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Research article

Existence of a solution of fractional differential equations using the fixed point technique in extended b-metric spaces

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Abstract: The purpose of the present paper is to prove some fixed point results for cyclic-type operators in extended b-metric spaces. The considered operators are generalized φ-contractions and α-φ contractions. The last section is devoted to applications to integral type equations and nonlinear fractional differential equations using the Atangana-Băleanu fractional operator.

Keywords: fixed point; extended b-metric space; φ-contraction; α-φ-contraction; well-posedness; integral equations; fractional differential equations
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1. Introduction and preliminaries

At the beginning of the 20th century, Banach gave a result in the context of metric spaces (see [10]), which is now known as the famous "Banach Contraction Principle". Since then, many other generalizations of this theorem were proved.

Recently, different fixed point theorems for operators that satisfy a cyclic type contraction condition were given. One of the first papers that introduced the fixed point theory for cyclic contractions is [24], where some fixed point results for cyclic contractions are proved. The theorems are then generalized in the paper [16], where the authors considered a generalization of the contraction condition. George et al. in [31] gave some important fixed point results using various types of cyclic contractions. They studied the existence and uniqueness of fixed points for the cyclic type operators. Other results considering the cyclic type contractions and applications are given in [3, 32, 37, 38, 46, 47, 52].

Another direction of generalizing the "Banach Contraction Principle" is changing the working space. Considering this direction, an usual space is that of b-metric space. This notion was given in 1989 by Bakhtin (see [8]) and formally defined in 1993 by Czerwik (see [13]). This notion was the starting point for developing the fixed point theory in b-metric spaces (see [4, 25, 26, 36], etc). Searching the differences between the concepts of metric and the b-metric, one of them is the fact that the b-metric is not necessary continuous (see [34, 35]).

The purpose of the present paper is to prove some fixed point results for cyclic-type operators in extended b-metric spaces. The considered operators are generalized φ-contractions and α-φ contractions. The last section is devoted to applications to integral type equations and nonlinear fractional differential equations using the Atangana-Băleanu fractional operator.

Throughout this paper, standard notations and terminologies of the nonlinear analysis are used. For the convenience of the reader we recollect some well-known definitions and essential results.

We begin with the definition of extended b-metric space.

Definition 1.1. *(Bakhtin [8], Czerwik [14]) Let X be a non-empty set and let $s \geq 1$ be a given real number. A function $d : X \times X \to [0, +\infty)$ is said to be a b-metric if the following conditions are satisfied:*

(1) $d(x, y) = 0$ *if and only if* $x = y$,
(2) $d(x, y) = d(y, x)$,
(3) $d(x, z) \leq s[d(x, y) + d(y, z)]$,

for all $x, y, z \in X$. A pair (X, d) is called a b-metric space.

We notice that the notion reduces to that of a metric space if $s = 1$. Hence, this notion is a generalization of that of the metric space.

In [23], Kamran et al. introduced the notion of an extended b-metric space as follows:

Definition 1.2. *Let X be a nonempty set and $\theta : X \times X \to [1, +\infty)$. The function $d_\theta : X \times X \to [0, +\infty)$ is said to be an extended b-metric if the following conditions are satisfied:*

(1) $d_\theta(x, y) = 0$ *if and only if* $x = y$;
(2) $d_\theta(x, y) = d_\theta(y, x)$;
(3) $d_\theta(x, z) \leq \theta(x, z)[d_\theta(x, y) + d_\theta(y, z)]$.

for all $x, y, z \in X$. A pair (X, d_θ) is called an extended b-metric space.

As a remark, for $\theta(x, z) = s$ with $s \geq 1$, the notion reduces to that of b-metric space. We must emphasise the property of symmetry of the extended b-metric, which appears in the second axiom of Definition 1.2.

Any other results and examples concerning the b-metric and extended b-metric are presented in [4, 6, 8, 11, 14].

Kirk et al. [24] gave a new generalization of the Banach Contraction Principle using the notion of cyclic representation. Then he introduced inductively the notion of cyclic operator as follows.

