

$$\times \int_0^{\eta} \int_0^{\frac{y}{p^{-\theta-1}}} \int_0^{\frac{x}{p^{\beta-1}}} (\eta - qy)_{p,q}^{\underline{-\theta-1}} \frac{\left(\frac{y}{p^{-\theta-1}} - qx\right)_{p,q}^{\underline{\beta-1}}}{1+\rho\left(\frac{y}{p^{-\theta-1}}\right)} \left(\frac{x}{p^{\beta-1}} - qs\right)_{p,q}^{\underline{\alpha-1}}$$

$$\times g\left(\frac{x}{p^{-\theta-1}}\right) h\left(\frac{s}{p^{\alpha-1}}\right) d_{p,q}s\, d_{p,q}x\, d_{p,q}y, \tag{2.5}$$

$$\mathbb{O}_T[h] := \frac{1}{\left(1+\rho\left(\frac{T}{p}\right)\right) p^{\binom{\alpha}{2}+\binom{\beta}{2}} \Gamma_{p,q}(\alpha)\Gamma_{p,q}(\beta)}$$

$$\times \int_0^{\frac{T}{p}} \int_0^{\frac{x}{p^{\beta-1}}} \left(\frac{T}{p} - qx\right)_{p,q}^{\underline{\beta-1}} \left(\frac{x}{p^{\beta-1}} - qs\right)_{p,q}^{\underline{\alpha-1}} h\left(\frac{s}{p^{\alpha-1}}\right) d_{p,q}s\, d_{p,q}x, \tag{2.6}$$

respectively, and the constants $\boldsymbol{A}, \boldsymbol{B}$ *are defined by*

$$\boldsymbol{A} := \frac{1}{1+\rho\left(\frac{T}{p}\right)} - \frac{1}{p^{\binom{-\theta}{2}}\Gamma_{p,q}(-\theta)} \int_0^{\eta} (\eta - qs)_{p,q}^{\underline{-\theta-1}} \frac{g\left(\frac{s}{p^{-\theta-1}}\right)}{1+\rho\left(\frac{s}{p^{-\theta-1}}\right)} d_{p,q}s \tag{2.7}$$

$$\boldsymbol{B} := \frac{\left(\frac{T}{p}\right)^{\beta}}{\left(1+\rho\left(\frac{T}{p}\right)\right)\Gamma_{p,q}(\beta+1)} - \frac{1}{p^{\binom{-\theta}{2}}\Gamma_{p,q}(-\theta)\Gamma_{p,q}(\beta+1)} \times$$

$$\int_0^{\eta} (\eta - qs)_{p,q}^{\underline{-\theta-1}} \frac{g\left(\frac{s}{p^{-\theta-1}}\right)}{1+\rho\left(\frac{s}{p^{-\theta-1}}\right)} \left(\frac{s}{p^{-\theta-1}}\right)^{\beta} d_{p,q}s, \tag{2.8}$$

respectively.

Proof. Take fractional (p, q)-integral of order α for (2.1). Then, we have

$$\left[{}^{C}D_{p,q}^{\beta}\left(1+\rho(t)\right)\right] u(t) = C_1 + \mathcal{I}_{p,q}^{\alpha} h(t)$$

$$= C_1 + \frac{1}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} \int_0^t (t - qs)_{p,q}^{\underline{\alpha-1}} h\left(\frac{s}{p^{\alpha-1}}\right) d_{p,q}s. \tag{2.9}$$

Taking fractional (p, q)-integral of order β for (2.9), we obtain

$$(1+\rho(t))\, u(t) = C_0 + C_1 \mathcal{I}_{p,q}^{\beta}(1) + \mathcal{I}_{p,q}^{\beta}\mathcal{I}_{p,q}^{\alpha} h(t), \tag{2.10}$$

Therefore,

$$u(t) = \frac{C_0}{1+\rho(t)} + \frac{C_1}{(1+\rho(t))\, p^{\binom{\beta}{2}}\Gamma_{p,q}(\beta)} \int_0^t (t - qs)_{p,q}^{\underline{\beta-1}} d_{p,q}s$$

$$+ \frac{1}{(1+\rho(t))\, p^{\binom{\alpha}{2}+\binom{\beta}{2}}\Gamma_{p,q}(\alpha)\Gamma_{p,q}(\beta)}$$

$$\times \int_0^t \int_0^{\frac{x}{p^{\beta-1}}} (t - qx)_{p,q}^{\underline{\beta-1}} \left(\frac{x}{p^{\beta-1}} - qs\right)_{p,q}^{\underline{\alpha-1}} h\left(\frac{s}{p^{\alpha-1}}\right) d_{p,q}s\, d_{p,q}x. \tag{2.11}$$

Substituting the condition (2.2) to (2.11), we obtain

$$C_0 = (1+\rho(0))\, \phi(u). \tag{2.12}$$

Taking fractional Riemann-Liouville (p,q)-difference of order θ for (2.11), we get

$$\begin{aligned}
D^{\theta}_{p,q}u(t) \ &= \frac{C_0}{p^{\binom{-\theta}{2}}\Gamma_{p,q}(-\theta)}\int_0^t (t-qs)_{p,q}^{\underline{-\theta-1}}\frac{1}{1+\rho\left(\frac{s}{p^{-\theta-1}}\right)}d_{p,q}s \\
&+ \frac{C_1}{p^{\binom{-\theta}{2}}\Gamma_{p,q}(-\theta)\Gamma_{p,q}(\beta+1)}\int_0^t (t-qs)_{p,q}^{\underline{-\theta-1}}\frac{\left(\frac{s}{p^{-\theta-1}}\right)^{\beta}}{1+\rho\left(\frac{s}{p^{-\theta-1}}\right)}d_{p,q}s \\
&+ \frac{1}{p^{\binom{\alpha}{2}+\binom{\beta}{2}+\binom{-\theta}{2}}\Gamma_{p,q}(\alpha)\Gamma_{p,q}(\beta)\Gamma_{p,q}(-\theta)} \\
&\times \int_0^t\int_0^{\frac{y}{p^{-\theta-1}}}\int_0^{\frac{x}{p^{\beta-1}}}(t-qy)_{p,q}^{\underline{-\theta-1}}\frac{\left(\frac{y}{p^{-\theta-1}}-qx\right)_{p,q}^{\underline{\beta-1}}}{1+\rho\left(\frac{y}{p^{-\theta-1}}\right)}\left(\frac{x}{p^{\beta-1}}-qs\right)_{p,q}^{\underline{\alpha-1}} \\
&\times h\left(\frac{s}{p^{\alpha-1}}\right)d_{p,q}s\,d_{p,q}x\,d_{p,q}y. \qquad (2.13)
\end{aligned}$$

