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Research article

Hadamard and Fejér-Hadamard inequalities for generalized k-fractional integrals involving further extension of Mittag-Leffler function

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Abstract: In this paper, k-fractional integral operators containing further extension of Mittag-Leffler function are defined firstly. Then, the first and second version of Hadamard and Fejér-Hadamard inequalities for generalized k-fractional integrals are obtained. Finally, by using these generalized k-fractional integrals containing Mittag-Leffler functions, results for p-convex functions are obtained. The results for convex functions can be deduced by taking $p = 1$.

Keywords: Hadamard inequality; Fejér-Hadamard inequality; convex function; p-convex function; k-fractional integrals; Mittag-Leffler function
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1. Introduction

We first give the definitions of fractional integral operators and some theorems which establish the basis for the second and third part of the paper.

Definition 1. *(see [1]) Let* $w, \alpha, l, \gamma, c \in \mathbb{C}, \Re(\alpha), \Re(l) > 0, \Re(c) > \Re(\gamma) > 0$ *with* $\tilde{p} \geq 0, \mu, \delta > 0$ *and* $0 < \nu \leq \delta + \mu$*. Let* $f \in L_1[a,b]$ *and* $x \in [a,b]$*. Then, the generalized fractional integral operators* $F^{\gamma,\delta,\nu,c}_{\mu,\alpha,l,w,a+}f$ *and* $F^{\gamma,\delta,\nu,c}_{\mu,\alpha,l,w,b-}f$ *are defined by:*

$$\begin{aligned}\left(F^{\gamma,\delta,\nu,c}_{\mu,\alpha,l,w,a+}f\right)(x;\tilde{p}) &= \int_a^x (x-t)^{\alpha-1} E^{\gamma,\delta,\nu,c}_{\mu,\alpha,l}\left(w(x-t)^{\mu};\tilde{p}\right) f(t)dt,\\ \left(F^{\gamma,\delta,\nu,c}_{\mu,\alpha,l,w,b-}f\right)(x;\tilde{p}) &= \int_x^b (t-x)^{\alpha-1} E^{\gamma,\delta,\nu,c}_{\mu,\alpha,l}\left(w(t-x)^{\mu};\tilde{p}\right) f(t)dt,\end{aligned} \tag{1.1}$$

where

$$E_{\mu,\alpha,l}^{\gamma,\delta,v,c}(t;\tilde{p})=\sum_{n=0}^{\infty}\frac{\beta_{\tilde{p}}(\gamma+nv,c-\gamma)}{\beta(\gamma,c-\gamma)}\frac{(c)_{nv}}{\Gamma(\mu n+\alpha)}\frac{t^n}{(l)_{n\delta}} \tag{1.2}$$

is the extended generalized Mittag-Leffler function and $\beta_{\tilde{p}}$ *is the extension of beta function is defined as follows:*

$$\beta_{\tilde{p}}(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}e^{-\left(\frac{\tilde{p}}{t(1-t)}\right)}dt, \tag{1.3}$$

where $\Re(x),\Re(y),\Re(\tilde{p})>0$.

Definition 2. *(see [2]) Let* $f,g:[a,b]\to\mathbb{R}, 0<a<b$, *be the functions such that* f *be positive and* $f\in L_1[a,b]$ *and* g *be differentiable and strictly increasing. Also let* $\frac{\phi}{x}$ *be an increasing function on* $[a,\infty)$ *and* $w,\alpha,l,\gamma,c\in\mathbb{C},\Re(\alpha),\Re(l)>0,\Re(c)>\Re(\gamma)>0$ *with* $\tilde{p}\geq 0,\mu,\delta>0$ *and* $0<v\leq\delta+\mu$. *Then for* $x\in[a,b]$ *the fractional integral operators are defined by:*

$$\begin{aligned}\left({}_gF_{\mu,\alpha,l,w,a+}^{\phi,\gamma,\delta,v,c}f\right)(x;\tilde{p})&=\int_a^x\frac{\phi(g(x)-g(t))}{g(x)-g(t)}E_{\mu,\alpha,l}^{\gamma,\delta,v,c}(w(g(x)-g(t))^{\mu};\tilde{p})\,g'(t)f(t)dt,\\ \left({}_gF_{\mu,\alpha,l,w,b-}^{\phi,\gamma,\delta,v,c}f\right)(x;\tilde{p})&=\int_x^b\frac{\phi(g(t)-g(x))}{g(t)-g(x)}E_{\mu,\alpha,l}^{\gamma,\delta,v,c}(w(g(t)-g(x))^{\mu};\tilde{p})\,g'(t)f(t)dt.\end{aligned} \tag{1.4}$$

Definition 3. *(see [3]) Let* $f,g:[a,b]\to\mathbb{R},0<a<b$, *be the functions such that* f *be positive and* $f\in L_1[a,b]$ *and* g *be differentiable and strictly increasing. Also, let* $w,\alpha,l,\gamma,c\in\mathbb{C},\Re(\alpha),\Re(l)>0,\Re(c)>\Re(\gamma)>0$ *with* $\tilde{p}\geq 0,\mu,\delta>0$ *and* $0<v\leq\delta+\mu$. *Then for* $x\in[a,b]$ *the unified integral operators are defined by:*

$$\begin{aligned}\left({}_gF_{\mu,\alpha,l,w,a+}^{\gamma,\delta,v,c}f\right)(x;\tilde{p})&=\int_a^x(g(x)-g(t))^{\alpha-1}E_{\mu,\alpha,l}^{\gamma,\delta,v,c}(w(g(x)-g(t))^{\mu};\tilde{p})\,g'(t)f(t)dt,\\ \left({}_gF_{\mu,\alpha,l,w,b-}^{\gamma,\delta,v,c}f\right)(x;\tilde{p})&=\int_x^b(g(t)-g(x))^{\alpha-1}E_{\mu,\alpha,l}^{\gamma,\delta,v,c}(w(g(t)-g(x))^{\mu};\tilde{p})\,g'(t)f(t)dt.\end{aligned} \tag{1.5}$$

Definition 4. *(see [4]) A function* $f:[a,b]\to\mathbb{R}$ *is said to be convex, if*

$$f(tx+(1-t)y)\leq tf(x)+(1-t)f(y), \tag{1.6}$$

holds for all $x,y\in[a,b]$ *and* $t\in[0,1]$.

Definition 5. *(see [5]) Let* $I\subset(0,\infty)$ *be a real interval and* $p\in\mathbb{R}\setminus\{0\}$. *A function* $f:I\to\mathbb{R}$ *is said to be p-convex function, if*

$$f\left([tx^p+(1-t)y^p]^{\frac{1}{p}}\right)\leq tf(x)+(1-t)f(y), \tag{1.7}$$

holds for all $x,y\in I$ *and* $t\in[0,1]$.

Definition 6. *(see [6]) Let $p \in \mathbb{R} \setminus \{0\}$. Then a function $f : [a,b] \subset (0,\infty) \to \mathbb{R}$ is said to be p-symmetric with respect to $\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}$, if*

$$f\left(t^{\frac{1}{p}}\right) = f\left([a^p - b^p - t]^{\frac{1}{p}}\right), \tag{1.8}$$

for $t \in [0,1]$.

Following inequality is the well-known Hadamard inequality.

Theorem 1. *(see [7]) Let $f : [a,b] \to \mathbb{R}$ be a convex function for $a < b$. Then the following inequality holds:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.9}$$

The Fejér-Hadamard inequality is a weighted version of the Hadamard inequality given by Fejér in [8].

Theorem 2. *Let $f : [a,b] \to \mathbb{R}$ be a convex function and $g : [a,b] \to \mathbb{R}$ be non-negative, integrable and symmetric with respect to $\frac{a+b}{2}$. Then the following inequality holds:*

$$f\left(\frac{a+b}{2}\right)\int_a^b g(x)dx \leq \int_a^b f(x)g(x)dx \leq \frac{f(a)+f(b)}{2}\int_a^b g(x)dx. \tag{1.10}$$

The aim of this paper is to study the Hadamard and the Fejér-Hadamard inequalities for generalized k-fractional integrals containing Mittag-Leffler functions. In the subsequent section we deduce k-fractional integral operators containing Mittag-Leffler functions. In Section 3, we utilize these k-fractional integral operators to construct the k-fractional Hadamard and Fejér-Hadamard type inequalities.

2. k-fractional integral operators containing Mittag-Leffler function

We define k-fractional integral operators containing Mittag-Leffler function by setting $\phi(x) = x^{\alpha/k}$, $\alpha > k > 0$ in (1.4). Here we consider all involved parameters as real numbers.

Definition 7. *Let $f, g : [a,b] \to \mathbb{R}, 0 < a < b$, be the functions such that f be positive and $f \in L_1[a,b]$ and g be differentiable and strictly increasing. Also, let $\alpha > k > 0$ and $w, l, \gamma, c \in \mathbb{R}, c > \gamma > 0$ with $\tilde{p} \geq 0, \mu, \delta, l > 0$ and $0 < \nu \leq \delta + \mu$. Then for $x \in [a,b]$ the left and right generalized k-fractional integral operators $\left({}_g^kF^{\gamma,\delta,\nu,c}_{\mu,\alpha,l,w,a+}f\right)$ and $\left({}_g^kF^{\gamma,\delta,\nu,c}_{\mu,\alpha,l,w,b-}f\right)$ are defined by*

$$\left({}_g^kF^{\gamma,\delta,\nu,c}_{\mu,\alpha,l,w,a+}f\right)(x;\tilde{p}) = \int_a^x (g(x)-g(t))^{\frac{\alpha}{k}-1}E^{\gamma,\delta,\nu,c}_{\mu,\alpha,l}(w(g(x)-g(t))^{\mu};\tilde{p})\,g'(t)f(t)dt \tag{2.1}$$

and

$$\left({}_g^kF^{\gamma,\delta,\nu,c}_{\mu,\alpha,l,w,b-}f\right)(x;\tilde{p}) = \int_x^b (g(t)-g(x))^{\frac{\alpha}{k}-1}E^{\gamma,\delta,\nu,c}_{\mu,\alpha,l}(w(g(t)-g(x))^{\mu};\tilde{p})\,g'(t)f(t)dt. \tag{2.2}$$

Remark 1. *The following k-fractional integrals can be deduced from (2.1) and (2.2).*

(i) If we set $\tilde{p} = 0$ and $g(x) = x$ in Eqs (2.1) and (2.2), then we have

$$\left(F^{\gamma,\delta,\nu,c,k}_{\mu,\alpha,l,w,a+}f\right)(x) = \int_a^x (x-t)^{\frac{\alpha}{k}-1} E^{\gamma,\delta,\nu,c}_{\mu,\alpha,l}\left(w(x-t)^{\mu}\right) f(t)dt$$

and

$$\left(F^{\gamma,\delta,\nu,c,k}_{\mu,\alpha,l,w,b-}f\right)(x) = \int_x^b (t-x)^{\frac{\alpha}{k}-1} E^{\gamma,\delta,\nu,c}_{\mu,\alpha,l}\left(w(t-x)^{\mu}\right) f(t)dt.$$

(ii) If we set $l = \delta = 1$ and $g(x) = x$ in Eqs (2.1) and (2.2), then we have

$$\left(F^{\gamma,\nu,c,k}_{\mu,\alpha,w,a+}f\right)(x;\tilde{p}) = \int_a^x (x-t)^{\frac{\alpha}{k}-1} E^{\gamma,\nu,c}_{\mu,\alpha}\left(w(x-t)^{\mu};\tilde{p}\right) f(t)dt$$

and

$$\left(F^{\gamma,\nu,c,k}_{\mu,\alpha,w,b-}f\right)(x;\tilde{p}) = \int_x^b (t-x)^{\frac{\alpha}{k}-1} E^{\gamma,\nu,c}_{\mu,\alpha}\left(w(t-x)^{\mu};\tilde{p}\right) f(t)dt.$$

(iii) If we set $\tilde{p} = 0$, $l = \delta = 1$ and $g(x) = x$ in Eqs (2.1) and (2.2), then we have

$$\left(F^{\gamma,\nu,c,k}_{\mu,\alpha,w,a+}f\right)(x) = \int_a^x (x-t)^{\frac{\alpha}{k}-1} E^{\gamma,\nu,c}_{\mu,\alpha}\left(w(x-t)^{\mu}\right) f(t)dt$$

and

$$\left(F^{\gamma,\nu,c,k}_{\mu,\alpha,w,b-}f\right)(x) = \int_x^b (t-x)^{\frac{\alpha}{k}-1} E^{\gamma,\nu,c}_{\mu,\alpha}\left(w(t-x)^{\mu}\right) f(t)dt.$$

(iv) If we set $\tilde{p} = 0$, $l = \delta = \nu = 1$ and $g(x) = x$ in Eqs (2.1) and (2.2), then we have

$$\left(F^{\gamma,c,k}_{\mu,\alpha,w,a+}f\right)(x) = \int_a^x (x-t)^{\frac{\alpha}{k}-1} E^{\gamma,c}_{\mu,\alpha}\left(w(x-t)^{\mu}\right) f(t)dt$$

and

$$\left(F^{\gamma,c,k}_{\mu,\alpha,w,b-}f\right)(x) = \int_x^b (t-x)^{\frac{\alpha}{k}-1} E^{\gamma,c}_{\mu,\alpha}\left(w(t-x)^{\mu}\right) f(t)dt.$$

(v) For $\tilde{p} = w = 0$, $g(x) = x$ and $k = 1$ the Eqs (2.1) and (2.2) reduce to classical Riemann–Liouville fractional integral operators.

(vi) If we set $\gamma = \delta = l = \nu = 1$ and $w = \tilde{p} = 0$ in Eqs (2.1) and (2.2), Definition 1 of [9] is obtained.

Remark 2. *Some more definitions of integral operators are composed as follows:*

(i) In Remark 1 (i), if we take $w = 0$, (8) of [10] is obtained.