Definition 1.3. *Let X be a nonempty set and $Y = \bigcup_{i=1}^{p} A_i \subseteq X$, for p a positive integer, $p \geq 2$. We say that an operator $T : Y \to Y$ is a cyclic operator if*

(i) $A_i, i \in \{1, 2, ...p\}$ *are nonempty subsets;*
(ii) $T(A_1) \subseteq A_2 \subseteq T(A_2) \subseteq A_3 \subseteq ... \subseteq T(A_{p-1}) \subseteq A_p, T(A_p) \subseteq A_1$.

Connecting both notions: cyclic operator and extended b-metric space, we find interesting results and applications in [22, 30].

Throughout this paper, for the mapping $T : X \to X$ and $x_0 \in X$, $O(x_0) = \{x_0, Tx_0, T^2x_0, T^3x_0, ...\}$. represents the orbit of x_0.

A function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ which is increasing and satisfies the property $\lim_{n\to+\infty} \varphi^n(t) = 0$ for all $t \geq 0$ is said to be a comparison function (see Matkowski [33]).

The notion of b-comparison function was first given by Berinde in [12]. Regarding this he stated the following remark.

Remark 1.1. *[12] Let (X, d) be a b-metric with $s \geq 1$. A function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is called a b-comparison function if it is increasing and satisfies the property that $\sum_{n=0}^{+\infty} s^n\varphi^n(t)$ converges for all $t \in \mathbb{R}_+$ and $n \in \mathbb{N}$.*

In [50] Samreen et al. extended the previous notion to the case of extended comparison function. In what follows, we introduce the notion for a cyclic operator.

Definition 1.4. *Let (X, d_θ) be an extended b-metric space. We say that a function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is an extended comparison function if for all $t \in \mathbb{R}_+$;*

(i) φ is monotone increasing;

(ii) there exists a cyclic operator $T : Y \subset X \to X$, where $Y = \bigcup_{i=1}^{p} A_i$ such that for some $x_0 \in X$, $O(x_0) \subset Y$, the sum $\sum_{n=0}^{+\infty} \varphi^n(t) \prod_{i=1}^{n} \theta(x_i, x_m)$ converges for every $m \in \mathbb{N}$. We notice that $x_n = T^n x_0$ with $n = 1, 2, ...$

Remark 1.2. *By (i) and (ii) from Definition (1.4) it follows that $\varphi(t) < t$, for every $t \in \mathbb{R}^+$.*

Remark 1.3. *The following hold:*

(i) For $\theta(x, y) = s \geq 1$, for every $x, y \in X$, the Definition 1.1 coincide with the Definition 1.4, for any cyclic operator T on $Y \subset X$.

(ii) Since $\theta(x, y) \geq 1$ for every $x, y \in X$, then choosing $s = \inf_{x,y\in X} \theta(x, y)$ we have

$$\sum_{n=0}^{+\infty} s^n\varphi^n(t) \leq \sum_{n=0}^{+\infty} \varphi^n(t) \prod_{i=1}^{n} \theta(x_i, x_m).$$

Then we conclude that every extended b-comparison function is also a b-comparison function for some $s \geq 1$.

The following lemma is first given in [50]. We present it for the case of cyclic operators.

Lemma 1.1. *Let (X, d_θ) be an extended b-metric space, $Y = \bigcup_{i=1}^{p} A_i \subseteq X$, $T : Y \subset X \to X$ a cyclic operator, $x_0 \in X$ and $\lim_{n,m\to+\infty} \theta(x_n, x_m) < \frac{1}{\lambda}$ where $\lambda \in (0, 1)$ and $x_n = T^n x_0$ for $n = 1, 2, ...$ Assume that ψ is a comparison function. Then $\varphi(t) = \lambda\psi$ is an extended b-comparison function for T at x_0.*

Next we will give the definition of orbital lower semicontinuity with respect to a cyclic operator T.

Definition 1.5. *Let X a nonempty set and $Y = \bigcup_{i=1}^{p} A_i \subseteq X$, $T : Y \to Y$ be a cyclic operator and for some $x_0 \in X$ such that the orbit of x_0, $O(x_0) \subset Y$. A function $S : X \to \mathbb{R}$ is T-orbitally lower semicontinuous at $t \in X$ if $\{x_n\} \subset O(x_0)$ and $x_n \to t$ implies $S(t) \leq \lim_{n\to+\infty} \inf S(x_n)$.*

In the following the concepts of convergence, Cauchyness and completeness are considered in an extended b-metric space (see [23]).

