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Research article

Applications of q-difference symmetric operator in harmonic univalent functions

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Abstract: In this paper, for the first time, we apply symmetric q-calculus operator theory to define symmetric Salagean q-differential operator. We introduce a new class $\widetilde{\mathcal{H}}_q^m(\alpha)$ of harmonic univalent functions f associated with newly defined symmetric Salagean q-differential operator for complex harmonic functions. A sufficient coefficient condition for the functions f to be sense preserving and univalent and in the same class is obtained. It is proved that this coefficient condition is necessary for the functions in its subclass $\overline{\widetilde{\mathcal{H}}_q^m(\alpha)}$ and obtain sharp coefficient bounds, distortion theorems and covering results. Furthermore, we also highlight some known consequence of our main results.

Keywords: univalent functions; harmonic functions; symmetric q-derivative operator; symmetric Salagean q-differential operator
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1. Introduction, definitions and motivation

Let $\mathcal{A}$ denote the class of functions that are analytic in the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$ and let $\mathcal{A}^0$ be the subclass of $\mathcal{A}$ consisting of functions h with the normalization $h(0) = h'(0) - 1 = 0$ and has the following series expansion,

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

A continuous function $f = u + iv$ is a complex valued harmonic function defined in U, where u and v are real harmonic functions in U. We can write $f(z) = h + \overline{g}$ where h and g are analytic in U (see [2]). We call h the analytic part and g the co-analytic part of f, where $h \in \mathcal{A}^0$ is given by (1.1) and $g \in \mathcal{A}$ has the following power series expansion (see, for details, [4, 5, 10]):

$$g(z) = \sum_{n=1}^{\infty} b_n z^n, \ \ |b_1| < 1. \tag{1.2}$$

A necessary and sufficient condition for f to be locally univalent and sense preserving in U is that $\left|h'(z)\right| > \left|g'(z)\right|$ in U.

We denote by $\mathcal{S}^*\mathcal{H}$ the class of functions $f = h+\overline{g}$, that are harmonic univalent and sense preserving in U and satisfies the normalization conditions. For $f = h + \overline{g} \in \mathcal{S}^*\mathcal{H}$, where $h(z)$ and $g(z)$ are given in (1.1) and (1.2). Note that $\mathcal{S}^*\mathcal{H}$ reduces to $\mathcal{S}^*$, the class of normalized analytic univalent functions if the co-analytic part of $f = h + \overline{g}$ is identically zero. Also, $\mathcal{SH}$ is the subclass of $\mathcal{S}^*\mathcal{H}$ consisting of functions f that map $U = \{z : |z| < 1\}$ onto starlike domain (see [27]) .

The fractional q-calculus is the q-extension of the ordinary fractional calculus and dates back to early 20th century. The theory of q-calculus operators are used in various areas of science such as ordinary fractional calculus, optimal control, q-difference and q-integral equations, and also in the Geometric Function Theory of complex analysis. Initially in 1908, Jackson [7] defined the q-analogue of derivative and integral operator as well as provided some of their applications. Further in [6] Ismail et al. gave the idea of q-extension of class of q-starlike functions after that Srivastava [28] studied q-calculus in the context of univalent functions theory. Kanas and Raducanu [14] introduced the q-analogue of Ruscheweyh differential operator and Srivastava in [30] studied q-starlike functions related with generalized conic domain. By using the concept of convolution Srivastava et al. [31] introduced q-analogue of Noor integral operator and studied some of its applications, also Srivastava et al. also published a set of articles in which they concentrated class of q-starlike functions from different aspects (see [32, 33, 35, 36]). Additionally, a recently published survey-cum-expository review article by Srivastava [29] is potentially useful for researchers and scholars working on these topics. For some more recent investigation about q-calculus we may refer to [20, 22, 23, 37–39].

The theory of symmetric q-calculus has been applied to many areas of mathematics and physics such as fractional calculus and quantum physics. The symmetric q-calculus has proven to be valuable in a few areas, specially in quantum mechanics [3, 24]. Recently in [13] Kanas et al. defined a symmetric operator of q-derivative and introduced a new family of univalent functions. Furthermore Shahid et al. [21] used the concepts of symmetric q-calculus operator theory and defined symmetric conic domains. But research on symmetric q-calculus in connection with Geometric Function Theory and especially harmonic univalent functions is fairly new and not much is published on this topic.

Jahangiri in [9] applied certain q-calculus operators to complex harmonic functions, while Porwal and Gupta discussed applications of q-calculus to harmonic univalent functions in [25]. Srivastava [34] defined the q-analogue of derivative operator as well as provided some of its applications to complex harmonic functions. In this article, first time we apply symmetric q-calculus theory in order to define symmetric Salagean q-differential operator to complex harmonic functions and introduce a new class of harmonic univalent functions.

For better understanding of the article, we recall some concept details and definitions of the symmetric q-difference calculus. We suppose throughout in this paper that $0 < q < 1$ and that

$$\mathbb{N} = \{1, 2, 3, ...\} = \mathbb{N}_0 \backslash \{0\} \quad (\mathbb{N}_0 = \{0, 1, 2, 3, ...\}).$$

Definition 1.1. For $n \in \mathbb{N}$, the symmetric q-number is defined by:

$$\widetilde{[n]}_q = \frac{q^{-n} - q^n}{q^{-1} - q}, \quad \widetilde{[0]}_q = 0.$$

We note that the symmetric q-number do not reduce to the q-number, and frequently occurs in the study of q-deformed quantum mechanical simple harmonic oscillator (see [1]).