(ii) In Remark 1 (iii), if we take $\gamma = k = 1$, (2.2) of [11] is obtained.

(iii) In Remark 1 (iv), if we take $w = 0$ and $k = 1$, we get classical Riemann-Liouville fractional integral.

Remark 3.

(i) In Remark 2, if we take $w = 0$ and $c = 1$, Definition 4 of [12] is obtained.

3. k-fractional integral inequalities of Hadamard and Fejér-Hadamard type for p-convex function

First, we give the following generalized k-fractional Hadamard inequality.

Theorem 3. *Let $f, g : [a, b] \to \mathbb{R}$, $0 < a < b$, be the functions such that f positive and $f \in L_1[a, b]$ and g be differentiable and strictly increasing. If f is p-convex, $p \in \mathbb{R}\setminus\{0\}$, then the following inequalities for k-fractional integral operators (2.1) and (2.2) hold:*

(i) If $p > 0$, then

$$\begin{aligned} & f\left(\left[\frac{g^p(a)+g^p(b)}{2}\right]^{\frac{1}{p}}\right)\left({}_g^kF^{\gamma,\delta,v,c}_{\mu,\alpha,l,\bar{w},g^{-1}(g^p(a))+}1\right)\left(g^{-1}\left(g^p(b)\right);\tilde{p}\right) \\ & \leq \frac{1}{2}\left[\left({}_g^kF^{\gamma,\delta,v,c}_{\mu,\alpha,l,\bar{w},g^{-1}(g^p(a))+}f\circ\theta\right)\left(g^{-1}\left(g^p(b)\right);\tilde{p}\right)+\left({}_g^kF^{\gamma,\delta,v,c}_{\mu,\alpha,l,\bar{w},g^{-1}(g^p(b))-}f\circ\theta\right)\left(g^{-1}\left(g^p(a)\right);\tilde{p}\right)\right] \\ & \leq \frac{f(g(a))+f(g(b))}{2}\left({}_g^kF^{\gamma,\delta,v,c}_{\mu,\alpha,l,\bar{w},g^{-1}(g^p(a))+}1\right)\left(g^{-1}\left(g^p(b)\right);\tilde{p}\right), \end{aligned} \tag{3.1}$$

where $\bar{w} = \frac{w}{(g^p(b)-g^p(a))^\mu}$ and $\theta(t) = g^{\frac{1}{p}}(t)$ for all $t \in [a^p, b^p]$.

(ii) If $p < 0$, then

$$\begin{aligned} & f\left(\left[\frac{g^p(a)+g^p(b)}{2}\right]^{\frac{1}{p}}\right)\left({}_g^kF^{\gamma,\delta,v,c}_{\mu,\alpha,l,\bar{w},g^{-1}(g^p(a))-}1\right)\left(g^{-1}(g^p(b));\tilde{p}\right) \\ & \leq \frac{1}{2}\left[\left({}_g^kF^{\gamma,\delta,v,c}_{\mu,\alpha,l,\bar{w},g^{-1}(g^p(b))+}f\circ\theta\right)\left(g^{-1}\left(g^p(b)\right);\tilde{p}\right)+\left({}_g^kF^{\gamma,\delta,v,c}_{\mu,\alpha,l,\bar{w},g^{-1}(g^p(a))-}f\circ\theta\right)\left(g^{-1}\left(g^p(a)\right);\tilde{p}\right)\right] \\ & \leq \frac{f(g(a))+f(g(b))}{2}\left({}_g^kF^{\gamma,\delta,v,c}_{\mu,\alpha,l,\bar{w},g^{-1}(g^p(a))-}1\right)\left(g^{-1}\left(g^p(b)\right);\tilde{p}\right), \end{aligned} \tag{3.2}$$

where $\bar{w} = \frac{w}{(g^p(a)-g^p(b))^\mu}$ and $\theta(t) = g^{\frac{1}{p}}(t)$ for all $t \in [b^p, a^p]$.

Proof. We prove the assertion (i) as follows:

(i) Since f is p-convex on $[a, b]$, for all $x, y \in I$, we have

$$f\left(\left[\frac{g^p(x)+g^p(y)}{2}\right]^{\frac{1}{p}}\right) \leq \frac{f(g(x))+f(g(y))}{2}. \tag{3.3}$$

Setting $g(x) = (tg^p(a) + (1-t)g^p(b))^{\frac{1}{p}}$ and $g(y) = (tg^p(b) + (1-t)g^p(a))^{\frac{1}{p}}$ in above inequality, we have

$$f\left(\left[\frac{g^p(a)+g^p(b)}{2}\right]^{\frac{1}{p}}\right) \leq \frac{f\left(\left[tg^p(a)+(1-t)g^p(b)\right]^{\frac{1}{p}}\right)+f\left(\left[tg^p(b)+(1-t)g^p(a)\right]^{\frac{1}{p}}\right)}{2}. \tag{3.4}$$

Multiplying both sides of (3.4) by $2t^{\frac{\alpha}{k}-1}E^{\gamma,\delta,v,c}_{\mu,\alpha,l}(wt^\mu;\tilde{p})$ and then integrating over $[0, 1]$, we have

$$
\begin{aligned}
&2f\left(\left[\frac{g^{p}(a)+g^{p}(b)}{2}\right]^{\frac{1}{p}}\right)\int_{0}^{1}t^{\frac{\alpha}{k}-1}E_{\mu,\alpha,l}^{\gamma,\delta,v,c}(wt^{\mu};\tilde{p})\,dt \\
&\leq\int_{0}^{1}t^{\frac{\alpha}{k}-1}E_{\mu,\alpha,l}^{\gamma,\delta,v,c}(wt^{\mu};\tilde{p})\,f\left(\left[tg^{p}(a)+(1-t)g^{p}(b)\right]^{\frac{1}{p}}\right)dt \\
&+\int_{0}^{1}t^{\frac{\alpha}{k}-1}E_{\mu,\alpha,l}^{\gamma,\delta,v,c}(wt^{\mu};\tilde{p})\,f\left(\left[tg^{p}(b)+(1-t)g^{p}(a)\right]^{\frac{1}{p}}\right)dt.
\end{aligned}
\tag{3.5}
$$

Setting $g(m)=tg^{p}(a)+(1-t)g^{p}(b)$ and $g(n)=tg^{p}(b)+(1-t)g^{p}(a)$ in (3.5), we have

$$
\begin{aligned}
&2f\left(\left[\frac{g^{p}(a)+g^{p}(b)}{2}\right]^{\frac{1}{p}}\right)\int_{g^{-1}(g^{p}(a))}^{g^{-1}(g^{p}(b))}\left(g^{p}(b)-g(m)\right)^{\frac{\alpha}{k}-1}E_{\mu,\alpha,l}^{\gamma,\delta,v,c}\left(\bar{w}\left(g^{p}(b)-g(m)\right)^{\mu};\tilde{p}\right)g'(m)\,dm \\
&\leq\int_{g^{-1}(g^{p}(a))}^{g^{-1}(g^{p}(b))}\left(g^{p}(b)-g(m)\right)^{\frac{\alpha}{k}-1}E_{\mu,\alpha,l}^{\gamma,\delta,v,c}\left(\bar{w}\left(g^{p}(b)-g(m)\right)^{\mu};\tilde{p}\right)f\left(g^{\frac{1}{p}}(m)\right)g'(m)\,dm \\
&+\int_{g^{-1}(g^{p}(a))}^{g^{-1}(g^{p}(b))}\left(g(n)-g^{p}(a)\right)^{\frac{\alpha}{k}-1}E_{\mu,\alpha,l}^{\gamma,\delta,v,c}\left(\bar{w}\left(g(n)-g^{p}(a)\right)^{\mu};\tilde{p}\right)f\left(g^{\frac{1}{p}}(n)\right)g'(n)\,dn,
\end{aligned}
\tag{3.6}
$$

by using k-fractional integral operators (2.1) and (2.2), the first inequality of (3.1) is obtained.

Now to prove the second inequality of (3.1), again from p-convexity of f over $[0,1]$ and for $t\in[0,1]$, we have

$$
f\left(\left[tg^{p}(a)+(1-t)g^{p}(b)\right]^{\frac{1}{p}}\right)+f\left(\left[tg^{p}(b)+(1-t)g^{p}(a)\right]^{\frac{1}{p}}\right)\leq f(g(a))+f(g(b)).
\tag{3.7}
$$

Multiplying both sides of (3.7) by $t^{\frac{\alpha}{k}-1}E_{\mu,\alpha,l}^{\gamma,\delta,v,c}(wt^{\mu};\tilde{p})$ and then integrating over $[0,1]$, we have

$$
\begin{aligned}
&\int_{0}^{1}t^{\frac{\alpha}{k}-1}E_{\mu,\alpha,l}^{\gamma,\delta,v,c}(wt^{\mu};\tilde{p})\,f\left(\left[tg^{p}(a)+(1-t)g^{p}(b)\right]^{\frac{1}{p}}\right)dt \\
&+\int_{0}^{1}t^{\frac{\alpha}{k}-1}E_{\mu,\alpha,l}^{\gamma,\delta,v,c}(wt^{\mu};\tilde{p})\,f\left(\left[tg^{p}(b)+(1-t)g^{p}(a)\right]^{\frac{1}{p}}\right)dt \\
&\leq\left[f(g(a))+f(g(b))\right]\int_{0}^{1}t^{\frac{\alpha}{k}-1}E_{\mu,\alpha,l}^{\gamma,\delta,v,c}(wt^{\mu};\tilde{p})\,dt.
\end{aligned}
\tag{3.8}
$$

Setting $g(m)=tg^{p}(a)+(1-t)g^{p}(b)$ and $g(n)=tg^{p}(b)+(1-t)g^{p}(a)$ in (3.8), then by using k-fractional integral operators (2.1) and (2.2), the second inequality of (3.1) is obtained.

(ii) Proof is similar to the proof of *(i)*. □

Remark 4. *By setting $\tilde{p}=w=0$ in (3.1) and (3.2), Corollary 23 of [13] is obtained.*

Remark 5. *By using (3.1) and (3.2), some more k-fractional inequalities are presented as follows:*

(i) By setting $\tilde{p} = w = 0$ and $g = I$, we get

$$\begin{aligned}
&f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right)\int_{a^p}^{b^p}(b^p-x)^{\frac{\alpha}{k}-1}dx\\
&\le \frac{1}{2}\left[\int_{a^p}^{b^p}(b^p-x)^{\frac{\alpha}{k}-1}f\left(x^{\frac{1}{p}}\right)dx+\int_{a^p}^{b^p}(y-a^p)^{\frac{\alpha}{k}-1}f\left(y^{\frac{1}{p}}\right)dy\right]\\
&\le \frac{f(a)+f(b)}{2}\int_{a^p}^{b^p}(b^p-x)^{\frac{\alpha}{k}-1}dx.
\end{aligned}$$

(ii) By setting $\tilde{p} = 0$, $p = -1$ and $g = I$, we get

$$\begin{aligned}
&f\left(\frac{2ab}{a+b}\right)\int_{\frac{1}{a}}^{\frac{1}{b}}\left(\frac{1}{b}-x\right)^{\frac{\alpha}{k}-1}E_{\mu,\alpha,l}^{\gamma,\delta,v,c}\left(\bar{w}\left(\frac{1}{b}-x\right)^{\mu}\right)dx\\
&\le \frac{1}{2}\Bigg[\int_{\frac{1}{a}}^{\frac{1}{b}}\left(\frac{1}{b}-x\right)^{\frac{\alpha}{k}-1}E_{\mu,\alpha,l}^{\gamma,\delta,v,c}\left(\bar{w}\left(\frac{1}{b}-x\right)^{\mu}\right)f\left(x^{\frac{1}{p}}\right)dx\\
&+\int_{\frac{1}{a}}^{\frac{1}{b}}\left(y-\frac{1}{a}\right)^{\frac{\alpha}{k}-1}E_{\mu,\alpha,l}^{\gamma,\delta,v,c}\left(\bar{w}\left(y-\frac{1}{a}\right)^{\mu}\right)f\left(y^{\frac{1}{p}}\right)dx\Bigg]\\
&\le \frac{f(a)+f(b)}{2}\int_{\frac{1}{a}}^{\frac{1}{b}}\left(\frac{1}{b}-x\right)^{\frac{\alpha}{k}-1}E_{\mu,\alpha,l}^{\gamma,\delta,v,c}\left(\bar{w}\left(\frac{1}{b}-x\right)^{\mu}\right)dx.
\end{aligned}$$

(iii) By setting $g = I$ and $p = -1$, we get

$$\begin{aligned}
&f\left(\frac{2ab}{a+b}\right)\int_{\frac{1}{a}}^{\frac{1}{b}}\left(\frac{1}{b}-x\right)^{\frac{\alpha}{k}-1}E_{\mu,\alpha,l}^{\gamma,\delta,v,c}\left(\bar{w}\left(\frac{1}{b}-x\right)^{\mu};\tilde{p}\right)dx\\
&\le \frac{1}{2}\Bigg[\int_{\frac{1}{a}}^{\frac{1}{b}}\left(\frac{1}{b}-x\right)^{\frac{\alpha}{k}-1}E_{\mu,\alpha,l}^{\gamma,\delta,v,c}\left(\bar{w}\left(\frac{1}{b}-x\right)^{\mu};\tilde{p}\right)f\left(x^{\frac{1}{p}}\right)dx\\
&+\int_{\frac{1}{a}}^{\frac{1}{b}}\left(y-\frac{1}{a}\right)^{\frac{\alpha}{k}-1}E_{\mu,\alpha,l}^{\gamma,\delta,v,c}\left(\bar{w}\left(y-\frac{1}{a}\right)^{\mu};\tilde{p}\right)f\left(y^{\frac{1}{p}}\right)dx\Bigg]\\
&\le \frac{f(a)+f(b)}{2}\int_{\frac{1}{a}}^{\frac{1}{b}}\left(\frac{1}{b}-x\right)^{\frac{\alpha}{k}-1}E_{\mu,\alpha,l}^{\gamma,\delta,v,c}\left(\bar{w}\left(\frac{1}{b}-x\right)^{\mu};\tilde{p}\right)dx.
\end{aligned}$$