From the condition (2.3) and by (2.12), we obtain

$$C_1 = \frac{\mathbb{O}_{\eta}[h]-\mathbb{O}_T[h]-\phi(u)\left(1+\rho(0)\right)\mathbf{A}}{\mathbf{B}}. \qquad (2.14)$$

After substituting C_0, C_1 into (2.11), we obtain (2.4). The converse can be proved by direct computation. Our proof is complete. □

3. Existence and uniqueness result

In this section, we aim to prove the existence and uniqueness result for problem (1.1) by using the Banach fixed point theorem. Let $C = C\left(I^T_{p,q},\mathbb{R}\right)$ be a Banach space of all function u. The norm defined by

$$\|u\|_C = \|u\| + \left\|D^{\nu}_{p,q}u\right\|,$$

where $\|u\| = \max_{t\in I^T_{p,q}}\left\{|u(t)|\right\}$ and $\|D^{\nu}_{p,q}u\| = \max_{t\in I^T_{p,q}}\left\{\left|D^{\nu}_{p,q}u(t)\right|\right\}$.

By Lemma 2.6, replacing $h(t)$ by $F\left[t,u(t),\left(\Psi^{\gamma}_{p,q}u\right)(t),\left(\Upsilon^{\nu}_{p,q}u\right)(t)\right]$, we define an operator $\mathcal{F}: C\to C$ is defined by

$$\begin{aligned}
(\mathcal{F}u)(t) \ &:= \phi(u)\left[\frac{1+\rho(0)}{1+\rho(t)}\right]\left[1-\frac{\mathbf{A}t^{\beta}}{\mathbf{B}\Gamma_{p,q}(\beta+1)}\right]+\left[\frac{\mathbb{O}^*_{\eta}[F,u]-\mathbb{O}^*_T[F,u]}{1+\rho(t)}\right]\frac{t^{\beta}}{\mathbf{B}\Gamma_{p,q}(\beta+1)} \\
&+ \frac{1}{(1+\rho(t))\,p^{\binom{\alpha}{2}+\binom{\beta}{2}}\Gamma_{p,q}(\alpha)\Gamma_{p,q}(\beta)}\int_0^t\int_0^{\frac{x}{p^{\beta-1}}}(t-qx)_{p,q}^{\underline{\beta-1}}\left(\frac{x}{p^{\beta-1}}-qs\right)_{p,q}^{\underline{\alpha-1}} \\
&\times F\left[\frac{s}{p^{\alpha-1}},u\left(\frac{s}{p^{\alpha-1}}\right),\left(\Psi^{\gamma}_{p,q}u\right)\left(\frac{s}{p^{\alpha-1}}\right),\left(\Upsilon^{\nu}_{p,q}u\right)\left(\frac{s}{p^{\alpha-1}}\right)\right]d_{p,q}s\,d_{p,q}x \qquad (3.1)
\end{aligned}$$

where the functionals $\mathbb{O}^*_{\eta}[F,u]$ and $\mathbb{O}^*_T[F,u]$ are defined by

$$\mathbb{O}^*_{\eta}[F,u] \ := \ \frac{1}{p^{\binom{\alpha}{2}+\binom{\beta}{2}+\binom{-\theta}{2}}\Gamma_{p,q}(\alpha)\Gamma_{p,q}(\beta)\Gamma_{p,q}(-\theta)} \qquad (3.2)$$

$$\times \int_0^{\eta} \int_0^{\frac{y}{p^{-\theta-1}}} \int_0^{\frac{x}{p^{\beta-1}}} (\eta - qy)_{p,q}^{\underline{-\theta-1}} \frac{\left(\frac{y}{p^{-\theta-1}} - qx\right)_{p,q}^{\underline{\beta-1}}}{1 + \rho\left(\frac{y}{p^{-\theta-1}}\right)} \left(\frac{x}{p^{\beta-1}} - qs\right)_{p,q}^{\underline{\alpha-1}} g\left(\frac{x}{p^{\beta-1}}\right)$$

$$\times F\left[\frac{s}{p^{\alpha-1}}, u\left(\frac{s}{p^{\alpha-1}}\right), \left(\Psi_{p,q}^{\gamma} u\right)\left(\frac{s}{p^{\alpha-1}}\right), \left(\Upsilon_{p,q}^{\nu} u\right)\left(\frac{s}{p^{\alpha-1}}\right)\right] d_{p,q}s\, d_{p,q}x\, d_{p,q}y$$

$$\mathbb{O}_T^*[F, u] := \frac{1}{\left(1 + \rho\left(\frac{T}{p}\right)\right) p^{\binom{\alpha}{2}+\binom{\beta}{2}} \Gamma_{p,q}(\alpha)\Gamma_{p,q}(\beta)} \tag{3.3}$$

$$\times \int_0^{\frac{T}{p}} \int_0^{\frac{x}{p^{\beta-1}}} \left(\frac{T}{p} - qx\right)_{p,q}^{\underline{\beta-1}} \left(\frac{x}{p^{\beta-1}} - qs\right)_{p,q}^{\underline{\alpha-1}}$$

$$\times F\left[\frac{s}{p^{\alpha-1}}, u\left(\frac{s}{p^{\alpha-1}}\right), \left(\Psi_{p,q}^{\gamma} u\right)\left(\frac{s}{p^{\alpha-1}}\right), \left(\Upsilon_{p,q}^{\nu} u\right)\left(\frac{s}{p^{\alpha-1}}\right)\right] d_{p,q}s\, d_{p,q}x$$

and **A** and **B** are defined by (2.7) and (2.8), respectively.

Since the problem (1.1) has solution if and only if the operator $\mathcal{F}$ has fixed point, we next prove that $\mathcal{F}$ has fixed point as shawn in the following theorem.