Definition 1.2. For any $n \in \mathbb{Z}^+ \cup \{0\}$, the symmetric q-number shift factorial is defined by:

$$\widetilde{[n]}_q! = \begin{cases} \widetilde{[n]}_q \widetilde{[n-1]}_q \widetilde{[n-2]}_q ... \widetilde{[2]}_q \widetilde{[1]}_q, & n \geq 1, \\ \\ 1 & n = 0. \end{cases}$$

Note That

$$\lim_{q \to 1-} \widetilde{[n]}_q! = n!.$$

Definition 1.3. The symmetric q-derivative (q-difference) operator of symmetric q-calculus operated on the function h ([12]) is defined by

$$\begin{aligned} \widetilde{\partial}_q h(z) &= \frac{1}{z}\left(\frac{h(qz) - h(q^{-1}z)}{q - q^{-1}}\right), \quad z \in U, \qquad (1.3) \\ &= 1 + \sum_{n=1}^{\infty} \widetilde{[n]}_q a_n z^{n-1}, \quad (z \neq 0, \ q \neq 1), \end{aligned}$$

and

$$\widetilde{\partial}_q z^n = \widetilde{[n]}_q z^{n-1}, \quad \widetilde{\partial}_q \left\{\sum_{n=1}^{\infty} a_n z^n\right\} = \sum_{n=1}^{\infty} \widetilde{[n]}_q a_n z^{n-1}.$$

We can observe that

$$\lim_{q \to 1-} \widetilde{\partial}_q h(z) = h'(z).$$

A successive application of the symmetric q-derivative (q-difference) operator of symmetric q-calculus as defined in (1.3) leads to symmetric Salagean q-differential operator which is define as:

Definition 1.4. The symmetric Salagean q-differential operator of h is defined by

$$\begin{aligned}
\widetilde{D}_q^0 h(z) &= h(z),\ \widetilde{D}_q^1 h(z) = z\widetilde{\partial}_q h(z) = \frac{h(qz)-h(q^{-1}z)}{q-q^{-1}}, ..., \\
\widetilde{D}_q^m h(z) &= z\widetilde{\partial}_q \widetilde{D}_q^{m-1} h(z) = h(z) * \left(z + \sum_{n=2}^{\infty} \widetilde{[n]}_q^m z^n \right) \\
&= z + \sum_{n=2}^{\infty} \widetilde{[n]}_q^m a_n z^n,
\end{aligned}$$

where m is a positive integer and the operator $*$ stands for the Hadamard product or convolution of two analytic power series. The operator $\widetilde{D}_q^m h(z)$ is called symmetric Salagean q-differential operator.

Note that

$$\lim_{q\to 1-} \widetilde{D}_q^m h(z) = z + \sum_{n=2}^{\infty} n^m a_n z^n,$$

which is the famous Salagean operator defined in [26].

It is the aim of this article to define the symmetric q-derivative (q-difference) operator by using symmetric q-calculus on the complex functions that are harmonic in U and obtain sharp coefficient bounds, distortion theorems and covering results.

Definition 1.5. We define the symmetric Salagean q-differential operator for harmonic function $f = h + \overline{g}$ as follows:

$$\widetilde{D}_q^m f(z) = \widetilde{D}_q^m h(z) + (-1)^m \, \widetilde{D}_q^m \overline{g(z)}, \tag{1.4}$$

where

$$\begin{aligned}
\widetilde{D}_q^m h(z) &= z + \sum_{n=2}^{\infty} \widetilde{[n]}_q^m a_n z^n, \\
\widetilde{D}_q^m g(z) &= \sum_{n=1}^{\infty} \widetilde{[n]}_q^m b_n z^n.
\end{aligned}$$

Remark 1.6. It is easy to see that, for $q \to 1-$, we obtain symmetric Salagean differential operator for harmonic functions $f = h + \overline{g}$.

Definition 1.7. For $0 \le \alpha < 1$, let $\widetilde{\mathcal{H}}_q^m(\alpha)$ denote the family of harmonic functions $f = h + \overline{g}$ which satisfy the condition

$$\Re \left\{ \frac{\widetilde{D}_q^{m+1} f(z)}{\widetilde{D}_q^m f(z)} \right\} \ge \alpha, \tag{1.5}$$

where $\widetilde{D}_q^m f(z)$ is given by (1.4). Further, denote by $\overline{\widetilde{\mathcal{H}}_q^m(\alpha)}$ the subclass of $\widetilde{\mathcal{H}}_q^m(\alpha)$ consisting of harmonic functions $f = h + \overline{g}$ so that h and g are of the form

$$h(z) = z - \sum_{n=2}^{\infty} a_n z^n, \text{ and } g(z) = \sum_{n=1}^{\infty} b_n z^n, \ \ |b_1| < 1. \tag{1.6}$$

where $a_n, b_n \ge 0$.

2. Main results

In the following theorem we shall determine coefficient bounds for harmonic functions belonging to the classes $\widetilde{\mathcal{H}}_q^m(\alpha)$ and $\overline{\widetilde{\mathcal{H}}_q^m}(\alpha)$.

Theorem 2.1. *For* $0 \le \alpha < 1$ *and* $f = h + \overline{g}$*, let*

$$\sum_{n=2}^{\infty} \widetilde{[n]}_q^m \left(\widetilde{[n]}_q - \alpha\right) |a_n| + \sum_{n=1}^{\infty} \widetilde{[n]}_q^m \left(\widetilde{[n]}_q + \alpha\right) |b_n| \le 1 - \alpha, \tag{2.1}$$

where h and g are respectively given by (1.1) and (1.2). Then

(i) f is harmonic univalent in U and $f \in \widetilde{\mathcal{H}}_q^m(\alpha)$ if the inequality (2.1) holds.