(iv) By setting $w = \tilde{p} = 0$; $p = -1$ and $g = I$, we get

$$\begin{aligned}
&f\left(\frac{2ab}{a+b}\right)\int_{\frac{1}{a}}^{\frac{1}{b}}\left(\frac{1}{b}-x\right)^{\frac{\alpha}{k}-1}dx\\
&\le \frac{1}{2}\left[\int_{\frac{1}{a}}^{\frac{1}{b}}\left(\frac{1}{b}-x\right)^{\frac{\alpha}{k}-1}f\left(x^{\frac{1}{p}}\right)dx+\int_{\frac{1}{a}}^{\frac{1}{b}}\left(y-\frac{1}{a}\right)^{\frac{\alpha}{k}-1}f\left(y^{\frac{1}{p}}\right)dy\right]\\
&\le \frac{f(a)+f(b)}{2}\int_{\frac{1}{a}}^{\frac{1}{b}}\left(\frac{1}{b}-x\right)^{\frac{\alpha}{k}-1}dx.
\end{aligned}$$

Corollary 1. *By using (3.1) and (3.2), some more k-fractional inequalities are presented as follows:*

(i) By setting $p = 1$, we get

$$f\left(\frac{g(a)+g(b)}{2}\right) \cdot \int_a^b (g(b)-g(x))^{\frac{\alpha}{k}-1} E_{\mu,\alpha,l}^{\gamma,\delta,v,c}(\bar{w}(g(b)-g(x))^{\mu};\tilde{p})\, g'(x)\,dx$$
$$\leq \frac{1}{2}\Big[\int_a^b (g(b)-g(x))^{\frac{\alpha}{k}-1} E_{\mu,\alpha,l}^{\gamma,\delta,v,c}(\bar{w}(g(b)-g(x))^{\mu};\tilde{p})\, f(g(x))\, g'(x)dx$$
$$+\int_a^b (g(y)-g(a))^{\frac{\alpha}{k}-1} E_{\mu,\alpha,l}^{\gamma,\delta,v,c}(\bar{w}(g(y)-g(a))^{\mu};\tilde{p})\, f(g(y))\, g'(y)\,dy\Big]$$
$$\leq \frac{f(g(a))+f(g(b))}{2}\int_a^b (g(b)-g(x))^{\frac{\alpha}{k}-1} E_{\mu,\alpha,l}^{\gamma,\delta,v,c}(\bar{w}(g(b)-g(x))^{\mu};\tilde{p})\, g'(x)\,dx.$$

(ii) By setting $p = -1$, we get

$$f\left(\left[\frac{g^{-1}(a)+g^{-1}(b)}{2}\right]^{-1}\right) \cdot \int_{g^{-1}(g^{-1}(a))}^{g^{-1}(g^{-1}(b))} \left(g^{-1}(b)-g(x)\right)^{\frac{\alpha}{k}-1} E_{\mu,\alpha,l}^{\gamma,\delta,v,c}\left(\bar{w}\left(g^{-1}(b)-g(x)\right)^{\mu};\tilde{p}\right) g'(x)\,dx$$
$$\leq \frac{1}{2}\Big[\int_{g^{-1}(g^{-1}(a))}^{g^{-1}(g^{-1}(b))} \left(g^{-1}(b)-g(x)\right)^{\frac{\alpha}{k}-1} E_{\mu,\alpha,l}^{\gamma,\delta,v,c}\left(\bar{w}\left(g^{-1}(b)-g(x)\right)^{\mu};\tilde{p}\right) f\left(g^{-1}(x)\right) g'(x)dx$$
$$+\int_{g^{-1}(g^{-1}(a))}^{g^{-1}(g^{-1}(b))} \left(g(y)-g^{-1}(a)\right)^{\frac{\alpha}{k}-1} E_{\mu,\alpha,l}^{\gamma,\delta,v,c}\left(\bar{w}\left(g(y)-g^{-1}(a)\right)^{\mu};\tilde{p}\right) f\left(g^{-1}(y)\right) g'(y)\,dy\Big]$$
$$\leq \frac{f(g(a))+f(g(b))}{2}\int_{g^{-1}(g^{-1}(a))}^{g^{-1}(g^{-1}(b))} \left(g^{-1}(b)-g(x)\right)^{\frac{\alpha}{k}-1} E_{\mu,\alpha,l}^{\gamma,\delta,v,c}\left(\bar{w}\left(g^{-1}(b)-g(x)\right)^{\mu};\tilde{p}\right) g'(x)\,dx.$$

(iii) By setting $p = -2$, we get

$$f\left(\sqrt{\frac{2g^2(a)g^2(b)}{g^2(a)+g^2(b)}}\right) \cdot \int_{g^{-1}(g^{-2}(a))}^{g^{-1}(g^{-2}(b))} \left(g^{-2}(b)-g(x)\right)^{\frac{\alpha}{k}-1} E_{\mu,\alpha,l}^{\gamma,\delta,v,c}\left(\bar{w}\left(g^{-2}(b)-g(x)\right)^{\mu};\tilde{p}\right) g'(x)\,dx$$
$$\leq \frac{1}{2}\Big[\int_{g^{-1}(g^{-2}(a))}^{g^{-1}(g^{-2}(b))} \left(g^{-2}(b)-g(x)\right)^{\frac{\alpha}{k}-1} E_{\mu,\alpha,l}^{\gamma,\delta,v,c}\left(\bar{w}\left(g^{-2}(b)-g(x)\right)^{\mu};\tilde{p}\right) f\left(\left(\sqrt{g(x)}\right)^{-1}\right) g'(x)dx$$
$$+\int_{g^{-1}(g^{-2}(a))}^{g^{-1}(g^{-2}(b))} \left(g(y)-g^{-2}(a)\right)^{\frac{\alpha}{k}-1} E_{\mu,\alpha,l}^{\gamma,\delta,v,c}\left(\bar{w}\left(g(y)-g^{-2}(a)\right)^{\mu};\tilde{p}\right) f\left(\left(\sqrt{g(y)}\right)^{-1}\right) g'(y)\,dy\Big]$$
$$\leq \frac{f(g(a))+f(g(b))}{2}\int_{g^{-1}(g^{-2}(a))}^{g^{-1}(g^{-2}(b))} \left(g^{-2}(b)-g(x)\right)^{\frac{\alpha}{k}-1} E_{\mu,\alpha,l}^{\gamma,\delta,v,c}\left(\bar{w}\left(g^{-2}(b)-g(x)\right)^{\mu};\tilde{p}\right) g'(x)\,dx.$$

Corollary 2. *The aforementioned k-fractional inequalities are further connected with already known results as follows:*

(i) By setting $k = 1$ in Remark 5 (i), Theorem 9 of [6] is obtained.

(ii) By setting $k = 1$ in Remark 5 (ii), Theorem 2.1 of [14] is obtained.

(iii) By setting $k = 1$ in Remark 5 (iii), Theorem 2.1 of [15] is obtained.

(iv) By setting $k = 1$ in Remark 5 (iv), Theorem 4 of [16] is obtained.

(v) By setting $k = 1$ in Remark 5 (v), Theorem 2.1 of [17] is obtained.

The following lemma is useful to present the Fejér-Hadamard inequality for generalized k-fractional integrals.

Lemma 1. *Let $f, g : [a, b] \to \mathbb{R}$, $0 < a < b$, be the functions such that f positive and $f \in L_1[a, b]$ and g be differentiable and strictly increasing. If f is p-convex, $p \in \mathbb{R}\setminus\{0\}$ and $f\left(g^{\frac{1}{p}}(x)\right) = f\left(\left[g^p(a) + g^p(b) - g(x)\right]^{\frac{1}{p}}\right)$, then for generalized k-fractional integral operators (2.1) and (2.2), we have:*

(i) If $p > 0$, then

$$\begin{aligned}
&\left({}_g^kF^{\gamma,\delta,\nu,c}_{\mu,\alpha,l,\bar{w},g^{-1}(g^p(a))+}f\circ\theta\right)\left(g^{-1}(g^p(b));\tilde{p}\right) = \left({}_g^kF^{\gamma,\delta,\nu,c}_{\mu,\alpha,l,\bar{w},g^{-1}(g^p(b))-}f\circ\theta\right)\left(g^{-1}(g^p(a));\tilde{p}\right)\\
&= \frac{1}{2}\left[\left({}_g^kF^{\gamma,\delta,\nu,c}_{\mu,\alpha,l,\bar{w},g^{-1}(g^p(a))+}f\circ\theta\right)\left(g^{-1}(g^p(b));\tilde{p}\right) + \left({}_g^kF^{\gamma,\delta,\nu,c}_{\mu,\alpha,l,\bar{w},g^{-1}(g^p(b))-}f\circ\theta\right)\left(g^{-1}(g^p(a));\tilde{p}\right)\right],
\end{aligned} \tag{3.9}$$

with $\theta(t) = g^{\frac{1}{p}}(t)$ for all $t \in [a^p, b^p]$.

(ii) If $p < 0$, then

$$\begin{aligned}
&\left({}_g^kF^{\gamma,\delta,\nu,c}_{\mu,\alpha,l,\bar{w},g^{-1}(g^p(b))+}f\circ\theta\right)\left(g^{-1}(g^p(a));\tilde{p}\right) = \left({}_g^kF^{\gamma,\delta,\nu,c}_{\mu,\alpha,l,\bar{w},g^{-1}(g^p(a))-}f\circ\theta\right)\left(g^{-1}(g^p(b));\tilde{p}\right)\\
&= \frac{1}{2}\left[\left({}_g^kF^{\gamma,\delta,\nu,c}_{\mu,\alpha,l,\bar{w},g^{-1}(g^p(b))+}f\circ\theta\right)\left(g^{-1}(g^p(a));\tilde{p}\right) + \left({}_g^kF^{\gamma,\delta,\nu,c}_{\mu,\alpha,l,\bar{w},g^{-1}(g^p(a))-}f\circ\theta\right)\left(g^{-1}(g^p(b));\tilde{p}\right)\right],
\end{aligned} \tag{3.10}$$

with $\theta(t) = g^{\frac{1}{p}}(t)$ for all $t \in [b^p, a^p]$.

Proof. We prove the assertion (i) as follows:

(i) By definition of generalized k-fractional integral operators (2.1) and (2.2), we have

$$\begin{aligned}
&\left({}_g^kF^{\gamma,\delta,\nu,c}_{\mu,\alpha,l,\bar{w},g^{-1}(g^p(a))+}f\circ\theta\right)\left(g^{-1}(g^p(b));\tilde{p}\right)\\
&= \int_{g^{-1}(g^p(a))}^{g^{-1}(g^p(b))} (g^p(b) - g(x))^{\frac{\alpha}{k}-1} E^{\gamma,\delta,\nu,c}_{\mu,\alpha,l}\left(\bar{w}(g^p(b) - g(x))^{\mu};\tilde{p}\right)(f\circ\theta)(x)\,g'(x)\,dx\\
&= \int_{g^{-1}(g^p(a))}^{g^{-1}(g^p(b))} (g^p(b) - g(x))^{\frac{\alpha}{k}-1} E^{\gamma,\delta,\nu,c}_{\mu,\alpha,l}\left(\bar{w}(g^p(b) - g(x))^{\mu};\tilde{p}\right) f\left(g^{\frac{1}{p}}(x)\right) g'(x)\,dx\\
&= \int_{g^{-1}(g^p(a))}^{g^{-1}(g^p(b))} (g^p(b) - g(x))^{\frac{\alpha}{k}-1} E^{\gamma,\delta,\nu,c}_{\mu,\alpha,l}\left(\bar{w}(g^p(b) - g(x))^{\mu};\tilde{p}\right) f\left(\left[g^p(a) + g^p(b) - g(x)\right]^{\frac{1}{p}}\right) g'(x)\,dx.
\end{aligned} \tag{3.11}$$

Setting $g(t) = g^p(a) + g^p(b) - g(x)$ in the above equation and using $f\left(g^{\frac{1}{p}}(x)\right) = f\left(\left[g^p(a) + g^p(b) - g(x)\right]^{\frac{1}{p}}\right)$, we have:

$$\left({}_{g}^{k}F^{\gamma,\delta,v,c}_{\mu,\alpha,l,\bar{w},g^{-1}(g^{p}(a))+}f\circ\theta\right)\left(g^{-1}\left(g^{p}\left(b\right)\right);\tilde{p}\right)$$
$$=\int_{g^{-1}(g^{p}(a))}^{g^{-1}(g^{p}(b))}\left(g(t)-g^{p}(a)\right)^{\frac{\alpha}{k}-1}E^{\gamma,\delta,v,c}_{\mu,\alpha,l}\left(\bar{w}\left(g\left(t\right)-g^{p}\left(a\right)\right)^{\mu};\tilde{p}\right)f\left(g^{\frac{1}{p}}\left(t\right)\right)g'\left(t\right)dt \tag{3.12}$$
$$=\int_{g^{-1}(g^{p}(a))}^{g^{-1}(g^{p}(b))}\left(g\left(t\right)-g^{p}\left(a\right)\right)^{\frac{\alpha}{k}-1}E^{\gamma,\delta,v,c}_{\mu,\alpha,l}\left(\bar{w}\left(g\left(t\right)-g^{p}\left(a\right)\right)^{\mu};\tilde{p}\right)\left(f\circ\theta\right)\left(t\right)g'\left(t\right)dt.$$

This implies

$$\left({}_{g}^{k}F^{\gamma,\delta,v,c}_{\mu,\alpha,l,\bar{w},g^{-1}(g^{p}(a))+}f\circ\theta\right)\left(g^{-1}\left(g^{p}\left(b\right)\right);\tilde{p}\right)=\left({}_{g}^{k}F^{\gamma,\delta,v,c}_{\mu,\alpha,l,\bar{w},g^{-1}(g^{p}(b))-}f\circ\theta\right)\left(g^{-1}\left(g^{p}\left(a\right)\right);\tilde{p}\right). \tag{3.13}$$

By adding $\left({}_{g}^{k}F^{\gamma,\delta,v,c}_{\mu,\alpha,l,\bar{w},g^{-1}(g^{p}(a))+}f\circ\theta\right)\left(g^{-1}\left(g^{p}\left(b\right)\right);\tilde{p}\right)$ on both sides of (3.13), we have

$$2\left({}_{g}^{k}F^{\gamma,\delta,v,c}_{\mu,\alpha,l,\bar{w},g^{-1}(g^{p}(a))+}f\circ\theta\right)\left(g^{-1}\left(g^{p}\left(b\right)\right);\tilde{p}\right)$$
$$=\left({}_{g}^{k}F^{\gamma,\delta,v,c}_{\mu,\alpha,l,\bar{w},g^{-1}(g^{p}(a))+}f\circ\theta\right)\left(g^{-1}\left(g^{p}\left(b\right)\right);\tilde{p}\right)+\left({}_{g}^{k}F^{\gamma,\delta,v,c}_{\mu,\alpha,l,\bar{w},g^{-1}(g^{p}(b))-}f\circ\theta\right)\left(g^{-1}\left(g^{p}\left(a\right)\right);\tilde{p}\right). \tag{3.14}$$

From Eqs (3.13) and (3.14), the result (3.9) can be obtained.