Definition 1.6. *Let (X, d_θ) be an extended b-metric space. Then a sequence $\{x_n\}$ in X is said to be:*

(i) Convergent sequence if and only if there exists $x \in X$ such that $d_\theta(x_n, x) \to 0$ as $n \to +\infty$ we write $\lim_{n\to+\infty} (x_n) = x$.
(ii) Cauchy sequence if and only if $d_\theta(x_n, x_m) \to 0$ as $n, m \to +\infty$.

The extended b-metric space (X, d_θ) is complete if every Cauchy sequence converges in X. We note that the extended b-metric d_θ is not a continuous function in general.

Remark 1.4. *Let (X, d_θ) be an extended b-metric space. Then every convergent sequence has a unique limit.*

2. Fixed point results

The first main result of this paper is the following.

Theorem 2.1. *Let (X, d_θ) be a complete extended b-metric space with d_θ a continuous functional. Let $Y = \bigcup_{i=1}^{p} A_i \subseteq X$, where p is a positive integer, be the set of all nonempty closed subsets of X and suppose $T : Y \to Y$ be a cyclic operator such that*

(i) $T(A_i) \subseteq A_{i+1}$, for all $i \in \{1, 2, ...p\}$;
(ii) $d_\theta(Tx, Ty) \leq \varphi(d_\theta(x, y))$ where φ is a b-extended comparison function for all $x, y \in X$.

Then $T^n x_0 \to x^* \in \bigcap_{i=1}^{p} A_i$, as $n \to +\infty$. Moreover, x^* is a unique fixed point of T if and only if $S = d_\theta(x, Tx)$ is T-orbitally lower semicontinuous at x^*.

Proof. Let $x_0 \in Y$. Then there exists $i \in \{1, 2, ...p\}$ such that $x_0 \in A_i$.

From hypothesis (i) we have $x_1 = Tx_0 \in A_{i+1}$.

Thus, we define a sequence $\{x_n\}$ by $x_{n+1} = Tx_n$ for all $n \geq 0$. We shall show that $\{x_n\}$ is a Cauchy sequence.

If $x_n = x_{n+1}$ then x_n is a fixed point of T. We suppose that $x_n \neq x_{n+1}$ for all $n \geq 0$.

From hypothesis (ii) it follows

$$d_\theta(x_n, x_{n+1}) = d_\theta(Tx_{n-1}, Tx_n) \leq \varphi d_\theta(x_{n-1}, x_n).$$

Applying (ii) successively we get

$$d_\theta(x_n, x_{n+1}) \leq \varphi^n d_\theta(x_0, x_1). \tag{2.1}$$

Furthermore we assume that x_0 is a non periodic point of T. If $x_0 = x_n$ using (2) for any $n \geq 2$ we have

$$d_\theta(x_0, x_1) = d_\theta(x_0, Tx_0) = d_\theta(x_n, Tx_n).$$

Then $d_\theta(x_0, x_1) = d_\theta(x_n, x_{n+1})$. Then, $d_\theta(x_0, x_1) \leq \varphi^n d_\theta(x_0, x_1)$. Since $\varphi(t) < t$ we get a contradiction.

Therefore $d_\theta(x_0, x_1) = 0$ i.e., $x_0 = x_1$. Then x_0 is a fixed point of T. Thus, we assume that $x_n \neq x_m$ for all $n, m \in \mathbb{N}$ with $m \neq n$.

For any m, n with $m > n$ using the triangular inequality, we get

$$d_\theta(x_n, x_m) \leq \theta(x_n, x_m)\varphi^n d_\theta(x_0, x_1) + \theta(x_n, x_m)\theta(x_{n+1}, x_m)\varphi^{n+1}(d_\theta(x_0, x_1)) \tag{2.2}$$

$$\begin{aligned}
&+ \cdots + \theta(x_n, x_m)\theta(x_{n+1}, x_m) + \cdots \theta(x_{m-1}, x_m)\varphi^{m-1} d_\theta(x_0, x_1) \\
&\leq \theta(x_1, x_m)\theta(x_2, x_m) \cdots \theta(x_{n-1}, x_m)\theta(x_n, x_m)\varphi^n(d_\theta(x_0, x_1)) \\
&+ \theta(x_1, x_m)\theta(x_2, x_m) \cdots \theta(x_n, x_m)\theta(x_{n+1}, x_m)\varphi^{n+1}(d_\theta(x_0, x_1)) \\
&+ \theta(x_1, x_m)\theta(x_2, x_m) \cdots \theta(x_{n-1}, x_m)\theta(x_n, x_m)\theta(x_{n+1}, x_m) \\
&\cdots \theta(x_{m-1}, x_m)\varphi^{m-1}(d_\theta(x_0, x_1))].
\end{aligned}$$

The series $\sum\limits_{n=1}^{+\infty} \varphi^n \prod\limits_{r=1}^{n} \theta(x_r, x_m)$ converges by ratio test for each $m \in \mathbb{N}$.