(ii) f is harmonic univalent in U and $f \in \overline{\widetilde{\mathcal{H}}_q^m}(\alpha)$ if and only if the inequality (2.1) holds.

The equality in (2.1) occurs for harmonic function

$$f(z) = z + \sum_{n=2}^{\infty} \frac{1-\alpha}{\widetilde{[n]}_q^m \left(\widetilde{[n]}_q - \alpha\right)} x_n z^n + \overline{\sum_{n=1}^{\infty} \frac{1-\alpha}{\widetilde{[n]}_q^m \left(\widetilde{[n]}_q + \alpha\right)} y_n z^n},$$

where

$$\sum_{n=2}^{\infty} |x_n| + \sum_{n=1}^{\infty} |y_n| = 1.$$

Proof. For part (i): First we need to show that $f = h + \overline{g}$ is locally univalent and orientation-preserving in U. It suffices to show that the second complex dilatation w of f satisfies $|w| = \left|g'/h'\right| < 1$ in U. This is the case since for $z = re^{i\theta} \in U$. We have

$$\begin{aligned}
\left|\widetilde{\partial}_q h(z)\right| &\ge 1 - \sum_{n=2}^{\infty} \widetilde{[n]}_q |a_n| r^{n-1} > 1 - \sum_{n=2}^{\infty} \widetilde{[n]}_q |a_n| \ge 1 - \sum_{n=2}^{\infty} \frac{\widetilde{[n]}_q^m \left(\widetilde{[n]}_q - \alpha\right)}{1-\alpha} |a_n| \\
&\ge \sum_{n=1}^{\infty} \frac{\widetilde{[n]}_q^m \left(\widetilde{[n]}_q + \alpha\right)}{1-\alpha} |b_n| \ge \sum_{n=1}^{\infty} \widetilde{[n]}_q |b_n| \ge \sum_{n=1}^{\infty} \widetilde{[n]}_q |b_n| r^{n-1} \ge \left|\widetilde{\partial}_q g(z)\right|
\end{aligned}$$

in U which implies as $q \to 1^-$ that $\left|h'(z)\right| > \left|g'(z)\right|$ in U that is the function f is locally univalent and sense-preserving in U. To show that $f = h + \overline{g}$ is univalent in U we use an argument that is due to author [8]. Suppose z_1 and z_2 are in U so that $z_1 \ne z_2$. Since U is simply connected and convex, we have $z(t) = (1-t) z_1 + t z_2 \in U$ for $0 \le t \le 1$. Then for $z_1 - z_2 \ne 0$, we can write

$$\Re \frac{f(z_2) - f(z_1)}{z_2 - z_1} > \int_0^1 \left(\Re \widetilde{\partial}_q h(z(t)) - \left|\widetilde{\partial}_q g(z(t))\right|\right) dt.$$

On the other hand, we observe that

$$\Re\left(\widetilde{\partial}_q h(z)\right) - \left|\widetilde{\partial}_q g(z)\right| \ge \Re \widetilde{\partial}_q h(z) - \sum_{n=1}^{\infty} \widetilde{[n]}_q |b_n| \ge 1 - \sum_{n=2}^{\infty} \widetilde{[n]}_q |a_n| - \sum_{n=1}^{\infty} \widetilde{[n]}_q |b_n|$$

$$\geq \quad 1-\sum_{n=2}^{\infty}\frac{\widetilde{[n]}_q^m\left(\widetilde{[n]}_q-\alpha\right)}{1-\alpha}|a_n|-\sum_{n=1}^{\infty}\frac{\widetilde{[n]}_q^m\left(\widetilde{[n]}_q+\alpha\right)}{1-\alpha}|b_n|\geq 0.$$

Therefore, $f = h + \overline{g}$ is univalent in U. It remains to show that the inequality (1.5) holds if the coefficients of the univalent harmonic function $f = h + \overline{g}$ satisfy the condition (2.1). In other words, for $0 \leq \alpha < 1$, we need to show that

$$\Re\left(\frac{\widetilde{D}_q^{m+1}f(z)}{\widetilde{D}_q^m f(z)}\right)=\Re\left(\frac{\widetilde{D}_q^{m+1}h(z)+(-1)^{m+1}\,\widetilde{D}_q^{m+1}\overline{g(z)}}{\widetilde{D}_q^m h(z)+(-1)^m\,\widetilde{D}_q^m\overline{g(z)}}\right)\geq\alpha.$$

Using the fact that $\Re(w) \geq \alpha$ if and only if $|1-\alpha+w| \geq |1+\alpha-w|$, it suffices to show that

$$\left|\widetilde{D}_q^{m+1}f(z)+(1-\alpha)\,\widetilde{D}_q^m f(z)\right|-\left|\widetilde{D}_q^{m+1}f(z)-(1+\alpha)\,\widetilde{D}_q^m f(z)\right|\geq 0. \tag{2.2}$$

Substituting for

$$\widetilde{D}_q^m f(z)=z+\sum_{n=2}^{\infty}\widetilde{[n]}_q^m a_n z^n+(-1)^m\overline{\sum_{n=1}^{\infty}\widetilde{[n]}_q^m b_n z^n}$$

and

$$\widetilde{D}_q^{m+1} f(z)=z+\sum_{n=2}^{\infty}\widetilde{[n]}_q^{m+1} a_n z^n+(-1)^{m+1}\overline{\sum_{n=1}^{\infty}\widetilde{[n]}_q^{m+1} b_n z^n}.$$