(ii) Proof is similar to the proof of *(i)*. □

The first version of the Fejér-Hadamard inequality for generalized k-fractional integrals is given as follows:

Theorem 4. *Let $f,g:[a,b]\rightarrow\mathbb{R}$, $0<a<b$, be the functions such that f positive and $f\in L_1[a,b]$ and g be differentiable and strictly increasing, h is a non-negative and integrable function. If f is p-convex, $p\in \mathrm{R}\backslash\{0\}$ and $f\left(g^{\frac{1}{p}}\left(x\right)\right)=f\left(\left[g^{p}\left(a\right)+g^{p}\left(b\right)-g\left(x\right)\right]^{\frac{1}{p}}\right)$, then the following inequalities for generalized k-fractional integral operators (2.1) and (2.2) hold:*

(i) If $p>0$, then

$$f\left(\left[\frac{g^{p}\left(a\right)+g^{p}\left(b\right)}{2}\right]^{\frac{1}{p}}\right)\left[\left({}_{g}^{k}F^{\gamma,\delta,v,c}_{\mu,\alpha,l,\bar{w},g^{-1}(g^{p}(a))+}h\circ\theta\right)\left(g^{-1}\left(g^{p}\left(b\right)\right);\tilde{p}\right)+\left({}_{g}^{k}F^{\gamma,\delta,v,c}_{\mu,\alpha,l,\bar{w},g^{-1}(g^{p}(b))-}h\circ\theta\right)\left(g^{-1}\left(g^{p}\left(a\right)\right);\tilde{p}\right)\right]$$
$$\leq\left({}_{g}^{k}F^{\gamma,\delta,v,c}_{\mu,\alpha,l,\bar{w},g^{-1}(g^{p}(a))+}f\circ h\circ\theta\right)\left(g^{-1}\left(g^{p}\left(b\right)\right);\tilde{p}\right)+\left({}_{g}^{k}F^{\gamma,\delta,v,c}_{\mu,\alpha,l,\bar{w},g^{-1}(g^{p}(b))-}f\circ h\circ\theta\right)\left(g^{-1}\left(g^{p}\left(a\right)\right);\tilde{p}\right) \tag{3.15}$$
$$\leq\frac{f\left(g\left(a\right)\right)+f\left(g\left(b\right)\right)}{2}\left[\left({}_{g}^{k}F^{\gamma,\delta,v,c}_{\mu,\alpha,l,\bar{w},g^{-1}(g^{p}(a))+}h\circ\theta\right)\left(g^{-1}\left(g^{p}\left(b\right)\right);\tilde{p}\right)+\left({}_{g}^{k}F^{\gamma,\delta,v,c}_{\mu,\alpha,l,\bar{w},g^{-1}(g^{p}(b))-}h\circ\theta\right)\left(g^{-1}\left(g^{p}\left(a\right)\right);\tilde{p}\right)\right],$$

where $\bar{w}=\frac{w}{(g^{p}(b)-g^{p}(a))^{\mu}}$ and $\theta\left(t\right)=g^{\frac{1}{p}}\left(t\right)$ for all $t\in[a^{p},b^{p}]$.

(ii) If $p < 0$, then

$$\begin{aligned} & f\left(\left[\frac{g^p(a)+g^p(b)}{2}\right]^{\frac{1}{p}}\right)\left[\left({}_g^kF^{\gamma,\delta,v,c}_{\mu,\alpha,l,\bar{w},g^{-1}(g^p(b))+}h\circ\theta\right)\left(g^{-1}\left(g^p(a)\right);\tilde{p}\right)+\left({}_g^kF^{\gamma,\delta,v,c}_{\mu,\alpha,l,\bar{w},g^{-1}(g^p(a))-}h\circ\theta\right)\left(g^{-1}\left(g^p(b)\right);\tilde{p}\right)\right] \\ & \leq \left({}_g^kF^{\gamma,\delta,v,c}_{\mu,\alpha,l,\bar{w},g^{-1}(g^p(b))+}f\circ h\circ\theta\right)\left(g^{-1}\left(g^p(a)\right);\tilde{p}\right)+\left({}_g^kF^{\gamma,\delta,v,c}_{\mu,\alpha,l,\bar{w},g^{-1}(g^p(a))-}f\circ h\circ\theta\right)\left(g^{-1}\left(g^p(b)\right);\tilde{p}\right) \\ & \leq \frac{f(g(a))+f(g(b))}{2}\left[\left({}_g^kF^{\gamma,\delta,v,c}_{\mu,\alpha,l,\bar{w},g^{-1}(g^p(b))+}h\circ\theta\right)\left(g^{-1}\left(g^p(a)\right);\tilde{p}\right)+\left({}_g^kF^{\gamma,\delta,v,c}_{\mu,\alpha,l,\bar{w},g^{-1}(g^p(a))-}h\circ\theta\right)\left(g^{-1}\left(g^p(b)\right);\tilde{p}\right)\right]. \end{aligned} \tag{3.16}$$

where $\bar{w} = \frac{w}{(g^p(a)-g^p(b))^\mu}$ and $\theta(t) = g^{\frac{1}{p}}(t)$ for all $t \in [b^p, a^p]$.

Proof. We prove the assertion (i) as follows:

(i) Multiplying both sides of (3.4) by $2t^{\frac{\alpha}{k}-1}E^{\gamma,\delta,v,c}_{\mu,\alpha,l}(wt^\mu;\tilde{p})\,h\left(\left[tg^p(a)+(1-t)\,g^p(b)\right]^{\frac{1}{p}}\right)$ and then integrating over $[0,1]$, we have

$$\begin{aligned} & 2f\left(\left[\frac{g^p(a)+g^p(b)}{2}\right]^{\frac{1}{p}}\right)\int_0^1 t^{\frac{\alpha}{k}-1}E^{\gamma,\delta,v,c}_{\mu,\alpha,l}(wt^\mu;\tilde{p})\,h\left(\left[tg^p(a)+(1-t)\,g^p(b)\right]^{\frac{1}{p}}\right)dt \\ & \leq \int_0^1 t^{\frac{\alpha}{k}-1}E^{\gamma,\delta,v,c}_{\mu,\alpha,l}(wt^\mu;\tilde{p})\,f\left(\left[tg^p(a)+(1-t)g^p(b)\right]^{\frac{1}{p}}\right)h\left(\left[tg^p(a)+(1-t)\,g^p(b)\right]^{\frac{1}{p}}\right)dt \\ & + \int_0^1 t^{\frac{\alpha}{k}-1}E^{\gamma,\delta,v,c}_{\mu,\alpha,l}(wt^\mu;\tilde{p})\,f\left(\left[tg^p(b)+(1-t)g^p(a)\right]^{\frac{1}{p}}\right)h\left(\left[tg^p(a)+(1-t)\,g^p(b)\right]^{\frac{1}{p}}\right)dt. \end{aligned} \tag{3.17}$$

Setting $g(x) = tg^p(a) + (1-t)g^p(b)$, that is, $g^p(a) + g^p(b) - g(x) = tg^p(b) + (1-t)g^p(a)$, in (3.17) and using $f\left(g^{\frac{1}{p}}(x)\right) = f\left(\left[g^p(a)+g^p(b)-g(x)\right]^{\frac{1}{p}}\right)$, we have

$$\begin{aligned} & 2f\left(\left[\frac{g^p(a)+g^p(b)}{2}\right]^{\frac{1}{p}}\right)\int_{g^{-1}(g^p(a))}^{g^{-1}(g^p(b))}\left(g^p(b)-g(xt)\right)^{\frac{\alpha}{k}-1}E^{\gamma,\delta,v,c}_{\mu,\alpha,l}\left(\bar{w}\left(g^p(b)-g(x)\right)^\mu;\tilde{p}\right)(h\circ\theta)(x)g'(x)dx \\ & \leq \int_{g^{-1}(g^p(a))}^{g^{-1}(g^p(b))}\left(g^p(b)-g(x)\right)^{\frac{\alpha}{k}-1}E^{\gamma,\delta,v,c}_{\mu,\alpha,l}\left(\bar{w}\left(g^p(b)-g(x)\right)^\mu;\tilde{p}\right)(f\circ\theta)(x)(h\circ\theta)(x)\,g'(x)\,dx \\ & + \int_{g^{-1}(g^p(a))}^{g^{-1}(g^p(b))}\left(g(x)-g^p(a)\right)^{\frac{\alpha}{k}-1}E^{\gamma,\delta,v,c}_{\mu,\alpha,l}\left(\bar{w}\left(g(x)-g^p(a)\right)^\mu;\tilde{p}\right)(f\circ\theta)(x)(h\circ\theta)(x)\,g'(x)\,dx. \end{aligned} \tag{3.18}$$

This implies

$$\begin{aligned} & 2f\left(\left[\frac{g^p(a)+g^p(b)}{2}\right]^{\frac{1}{p}}\right)\left({}_g^kF^{\gamma,\delta,v,c}_{\mu,\alpha,l,\bar{w},g^{-1}(g^p(a))+}h\circ\theta\right)\left(g^{-1}\left(g^p(b)\right);\tilde{p}\right) \\ & \leq \left({}_g^kF^{\gamma,\delta,v,c}_{\mu,\alpha,l,\bar{w},g^{-1}(g^p(a))+}f\circ h\circ\theta\right)\left(g^{-1}\left(g^p(b)\right);\tilde{p}\right)+\left({}_g^kF^{\gamma,\delta,v,c}_{\mu,\alpha,l,\bar{w},g^{-1}(g^p(b))-}f\circ h\circ\theta\right)\left(g^{-1}\left(g^p(a)\right);\tilde{p}\right). \end{aligned} \tag{3.19}$$

Using Lemma 3.1 *(i)* in above inequality, then first inequality of (3.15) is obtained.

Now to prove second inequality of (3.15), multiplying both sides of (3.7) by $t^{\frac{\alpha}{k}-1}E^{\gamma,\delta,\nu,c}_{\mu,\alpha,l}(wt^{\mu};\tilde{p})$ $h\left(\left[tg^{p}(a)+(1-t)g^{p}(b)\right]^{\frac{1}{p}}\right)$ and then integrating over [0, 1], we have

$$\begin{aligned}
&\int_0^1 t^{\frac{\alpha}{k}-1}E^{\gamma,\delta,\nu,c}_{\mu,\alpha,l}(wt^{\mu};\tilde{p})\,h\left(\left[tg^{p}(a)+(1-t)g^{p}(b)\right]^{\frac{1}{p}}\right)f\left(\left[tg^{p}(a)+(1-t)g^{p}(b)\right]^{\frac{1}{p}}\right)dt\\
&+\int_0^1 t^{\frac{\alpha}{k}-1}E^{\gamma,\delta,\nu,c}_{\mu,\alpha,l}(wt^{\mu};\tilde{p})\,h\left(\left[tg^{p}(a)+(1-t)g^{p}(b)\right]^{\frac{1}{p}}\right)f\left(\left[tg^{p}(b)+(1-t)g^{p}(a)\right]^{\frac{1}{p}}\right)dt \qquad (3.20)\\
&\leq \left[f(g(a))+f(g(b))\right]\int_0^1 t^{\frac{\alpha}{k}-1}E^{\gamma,\delta,\nu,c}_{\mu,\alpha,l}(wt^{\mu};\tilde{p})\,h\left(\left[tg^{p}(a)+(1-t)g^{p}(b)\right]^{\frac{1}{p}}\right)dt.
\end{aligned}$$

Setting $g(x)=tg^{p}(a)+(1-t)g^{p}(b)$ and using $f\left(g^{\frac{1}{p}}(x)\right)=f\left(\left[g^{p}(a)+g^{p}(b)-g(x)\right]^{\frac{1}{p}}\right)$ in (3.20), we have

$$\begin{aligned}
&\left({}^{k}_{g}F^{\gamma,\delta,\nu,c}_{\mu,\alpha,l,\bar{w},g^{-1}(g^{p}(a))+}f\circ h\circ\theta\right)\left(g^{-1}(g^{p}(b));\tilde{p}\right)+\left({}^{k}_{g}F^{\gamma,\delta,\nu,c}_{\mu,\alpha,l,\bar{w},g^{-1}(g^{p}(b))-}f\circ h\circ\theta\right)\left(g^{-1}(g^{p}(a));\tilde{p}\right)\\
&\leq\left[f(g(a))+f(g(b))\right]\left({}^{k}_{g}F^{\gamma,\delta,\nu,c}_{\mu,\alpha,l,\bar{w},g^{-1}(g^{p}(a))+}h\circ\theta\right)\left(g^{-1}(g^{p}(b));\tilde{p}\right).
\end{aligned} \qquad (3.21)$$

Using Lemma 3.1 *(i)* in above inequality, then second inequality of (3.15) is obtained.