Let $\mathfrak{S} = \sum\limits_{n=1}^{+\infty} \varphi^n \prod\limits_{r=1}^{n} \theta(x_r, x_m)$, $\quad \mathfrak{S}_n = \sum\limits_{j=1}^{n} \varphi^j \prod\limits_{r=1}^{j} \theta(x_r, x_m)$.

Thus for $m > n$, in (2.2) we have $d_\theta(x_n, x_m) \leq [\mathfrak{S}_{m-1} - \mathfrak{S}_n]$.

Letting $n \to +\infty$ we conclude that $\{x_n\}$ is a Cauchy sequence in the subspace Y. Since X is complete, Y is complete too. Therefore there exists $x^* \in Y$ such that $d_\theta(x_n, x^*) \to 0$ as $n \to +\infty$. Then $x_n = T^n x_0 \to x^*$. The sequence $\{x_n\}$ has an infinite number of terms in each A_i for all $i \in \{1, 2, ...p\}$.

Therefore $x^* \in \bigcap\limits_{i=1}^{p} A_i$. Since $S = d_\theta(x, Tx)$ is T-orbitally lower semicontinuous at $x^* \in \bigcap\limits_{i=1}^{p} A_i$ we obtain

$$\begin{aligned}
d_\theta(x^*, Tx^*) &\leq \lim_{n\to+\infty} \inf d_\theta(x_n, x_{n+1}) \\
&\leq \lim_{n\to+\infty} \inf \varphi^n(d_\theta(x_0, x_1)) = 0.
\end{aligned}$$

Then $d_\theta(x^*, Tx^*) = 0$; results $x^* = Tx^*$.

Inversely, let $x^* = Tx^*$ and $x_n \in O \subseteq Y$ with $x_n \to x^*$. Then we have

$$S(x^*) = d_\theta(x^*, Tx^*) = 0 \leq \lim \inf_{n\to+\infty} d_\theta(T^n x_0, T^{n+1} x_0).$$

For the uniqueness of fixed point we suppose that there exists another fixed point $\tau = T(\tau) \in \bigcap\limits_{i=1}^{p} A_i$.
By (ii) we get $d_\theta(x^*, \tau) = d_\theta(Tx^*, T\tau) \leq \varphi(d_\theta(x^*, \tau)) < d_\theta(x^*, \tau)$.
Therefore $d_\theta(x^*, \tau) = 0$ which implies that $x^* = \tau$. □

Next let us give another general fixed point result.

Theorem 2.2. *Let (X, d_θ) be a complete extended b-metric space with d_θ a continuous functional. Let $Y = \bigcup_{i=1}^{p} A_i \subseteq X$, where p is a positive integer, be the set of all nonempty closed subsets of X and suppose $T : Y \to Y$ be a cyclic operator such that*

(*i*) *$T(A_i) \subseteq A_{i+1}$, for all $i \in \{1, 2, ...p\}$;*
(*ii*) *$d_\theta(Tx, Ty) \leq \varphi(d_\theta(x, y))$ where φ is a b-extended comparison function for all $x, y \in X$ with $\varphi(0) = 0$.*

Then T has a unique fixed point.

Proof. As in the proof of Theorem 2.1 we prove the existence of a Cauchy sequence $\{x_n\}$. Since Y is a complete subspace there exists $x^* \in Y$ such that $d_\theta(x_n, x^*) \to 0$ as $n \to +\infty$.

Since, the sequence $\{x_n\}$ has an infinite number of terms in each A_i for all $i \in \{1, 2, ...p\}$ we have $x^* \in \bigcap_{i=1}^{p} A_i$.