In the left hand side of the inequality (2.2), we obtain

$$\begin{aligned}
&\left|\widetilde{D}_q^{m+1}f(z)+(1-\alpha)\,\widetilde{D}_q^m f(z)\right|-\left|\widetilde{D}_q^{m+1}f(z)-(1+\alpha)\,\widetilde{D}_q^m f(z)\right|\\
\geq\quad & 2(1-\alpha)|z|\left\{\begin{array}{c}1-\sum\limits_{n=2}^{\infty}\frac{\widetilde{[n]}_q^m\left(\widetilde{[n]}_q-\alpha\right)}{1-\alpha}|a_n|\,|z|^{n-1}\\ -\sum\limits_{n=1}^{\infty}\frac{\widetilde{[n]}_q^m\left(\widetilde{[n]}_q+\alpha\right)}{1-\alpha}|b_n|\,|z|^{n-1}\end{array}\right\}\\
\geq\quad & 2(1-\alpha)\left\{1-\sum_{n=2}^{\infty}\frac{\widetilde{[n]}_q^m\left(\widetilde{[n]}_q-\alpha\right)}{1-\alpha}|a_n|-\sum_{n=1}^{\infty}\frac{\widetilde{[n]}_q^m\left(\widetilde{[n]}_q+\alpha\right)}{1-\alpha}|b_n|\right\}.
\end{aligned}$$

This last expression is non-negative by (2.1), and this completes the proof.

For part (ii): Since $\overline{\widetilde{\mathcal{H}}_q^m}(\alpha)$ is subset of $\widetilde{\mathcal{H}}_q^m(\alpha)$, we only need to prove the "only if" part of the theorem. Let $f \in \overline{\widetilde{\mathcal{H}}_q^m}(\alpha)$. Then by the required condition (1.5), we have

$$\Re\left(\frac{(1-\alpha)z-\sum\limits_{n=2}^{\infty}\widetilde{[n]}_q^m\left(\widetilde{[n]}_q-\alpha\right)a_n z^n-(-1)^{2m}\sum\limits_{n=1}^{\infty}\widetilde{[n]}_q^m\left(\widetilde{[n]}_q+\alpha\right)b_n z^n}{z-\sum\limits_{n=2}^{\infty}\widetilde{[n]}_q^m a_n z^n+(-1)^{2m}\sum\limits_{n=1}^{\infty}\widetilde{[n]}_q^m b_n z^n}\right)\geq 0.$$

This must hold for all values of z in U. So, upon choosing the values of z on the positive real axis where $0 \leq z = r < 1$, we have

$$\frac{1-\alpha-\sum\limits_{n=2}^{\infty}\widetilde{[n]}_q^m\left(\widetilde{[n]}_q-\alpha\right)a_n r^{n-1}-\sum\limits_{n=1}^{\infty}\widetilde{[n]}_q^m\left(\widetilde{[n]}_q+\alpha\right)b_n r^{n-1}}{1-\sum\limits_{n=2}^{\infty}\widetilde{[n]}_q^m a_n r^{n-1}+\sum\limits_{n=1}^{\infty}\widetilde{[n]}_q^m b_n r^{n-1}}\geq 0. \tag{2.3}$$

If the condition (2.1) does not hold, then the numerator in (2.3) is negative for r sufficiently close to 1. Hence there exists $z_0 = r_0$ in $(0, 1)$ for which the left hand side of the inequality (2.3) is negative. This contradicts the required condition that $f \in \overline{\widetilde{\mathcal{H}}_q^m}(\alpha)$ and so the proof is complete. □

Example 2.2. The function $f = h + \overline{g}$ given by

$$f(z) = z + \sum_{n=2}^{\infty} A_n z^n + \sum_{n=1}^{\infty} B_n \overline{z}^n,$$

where

$$\begin{aligned} A_n &= \frac{(2+\delta)(1-\alpha)\epsilon_n}{2(n+\delta)(n+1+\delta)\widetilde{[n]}_q^m\left(\widetilde{[n]}_q-\alpha\right)}, \\ B_n &= \frac{(1+\delta)(1-\alpha)\epsilon_n}{2(n+\delta)(n+1+\delta)\widetilde{[n]}_q^m\left(\widetilde{[n]}_q+\alpha\right)}, \end{aligned}$$

belonging to the class $\widetilde{\mathcal{H}}_q^m(\alpha)$, for $\delta > -2$, $0 \le \alpha < 1$, $q \in (0,1)$, $\epsilon_n \in \mathbb{C}$, $|\epsilon_n| = 1$. Because, we know that

$$\begin{aligned} & \sum_{n=2}^{\infty} \widetilde{[n]}_q^m\left(\widetilde{[n]}_q-\alpha\right)|A_n| + \sum_{n=1}^{\infty} \widetilde{[n]}_q^m\left(\widetilde{[n]}_q+\alpha\right)|B_n| \\ \le & \sum_{n=2}^{\infty} \frac{(2+\delta)(1-\alpha)}{2(n+\delta)(n+1+\delta)} + \sum_{n=1}^{\infty} \frac{(1+\delta)(1-\alpha)}{2(n+\delta)(n+1+\delta)} \\ = & \frac{(2+\delta)(1-\alpha)}{2} \sum_{n=2}^{\infty} \frac{1}{(n+\delta)(n+1+\delta)} \\ & + \frac{(1+\delta)(1-\alpha)}{2} \sum_{n=1}^{\infty} \frac{1}{(n+\delta)(n+1+\delta)} \\ = & \frac{(2+\delta)(1-\alpha)}{2} \sum_{n=2}^{\infty} \left(\frac{1}{(n+\delta)} - \frac{1}{(n+1+\delta)}\right) \\ & + \frac{(1+\delta)(1-\alpha)}{2} \sum_{n=1}^{\infty} \left(\frac{1}{(n+\delta)} - \frac{1}{(n+1+\delta)}\right) \\ = & 1-\alpha. \end{aligned}$$

For $q \to 1-$, then Theorem 2.1 reduces to following known results ([11, Theorems 1 and 2]).