(ii) Proof is similar to the proof of *(i)*. □

Remark 6. *By using (3.15) and (3.16), some more k-fractional inequalities are presented as follows:*
(i) By setting $\tilde{p}=0$ *and* $g=I$*, we get*

$$\begin{aligned}
&f\left(\left[\frac{a^{p}+b^{p}}{2}\right]^{\frac{1}{p}}\right)\int_{a^{p}}^{b^{p}}(b^{p}-x)^{\frac{\alpha}{k}-1}E^{\gamma,\delta,\nu,c}_{\mu,\alpha,l}\left(\bar{w}(b^{p}-x)^{\mu}\right)(h\circ\theta)(x)\,dx\\
&\leq\frac{1}{2}\Big[\int_{a^{p}}^{b^{p}}(b^{p}-x)^{\frac{\alpha}{k}-1}E^{\gamma,\delta,\nu,c}_{\mu,\alpha,l}\left(\bar{w}(b^{p}-x)^{\mu}\right)(f\circ\theta)(x)(h\circ\theta)(x)\,dx\\
&+\int_{a^{p}}^{b^{p}}(x-a^{p})^{\frac{\alpha}{k}-1}E^{\gamma,\delta,\nu,c}_{\mu,\alpha,l}\left(\bar{w}(x-a^{p})^{\mu};\tilde{p}\right)(f\circ\theta)(x)(h\circ\theta)(x)\,dx\Big]\\
&\leq\frac{f(a)+f(b)}{2}\int_{a^{p}}^{b^{p}}(b^{p}-x)^{\frac{\alpha}{k}-1}E^{\gamma,\delta,\nu,c}_{\mu,\alpha,l}\left(\bar{w}(b^{p}-x)^{\mu}\right)(h\circ\theta)(x)\,dx.
\end{aligned}$$

(ii) By setting $w=\tilde{p}=0$ *and* $g=I$*, we get*

$$\begin{aligned}
&f\left(\left[\frac{a^{p}+b^{p}}{2}\right]^{\frac{1}{p}}\right)\int_{a^{p}}^{b^{p}}(b^{p}-x)^{\frac{\alpha}{k}-1}(h\circ\theta)(x)\,dx\\
&\leq\frac{1}{2}\Big[\int_{a^{p}}^{b^{p}}(b^{p}-x)^{\frac{\alpha}{k}-1}(f\circ\theta)(x)(h\circ\theta)(x)\,dx\\
&+\int_{a^{p}}^{b^{p}}(x-a^{p})^{\frac{\alpha}{k}-1}(f\circ\theta)(x)(h\circ\theta)(x)\,dx\Big]\\
&\leq\frac{f(a)+f(b)}{2}\int_{a^{p}}^{b^{p}}(b^{p}-x)^{\frac{\alpha}{k}-1}(h\circ\theta)(x)\,dx.
\end{aligned}$$

(iii) By setting $\tilde{p}=0$, $h(x)=1$, $p=-1$ and $g=I$, we get

$$\begin{aligned}
&f\left(\frac{2ab}{a+b}\right)\int_{\frac{1}{a}}^{\frac{1}{b}}\left(\frac{1}{b}-x\right)^{\frac{\alpha}{k}-1}E_{\mu,\alpha,l}^{\gamma,\delta,v,c}\left(\bar{w}\left(\frac{1}{b}-x\right)^{\mu}\right)\theta(x)\,dx\\
&\leq\frac{1}{2}\Big[\int_{\frac{1}{a}}^{\frac{1}{b}}\left(\frac{1}{b}-x\right)^{\frac{\alpha}{k}-1}E_{\mu,\alpha,l}^{\gamma,\delta,v,c}\left(\bar{w}\left(\frac{1}{b}-x\right)^{\mu}\right)(f\circ\theta)(x)\,\theta(x)\,dx\\
&+\int_{\frac{1}{a}}^{\frac{1}{b}}\left(x-\frac{1}{a}\right)^{\frac{\alpha}{k}-1}E_{\mu,\alpha,l}^{\gamma,\delta,v,c}\left(\bar{w}\left(x-\frac{1}{a}\right)^{\mu}\right)(f\circ\theta)(x)\,\theta(x)\,dx\Big]\\
&\leq\frac{f(a)+f(b)}{2}\int_{\frac{1}{a}}^{\frac{1}{b}}\left(\frac{1}{b}-x\right)^{\frac{\alpha}{k}-1}E_{\mu,\alpha,l}^{\gamma,\delta,v,c}\left(\bar{w}\left(\frac{1}{b}-x\right)^{\mu}\right)\theta(x)\,dx.
\end{aligned}$$

(iv) By setting $g=I$, $h(x)=1$ and $p=-1$, we get

$$\begin{aligned}
&f\left(\frac{2ab}{a+b}\right)\int_{\frac{1}{a}}^{\frac{1}{b}}\left(\frac{1}{b}-x\right)^{\frac{\alpha}{k}-1}E_{\mu,\alpha,l}^{\gamma,\delta,v,c}\left(\bar{w}\left(\frac{1}{b}-x\right)^{\mu};\tilde{p}\right)\theta(x)\,dx\\
&\leq\frac{1}{2}\Big[\int_{\frac{1}{a}}^{\frac{1}{b}}\left(\frac{1}{b}-x\right)^{\frac{\alpha}{k}-1}E_{\mu,\alpha,l}^{\gamma,\delta,v,c}\left(\bar{w}\left(\frac{1}{b}-x\right)^{\mu};\tilde{p}\right)(f\circ\theta)(x)\theta(x)dx\\
&+\int_{\frac{1}{a}}^{\frac{1}{b}}\left(x-\frac{1}{a}\right)^{\frac{\alpha}{k}-1}E_{\mu,\alpha,l}^{\gamma,\delta,v,c}\left(\bar{w}\left(x-\frac{1}{a}\right)^{\mu};\tilde{p}\right)(f\circ\theta)(x)\,\theta(x)\,dx\Big]\\
&\leq\frac{f(a)+f(b)}{2}\int_{\frac{1}{a}}^{\frac{1}{b}}\left(\frac{1}{b}-x\right)^{\frac{\alpha}{k}-1}E_{\mu,\alpha,l}^{\gamma,\delta,v,c}\left(\bar{w}\left(\frac{1}{b}-x\right)^{\mu};\tilde{p}\right)\theta(x)\,dx.
\end{aligned}$$

(v) By setting $w=\tilde{p}=0$, $h(x)=1$, $p=-1$ and $g=I$, we get

$$\begin{aligned}
&f\left(\frac{2ab}{a+b}\right)\int_{\frac{1}{a}}^{\frac{1}{b}}\left(\frac{1}{b}-x\right)^{\frac{\alpha}{k}-1}\theta(x)\,dx\\
&\leq\frac{1}{2}\left[\int_{\frac{1}{a}}^{\frac{1}{b}}\left(\frac{1}{b}-x\right)^{\frac{\alpha}{k}-1}(f\circ\theta)(x)\,\theta(x)\,dx+\int_{\frac{1}{a}}^{\frac{1}{b}}\left(x-\frac{1}{a}\right)^{\frac{\alpha}{k}-1}(f\circ\theta)(x)\,\theta(x)\,dx\right]\\
&\leq\frac{f(a)+f(b)}{2}\int_{\frac{1}{a}}^{\frac{1}{b}}\left(\frac{1}{b}-x\right)^{\frac{\alpha}{k}-1}\theta(x)\,dx.
\end{aligned}$$

Corollary 3. *By using (3.15) and (3.16), some more k-fractional inequalities are presented as follows:*

(i) By setting $p = 1$, we get

$$f\left(\frac{g(a)+g(b)}{2}\right) \cdot \int_a^b (g(b)-g(x))^{\frac{\alpha}{k}-1} E_{\mu,\alpha,l}^{\gamma,\delta,v,c}(\bar{w}(g(b)-g(x))^{\mu};\tilde{p})(h\circ\theta)(x)g'(x)dx$$
$$\leq \frac{1}{2}\Big[\int_a^b (g(b)-g(x))^{\frac{\alpha}{k}-1} E_{\mu,\alpha,l}^{\gamma,\delta,v,c}(\bar{w}(g(b)-g(x))^{\mu};\tilde{p})(f\circ\theta)(x)(h\circ\theta)(x)g'(x)dx$$
$$+\int_a^b (g(x)-g(a))^{\frac{\alpha}{k}-1} E_{\mu,\alpha,l}^{\gamma,\delta,v,c}(\bar{w}(g(x)-g(a))^{\mu};\tilde{p})(f\circ\theta)(x)(h\circ\theta)(x)g'(x)dx\Big]$$
$$\leq \frac{f(g(a))+f(g(b))}{2} \cdot \int_a^b (g(b)-g(x))^{\frac{\alpha}{k}-1} E_{\mu,\alpha,l}^{\gamma,\delta,v,c}(\bar{w}(g(b)-g(x))^{\mu};\tilde{p})(h\circ\theta)(x)g'(x)dx.$$

(ii) By setting $p = -1$, we get

$$f\left(\left[\frac{g^{-1}(a)+g^{-1}(b)}{2}\right]^{-1}\right) \cdot \int_{g^{-1}(g^{-1}(a))}^{g^{-1}(g^{-1}(b))} \left(g^{-1}(b)-g(x)\right)^{\frac{\alpha}{k}-1} E_{\mu,\alpha,l}^{\gamma,\delta,v,c}\left(\bar{w}\left(g^{-1}(b)-g(x)\right)^{\mu};\tilde{p}\right)(h\circ\theta)(x)g'(x)dx$$
$$\leq \frac{1}{2}\Big[\int_{g^{-1}(g^{-1}(a))}^{g^{-1}(g^{-1}(b))} \left(g^{-1}(b)-g(x)\right)^{\frac{\alpha}{k}-1} E_{\mu,\alpha,l}^{\gamma,\delta,v,c}\left(\bar{w}\left(g^{-1}(b)-g(x)\right)^{\mu};\tilde{p}\right)(f\circ\theta)(x)(h\circ\theta)(x)g'(x)dx$$
$$+\int_{g^{-1}(g^{-1}(a))}^{g^{-1}(g^{-1}(b))} \left(g(x)-g^{-1}(a)\right)^{\frac{\alpha}{k}-1} E_{\mu,\alpha,l}^{\gamma,\delta,v,c}\left(\bar{w}\left(g(x)-g^{-1}(a)\right)^{\mu};\tilde{p}\right)(f\circ\theta)(x)(h\circ\theta)(x)g'(x)dx\Big]$$
$$\leq \frac{f(g(a))+f(g(b))}{2} \cdot \int_{g^{-1}(g^{-1}(a))}^{g^{-1}(g^{-1}(b))} \left(g^{-1}(b)-g(x)\right)^{\frac{\alpha}{k}-1} E_{\mu,\alpha,l}^{\gamma,\delta,v,c}\left(\bar{w}\left(g^{-1}(b)-g(x)\right)^{\mu};\tilde{p}\right)(h\circ\theta)(x)g'(x)dx.$$

(iii) By setting $p = -2$, we get

$$f\left(\left(\sqrt{\frac{2g^2(a)g^2(b)}{g^2(a)+g^2(b)}}\right)\right) \cdot \int_{g^{-1}(g^{-2}(a))}^{g^{-1}(g^{-2}(b))} \left(g^{-2}(b)-g(x)\right)^{\frac{\alpha}{k}-1} E_{\mu,\alpha,l}^{\gamma,\delta,v,c}\left(\bar{w}\left(g^{-2}(b)-g(x)\right)^{\mu};\tilde{p}\right)(h\circ\theta)(x)g'(x)dx$$
$$\leq \frac{1}{2}\Big[\int_{g^{-1}(g^{-2}(a))}^{g^{-1}(g^{-2}(b))} \left(g^{-2}(b)-g(x)\right)^{\frac{\alpha}{k}-1} E_{\mu,\alpha,l}^{\gamma,\delta,v,c}\left(\bar{w}\left(g^{-2}(b)-g(x)\right)^{\mu};\tilde{p}\right)(f\circ\theta)(x)(h\circ\theta)(x)g'(x)dx$$
$$+\int_{g^{-1}(g^{-2}(a))}^{g^{-1}(g^{-2}(b))} \left(g(x)-g^{-2}(a)\right)^{\frac{\alpha}{k}-1} E_{\mu,\alpha,l}^{\gamma,\delta,v,c}\left(\bar{w}\left(g(x)-g^{-2}(a)\right)^{\mu};\tilde{p}\right)(f\circ\theta)(x)(h\circ\theta)(x)g'(x)dx\Big]$$
$$\leq \frac{f(g(a))+f(g(b))}{2} \cdot \int_{g^{-1}(g^{-2}(a))}^{g^{-1}(g^{-2}(b))} \left(g^{-2}(b)-g(x)\right)^{\frac{\alpha}{k}-1} E_{\mu,\alpha,l}^{\gamma,\delta,v,c}\left(\bar{w}\left(g^{-2}(b)-g(x)\right)^{\mu};\tilde{p}\right)(h\circ\theta)(x)g'(x)dx.$$

Corollary 4. *The aforementioned k-fractional inequalities are further connected with already known results as follows:*

(i) By setting $k = 1$ in Remark 6 (i), Theorem 2.2 of [18] is obtained.

(ii) By setting $k = 1$ in Remark 6 (ii), Theorem 9 of [6] is obtained.