Theorem 3.1. *Assume that* $F : I_{p,q}^T \times \mathbb{R}^3 \to \mathbb{R}$ *and* $\rho, g : I_{p,q}^T \to \mathbb{R}^+$ *are continuous,* $\varphi, \psi : I_{p,q}^T \times I_{p,q}^T \to [0, \infty)$ *are continuous with* $\varphi_0 = \max\left\{\varphi(t, s) : (t, s) \in I_{p,q}^T \times I_{p,q}^T\right\}$ *and* $\psi_0 = \max\left\{\psi(t, s) : (t, s) \in I_{p,q}^T \times I_{p,q}^T\right\}$. *Suppose that the following conditions hold:*

(H_1) *There exist positive constants* L_i *such that for each* $t \in I_{p,q}^T$ *and* $u_i, v_i \in \mathbb{R},\ i = 1, 2, 3$

$$\left|F[t, u_1, u_2, u_3] - F[t, v_1, v_2, v_3]\right| \le L_1\left|u_1 - v_1\right| + L_2\left|u_2 - v_2\right| + L_3\left|u_3 - v_3\right|.$$

(H_2) *There exists a positive constant* ω *such that for each* $u, v \in \mathcal{C}$

$$|\phi(u) - \phi(v)| \le \omega \|u - v\|_{\mathcal{C}}.$$

(H_3) *For each* $t \in I_{p,q}^T$, $0 < \rho < \rho(t) < P$.
(H_4) *For each* $t \in I_{p,q}^T$, $0 < g < g(t) < G$.
(H_5) $\mathcal{X} := [\omega\mathcal{P} + \mathcal{L}\mathcal{Q}]\left(1 + \frac{\left(\frac{T}{p}\right)^{-\nu}}{\Gamma_{p,q}(1-\nu)}\right) + \mathcal{L}\left(\frac{\left(\frac{T}{p}\right)^{\alpha+\beta}}{\Gamma_{p,q}(\alpha+\beta+1)} + \frac{\left(\frac{T}{p}\right)^{\alpha+\beta-\nu}}{\Gamma_{p,q}(\alpha+\beta-\nu+1)}\right) < 1$

where

$$\mathcal{L} := L_1 + L_2\frac{\varphi_0\left(\frac{T}{p}\right)^{\gamma}}{\Gamma_{p,q}(\gamma + 1)} + L_3\frac{\varphi_0\left(\frac{T}{p}\right)^{-\nu}}{\Gamma_{p,q}(1 - \gamma)}, \tag{3.4}$$

$$\mathcal{P} := \left[\frac{1 + P}{1 + \rho}\right]\left[1 + \frac{\max|\boldsymbol{A}|\left(\frac{T}{p}\right)^{\beta}}{\min|\boldsymbol{B}|\,\Gamma_{p,q}(\beta + 1)}\right], \tag{3.5}$$

$$\mathcal{Q} := \frac{\left(\frac{T}{p}\right)^{\beta}}{(1 + \rho)\min|\boldsymbol{B}|\,\Gamma_{p,q}(\beta + 1)}$$

$$\times\left[\frac{G\eta^{\alpha+\beta-\theta}}{(1+\rho)\Gamma_{p,q}(\alpha+\beta-\theta+1)}+\frac{\left(\frac{T}{p}\right)^{\alpha+\beta}}{(1+\rho)\Gamma_{p,q}(\alpha+\beta+1)}\right]. \tag{3.6}$$

Then, problem (1.1) *has a unique solution in* $I^T_{p,q}$*.*

Proof. For each $t\in I^T_{p,q}$ and $u,v\in C$, we have

$$\begin{aligned}
\left|\left(\Psi^{\gamma}_{p,q}u\right)(t)-\left(\Psi^{\gamma}_{p,q}v\right)(t)\right| &\le \frac{\varphi_0}{p^{\binom{\gamma}{2}}\Gamma_{p,q}(\gamma)}\int_0^t (t-qs)^{\underline{\gamma-1}}_{p,q}\left|u\left(\frac{s}{p^{\gamma-1}}\right)-v\left(\frac{s}{p^{\gamma-1}}\right)\right| d_{p,q}s\\
&\le \frac{\varphi_0}{p^{\binom{\gamma}{2}}\Gamma_{p,q}(\gamma)}\|u-v\|\int_0^{\frac{T}{p}}\left(\frac{T}{p}-qs\right)^{\underline{\gamma-1}}_{p,q} d_{p,q}s\\
&= \frac{\varphi_0\left(\frac{T}{p}\right)^{\gamma}}{\Gamma_{p,q}(\gamma+1)}\|u-v\|,\\
\text{and}\quad \left|\left(\Upsilon^{\nu}_{p,q}u\right)(t)-\left(\Upsilon^{\nu}_{p,q}v\right)(t)\right| &\le \frac{\psi_0\left(\frac{T}{p}\right)^{-\nu}}{\Gamma_{p,q}(1-\nu)}\|u-v\|.
\end{aligned}$$

Denote that

$$\mathcal{H}|u-v|(t):=\left|F\left[t,u(t),\left(\Psi^{\gamma}_{p,q}\right)(t),\left(\Upsilon^{\nu}_{p,q}u\right)(t)\right]-F\left[t,v(t),\left(\Psi^{\gamma}_{p,q}v\right)(t),\left(\Upsilon^{\nu}_{p,q}v\right)(t)\right]\right|.$$

Then, we have

$$\begin{aligned}
&\left|\mathbb{O}^*_{\eta}[F,u]-\mathbb{O}^*_{\eta}[F,v]\right|\\
\le\ & \frac{1}{p^{\binom{\alpha}{2}+\binom{\beta}{2}+\binom{-\theta}{2}}\Gamma_{p,q}(\alpha)\Gamma_{p,q}(\beta)\Gamma_{p,q}(-\theta)}\\
&\times\int_0^{\eta}\int_0^{\frac{y}{p^{-\theta-1}}}\int_0^{\frac{x}{p^{\beta-1}}}(\eta-qy)^{\underline{-\theta-1}}_{p,q}\frac{\left(\frac{y}{p^{-\theta-1}}-qx\right)^{\underline{\beta-1}}_{p,q}}{1+\rho\left(\frac{y}{p^{-\theta-1}}\right)}\left(\frac{x}{p^{\beta-1}}-qs\right)^{\underline{\alpha-1}}_{p,q} g\left(\frac{x}{p^{\beta-1}}\right)\\
&\times\mathcal{H}|u-v|\left(\frac{s}{p^{\alpha-1}}\right)d_{p,q}s\,d_{p,q}x\,d_{p,q}y\\
\le\ & \frac{G\left[L_1\left|u-v\right|+L_2\left|\Psi^{\gamma}_{p,q}u-\Psi^{\gamma}_{p,q}v\right|+L_3\left|\Upsilon^{\nu}_{p,q}u-\Upsilon^{\nu}_{p,q}v\right|\right]}{(1+\rho)p^{\binom{\alpha}{2}+\binom{\beta}{2}+\binom{-\theta}{2}}\Gamma_{p,q}(\alpha)\Gamma_{p,q}(\beta)\Gamma_{p,q}(-\theta)}\\
&\times\int_0^{\eta}\int_0^{\frac{y}{p^{-\theta-1}}}\int_0^{\frac{x}{p^{\beta-1}}}(\eta-qy)^{\underline{-\theta-1}}_{p,q}\frac{\left(\frac{y}{p^{-\theta-1}}-qx\right)^{\underline{\beta-1}}_{p,q}}{1+\rho\left(\frac{y}{p^{-\theta-1}}\right)}\left(\frac{x}{p^{\beta-1}}-qs\right)^{\underline{\alpha-1}}_{p,q} d_{p,q}s\,d_{p,q}x\,d_{p,q}y\\
\le\ & \frac{G\eta^{\alpha+\beta-\theta}}{(1+\rho)\Gamma_{p,q}(\alpha+\beta-\theta+1)}\left[L_1+L_2\frac{\varphi_0\left(\frac{T}{p}\right)^{\gamma}}{\Gamma_{p,q}(\gamma+1)}+L_3\frac{\psi_0\left(\frac{T}{p}\right)^{-\nu}}{\Gamma_{p,q}(1-\nu)}\right]\|u-v\|_C\\
=\ & \frac{G\eta^{\alpha+\beta-\theta}\mathcal{L}}{(1+\rho)\Gamma_{p,q}(\alpha+\beta-\theta+1)}\|u-v\|_C.
\end{aligned}$$