Corollary 2.3. *For* $0 \le \alpha < 1$ *and* $f = h + \overline{g}$, *let*

$$\sum_{n=2}^{\infty} n^m (n-\alpha)|a_n| + \sum_{n=1}^{\infty} n^m (n+\alpha)|b_n| \le 1-\alpha, \tag{2.4}$$

where h and g are, respectively, given by (1.1) and (1.2). Then

(i) f is harmonic univalent in U and $f \in \mathcal{H}^m(\alpha)$ if the inequality (2.4) holds.
(ii) f is harmonic univalent in U and $f \in \overline{\mathcal{H}^m(\alpha)}$ if and only if the inequality (2.4) holds.
The equality in (2.4) occurs for harmonic functions

$$f(z) = z + \sum_{n=2}^{\infty} \frac{1-\alpha}{n^m(n-\alpha)} x_n z^n + \overline{\sum_{n=1}^{\infty} \frac{1-\alpha}{n^m(n+\alpha)} y_n z^n},$$

where

$$\sum_{n=2}^{\infty} |x_n| + \sum_{n=1}^{\infty} |y_n| = 1.$$

The closed convex hull of $\overline{\widetilde{\mathcal{H}}_q^m(\alpha)}$ denoted by $clco\overline{\widetilde{\mathcal{H}}_q^m(\alpha)}$, is the smallest closed set containing $\overline{\widetilde{\mathcal{H}}_q^m(\alpha)}$, it is the intersection of all closed convex sets containg $\overline{\widetilde{\mathcal{H}}_q^m(\alpha)}$, In the next theorem we determine the extreme points of the closed convex hull of $\overline{\widetilde{\mathcal{H}}_q^m(\alpha)}$.

Theorem 2.4. *If the functions $f = h + \overline{g} \in clco\overline{\widetilde{\mathcal{H}}_q^m(\alpha)}$ if and only if*

$$\begin{aligned} f(z) &= \sum_{n=1}^{\infty} \left(X_n h_n(z) + \overline{Y_n g_n(z)} \right), \\ h_1(z) &= z, \end{aligned}$$

$$\begin{aligned} h_n(z) &= z - \frac{1-\alpha}{\widetilde{[n]}_q^m \left(\widetilde{[n]}_q - \alpha \right)} z^n, \quad (n = 2, 3, \ldots), \\ g_n(z) &= z + (-1)^m \frac{1-\alpha}{\widetilde{[n]}_q^m \left(\widetilde{[n]}_q + \alpha \right)} z^n, \quad (n = 1, 2, 3, \ldots), \end{aligned}$$

$$\sum_{n=1}^{\infty} (X_n + Y_n) = 1, \; X_n \geq 0, \; Y_n \geq 0. \tag{2.5}$$

In particular, the extreme points of $\overline{\widetilde{\mathcal{H}}_q^m(\alpha)}$ are h_n and g_n.

Proof. For the functions of the form (2.5), we have

$$\begin{aligned} f(z) &= \sum_{n=1}^{\infty} (X_n + Y_n) z - \sum_{n=2}^{\infty} \frac{1-\alpha}{\widetilde{[n]}_q^m \left(\widetilde{[n]}_q - \alpha \right)} X_n z^n \\ &\quad + (-1)^m \sum_{n=1}^{\infty} \frac{1-\alpha}{\widetilde{[n]}_q^m \left(\widetilde{[n]}_q + \alpha \right)} \overline{Y_n z^n}. \end{aligned}$$

This yields

$$\sum_{n=2}^{\infty} \frac{\widetilde{[n]}_q^m \left(\widetilde{[n]}_q - \alpha \right)}{1-\alpha} a_n + \sum_{n=1}^{\infty} \frac{\widetilde{[n]}_q^m \left(\widetilde{[n]}_q + \alpha \right)}{1-\alpha} b_n$$

$$= \sum_{n=2}^{\infty} X_n + \sum_{n=1}^{\infty} Y_n = 1 - X_1 \leq 1.$$

and so $f = h + \overline{g} \in clco\widetilde{\mathcal{H}}_q^m(\alpha)$. Conversely, let $f = h + \overline{g} \in clco\widetilde{\mathcal{H}}_q^m(\alpha)$. Then by setting

$$X_n = \frac{\widetilde{[n]}_q^m \left(\widetilde{[n]}_q - \alpha\right)}{1-\alpha} a_n, \quad (n = 2, 3, ...),$$

$$Y_n = \frac{\widetilde{[n]}_q^m \left(\widetilde{[n]}_q + \alpha\right)}{1-\alpha} b_n, \quad (n = 1, 2, 3, ...),$$

where

$$\sum_{n=1}^{\infty} (X_n + Y_n) = 1.$$

We obtain the functions of the form (2.5) as required. □

Remark 2.5. For $q \to 1-$, then Theorem 2.4 reduces to the known results proved in [11].

Finally, we give the following distortion bounds which yields a covering result for $\overline{\widetilde{\mathcal{H}}_q^m(\alpha)}$.