(iii) By setting $k = 1$ in Remark 6 (iii), Theorem 2.1 of [14] is obtained.

(iv) By setting $k = 1$ in Remark 6 (iv), Theorem 2.1 of [15] is obtained.

(v) By setting $k = 1$ in Remark 6 (v), Theorem 4 of [16] is obtained.

(vi) By setting $k = 1$ in Remark 6 (vi), Theorem 2.5 of [17] is obtained.

In the next theorem we present another version of the Hadamard inequality.

Theorem 5. *Let $f, g : [a, b] \to \mathbb{R}$, $0 < a < b$, be the functions such that f positive and $f \in L_1[a, b]$ and g be differentiable and strictly increasing. If f is p-convex, $p \in \mathbb{R} \setminus \{0\}$, then for generalized k-fractional integral operators (2.1) and (2.2), we have:*

(i) If $p > 0$, then

$$\begin{aligned}
&f\left(\left[\frac{g^p(a)+g^p(b)}{2}\right]^{\frac{1}{p}}\right)\left({}_g^kF^{\gamma,\delta,v,c}_{\mu,\alpha,l,\bar{\bar{w}},g^{-1}((g^p(a)+g^p(b))/2)+}1\right)\left(g^{-1}\left(g^p(b)\right);\tilde{p}\right)\\
&\leq \left({}_g^kF^{\gamma,\delta,v,c}_{\mu,\alpha,l,\bar{\bar{w}},g^{-1}((g^p(a)+g^p(b))/2)+}f\circ\theta\right)\left(g^{-1}\left(g^p(b)\right);\tilde{p}\right)+\left({}_g^kF^{\gamma,\delta,v,c}_{\mu,\alpha,l,\bar{\bar{w}},g^{-1}((g^p(a)+g^p(b))/2)-}f\circ\theta\right)\left(g^{-1}\left(g^p(a)\right);\tilde{p}\right)\\
&\leq \frac{f(g(a))+f(g(b))}{2}\left({}_g^kF^{\gamma,\delta,v,c}_{\mu,\alpha,l,\bar{\bar{w}},g^{-1}((g^p(a)+g^p(b))/2)+}1\right)\left(g^{-1}\left(g^p(b)\right);\tilde{p}\right),
\end{aligned} \tag{3.22}$$

where $\bar{\bar{w}} = \frac{2^\mu w}{(g^p(b)-g^p(a))^\mu}$ and $\theta(t) = g^{\frac{1}{p}}(t)$ for all $t \in [a^p, b^p]$.

(ii) If $p < 0$, then

$$\begin{aligned}
&f\left(\left[\frac{g^p(a)+g^p(b)}{2}\right]^{\frac{1}{p}}\right)\left({}_g^kF^{\gamma,\delta,v,c}_{\mu,\alpha,l,\bar{\bar{w}},g^{-1}((g^p(a)+g^p(b))/2)-}1\right)\left(g^{-1}\left(g^p(b)\right);\tilde{p}\right)\\
&\leq \left({}_g^kF^{\gamma,\delta,v,c}_{\mu,\alpha,l,\bar{\bar{w}},g^{-1}((g^p(a)+g^p(b))/2)+}f\circ\theta\right)\left(g^{-1}\left(g^p(a)\right);\tilde{p}\right)+\left({}_g^kF^{\gamma,\delta,v,c}_{\mu,\alpha,l,\bar{\bar{w}},g^{-1}((g^p(a)+g^p(b))/2)-}f\circ\theta\right)\left(g^{-1}\left(g^p(b)\right);\tilde{p}\right)\\
&\leq \frac{f(g(a))+f(g(b))}{2}\left({}_g^kF^{\gamma,\delta,v,c}_{\mu,\alpha,l,\bar{\bar{w}},g^{-1}((g^p(a)+g^p(b))/2)-}1\right)\left(g^{-1}\left(g^p(b)\right);\tilde{p}\right),
\end{aligned} \tag{3.23}$$

where $\bar{\bar{w}} = \frac{2^\mu w}{(g^p(b)-g^p(a))^\mu}$ and $\theta(t) = g^{\frac{1}{p}}(t)$ for all $t \in [b^p, a^p]$.

Proof. We prove the assertion (i) as follows:

(i) Setting $g(x) = [(\frac{t}{2})g^p(a) + (\frac{2-t}{2})g^p(b)]^{\frac{1}{p}}$ and $g(y) = [(\frac{t}{2})g^p(b) + (\frac{2-t}{2})g^p(a)]^{\frac{1}{p}}$ in (3.3), we have

$$\begin{aligned}
&f\left(\left[\frac{g^p(a)+g^p(b)}{2}\right]^{\frac{1}{p}}\right)\\
&\leq \frac{f\left(\left[\left(\frac{t}{2}\right)g^p(a)+\left(\frac{(2-t)}{2}\right)g^p(b)\right]^{\frac{1}{p}}\right)+f\left(\left[\left(\frac{t}{2}\right)g^p(b)+\left(\frac{(2-t)}{2}\right)g^p(a)\right]^{\frac{1}{p}}\right)}{2}.
\end{aligned} \tag{3.24}$$

Multiplying both sides of (3.24) by $2t^{\frac{\alpha}{k}-1}E^{\gamma,\delta,\nu,c}_{\mu,\alpha,l}(wt^{\mu};\tilde{p})$ and then integrating over $[0,1]$, we have

$$\begin{aligned}
&2f\left(\left[\frac{g^{p}(a)+g^{p}(b)}{2}\right]^{\frac{1}{p}}\right)\int_{0}^{1}t^{\frac{\alpha}{k}-1}E^{\gamma,\delta,\nu,c}_{\mu,\alpha,l}(wt^{\mu};\tilde{p})\,dt\\
&\leq\int_{0}^{1}t^{\frac{\alpha}{k}-1}E^{\gamma,\delta,\nu,c}_{\mu,\alpha,l}(wt^{\mu};\tilde{p})\,f\left(\left[\frac{t}{2}g^{p}(a)+\left(\frac{2-t}{2}\right)g^{p}(b)\right]^{\frac{1}{p}}\right)dt\\
&+\int_{0}^{1}t^{\frac{\alpha}{k}-1}E^{\gamma,\delta,\nu,c}_{\mu,\alpha,l}(wt^{\mu};\tilde{p})\,f\left(\left[\frac{t}{2}g^{p}(b)+\left(\frac{2-t}{2}\right)g^{p}(a)\right]^{\frac{1}{p}}\right)dt.
\end{aligned}\tag{3.25}$$

Setting $g(x)=[(\frac{t}{2})g^{p}(a)+(\frac{2-t}{2})g^{p}(b)]^{\frac{1}{p}}$ and $g(y)=[(\frac{t}{2})g^{p}(b)+(\frac{2-t}{2})g^{p}(a)]^{\frac{1}{p}}$ in (3.25), then by using k-fractional integral operators (2.1) and (2.2), the first inequality of (3.22) is obtained.

Now to prove the second inequality of (3.22), again from p-convexity of f over $[a,b]$ and for $t\in[0,1]$, we have

$$f\left(\left[\frac{t}{2}g^{p}(a)+\left(\frac{2-t}{2}\right)g^{p}(b)\right]^{\frac{1}{p}}\right)+f\left(\left[\frac{t}{2}g^{p}(b)+\left(\frac{2-t}{2}\right)g^{p}(a)\right]^{\frac{1}{p}}\right)\leq f(g(a))+f(g(b)).\tag{3.26}$$

Multiplying both sides of (3.26) by $t^{\frac{\alpha}{k}-1}E^{\gamma,\delta,\nu,c}_{\mu,\alpha,l}(wt^{\mu};\tilde{p})$ and then integrating over $[0,1]$, we have

$$\begin{aligned}
&\int_{0}^{1}t^{\frac{\alpha}{k}-1}E^{\gamma,\delta,\nu,c}_{\mu,\alpha,l}(wt^{\mu};\tilde{p})\,f\left(\left[\frac{t}{2}g^{p}(a)+\left(\frac{2-t}{2}\right)g^{p}(b)\right]^{\frac{1}{p}}\right)dt\\
&+\int_{0}^{1}t^{\frac{\alpha}{k}-1}E^{\gamma,\delta,\nu,c}_{\mu,\alpha,l}(wt^{\mu};\tilde{p})\,f\left(\left[\frac{t}{2}g^{p}(b)+\left(\frac{2-t}{2}\right)g^{p}(a)\right]^{\frac{1}{p}}\right)dt\\
&\leq[f(g(a))+f(g(b))]\int_{0}^{1}t^{\frac{\alpha}{k}-1}E^{\gamma,\delta,\nu,c}_{\mu,\alpha,l}(wt^{\mu};\tilde{p})\,dt.
\end{aligned}\tag{3.27}$$

Setting $g(m)=(\frac{t}{2})g^{p}(a)+(\frac{2-t}{2})g^{p}(b)$ and $g(n)=(\frac{t}{2})g^{p}(b)+(\frac{2-t}{2})g^{p}(a)$ in (3.27), then by using k-fractional integral operators (2.1) and (2.2), the second inequality of (3.22) is obtained.

(ii) Proof is similar to the proof of *(i)*. □

Remark 7. *By using (3.22) and (3.23), some more k-fractional inequalities are presented as follows:*
(i) By setting $\tilde{p}=0$, $p=-1$ *and* $g=I$, *we get*

$$\begin{aligned}
&f\left(\frac{2ab}{a+b}\right)\int_{\frac{2ab}{a+b}}^{\frac{1}{b}}\left(\frac{1}{b}-x\right)^{\frac{\alpha}{k}-1}E^{\gamma,\delta,\nu,c}_{\mu,\alpha,l}\left(\bar{\bar{w}}\left(\frac{1}{b}-x\right)^{\mu}\right)dx\\
&\leq\frac{1}{2}\Big[\int_{\frac{2ab}{a+b}}^{\frac{1}{b}}\left(\frac{1}{b}-x\right)^{\frac{\alpha}{k}-1}E^{\gamma,\delta,\nu,c}_{\mu,\alpha,l}\left(\bar{\bar{w}}\left(\frac{1}{b}-x\right)^{\mu}\right)f\left(x^{-1}\right)dx\\
&+\int_{\frac{1}{a}}^{\frac{2ab}{a+b}}\left(y-\frac{1}{a}\right)^{\frac{\alpha}{k}-1}E^{\gamma,\delta,\nu,c}_{\mu,\alpha,l}\left(\bar{\bar{w}}\left(y-\frac{1}{a}\right)^{\mu}\right)f\left(y^{-1}\right)dy\Big]\\
&\leq\frac{f(a)+f(b)}{2}\int_{\frac{2ab}{a+b}}^{\frac{1}{b}}\left(\frac{1}{b}-x\right)^{\frac{\alpha}{k}-1}E^{\gamma,\delta,\nu,c}_{\mu,\alpha,l}\left(\bar{\bar{w}}\left(\frac{1}{b}-x\right)^{\mu}\right)dx.
\end{aligned}$$

(ii) By setting $g = I$ and $p = -1$, we get

$$
\begin{aligned}
& f\left(\frac{2ab}{a+b}\right)\int_{\frac{2ab}{a+b}}^{\frac{1}{b}}\left(\frac{1}{b}-x\right)^{\frac{\alpha}{k}-1} E_{\mu,\alpha,l}^{\gamma,\delta,\nu,c}\left(\bar{\bar{w}}\left(\frac{1}{b}-x\right)^{\mu};\tilde{p}\right)dx \\
\leq\; & \frac{1}{2}\Big[\int_{\frac{2ab}{a+b}}^{\frac{1}{b}}\left(\frac{1}{b}-x\right)^{\frac{\alpha}{k}-1} E_{\mu,\alpha,l}^{\gamma,\delta,\nu,c}\left(\bar{\bar{w}}\left(\frac{1}{b}-x\right)^{\mu};\tilde{p}\right)f\left(x^{-1}\right)dx \\
+\; & \int_{\frac{1}{a}}^{\frac{2ab}{a+b}}\left(y-\frac{1}{a}\right)^{\frac{\alpha}{k}-1} E_{\mu,\alpha,l}^{\gamma,\delta,\nu,c}\left(\bar{\bar{w}}\left(y-\frac{1}{a}\right)^{\mu};\tilde{p}\right)f\left(y^{-1}\right)dy\Big] \\
\leq\; & \frac{f(a)+f(b)}{2}\int_{\frac{2ab}{a+b}}^{\frac{1}{b}}\left(\frac{1}{b}-x\right)^{\frac{\alpha}{k}-1} E_{\mu,\alpha,l}^{\gamma,\delta,\nu,c}\left(\bar{\bar{w}}\left(\frac{1}{b}-x\right)^{\mu};\tilde{p}\right)dx.
\end{aligned}
$$

Corollary 5. *By using (3.22) and (3.23), some more k-fractional inequalities are presented as follows:*

(i) By setting $p = 1$, we get

$$
\begin{aligned}
& f\left(\frac{g(a)+g(b)}{2}\right) \\
\cdot\; & \int_a^b (g(b)-g(x))^{\frac{\alpha}{k}-1} E_{\mu,\alpha,l}^{\gamma,\delta,\nu,c}\left(\bar{\bar{w}}(g(b)-g(x))^{\mu};\tilde{p}\right)g'(x)\,dx \\
\leq\; & \frac{1}{2}\Big[\int_a^b (g(b)-g(x))^{\frac{\alpha}{k}-1} E_{\mu,\alpha,l}^{\gamma,\delta,\nu,c}\left(\bar{\bar{w}}(g(b)-g(x))^{\mu};\tilde{p}\right)f(g(x))\,g'(x)\,dx \\
+\; & \int_a^b (g(y)-g(a))^{\frac{\alpha}{k}-1} E_{\mu,\alpha,l}^{\gamma,\delta,\nu,c}\left(\bar{\bar{w}}(g(y)-g(a))^{\mu};\tilde{p}\right)f(g(y))\,g'(y)\,dy\Big] \\
\leq\; & \frac{f(g(a))+f(g(b))}{2}\int_a^b (g(b)-g(x))^{\frac{\alpha}{k}-1} E_{\mu,\alpha,l}^{\gamma,\delta,\nu,c}\left(\bar{\bar{w}}(g(b)-g(x))^{\mu};\tilde{p}\right)g'(x)\,dx.
\end{aligned}
$$