Similary,

$$
\begin{aligned}
&\left|\mathbb{O}_T^*[F,u]-\mathbb{O}_T^*[F,v]\right|\\
\le\ & \frac{1}{\left(1+\rho\left(\frac{T}{p}\right)\right)p^{\binom{\alpha}{2}+\binom{\beta}{2}}\Gamma_{p,q}(\alpha)\Gamma_{p,q}(\beta)}\\
&\times\int_0^{\frac{T}{p}}\int_0^{\frac{x}{p^{\beta-1}}}\left(\frac{T}{p}-qx\right)_{p,q}^{\underline{\beta-1}}\left(\frac{x}{p^{\beta-1}}-qs\right)_{p,q}^{\underline{\alpha-1}}\mathcal{H}|u-v|\left(\frac{s}{p^{\alpha-1}}\right)d_{p,q}s\,d_{p,q}x\\
\le\ & \frac{\left[L_1\left|u-v\right|+L_2\left|\Psi_{p,q}^{\gamma}u-\Psi_{p,q}^{\gamma}v\right|+L_3\left|\Upsilon_{p,q}^{\nu}u-\Upsilon_{p,q}^{\nu}v\right|\right]}{(1+\rho)p^{\binom{\alpha}{2}+\binom{\beta}{2}}\Gamma_{p,q}(\alpha)\Gamma_{p,q}(\beta)}\\
&\times\int_0^{\frac{T}{p}}\int_0^{\frac{x}{p^{\beta-1}}}\left(\frac{T}{p}-qx\right)_{p,q}^{\underline{\beta-1}}\left(\frac{x}{p^{\beta-1}}-qs\right)_{p,q}^{\underline{\alpha-1}}d_{p,q}s\,d_{p,q}x\\
\le\ & \frac{\left(\frac{T}{p}\right)^{\alpha+\beta}\mathcal{L}}{(1+\rho)\Gamma_{p,q}(\alpha+\beta+1)}\|u-v\|_C.
\end{aligned}
$$

Consider

$$
\begin{aligned}
&\left|(\mathcal{F}u)(t)-(\mathcal{F}v)(t)\right|\\
\le\ & \left[\frac{1+\rho(0)}{1+\rho}\right]\left[1+\frac{\max|\mathbf{A}|\left(\frac{T}{p}\right)^{\beta}}{\min|\mathbf{B}|\,\Gamma_{p,q}(\beta+1)}\right]|\phi(u)-\phi(v)|\\
&+\frac{\left(\frac{T}{p}\right)^{\beta}}{(1+\rho)\min|\mathbf{B}|\,\Gamma_{p,q}(\beta+1)}\left[\left|\mathbb{O}_\eta^*[F,u]-\mathbb{O}_\eta^*[F,v]\right|+\left|\mathbb{O}_T^*[F,u]-\mathbb{O}_T^*[F,v]\right|\right]\\
&+\frac{\left[L_1+L_2\frac{\varphi_0\left(\frac{T}{p}\right)^{\gamma}}{\Gamma_{p,q}(\gamma+1)}+L_3\frac{\psi_0\left(\frac{T}{p}\right)^{-\nu}}{\Gamma_{p,q}(1-\nu)}\right]\|u-v\|_C}{(1+\rho)p^{\binom{\alpha}{2}+\binom{\beta}{2}}\Gamma_{p,q}(\alpha)\Gamma_{p,q}(\beta)}\\
&\times\int_0^{\frac{T}{p}}\int_0^{\frac{x}{p^{\beta-1}}}\left(\frac{T}{p}-qx\right)_{p,q}^{\underline{\beta-1}}\left(\frac{x}{p^{\beta-1}}-qs\right)_{p,q}^{\underline{\alpha-1}}d_{p,q}s\,d_{p,q}x\\
\le\ & \left[\frac{1+P}{1+\rho}\right]\left[1+\frac{\max|\mathbf{A}|\left(\frac{T}{p}\right)^{\beta}}{\min|\mathbf{B}|\,\Gamma_{p,q}(\beta+1)}\right]\omega\|u-v\|_C\\
&+\left\{\frac{\left(\frac{T}{p}\right)^{\beta}}{(1+\rho)\min|\mathbf{B}|\,\Gamma_{p,q}(\beta+1)}\left[\frac{G\eta^{\alpha+\beta-\theta}}{(1+\rho)\Gamma_{p,q}(\alpha+\beta-\theta+1)}+\frac{\left(\frac{T}{p}\right)^{\alpha+\beta}}{(1+\rho)\Gamma_{p,q}(\alpha+\beta+1)}\right]\right.\\
&\left.+\frac{\left(\frac{T}{p}\right)^{\alpha+\beta}}{(1+\rho)\Gamma_{p,q}(\alpha+\beta+1)}\right\}\left[L_1+L_2\frac{\varphi_0\left(\frac{T}{p}\right)^{\gamma}}{\Gamma_{p,q}(\gamma+1)}+L_3\frac{\psi_0\left(\frac{T}{p}\right)^{-\nu}}{\Gamma_{p,q}(1-\nu)}\right]\|u-v\|_C\\
=\ & \left\{\mathcal{P}\omega+\mathcal{L}\left[\mathcal{Q}+\frac{\left(\frac{T}{p}\right)^{\alpha+\beta}}{(1+\rho)\Gamma_{p,q}(\alpha+\beta+1)}\right]\right\}\|u-v\|_C.
\end{aligned}
\tag{3.7}
$$