We must show that x^* is a fixed point for T. For any $n \in \mathbb{N}$ we have

$$\begin{aligned} d_\theta(Tx^*, x^*) &\leq \theta(Tx^*, x^*)[d_\theta(Tx^*, x_n) + d_\theta(x_n, x^*)] \\ &\leq \theta(Tx^*, x^*)[d_\theta(Tx^*, Tx_{n-1}) + d_\theta(x_n, x^*)] \\ &\leq \theta(Tx^*, x^*)[\varphi(d_\theta(x^*, x_{n-1})) + d_\theta(x_n, x^*)]. \end{aligned}$$

Since $\varphi(0) = 0$ for $n \to +\infty$ we get that $d_\theta(Tx^*, x^*) = 0$. This implies $Tx^* = x^*$, i.e., x^* is a fixed point of T.

For uniqueness we follow the same steps as in Theorem 2.1. □

Example 2.1. *Let $X = \mathbb{R}_+$ endowed with $d_\theta : X \times X \to \mathbb{R}_+$ defined by $d_\theta = |x - y|^3$ and let $\theta : X \times X \to [1, +\infty)$ defined by $\theta(x, y) = x + y + 1$. It is easy to check that (X, d_θ) is a complete extended b-metric space.*

Let $A_1 = [0, \frac{1}{2}]$, $A_2 = [0, \frac{1}{3}]$, $A_3 = [0, \frac{1}{5}]$ be three subsets of $X = \mathbb{R}^+$.

Define $T : \bigcup_{i=1}^{3} A_i \to \bigcup_{i=1}^{3} A_i$ by $Tx = \frac{x}{2}$. Obviously $T(A_1) \subseteq A_2, T(A_2) \subseteq A_3, T(A_3) \subseteq A_1$. Then $\bigcup_{i=1}^{3} A_i$ is a cyclic representation with respect to T.

Define $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ a comparison function by $\varphi(t) = \frac{1}{2}t$.

We verify the contraction condition.

$$d_\theta(Tx, Ty) = |\frac{x}{2} - \frac{y}{2}|^3 = |\frac{1}{2}(x - y)|^3 \leq \frac{1}{8}|x - y|^3 = \frac{1}{4}\psi(d_\theta(x, y)).$$

Taking into account that for each $x \in \bigcup_{i=1}^{3} A_i$, $T^n x = \frac{x}{2^n}$ we get

$$\lim_{n,m\to+\infty} \theta(x_n, x_m) = \lim_{n,m\to+\infty} \theta(\frac{x}{2^n}, \frac{x}{2^m}) = \lim_{n,m\to+\infty} (\frac{x}{2^n} + \frac{x}{2^m} + 1) = 1 < 4.$$

Then $d_\theta(Tx, Ty) \leq \frac{1}{4}\psi(d_\theta(x, y)) \leq \varphi(d_\theta(x, y))$, where $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$, defined by $\varphi = \frac{1}{4}\psi$, is a b-extended comparison function. Therefore, all conditions of Theorem 2.1 (respectively Theorem 2.2) are satisfied. Then $0 \in \bigcap_{i=1}^{3} A_i$ is the unique fixed point of T.

The following definition was introduced by Samet et al. in [49] and is used to define the notion of α-φ contraction.

Definition 2.1. *Let $\alpha : X \times X \to [0, +\infty)$. A self-map $T : X \to X$ is said to be α-admissible if*

$$\alpha(x, y) \geq 1 \text{ implies } \alpha(Tx, Ty) \geq 1, \text{ for every } x, y \in X.$$

Let us denote with Ψ the family of nondecreasing functions $\psi : [0, +\infty) \to [0, +\infty)$ such that $\sum_{n=1}^{+\infty} \psi^n(t) < +\infty$ for each $t > 0$, where ψ^n is the n-th iterate of ψ. Then we can recall the definition of α-φ contraction as follows.

Definition 2.2. *(Samet et al. [49]) Let (X, d) be a metric space and $T : X \to X$ be a given mapping. We say that T is an α-φ-contractive mapping if there exist two functions $\alpha : X \times X \to [0, +\infty)$ and $\psi \in \Psi$ such that*

$$\alpha(x, y)d(Tx, Ty) \leq \varphi(d(x, y)), \text{ for all } x, y \in X. \tag{2.3}$$

As a consequence of Theorem 2.1 let us give the following result.