(ii) By setting $p = -1$, we get

$$
\begin{aligned}
& f\left(\left[\frac{g^{-1}(a)+g^{-1}(b)}{2}\right]^{-1}\right) \\
\cdot\; & \int_{g^{-1}\left(\frac{g^{-1}(a)+g^{-1}(b)}{2}\right)}^{g^{-1}\left(g^{-1}(b)\right)}\left(g^{-1}(b)-g(x)\right)^{\frac{\alpha}{k}-1} E_{\mu,\alpha,l}^{\gamma,\delta,\nu,c}\left(\bar{\bar{w}}\left(g^{-1}(b)-g(x)\right)^{\mu};\tilde{p}\right)g'(x)\,dx \\
\leq\; & \frac{1}{2}\Big[\int_{g^{-1}\left(\frac{g^{-1}(a)+g^{-1}(b)}{2}\right)}^{g^{-1}\left(g^{-1}(b)\right)}\left(g^{-1}(b)-g(x)\right)^{\frac{\alpha}{k}-1} E_{\mu,\alpha,l}^{\gamma,\delta,\nu,c}\left(\bar{\bar{w}}\left(g^{-1}(b)-g(x)\right)^{\mu};\tilde{p}\right)f\left(g^{-1}(x)\right)g'(x)\,dx \\
+\; & \int_{g^{-1}\left(g^{-1}(a)\right)}^{g^{-1}\left(\frac{g^{-1}(a)+g^{-1}(b)}{2}\right)}\left(g(y)-g^{-1}(a)\right)^{\frac{\alpha}{k}-1} E_{\mu,\alpha,l}^{\gamma,\delta,\nu,c}\left(\bar{\bar{w}}\left(g(y)-g^{-1}(a)\right)^{\mu};\tilde{p}\right)f\left(g^{-1}(y)\right)g'(y)\,dy\Big] \\
\leq\; & \frac{f(g(a))+f(g(b))}{2}\int_{g^{-1}\left(\frac{g^{-1}(a)+g^{-1}(b)}{2}\right)}^{g^{-1}\left(g^{-1}(b)\right)}\left(g^{-1}(b)-g(x)\right)^{\frac{\alpha}{k}-1} E_{\mu,\alpha,l}^{\gamma,\delta,\nu,c}\left(\bar{\bar{w}}\left(g^{-1}(b)-g(x)\right)^{\mu};\tilde{p}\right)g'(x)\,dx.
\end{aligned}
$$

(iii) By setting $p = -2$, we get

$$
\begin{aligned}
& f\left(\sqrt{\frac{2g^2(a)g^2(b)}{g^2(a)+g^2(b)}}\right) \\
& \cdot \int_{g^{-1}\left(\frac{2g^2(a)g^2(b)}{g^2(a)+g^2(b)}\right)}^{g^{-1}\left(g^{-2}(b)\right)} \left(g^{-2}(b)-g(x)\right)^{\frac{\alpha}{k}-1} E_{\mu,\alpha,l}^{\gamma,\delta,v,c}\left(\bar{\bar{w}}\left(g^{-2}(b)-g(x)\right)^{\mu};\tilde{p}\right) g'(x)\,dx \\
& \leq \frac{1}{2}\Bigg[\int_{g^{-1}\left(\frac{2g^2(a)g^2(b)}{g^2(a)+g^2(b)}\right)}^{g^{-1}\left(g^{-2}(b)\right)} \left(g^{-2}(b)-g(x)\right)^{\frac{\alpha}{k}-1} E_{\mu,\alpha,l}^{\gamma,\delta,v,c}\left(\bar{\bar{w}}\left(g^{-2}(b)-g(x)\right)^{\mu};\tilde{p}\right) f\left(g^{-2}(x)\right) g'(x)\,dx \\
& + \int_{g^{-1}\left(g^{-2}(a)\right)}^{g^{-1}\left(\frac{2g^2(a)g^2(b)}{g^2(a)+g^2(b)}\right)} (g(y)-g(a))^{\frac{\alpha}{k}-1} E_{\mu,\alpha,l}^{\gamma,\delta,v,c}\left(\bar{\bar{w}}(g(y)-g(a))^{\mu};\tilde{p}\right) f\left(g^{-2}(y)\right) g'(y)\,dy\Bigg] \\
& \leq \frac{f(g(a))+f(g(b))}{2}\int_{g^{-1}\left(\frac{2g^2(a)g^2(b)}{g^2(a)+g^2(b)}\right)}^{g^{-1}\left(g^{-2}(b)\right)} \left(g^{-2}(b)-g(x)\right)^{\frac{\alpha}{k}-1} E_{\mu,\alpha,l}^{\gamma,\delta,v,c}\left(\bar{\bar{w}}\left(g^{-2}(b)-g(x)\right)^{\mu};\tilde{p}\right) g'(x)\,dx.
\end{aligned}
$$

Corollary 6. *The aforementioned k-fractional inequalities are further connected with already known results as follows:*

(i) By setting $k = 1$ in Remark 7 (i), Theorem 2.3 of [14] is obtained.

(ii) By setting $k = 1$ in Remark 7 (ii), Theorem 2.3 of [15] is obtained.

(iii) By setting $k = 1$ in Remark 7 (iii), Theorem 2.7 of [17] is obtained.

The second version of the Fejér-Hadamard inequality for generalized k-fractional integrals is given as follows:

Theorem 6. *Let $f, g : [a,b] \to \mathbb{R}$, $0 < a < b$, be the functions such that f positive and $f \in L_1[a,b]$ and g be differentiable and strictly increasing, h is a non-negative and integrable function. If f is p-convex, $p \in \mathbb{R}\backslash\{0\}$ and $f\left(g^{\frac{1}{p}}(x)\right) = f\left(\left[g^p(a)+g^p(b)-g(x)\right]^{\frac{1}{p}}\right)$, then the following inequalities for generalized k-fractional integral operators (2.1) and (2.2) hold:*

(i) If $p > 0$, then

$$
\begin{aligned}
& f\left(\left[\frac{g^p(a)+g^p(b)}{2}\right]^{\frac{1}{p}}\right)\left({}_g^kF_{\mu,\alpha,l,\bar{\bar{w}},g^{-1}\left(\frac{g^p(a)+g^p(b)}{2}\right)+}^{\gamma,\delta,v,c} h\circ\theta\right)\left(g^{-1}(g^p(b));\tilde{p}\right) \\
& \leq \left({}_g^kF_{\mu,\alpha,l,\bar{\bar{w}},g^{-1}\left(\frac{g^p(a)+g^p(b)}{2}\right)+}^{\gamma,\delta,v,c} f\circ h\circ\theta\right)\left(g^{-1}(g^p(b));\tilde{p}\right) \\
& + \left({}_g^kF_{\mu,\alpha,l,\bar{\bar{w}},g^{-1}\left(\frac{g^p(a)+g^p(b)}{2}\right)-}^{\gamma,\delta,v,c} f\circ h\circ\theta\right)\left(g^{-1}(g^p(a));\tilde{p}\right) \\
& \leq \frac{f(g(a))+f(g(b))}{2}\left({}_g^kF_{\mu,\alpha,l,\bar{\bar{w}},g^{-1}\left(\frac{g^p(a)+g^p(b)}{2}\right)+}^{\gamma,\delta,v,c} h\circ\theta\right)\left(g^{-1}(g^p(b));\tilde{p}\right),
\end{aligned}
\tag{3.28}
$$

where $\bar{\bar{w}} = \frac{2^{\mu}w}{(g^p(b)-g^p(a))^{\mu}}$ and $\theta(t) = g^{\frac{1}{p}}(t)$ for all $t \in [a^p, b^p]$.

(ii) If $p < 0$, then

$$
\begin{aligned}
& f\left(\left[\frac{g^{p}(a)+g^{p}(b)}{2}\right]^{\frac{1}{p}}\right)\left({}_{g}^{k}F_{\mu,\alpha,l,\bar{\bar{w}},g^{-1}\left(\frac{g^{p}(a)+g^{p}(b)}{2}\right)-}^{\gamma,\delta,v,c} h\circ\theta\right)\left(g^{-1}\left(g^{p}(b)\right);\tilde{p}\right) \\
& \leq \left({}_{g}^{k}F_{\mu,\alpha,l,\bar{\bar{w}},g^{-1}\left(\frac{g^{p}(a)+g^{p}(b)}{2}\right)+}^{\gamma,\delta,v,c} f\circ h\circ\theta\right)\left(g^{-1}\left(g^{p}(a)\right);\tilde{p}\right) \\
& + \left({}_{g}^{k}F_{\mu,\alpha,l,\bar{\bar{w}},g^{-1}\left(\frac{g^{p}(a)+g^{p}(b)}{2}\right)-}^{\gamma,\delta,v,c} f\circ h\circ\theta\right)\left(g^{-1}\left(g^{p}(b)\right);\tilde{p}\right) \\
& \leq \frac{f(g(a))+f(g(b))}{2}\left({}_{g}^{k}F_{\mu,\alpha,l,\bar{\bar{w}},g^{-1}\left(\frac{g^{p}(a)+g^{p}(b)}{2}\right)-}^{\gamma,\delta,v,c} h\circ\theta\right)\left(g^{-1}\left(g^{p}(b)\right);\tilde{p}\right),
\end{aligned}
\tag{3.29}
$$

where $\bar{\bar{w}} = \frac{2^{\mu}w}{(g^{p}(a)-g^{p}(b))^{\mu}}$ and $\theta(t) = g^{\frac{1}{p}}(t)$ for all $t \in [b^{p}, a^{p}]$.

Proof. We prove the first assertion as follows:

(i) Multiplying (3.24) by $2t^{\frac{\alpha}{k}-1}E_{\mu,\alpha,l}^{\gamma,\delta,v,c}(wt^{\mu};\tilde{p})\,h\left([(\frac{t}{2})g^{p}(a)+(\frac{2-t}{2})g^{p}(b)]^{\frac{1}{p}}\right)$ and then integrating over $[0,1]$, we have

$$
\begin{aligned}
& 2f\left(\left[\frac{g^{p}(a)+g^{p}(b)}{2}\right]^{\frac{1}{p}}\right) \\
& \cdot\int_{0}^{1} t^{\frac{\alpha}{k}-1}E_{\mu,\alpha,l}^{\gamma,\delta,v,c}(wt^{\mu};\tilde{p})\,h\left(\left[\frac{t}{2}g^{p}(a)+\left(\frac{2-t}{2}\right)g^{p}(b)\right]^{\frac{1}{p}}\right)dt \\
& \leq \int_{0}^{1} t^{\frac{\alpha}{k}-1}E_{\mu,\alpha,l}^{\gamma,\delta,v,c}(wt^{\mu};\tilde{p})\,f\left(\left[\frac{t}{2}g^{p}(a)+\left(\frac{2-t}{2}\right)g^{p}(b)\right]^{\frac{1}{p}}\right)h\left(\left[\frac{t}{2}g^{p}(a)+\left(\frac{2-t}{2}\right)g^{p}(b)\right]^{\frac{1}{p}}\right)dt \\
& + \int_{0}^{1} t^{\frac{\alpha}{k}-1}E_{\mu,\alpha,l}^{\gamma,\delta,v,c}(wt^{\mu};\tilde{p})\,f\left(\left[\frac{t}{2}g^{p}(b)+\left(\frac{2-t}{2}\right)g^{p}(a)\right]^{\frac{1}{p}}\right)h\left(\left[\frac{t}{2}g^{p}(a)+\left(\frac{2-t}{2}\right)g^{p}(b)\right]^{\frac{1}{p}}\right)dt.
\end{aligned}
\tag{3.30}
$$

Setting $g(x) = (\frac{t}{2})g^{p}(a) + (\frac{2-t}{2})g^{p}(b)$ and $g(y) = (\frac{t}{2})g^{p}(b) + (\frac{2-t}{2})g^{p}(a)$, that is, $g^{p}(a)+g^{p}(b)-g(x) = \left(\frac{t}{2}\right)g^{p}(b)+\left(\frac{2-t}{2}\right)g^{p}(a)$ in (3.30), then by using $f\left(g^{\frac{1}{p}}(x)\right) = f\left(\left[g^{p}(a)+g^{p}(b)-g(x)\right]^{\frac{1}{p}}\right)$ and k-fractional integral operators (2.1) and (2.2), the first inequality of (3.28) is obtained.
Now to prove the second inequality of (3.28), multiplying both sides of (3.26) by $2t^{\frac{\alpha}{k}-1}E_{\mu,\alpha,l}^{\gamma,\delta,v,c}(wt^{\mu};\tilde{p})$ $h\left([(\frac{t}{2})g^{p}(a)+(\frac{2-t}{2})g^{p}(b)]^{\frac{1}{p}}\right)$ and then integrating over $[0,1]$, we have

$$
\begin{aligned}
& \int_{0}^{1} t^{\frac{\alpha}{k}-1}E_{\mu,\alpha,l}^{\gamma,\delta,v,c}(wt^{\mu};\tilde{p})\,f\left(\left[\frac{t}{2}g^{p}(a)+\left(\frac{2-t}{2}\right)g^{p}(b)\right]^{\frac{1}{p}}\right)h\left(\left[\frac{t}{2}g^{p}(a)+\left(\frac{2-t}{2}\right)g^{p}(b)\right]^{\frac{1}{p}}\right)dt \\
& + \int_{0}^{1} t^{\frac{\alpha}{k}-1}E_{\mu,\alpha,l}^{\gamma,\delta,v,c}(wt^{\mu};\tilde{p})\,f\left(\left[\frac{t}{2}g^{p}(b)+\left(\frac{2-t}{2}\right)g^{p}(a)\right]^{\frac{1}{p}}\right)h\left(\left[\frac{t}{2}g^{p}(a)+\left(\frac{2-t}{2}\right)g^{p}(b)\right]^{\frac{1}{p}}\right)dt \\
& \leq [f(g(a))+f(g(b))]\int_{0}^{1} t^{\frac{\alpha}{k}-1}E_{\mu,\alpha,l}^{\gamma,\delta,v,c}(wt^{\mu};\tilde{p})\,h\left(\left[\frac{t}{2}g^{p}(a)+\left(\frac{2-t}{2}\right)g^{p}(b)\right]^{\frac{1}{p}}\right)dt.
\end{aligned}
\tag{3.31}
$$

Setting $g(x) = (\frac{t}{2})g^p(a) + (\frac{2-t}{2})g^p(b)$ and $g(y) = (\frac{t}{2})g^p(b) + (\frac{2-t}{2})g^p(a)$, that is, $g^p(a) + g^p(b) - g(x) = \left(\frac{t}{2}\right)g^p(b) + \left(\frac{2-t}{2}\right)g^p(a)$ in (3.31), then by using $f\left(g^{\frac{1}{p}}(x)\right) = f\left(\left[g^p(a) + g^p(b) - g(x)\right]^{\frac{1}{p}}\right)$ and k-fractional integral operators (2.1) and (2.2), the second inequality of (3.28) is obtained.