Next, we provide $(D^{\nu}_{p,q}\mathcal{F}u)$ as

$$\begin{aligned}
&(D^{\nu}_{p,q}\mathcal{F}u)(t) \qquad (3.8)\\
&= \frac{(1+\rho(0))\,\phi(u)}{p^{\binom{-\nu}{2}}\Gamma_{p,q}(-\nu)}\left[\int_0^t (t-qs)_{p,q}^{\underline{-\nu-1}}\frac{1}{1+\rho\left(\frac{s}{p^{-\nu-1}}\right)}\left(1-\frac{\mathbf{A}\left(\frac{s}{p^{-\nu-1}}\right)^{\beta}}{\mathbf{B}\Gamma_{p,q}(\beta+1)}\right)d_{p,q}s\right]\\
&+\frac{\mathbb{O}^*_{\eta}[F,u]-\mathbb{O}^*_{T}[F,u]}{\mathbf{B}\Gamma_{p,q}(\beta+1)p^{\binom{-\nu}{2}}\Gamma_{p,q}(-\nu)}\int_0^t (t-qs)_{p,q}^{\underline{-\nu-1}}\frac{\left(\frac{s}{p^{-\nu-1}}\right)^{\beta}}{1+\rho\left(\frac{s}{p^{-\nu-1}}\right)}d_{p,q}s\\
&+\frac{1}{p^{\binom{\alpha}{2}+\binom{\beta}{2}+\binom{-\nu}{2}}\Gamma_{p,q}(\alpha)\Gamma_{p,q}(\beta)\Gamma_{p,q}(-\nu)}\\
&\times\int_0^t\int_0^{\frac{y}{p^{-\nu-1}}}\int_0^{\frac{x}{p^{\beta-1}}}(t-qy)_{p,q}^{\underline{-\nu-1}}\frac{\left(\frac{y}{p^{-\nu-1}}-qx\right)_{p,q}^{\underline{\beta-1}}}{1+\rho\left(\frac{y}{p^{-\nu-1}}\right)}\left(\frac{x}{p^{\beta-1}}-qs\right)_{p,q}^{\underline{\alpha-1}}\\
&\times F\left[\frac{s}{p^{\alpha-1}},u\left(\frac{s}{p^{\alpha-1}}\right),\left(\Psi^{\gamma}_{p,q}u\right)\left(\frac{s}{p^{\alpha-1}}\right),\left(\Upsilon^{\nu}_{p,q}u\right)\left(\frac{s}{p^{\alpha-1}}\right)\right]d_{p,q}s\,d_{p,q}x\,d_{p,q}y,
\end{aligned}$$

and

$$\begin{aligned}
&\left|(D^{\nu}_{p,q}\mathcal{F}u)(t)-(D^{\nu}_{p,q}\mathcal{F}v)(t)\right| \qquad (3.9)\\
&\le\frac{|\phi(u)-\phi(v)|}{p^{\binom{-\nu}{2}}\Gamma_{p,q}(-\nu)}\left[\frac{1+\rho(0)}{1+\rho}\right]\left[1+\frac{\max|\mathbf{A}|\left(\frac{T}{p}\right)^{\beta}}{\min|\mathbf{B}|\Gamma_{p,q}(\beta+1)}\right]\int_0^{\frac{T}{p}}\left(\frac{T}{p}-qs\right)_{p,q}^{\underline{-\nu-1}}d_{p,q}s\\
&+\frac{|\mathbb{O}^*_{\eta}[F,u]-\mathbb{O}^*_{\eta}[F,v]|+|\mathbb{O}^*_{T}[F,u]-\mathbb{O}^*_{T}[F,v]|}{\min|\mathbf{B}|p^{\binom{-\nu}{2}}\Gamma_{p,q}(\beta+1)\Gamma_{p,q}(-\nu)}\frac{\left(\frac{T}{p}\right)^{\beta}}{(1+\rho)}\int_0^{\frac{T}{p}}\left(\frac{T}{p}-qs\right)_{p,q}^{\underline{-\nu-1}}d_{p,q}s\\
&+\frac{\left[L_1+L_2\frac{\varphi_0\left(\frac{T}{p}\right)^{\gamma}}{\Gamma_{p,q}(\gamma+1)}+L_3\frac{\psi_0\left(\frac{T}{p}\right)^{-\nu}}{\Gamma_{p,q}(-\nu)}\right]}{(1+\rho)\,p^{\binom{\alpha}{2}+\binom{\beta}{2}+\binom{-\nu}{2}}\Gamma_{p,q}(\alpha)\Gamma_{p,q}(\beta)\Gamma_{p,q}(-\nu)}\\
&\times\int_0^{\frac{T}{p}}\int_0^{\frac{y}{p^{-\nu-1}}}\int_0^{\frac{x}{p^{\beta-1}}}\left(\frac{T}{p}-qy\right)_{p,q}^{\underline{-\nu-1}}\left(\frac{y}{p^{-\nu-1}}-qx\right)_{p,q}^{\underline{\beta-1}}\left(\frac{x}{p^{\beta-1}}-qs\right)_{p,q}^{\underline{\alpha-1}}d_{p,q}s\,d_{p,q}x\,d_{p,q}y\\
&\le\left\{\mathcal{P}\omega\frac{\left(\frac{T}{p}\right)^{-\nu}}{\Gamma_{p,q}(1-\nu)}+\mathcal{L}\left[\mathcal{Q}\frac{\left(\frac{T}{p}\right)^{-\nu}}{\Gamma_{p,q}(1-\nu)}+\frac{\left(\frac{T}{p}\right)^{\alpha+\beta-\nu}}{(1+\rho)\Gamma_{p,q}(\alpha+\beta-\nu+1)}\right]\right\}\|u-v\|_C.
\end{aligned}$$

From (3.7) and (3.9), we get

$$\begin{aligned}
&\|(\mathcal{F}u)(t)-(\mathcal{F}v)(t)\|_C\\
&\le[\omega\mathcal{P}+\mathcal{L}\mathcal{Q}]\left(1+\frac{\left(\frac{T}{p}\right)^{-\nu}}{\Gamma_{p,q}(1-\nu)}\right)+\mathcal{L}\left(\frac{\left(\frac{T}{p}\right)^{\alpha+\beta}}{\Gamma_{p,q}(\alpha+\beta+1)}+\frac{\left(\frac{T}{p}\right)^{\alpha+\beta-\nu}}{\Gamma_{p,q}(\alpha+\beta-\nu+1)}\right)\\
&=\mathcal{X}\,\|u-v\|_C.
\end{aligned}$$

Thus, from (H_5), $\mathcal{F}$ is a contraction. Therefore, by using Banach fixed point theorem, $\mathcal{F}$ has a fixed point which is a unique solution of problem (1.1) on $I_{p,q}^{T}$. □

4. Existence of at least one solution

In this section, we first introduce Schauder's fixed point theorem that is used to study a solution of (1.1).