Theorem 2.6. *If $f = h + \overline{g} \in \overline{\widetilde{\mathcal{H}}_q^m(\alpha)}$, then for $|z| = r < 1$, we have the distortion bounds*

$$|f(z)| \geq (1 - |b_1|) r - \frac{1}{\widetilde{[2]}_q^m} \left(\frac{1-\alpha}{\widetilde{[2]}_q - \alpha} - \frac{1+\alpha}{\widetilde{[2]}_q - \alpha} |b_1| \right) r^2, \tag{2.6}$$

$$|f(z)| \leq (1 + |b_1|) r + \frac{1}{\widetilde{[2]}_q^m} \left(\frac{1-\alpha}{\widetilde{[2]}_q - \alpha} - \frac{1+\alpha}{\widetilde{[2]}_q - \alpha} |b_1| \right) r^2. \tag{2.7}$$

The bounds given in (2.6) and (2.7) for the function $f(z) = h + \overline{g}$ of the form (1.6) also hold for functions of the form (1.1), (1.2), if the coefficient condition (2.1) is satisfied. Let the

$$f(z) = (1 + |b_1|) \overline{z} + \frac{1}{\widetilde{[2]}_q^m} \left(\frac{1-\alpha}{\widetilde{[2]}_q - \alpha} - \frac{1+\alpha}{\widetilde{[2]}_q - \alpha} |b_1| \right) \overline{z}^2$$

and

$$f(z) = (1 - |b_1|) z - \frac{1}{\widetilde{[2]}_q^m} \left(\frac{1-\alpha}{\widetilde{[2]}_q - \alpha} - \frac{1+\alpha}{\widetilde{[2]}_q - \alpha} |b_1| \right) z^2,$$

for

$$|b_1| \leq \frac{1-\alpha}{1+\alpha},$$

show that the bounds given in Theorem 2.6 are sharp.

Proof. We will only prove the inequality (2.7) of the Theorem 2.6. The arguments for the inequality (2.6) is similar and so we omit it. Let $f \in \overline{\widetilde{\mathcal{H}}_q^m(\alpha)}$. Taking the absolute value of $f(z)$, we obtain

$$|f(z)| \leq (1 + |b_1|) r + \sum_{n=2}^{\infty} (|a_n| + |b_n|) r^n$$

$$
\begin{aligned}
&\leq \ (1+|b_1|)\, r + \sum_{n=2}^{\infty} (|a_n| + |b_n|)\, r^2 \\
&\leq \ (1+|b_1|)\, r + \frac{1-\alpha}{\widetilde{[2]}_q^m \left(\widetilde{[2]}_q - \alpha\right)} \\
&\qquad \times \sum_{n=2}^{\infty} \left(\frac{\widetilde{[2]}_q^m \left(\widetilde{[2]}_q - \alpha\right)}{1-\alpha} |a_n| + \frac{\widetilde{[2]}_q^m \left(\widetilde{[2]}_q + \alpha\right)}{1-\alpha} |b_n| \right) r^2 \\
&\leq \ (1+|b_1|)\, r + \frac{1-\alpha}{\widetilde{[2]}_q^m \left(\widetilde{[2]}_q - \alpha\right)} \\
&\qquad \times \sum_{n=2}^{\infty} \left(\frac{\widetilde{[2]}_q^m \left(\widetilde{[n]}_q - \alpha\right)}{1-\alpha} |a_n| + \frac{\widetilde{[2]}_q^m \left(\widetilde{[n]}_q + \alpha\right)}{1-\alpha} |b_n| \right) r^2 \\
&\leq \ (1+|b_1|)\, r + \frac{1-\alpha}{\widetilde{[2]}_q^m \left(\widetilde{[2]}_q - \alpha\right)} \sum_{n=2}^{\infty} \left(1 - \frac{1+\alpha}{1-\alpha} |b_1| \right) r^2 \\
&\leq \ (1+|b_1|)\, r + \frac{1}{\widetilde{[2]}_q^m} \sum_{n=2}^{\infty} \left(\frac{1-\alpha}{\widetilde{[2]}_q - \alpha} - \frac{1+\alpha}{\widetilde{[2]}_q - \alpha} |b_1| \right) r^2.
\end{aligned}
$$

The following covering result follows the inequality (2.7) in Theorem (2.6). □

Theorem 2.7. *If $f \in \overline{\mathcal{H}_q^m(\alpha)}$, then for $|z| = r < 1$, we have*

$$
\left\{ w : |w| < \frac{\widetilde{[2]}_q^m \left(\widetilde{[2]}_q - \alpha\right) - (1-\alpha)}{\widetilde{[2]}_q^m \left(\widetilde{[2]}_q - \alpha\right)} - \frac{\left\{\widetilde{[2]}_q^m \left(\widetilde{[2]}_q - \alpha\right) - (1+\alpha)\right\} |b_1|}{\widetilde{[2]}_q^m \left(\widetilde{[2]}_q - \alpha\right)} \right\} \subset f(U).
$$

Proof. Using the inequality (2.6) of Theorem 2.6 and letting $r \to 1$, it follows that

$$
\begin{aligned}
& (1-|b_1|) - \frac{1}{\widetilde{[2]}_q^m} \left(\frac{1-\alpha}{\widetilde{[2]}_q - \alpha} - \frac{1+\alpha}{\widetilde{[2]}_q - \alpha} |b_1| \right) \\
= \ & (1-|b_1|) - \frac{1}{\widetilde{[2]}_q^m \left(\widetilde{[2]}_q - \alpha\right)} \left\{ 1 - \alpha - (1+\alpha) |b_1| \right\} \\
= \ & \frac{(1-|b_1|)\, \widetilde{[2]}_q^m \left(\widetilde{[2]}_q - \alpha\right) - (1-\alpha) + (1+\alpha) |b_1|}{\widetilde{[2]}_q^m \left(\widetilde{[2]}_q - \alpha\right)} \\
= \ & \frac{\widetilde{[2]}_q^m \left(\widetilde{[2]}_q - \alpha\right) - (1-\alpha)}{\widetilde{[2]}_q^m \left(\widetilde{[2]}_q - \alpha\right)} - \frac{\left\{\widetilde{[2]}_q^m \left(\widetilde{[2]}_q - \alpha\right) - (1+\alpha)\right\} |b_1|}{\widetilde{[2]}_q^m \left(\widetilde{[2]}_q - \alpha\right)} \\
\subset \ & f(U).
\end{aligned}
$$