(ii) Proof is similar to the proof of *(i)*. □

Remark 8. *By setting $\tilde{p} = w = 0$ in (3.28) and (3.29), Corollary 31 of [13] is obtained.*

Remark 9. *By using (3.28) and (3.29), some more k-fractional inequalities are presented as follows:*

(i) By setting $\tilde{p} = 0$, $p = -1$ and $g = I$, we get

$$\begin{aligned}
&f\left(\frac{2ab}{a+b}\right)\int_{\frac{2ab}{a+b}}^{\frac{1}{b}}\left(\frac{1}{b}-x\right)^{\frac{\alpha}{k}-1}E_{\mu,\alpha,l}^{\gamma,\delta,\nu,c}\left(\bar{\bar{w}}\left(\frac{1}{b}-x\right)^{\mu}\right)h\left(x^{-1}\right)dx\\
&\leq\frac{1}{2}\Big[\int_{\frac{2ab}{a+b}}^{\frac{1}{b}}\left(\frac{1}{b}-x\right)^{\frac{\alpha}{k}-1}E_{\mu,\alpha,l}^{\gamma,\delta,\nu,c}\left(\bar{\bar{w}}\left(\frac{1}{b}-x\right)^{\mu}\right)f\left(x^{-1}\right)h\left(x^{-1}\right)dx\\
&+\int_{\frac{1}{a}}^{\frac{2ab}{a+b}}\left(x-\frac{1}{a}\right)^{\frac{\alpha}{k}-1}E_{\mu,\alpha,l}^{\gamma,\delta,\nu,c}\left(\bar{\bar{w}}\left(x-\frac{1}{a}\right)^{\mu}\right)f\left(x^{-1}\right)h\left(x^{-1}\right)dx\Big]\\
&\leq\frac{f(a)+f(b)}{2}\int_{\frac{2ab}{a+b}}^{\frac{1}{b}}\left(\frac{1}{b}-x\right)^{\frac{\alpha}{k}-1}E_{\mu,\alpha,l}^{\gamma,\delta,\nu,c}\left(\bar{\bar{w}}\left(\frac{1}{b}-x\right)^{\mu}\right)h\left(x^{-1}\right)dx.
\end{aligned}$$

(ii) By setting $g = I$ and $p = -1$, we get

$$\begin{aligned}
&f\left(\frac{2ab}{a+b}\right)\int_{\frac{2ab}{a+b}}^{\frac{1}{b}}\left(\frac{1}{b}-x\right)^{\frac{\alpha}{k}-1}E_{\mu,\alpha,l}^{\gamma,\delta,\nu,c}\left(\bar{\bar{w}}\left(\frac{1}{b}-x\right)^{\mu};\tilde{p}\right)h\left(x^{-1}\right)dx\\
&\leq\frac{1}{2}\Big[\int_{\frac{2ab}{a+b}}^{\frac{1}{b}}\left(\frac{1}{b}-x\right)^{\frac{\alpha}{k}-1}E_{\mu,\alpha,l}^{\gamma,\delta,\nu,c}\left(\bar{\bar{w}}\left(\frac{1}{b}-x\right)^{\mu};\tilde{p}\right)f\left(x^{-1}\right)h\left(x^{-1}\right)dx\\
&+\int_{\frac{1}{a}}^{\frac{2ab}{a+b}}\left(x-\frac{1}{a}\right)^{\frac{\alpha}{k}-1}E_{\mu,\alpha,l}^{\gamma,\delta,\nu,c}\left(\bar{\bar{w}}\left(x-\frac{1}{a}\right)^{\mu};\tilde{p}\right)f\left(x^{-1}\right)h\left(x^{-1}\right)dx\Big]\\
&\leq\frac{f(a)+f(b)}{2}\int_{\frac{2ab}{a+b}}^{\frac{1}{b}}\left(\frac{1}{b}-x\right)^{\frac{\alpha}{k}-1}E_{\mu,\alpha,l}^{\gamma,\delta,\nu,c}\left(\bar{\bar{w}}\left(\frac{1}{b}-x\right)^{\mu};\tilde{p}\right)h\left(x^{-1}\right)dx.
\end{aligned}$$

Corollary 7. *By using (3.28) and (3.29), some more k-fractional inequalities are presented as follows:*

(i) By setting $p = 1$, we get

$$f\left(\frac{g(a)+g(b)}{2}\right) \cdot \int_a^b (g(b)-g(x))^{\frac{\alpha}{k}-1} E_{\mu,\alpha,l}^{\gamma,\delta,v,c}(\bar{\bar{w}}(g(b)-g(x))^{\mu};\tilde{p})\, h(g(x))\, g'(x)dx$$
$$\leq \frac{1}{2}\Big[\int_{\frac{a+b}{2}}^b (g(b)-g(x))^{\frac{\alpha}{k}-1} E_{\mu,\alpha,l}^{\gamma,\delta,v,c}(\bar{\bar{w}}(g(b)-g(x))^{\mu};\tilde{p})\, f(g(x))\, h(g(x))\, g'(x)\,dx$$
$$+\int_a^{\frac{a+b}{2}} (g(x)-g(a))^{\frac{\alpha}{k}-1} E_{\mu,\alpha,l}^{\gamma,\delta,v,c}(\bar{\bar{w}}(g(x)-g(a))^{\mu};\tilde{p})\, f(g(x))\, h(g(x))\, g'(x)\,dx\Big]$$
$$\leq \frac{f(g(a))+f(g(b))}{2} \cdot \int_{\frac{a+b}{2}}^b (g(b)-g(x))^{\frac{\alpha}{k}-1} E_{\mu,\alpha,l}^{\gamma,\delta,v,c}(\bar{\bar{w}}(g(b)-g(x))^{\mu};\tilde{p})\, h(g(x))\, g'(x)\,dx.$$

(ii) By setting $p = -1$, we get

$$f\left(\left[\frac{g^{-1}(a)+g^{-1}(b)}{2}\right]^{-1}\right) \cdot \int_{g^{-1}\left(\frac{g^{-1}(a)+g^{-1}(b)}{2}\right)}^{g^{-1}(g^{-1}(b))} \left(g^{-1}(b)-g(x)\right)^{\frac{\alpha}{k}-1} E_{\mu,\alpha,l}^{\gamma,\delta,v,c}\left(\bar{\bar{w}}\left(g^{-1}(b)-g(x)\right)^{\mu};\tilde{p}\right) h\left(g^{-1}(x)\right) g'(x)dx$$
$$\leq \frac{1}{2}\Big[\int_{g^{-1}\left(\frac{g^{-1}(a)+g^{-1}(b)}{2}\right)}^{g^{-1}(g^{-1}(b))} \left(g^{-1}(b)-g(x)\right)^{\frac{\alpha}{k}-1} E_{\mu,\alpha,l}^{\gamma,\delta,v,c}\left(\bar{\bar{w}}\left(g^{-1}(b)-g(x)\right)^{\mu};\tilde{p}\right) f\left(g^{-1}(x)\right) h\left(g^{-1}(x)\right) g'(x)\,dx$$
$$+\int_{g^{-1}(g^{-1}(a))}^{g^{-1}\left(\frac{g^{-1}(a)+g^{-1}(b)}{2}\right)} \left(g(x)-g^{-1}(a)\right)^{\frac{\alpha}{k}-1} E_{\mu,\alpha,l}^{\gamma,\delta,v,c}\left(\bar{\bar{w}}\left(g(x)-g^{-1}(a)\right)^{\mu};\tilde{p}\right) f\left(g^{-1}(x)\right) h\left(g^{-1}(x)\right) g'(x)\,dx\Big]$$
$$\leq \frac{f(g(a))+f(g(b))}{2} \cdot \int_{g^{-1}\left(\frac{g^{-1}(a)+g^{-1}(b)}{2}\right)}^{g^{-1}(g^{-1}(b))} \left(g^{-1}(b)-g(x)\right)^{\frac{\alpha}{k}-1} E_{\mu,\alpha,l}^{\gamma,\delta,v,c}\left(\bar{\bar{w}}\left(g^{-1}(b)-g(x)\right)^{\mu};\tilde{p}\right) h\left(g^{-1}(x)\right) g'(x)\,dx.$$

(iii) By setting $p = -2$, we get

$$f\left(\sqrt{\frac{2g^2(a)g^2(b)}{g^2(a)+g^2(b)}}\right) \cdot \int_{g^{-1}\left(\frac{2g^2(a)g^2(b)}{g^2(a)+g^2(b)}\right)}^{g^{-1}(g^{-2}(b))} \left(g^{-2}(b)-g(x)\right)^{\frac{\alpha}{k}-1} E_{\mu,\alpha,l}^{\gamma,\delta,v,c}\left(\bar{\bar{w}}\left(g^{-2}(b)-g(x)\right)^{\mu};\tilde{p}\right) h\left(g^{-2}(x)\right) g'(x)dx$$
$$\leq \frac{1}{2}\Big[\int_{g^{-1}\left(\frac{2g^2(a)g^2(b)}{g^2(a)+g^2(b)}\right)}^{g^{-1}(g^{-2}(b))} \left(g^{-2}(b)-g(x)\right)^{\frac{\alpha}{k}-1} E_{\mu,\alpha,l}^{\gamma,\delta,v,c}\left(\bar{\bar{w}}\left(g^{-2}(b)-g(x)\right)^{\mu};\tilde{p}\right) f\left(g^{-2}(x)\right) h\left(g^{-2}(x)\right) g'(x)\,dx$$
$$+\int_{g^{-1}(g^{-2}(a))}^{g^{-1}\left(\frac{2g^2(a)g^2(b)}{g^2(a)+g^2(b)}\right)} \left(g(x)-g^{-2}(a)\right)^{\frac{\alpha}{k}-1} E_{\mu,\alpha,l}^{\gamma,\delta,v,c}\left(\bar{\bar{w}}\left(g(x)-g^{-2}(a)\right)^{\mu};\tilde{p}\right) f\left(g^{-2}(x)\right) h\left(g^{-2}(x)\right) g'(x)\,dx\Big]$$
$$\leq \frac{f(g(a))+f(g(b))}{2} \cdot \int_{g^{-1}\left(\frac{2g^2(a)g^2(b)}{g^2(a)+g^2(b)}\right)}^{g^{-1}(g^{-2}(b))} \left(g^{-2}(b)-g(x)\right)^{\frac{\alpha}{k}-1} E_{\mu,\alpha,l}^{\gamma,\delta,v,c}\left(\bar{\bar{w}}\left(g^{-2}(b)-g(x)\right)^{\mu};\tilde{p}\right) h\left(g^{-2}(x)\right) g'(x)\,dx.$$

Corollary 8. *The aforementioned k-fractional inequalities are further connected with already known results as follows:*

(i) By setting $k = 1$ in Remark 9 (i), Theorem 2.6 of [14] is obtained.

(ii) By setting $k = 1$ in Remark 9 (ii), Theorem 2.6 of [15] is obtained.

(iii) By setting $k = 1$ in Remark 9 (iii), Theorem 2.10 of [17] is obtained.

Corollary 9. *When we set $w = \tilde{p} = 0$, $p = -1$ and $g = I$ in Theorem 2.4, then we get the following inequalities.*

$$\begin{aligned} & f\left(\frac{2ab}{a+b}\right)\int_{\frac{2ab}{a+b}}^{\frac{1}{b}}\left(\frac{1}{b}-x\right)^{\frac{\alpha}{k}-1} h\left(x^{-1}\right)dx \\ & \leq \frac{1}{2}\left[\int_{\frac{2ab}{a+b}}^{\frac{1}{b}}\left(\frac{1}{b}-x\right)^{\frac{\alpha}{k}-1} f\left(x^{-1}\right)h\left(x^{-1}\right)dx + \int_{\frac{1}{a}}^{\frac{2ab}{a+b}}\left(x-\frac{1}{a}\right)^{\frac{\alpha}{k}-1} f\left(x^{-1}\right)h\left(x^{-1}\right)dx\right] \\ & \leq \frac{f(a)+f(b)}{2}\int_{\frac{2ab}{a+b}}^{\frac{1}{b}}\left(\frac{1}{b}-x\right)^{\frac{\alpha}{k}-1} h\left(x^{-1}\right)dx. \end{aligned} \tag{3.32}$$

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