Lemma 4.1. *[48] (Arzelá-Ascoli theorem) A collection of functions in $C[a,b]$ with the sup norm, is relatively compact if and only if it is uniformly bounded and equicontinuous on $[a,b]$.*

Lemma 4.2. *[48] If a set is closed and relatively compact then it is compact.*

Lemma 4.3. *[49] (Schauder's fixed point theorem) Let (D,d) be a complete metric space, U be a closed convex subset of D, and $T : D \to D$ be the map such that the set $Tu : u \in U$ is relatively compact in D. Then the operator T has at least one fixed point $u^* \in U$: $Tu^* = u^*$.*

Next, we prove that (1.1) has at least one solution by using above lemmas as follows.

Theorem 4.1. *Suppose that (H_1) and $(H_3)-(H_5)$ hold. Then problem (1.1) has at least one solution on $I_{p,q}^{T}$.*

Proof. The proof is organized as follows.

Step I. Verify $\mathcal{F}$ maps bounded sets into bounded sets in $B_R = \{u \in C : \|u\|_C \leq R\}$. Let $\max_{t\in I_{p,q}^{T}} |F(t,0,0,0)| = F$, $\sup_{u\in C} |\phi(u)| = \phi$. We choose a constant

$$R \geq \frac{[\phi\mathcal{P} + F]\left(1 + \frac{\left(\frac{T}{p}\right)^{-\nu}}{\Gamma_{p,q}(1-\nu)}\right) + F\left(\frac{\left(\frac{T}{p}\right)^{\alpha+\beta}}{\Gamma_{p,q}(\alpha+\beta+1)} + \frac{\left(\frac{T}{p}\right)^{\alpha+\beta-\nu}}{\Gamma_{p,q}(\alpha+\beta-\nu+1)}\right)}{1 - \mathcal{L}\left[\mathcal{Q}\left(1 + \frac{\left(\frac{T}{p}\right)^{-\nu}}{\Gamma_{p,q}(1-\nu)}\right) + \left(\frac{\left(\frac{T}{p}\right)^{\alpha+\beta}}{\Gamma_{p,q}(\alpha+\beta+1)} + \frac{\left(\frac{T}{p}\right)^{\alpha+\beta-\nu}}{\Gamma_{p,q}(\alpha+\beta-\nu+1)}\right)\right]}. \tag{4.1}$$

Denote that $\left|\mathcal{T}(t,u,0)\right| = \left|F[t,u(t),\left(\Psi_{p,q}^{\gamma}u\right)(t),\left(\Upsilon_{p,q}^{\nu}u\right)(t)] - F[t,0,0,0]\right| + \left|F[t,0,0,0]\right|$. For each $t \in I_{p,q}^{T}$ and $u \in B_R$, we have

$$\begin{aligned}
&\left|\mathbb{O}_{\eta}^{*}[F,u]\right| \qquad (4.2)\\
\leq\ & \frac{G}{p^{\binom{\alpha}{2}+\binom{\beta}{2}+\binom{-\theta}{2}}(1+\rho)} \int_0^{\eta}\int_0^{\frac{y}{p^{-\theta-1}}}\int_0^{\frac{x}{p^{\beta-1}}} (\eta - qy)_{p,q}^{\underline{-\theta-1}} \left(\frac{y}{p^{-\theta-1}} - yx\right)_{p,q}^{\underline{\beta-1}} \left(\frac{x}{p^{\beta-1}} - qs\right)_{p,q}^{\underline{\alpha-1}}\\
&\times \left|\mathcal{T}\left(\frac{s}{p^{\alpha-1}},u,0\right)\right| d_{p,q}s\, d_{p,q}x\, d_{p,q}y\\
\leq\ & \frac{G}{(1+\rho)}\left(\left[L_1 + L_2\varphi_0\frac{\left(\frac{T}{p}\right)^{\gamma}}{\Gamma_{p,q}(\gamma+1)} + L_3\psi_0\frac{\left(\frac{T}{p}\right)^{-\nu}}{\Gamma_{p,q}(1-\nu)}\right]\|u\|_C + F\right)\frac{\eta^{\alpha+\beta-\theta}}{\Gamma_{p,q}(\alpha+\beta-\theta+1)},
\end{aligned}$$

and

$$\left|\mathbb{O}_{T}^{*}[F,u]\right| \tag{4.3}$$

$$\leq \frac{1}{(1+\rho)}\left(\left[L_1 + L_2\varphi_0\frac{\left(\frac{T}{p}\right)^{\gamma}}{\Gamma_{p,q}(\gamma+1)} + L_3\psi_0\frac{\left(\frac{T}{p}\right)^{-\nu}}{\Gamma_{p,q}(1-\nu)}\right]\|u\|_C + F\right)\frac{\left(\frac{T}{p}\right)^{\alpha+\beta}}{\Gamma_{p,q}(\alpha+\beta+1)}.$$

From (4.2) and (4.3), we find that

$$\begin{aligned}\left|(\mathcal{F}u)(t)\right| \leq\ & \phi\left[\frac{1+\rho(0)}{1+\rho}\right]\left[1+\frac{\max|\mathbf{A}|\left(\frac{T}{p}\right)^{\beta}}{\min|\mathbf{B}|\,\Gamma_{p,q}(\beta+1)}\right] \\ & + \left[\frac{\mathbb{O}^*_{\eta}[F,u] + \mathbb{O}^*_{T}[F,u]}{1+\rho}\right]\cdot\frac{\left(\frac{T}{p}\right)^{\beta}}{\min|\mathbf{B}|\,\Gamma_{p,q}(\beta+1)} \\ & + \frac{1}{(1+\rho)\,p^{\binom{\alpha}{2}+\binom{\beta}{2}}\Gamma_{p,q}(\alpha)\Gamma_{p,q}(\beta)}\int_0^{\frac{T}{p}}\int_0^{\frac{x}{p^{\beta-1}}}\left(\frac{T}{p}-qx\right)_{p,q}^{\underline{\beta-1}}\left(\frac{x}{p^{\beta-1}}-qs\right)_{p,q}^{\underline{\alpha-1}} \\ & \times\left|\mathcal{T}\left(\frac{s}{p^{\alpha-1}},u,0\right)\right| d_{p,q}s\, d_{p,q}x \\ \leq\ & \phi\mathcal{P} + [\mathcal{L}\|u\|_C + F]\left[Q + \frac{\left(\frac{T}{p}\right)^{\alpha+\beta}}{(1+\rho)\,\Gamma_{p,q}(\alpha+\beta+1)}\right]. \end{aligned} \tag{4.4}$$