□

3. Conclusions

Research on symmetric q-calculus in connection with Geometric Function Theory and especially harmonic univalent functions is fairly new and not much is published on this topic. In this paper we have made use of the symmetric quantum (or q-) calculus to defined and investigated new classes of harmonic univalent functions by using newly defined symmetric Salagean q-differential operator for complex harmonic functions and obtained sharp coefficient bounds, distortion theorems and covering results. Furthermore, we also highlighted some known consequence of our main results.

Basic (or q-) series and basic (or q-) polynomials, especially the basic (or q-) hypergeometric functions and basic (or q-) hypergeometric polynomials, are applicable particularly in several diverse areas (see, for example, [28, p.351–352] and [29, p.328]). Moreover, in this recently-published survey-cum-expository review article by Srivastava [29], the so-called (p, q)-calculus was exposed to be a rather trivial and inconsequential variation of the classical q-calculus, the additional parameter p being redundant (see, for details, [29, p.340], see also [15–19]). This observation by Srivastava [29] will indeed apply also to any attempt to produce the rather straightforward (p, q)-variations of the results which we have presented in this paper.

Conflict of interest

The authors declare no conflict of interest.

References

1. L. C. Biedenharn, The quantum group $SUq(2)$ and a q-analogue of the boson operators, *J. Phys. A*, **22** (1984), 873–878. doi: 10.1088/0305-4470/22/18/004.
2. J. Clunie, T. Sheil-Small, Harmonic univalent functions, *Ann. Acad. Sci. Fenn. Ser. AI Math.*, **9** (1984), 3–25. doi: 10.5186/aasfm.1984.0905.
3. A. M. Da Cruz, N. Martins, The q-symmetric variational calculus, *Comput. Math. Appl.*, **64** (2012), 2241–2250. doi: 10.1016/j.camwa.2012.01.076.
4. P. L. Duren, *Harmonic Mappings in the Plane*, Cambridge Tracts in Mathematics, Vol. 156, Cambridge University Press, Cambridge, London and New York, 2004. doi: 10.1017/CBO9780511546600.
5. W. Hengartner, G. Schober, Univalent harmonic functions, *Trans. Am. Math. Soc.*, **299** (1987), 1–31. doi: 10.2307/2000478.
6. M. E. H. Ismail, E. Merkes, D. Styer, A generalization of starlike functions, *Complex Var. Theory Appl.*, **14** (1990), 77–84. doi: 10.1080/17476939008814407.
7. F. H. Jackson, On q-functions and a certain difference operator, *Trans. R. Soc. Edinburgh*, **46** (1908), 253–281. doi: 10.1017/S0080456800002751.
8. J. M. Jahangiri, Harmonic functions starlike in the unit disk, *J. Math. Anal. Appl.*, **235** (1999), 470–477.
9. J. M. Jahangiri, Harmonic univalent functions defined by q-calculus operators, *Int. J. Math. Anal. Appl.*, **5** (2018), 39–43.

10. J. M. Jahangiri, Y. C. Kim, H. M. Srivastava, Construction of a certain class of harmonic close-to-convex functions associated with the Alexander integral transform, *Integr. Transforms Spec. Funct.*, **14** (2003), 237–242. doi: 10.1080/1065246031000074380.
11. J. M. Jahangiri, G. Murugusundaramoorthy, K. Vijaya, Salagean-type harmonic univalent functions, *Southwest J. Pure Appl. Math.*, **2** (2002), 77–82.
12. B. Kamel, S. Yosr, On some symmetric q-special functions, *Le Mat.*, **68** (2013), 107–122. doi: 10.4418/2013.68.2.8.
13. S. Kanas, S. Altinkaya, S. Yalcin, Subclass of k uniformly starlike functions defined by symmetric q-derivative operator, *Ukr. Math. J.*, **70** (2019), 1727–1740.
14. S. Kanas, D. Raducanu, Some class of analytic functions related to conic domains, *Math. Slovaca*, **64** (2014), 1183–1196. doi: 10.2478/s12175-014-0268-9.
15. B. Khan, Z. G. Liu, H. M. Srivastava, N. Khan, M. Darus, M. Tahir, A study of some families of multivalent q-starlike functions involving higher-order q-Derivatives, *Mathematics*, **8** (2020), 1470. doi: 10.3390/math8091470.
16. B. Khan, Z. G. Liu, H. M. Srivastava, N. Khan, M. Tahir, Applications of higher-order derivatives to subclasses of multivalent q-starlike functions, *Maejo Int. J. Sci. Technol.,* **15** (2021), 61–72.
17. B. Khan, H. M. Srivastava, N. Khan, M. Darus, M. Tahir, Q. Z. Ahmad, Coefficient estimates for a subclass of analytic functions associated with a certain leaf-like domain, *Mathematics*, **8** (2020), 1334. doi: 10.3390/math8081334.
18. B. Khan, H. M. Srivastava, M. Tahir, M. Darus, Q. Z. Ahmad, N. Khan, Applications of a certain integral operator to the subclasses of analytic and bi-univalent functions, *AIMS Math.*, **6** (2021), 1024–1039. doi: 10.3934/math.2021061.
19. B. Khan, H. M. Srivastava, N. Khan, M. Darus, Q. Z. Ahmad, M. Tahir, Applications of certain conic domains to a subclass of q-starlike functions associated with the Janowski functions, *Symmetry*, **13** (2021), 574. doi: 10.3390/sym13040574.
20. S. Khan, S. Hussain, M. Darus, Inclusion relations of q-Bessel functions associated with generalized conic domain, *AIMS Math.*, **6** (2021), 3624–3640. doi: 10.3934/math.2021216.
21. S. Khan, S. Hussain, M. Naeem, M. Darus, A. Rasheed, A subclass of q-starlike functions defined by using a symmetric q-derivative operator and related with generalized symmetric conic domains, *Mathematics*, **9** (2021), 917. doi: 10.3390/math9090917.
22. S. Khan, S. Hussain, M. A. Zaighum, M. Darus, A subclass of uniformly convex functions and a corresponding subclass of starlike function with fixed coefficient associated with q-Analogus of Ruscheweyh operator, *Math. Slovaca,* **69** (2019), 825–832. doi: 10.1515/ms-2017-0271.
23. O. S. Kwon, S. Khan, Y. J. Sim, S. Hussain, Bounds for the coefficient of Faber polynomial of meromorphic starlike and convex functions, *Symmetry*, **1** (2019), 1368–1381. doi: 10.3390/sym11111368.
24. A. Lavagno, Basic-deformed quantum mechanics, *Rep. Math. Phys.*, **64** (2009), 79–88. doi: 10.1016/S0034-4877(09)90021-0.
25. S. Porwal, A. Gupta, An application of q-calculus to harmonic univalent functions, *J. Qual. Meas. Anal.*, **14** (2018), 81–90.