Similarly as above, we have

$$\begin{aligned}&\left|\left(\mathcal{D}^{\nu}_{p,q}\mathcal{F}u\right)(t)\right| \\ &\leq \phi\mathcal{P}\frac{\left(\frac{T}{p}\right)^{-\nu}}{\Gamma_{p,q}(1-\nu)} + [\mathcal{L}\|u\|_C + F]\left[Q\frac{\left(\frac{T}{p}\right)^{-\nu}}{\Gamma_{p,q}(1-\nu)} + \frac{\left(\frac{T}{p}\right)^{\alpha+\beta-\nu}}{(1+\rho)\Gamma_{p,q}(\alpha+\beta-\nu+1)}\right]. \end{aligned} \tag{4.5}$$

Using (4.4) and (4.5), we get

$$\begin{aligned}\|\mathcal{F}u\| \leq \mathcal{L}\|u\|_C & \left\{Q\left(1+\frac{\left(\frac{T}{p}\right)^{-\nu}}{\Gamma_{p,q}(1-\nu)}\right) + \left(\frac{\left(\frac{T}{p}\right)^{\alpha+\beta}}{\Gamma_{p,q}(\alpha+\beta+1)} + \frac{\left(\frac{T}{p}\right)^{\alpha+\beta-\nu}}{\Gamma_{p,q}(\alpha+\beta-\nu+1)}\right)\right\} \\ & + [\phi\mathcal{P} + F]\left(1+\frac{\left(\frac{T}{p}\right)^{-\nu}}{\Gamma_{p,q}(1-\nu)}\right) + F\left(\frac{\left(\frac{T}{p}\right)^{\alpha+\beta}}{\Gamma_{p,q}(\alpha+\beta+1)} + \frac{\left(\frac{T}{p}\right)^{\alpha+\beta-\nu}}{\Gamma_{p,q}(\alpha+\beta-\nu+1)}\right) \\ & \leq R. \end{aligned} \tag{4.6}$$

We find that $\|\mathcal{F}u\| \leq R$. Thus, $\mathcal{F}$ is uniformly bounded.

Step II. Since F is continuous, the operator $\mathcal{F}$ is continuous on B_R.

Step III. We examine that $\mathcal{F}$ is equicontinuous on B_R. For any $t_1, t_2 \in I^T_{p,q}$ with $t_1 < t_2$, we find that

$$\begin{aligned}\left|(\mathcal{F}u)(t_2) - (\mathcal{F}u)(t_1)\right| \leq\ & \phi\left(\frac{1+P}{1+\rho}\right)\left[1+\frac{|\mathbf{A}|\left|t_2^{\beta}-t_1^{\beta}\right|}{|\mathbf{B}|\,\Gamma_{p,q}(\beta+1)}\right] \\ & + \frac{\left|t_2^{\beta}-t_1^{\beta}\right|}{(1+\rho)|\mathbf{B}|\,\Gamma_{p,q}(\beta+1)}\left[\mathbb{O}^*_{\eta}[F,u] + \mathbb{O}^*_{T}[F,u]\right]\end{aligned}$$

$$+\frac{\|F\|}{(1+\rho)}\frac{\left|t_2^{\alpha+\beta}-t_1^{\alpha+\beta}\right|}{\Gamma_{p,q}(\alpha+\beta+1)}, \tag{4.7}$$

and

$$\begin{aligned}\left|(D^{\nu}_{p,q}\mathcal{F}u)(t_2)-(D^{\nu}_{p,q}\mathcal{F}u)(t_1)\right| &\le \phi\left(\frac{1+P}{1+\rho}\right)\left[1+\frac{|\mathbf{A}|\left(\frac{T}{p}\right)^{\beta}}{|\mathbf{B}|\,\Gamma_{p,q}(\beta+1)}\right]\frac{\left|t_2^{-\nu}-t_1^{-\nu}\right|}{\Gamma_{p,q}(1-\nu)}\\ &+\frac{\left|t_2^{-\nu}-t_1^{-\nu}\right|\left(\frac{T}{p}\right)^{\beta}}{(1+\rho)\,|\mathbf{B}|\,\Gamma_{p,q}(\beta+1)\Gamma_{p,q}(1-\nu)}\left[\mathbb{O}^*_{\eta}[F,u]+\mathbb{O}^*_{T}[F,u]\right]\\ &+\frac{\|F\|}{1+\rho}\,\frac{\left|t_2^{\alpha+\beta-\nu}-t_1^{\alpha+\beta-\nu}\right|}{\Gamma_{p,q}(\alpha+\beta+-\nu+1)}. \end{aligned}\tag{4.8}$$

When $|t_2-t_1|\to 0$, the right-hand side of (4.7) and (4.8) tends to be zero. Thus, $\mathcal{F}$ is relatively compact on B_R.
Then, $\mathcal{F}(B_R)$ is an equicontinuous set. From Steps I to III together with the Arzelá-Ascoli theorem, $\mathcal{F}: C\to C$ is completely continuous. Therefore, by Schauder fixed point theorem, we can conclude that problem (1.1) has at least one solution. □