26. G. S. Salagean, Subclasses of univalent functions, *Complex Analysis - Fifth Romanian Finish Seminar, Bucharest,* 1981, 362–372. doi: 10.1007/BFb0066543.

27. T. Sheil-Small, Constants for planar harmonic mappings, *J. London Math. Soc.*, **42** (1990), 237–248. doi: 10.1112/jlms/s2-42.2.237.

28. H. M. Srivastava, Univalent functions, fractional calculus, and associated generalized hypergeometric functions, In: H. M. Srivastava and S. Owa, Editors, *Univalent Functions*, *Fractional Calculus*, *and Their Applications*, Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, Chichester, Brisbane and Toronto, 1989, 329–354.

29. H. M. Srivastava, Operators of basic (or q-) calculus and fractional q-calculus and their applications in geometric function theory of complex analysis, *Iran. J. Sci. Technol. Trans. A: Sci.*, **44** (2020), 327–344.

30. H. M. Srivastava, Q. Z. Ahmad, N. Khan, B. Khan, Hankel and Toeplitz determinants for a subclass of q-starlike functions associated with a general conic domain, *Mathematics,* **7** (2019), 181. doi: 10.3390/math7020181.

31. H. M. Srivastava, S. Khan, Q. Z. Ahmad, N. Khan, S. Hussain, The Faber polynomial expansion method and its application to the general coefficient problem for some subclasses of bi-univalent functions associated with a certain q-integral operator, *Stud. Univ. Babe s-Bolyai Math.*, **63** (2018), 419–436. doi: 10.24193/subbmath.2018.4.01.

32. H. M. Srivastava, N. Khan, M. Darus, S. Khan, Q. Z. Ahmad, S. Hussain, Fekete-Szegö type problems and their applications for a subclass of q-starlike functions with respect to symmetrical points, *Mathematics*, **8** (2020), 842. doi: 10.3390/math8050842.

33. H. M. Srivastava, B. Khan, N. Khan, Q. Z. Ahmad, Coefficient inequalities for q-starlike functions associated with the Janowski functions, *Hokkaido Math. J.*, **48** (2019), 407–425. doi: 10.14492/hokmj/1562810517.

34. H. M. Srivastava, N. Khan, S. Khan, Q. Z. Ahmad, B. Khan, A class of k-symmetric harmonic functions involving a certain q-derivative operator, *Mathematics,* **9** (2021), 1812. doi: 10.3390/math91518.

35. H. M. Srivastava, M. Tahir, B. Khan, Q. Z. Ahmad, N. Khan, Some general classes of q-starlike functions associated with the Janowski functions, *Symmetry*, **11** (2019), 1–14. doi: 10.3390/sym11020292.

36. H. M. Srivastava, M. Tahir, B. Khan, Q. Z. Ahmad, N. Khan, Some general families of q-starlike functions associated with the Janowski functions, *Filomat*, **33** (2019), 2613–2626. doi: 10.2298/FIL1909613S.

37. H. Tang, S. Khan, S. Hussain, N. Khan, Hankel and Toeplitz determinant for a subclass of multivalent q-starlike functions of order α, *AIMS Math.*, **6** (2021), 5421–5439. doi: 10.3934/math.2021320.

38. Z. G. Wang, S. Hussain, M. Naeem, T. Mahmood, S. Khan, A subclass of univalent functions associated with q-analogue of Choi-Saigo-Srivastava operator, *Hacet. J. Math. Stat.*, **49** (2020), 1471–1479. doi: 10.15672/hujms.576878.

39. X. Zhang, S. Khan, S. Hussain, H. Tang, Z. Shareef, New subclass of q-starlike functions associated with generalized conic domain, *AIMS Math.*, **5** (2020), 4830–4848. doi: 10.3934/math.2020308.

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