5. An example

In this section, we provide an example to show our results. We let $\alpha=\frac{3}{4}$, $\beta=\frac{1}{2}$, $\gamma=\frac{1}{3}$, $\nu=\frac{1}{4}$, $\theta=\frac{2}{3}$, $p=\frac{2}{3}$, $q=\frac{1}{2}$, $T=10$, $\eta=\sigma^4_{\frac{2}{3},\frac{1}{2}}(10)=\frac{1215}{256}$, $g(t)=(10\pi+\cos t)^2$, $\rho(t)=e^{\cos 2\pi t}$, $\phi(u)=\sum_{i=1}^{\infty}C_iu(t_i)$ and $F\left[t,u(t),\left(\Psi^{\gamma}_{p,q}u\right)(t),\left(\Upsilon^{\nu}_{p,q}u\right)(t)\right]=\frac{e^{\cos^2 2\pi t}}{\left(100+e^{sin2\pi t}\right)(1+|u(t)|)}\left[e^{-(\pi+t^2)}\left(u^2+2|u|\right)+e^{-(10t+\frac{\pi}{2})}\left|\left(\Psi^{\frac{1}{3}}_{\frac{2}{3},\frac{1}{2}}u\right)(t)\right|e^{-(2\pi+\cos\pi t)}\left|\left(\Upsilon^{\frac{1}{4}}_{\frac{2}{3},\frac{1}{2}}u\right)(t)\right|\right]$ which are satisfied with the conditions of the problem (1.1). Therefore the problem (1.1) is represented by

$$\begin{aligned}{}^{C}D^{\frac{3}{4}}_{\frac{2}{3},\frac{1}{2}}\left[{}^{C}D^{\frac{1}{2}}_{\frac{2}{3},\frac{1}{2}}\left(1+e^{\cos 2\pi t}\right)\right]u(t) &= \frac{e^{\cos^2 2\pi t}}{(100+e^{\sin 2\pi t})(1+|u(t)|)}\Bigg[e^{-(\pi+t^2)}\left(u^2+2|u|\right)\\ &+e^{-(10t+\frac{\pi}{2})}\left|\left(\Psi^{\frac{1}{3}}_{\frac{2}{3},\frac{1}{2}}u\right)(t)\right|+e^{-(2\pi+\cos\pi t)}\left|\left(\Upsilon^{\frac{1}{4}}_{\frac{2}{3},\frac{1}{2}}u\right)(t)\right|\Bigg],\end{aligned}$$

where $t\in I^{10}_{\frac{2}{3},\frac{1}{2}}=\left\{\frac{10\left(\frac{1}{2}\right)^k}{\left(\frac{2}{3}\right)^{k+1}}:k\in\mathbb{N}_0\right\}\cup\{0\}$, with three-point fractional (p,q)-difference boundary condition

$$\begin{aligned}u(0) &= \sum_{i=1}^{\infty}\frac{C_i|u(t_i)|}{1+|u(t_i)|},\quad t_i\in\sigma^i_{\frac{2}{3},\frac{1}{2}}(10),\\ u(15) &= D^{\frac{2}{3}}_{\frac{2}{3},\frac{1}{2}}\left(10\pi+\cos\left(\frac{1215}{256}\right)\right)^2u\left(\frac{1215}{256}\right)\end{aligned}$$

where C_i are given positive constants with $\frac{1}{e}<\sum_{i=1}^{\infty}C_i<\frac{1}{e^5}$, $\varphi(t,s)=\frac{e^{-|t-s|}}{(t+10)^3}$ and $\psi(t,s)=\frac{e^{-2|t-s|}}{(t+20)^2}$.

To investigate the values of $L_1, L_1, L_3, \mathcal{L}, \omega, g, G, \rho, P, \mathcal{X}, \mathcal{P}$ and $\mathcal{Q}$, we employ the assumptions (H_1)–(H_5) to get the results as follows. For all $t \in I^{10}_{\frac{2}{3},\frac{11}{2}}$ and $u, v \in \mathbb{R}$, we find that

$$\begin{aligned}&\left|F\left[t,u,\Psi^{\gamma}_{p,q}u,\Upsilon^{\nu}_{p,q}u\right]-F\left[t,v,\Psi^{\gamma}_{p,q}v,\Upsilon^{\nu}_{p,q}v\right]\right|\\ \le\ &\frac{1}{(100+\frac{1}{e})e^{\pi-1}}|u-v|+\frac{1}{(100+\frac{1}{e})e^{\frac{\pi}{2}-1}}\left|\Psi^{\gamma}_{p,q}u-\Psi^{\gamma}_{p,q}v\right|+\frac{1}{(100+\frac{1}{e})e^{2\pi-2}}\left|\Upsilon^{\nu}_{p,q}u-\Upsilon^{\nu}_{p,q}v\right|.\end{aligned}$$

Thus, (H_1) holds with $L_1 = 0.00117$, $L_2 = 0.00563$ and $L_3 = 0.000137$. So $\mathcal{L} = 0.001185$. For all $u, v \in C$,

$$|\phi_1(u)-\phi_1(v)| = \left|\sum_{i=1}^{\infty}C_i u(t_i)-\sum_{i=1}^{\infty}C_i v(t_i)\right| \le \sum_{i=1}^{\infty}C_i\|u-v\|_C \le \frac{1}{e^5}\|u-v\|_C.$$

Thus, (H_2) holds with $\omega = \frac{1}{e^5} = 0.00674$.

For all $t \in I^{10}_{\frac{2}{3},\frac{11}{2}}$, (H_3) and (H_4) hold with $925.129 = g \le g(t) \le G = 1050.792$ and $\frac{1}{e} = \rho \le \rho(t) \le P = e$.

We calculate and find that

$$|\mathbf{A}| \le 138.177, \quad |\mathbf{B}| \ge 243.229, \quad \varphi_0 = 0.001 \quad \text{and} \quad \psi_0 = 0.0025.$$

Next, we get

$$\mathcal{P} = 9.02715 \quad \text{and} \quad \mathcal{Q} = 0.84825.$$

Therefore (H_5) holds with

$$\mathcal{X} = 0.35757 < 1.$$

Hence, by Theorem 3.1, this problem has a unique solution. In addition, by Theorem 4.1 it has at least one solution.

6. Conclusions

A sequential fractional Caputo (p, q)-integrodifference equation with three-point fractional Riemann-Liouville (p, q)-difference boundary condition (1.1) is studied. Our problem contains two fractional Caputo (p, q)-difference operators, two fractional Riemann-Liouville (p, q)-difference operators, and one fractional (p, q)-integral operator. After proving an auxiliary result concerning a linear variant of the considered problem, the problem at hand is transformed into a fixed point problem related to (1.1). We establish the conditions for the existence and uniqueness of solution for problem (1.1) by using the Banach fixed point theorem, and the conditions of at least one solution by using the Schauder's fixed point theorem. Some properties of fractional (p, q)-integral needed in our study are also discussed. The results of the paper are new and enrich the subject of boundary value problems for fractional (p, q)- integrodifference equations. In the future work, we may extend this work by considering new boundary value problems.

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Conflict of interest

The authors declare no conflict of interest.

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