Theorem 2.3. *Let (X, d_θ) be a complete extended b-metric space such that d_θ be a continuous function and $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be an extended b-comparison function. Let $T : Y \to Y$ be a cyclic operator where $Y = \bigcup_{i=1}^{p} A_i \subseteq X$, with p integer $i = \{1, 2, ..., p\}$, be the set of nonempty closed subsets of X, such that*

(i) *$\alpha(x, y)d_\theta(Tx, Ty) \leq \varphi(d_\theta(x, y))$, for every $x, y \in Y$;*
(ii) *T is α-admissible;*
(iii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$.*

Then x^* is a fixed point of T.
Moreover, the fixed point x^* is unique, provides that

$$(H): \text{ for all } x, y \in X, \text{ there exists } z \in X \text{ such that } \alpha(x, z) \geq 1 \text{ and } \alpha(y, z) \geq 1.$$

Proof. By conditions (ii) and (iii) we obtain $\alpha(T^n x_0, T^{n+1} x_0) \geq 1$.
For $n \in \mathbb{N}$ using (i) we get

$$d_\theta(T^n x_0, T^{n+1} x_0) \leq \alpha(T^n x_0, T^{n+1} x_0)d_\theta(T^n x_0, T^{n+1} x_0) \tag{2.4}$$

$$\leq \varphi(d_\theta(T^{n-1} x_0, T^n x_0)).$$

Since φ is an increasing function, we have that

$$d_\theta(T^n x_0, T^{n+1} x_0) \leq \varphi^n(d_\theta(x_0, Tx_0)). \tag{2.5}$$

This inequality is equivalent with (2.1) in Theorem 2.2. Thus all the hypotheses of Theorem 2.2 are satisfied. Thus there exists a fixed point.

In order to prove the uniqueness of the fixed point let us suppose that x^* and y^* are two fixed points of T.

From the hypothesis (H), there exists $z \in X$ such that

$$\alpha(x^*, z) \geq 1 \quad \text{and} \quad \alpha(y^*, z) \geq 1. \tag{2.6}$$

Since T is α-admissible, from (2.6), we obtain

$$\alpha(x^*, T^n z) \geq 1 \quad \text{and} \quad \alpha(y^*, T^n z) \geq 1. \tag{2.7}$$

Using (2.7) and the hypothesis (i), we have

$$d_\theta(x^*, T^n z) = d_\theta(Tx^*, T(T^{n-1}z)) \leq \alpha(x^*, T^{n-1}z) d_\theta(Tx^*, T(T^{n-1}z)) \leq$$

$$\leq \psi(d_\theta(x^*, T^{n-1}z)).$$

This implies that

$$d_\theta(x^*, T^n z) \leq \psi^n(d_\theta(x^*, z)), \text{ for all } n \in \mathbb{N}.$$

Then, letting $n \to +\infty$, we have

$$T^n z \to x^*. \tag{2.8}$$

Similarly, using (2.7) and hypothesis (i), we obtain

$$T^n z \to y^* \text{ as } n \to +\infty. \tag{2.9}$$

Using (2.8) and (2.9), the uniqueness of the limit gives us $x^* = y^*$. The conclusion follows. □

The Ulam-Hyers stability, the well-posedness of the fixed point equation and the data dependence of the fixed point can be proved using the previous theorem. We shall give first the following definitions.

Definition 2.3. *Let (X, d_θ) be an extended b-metric space. An operator $T : X \to X$ is a Picard operator if the following conditions are satisfied:*

(a) $FixT = \{x^*\}$;
(b) $(Tx_n)_{n\in\mathbb{N}} \to x^*$, *as* $n \to +\infty$, *for all* $x \in X$.

Definition 2.4. *Let (X, d_θ) be an extended b-metric space and $T : X \to X$ be a singlevalued operator. The fixed point equation*

$$x = Tx, \tag{2.10}$$

is called generalized Ulam-Hyers stable if and only if there exists $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ increasing, continuous in 0 and $\psi(0) = 0$ such that for every $\varepsilon > 0$ and for every ε-solution of the fixed point Eq (2.10) $w^* \in X$, i.e., w^* satisfies the inequality

$$d_\theta(w^*, Tw^*) \leq \varepsilon, \tag{2.11}$$

there exists a solution $x^* \in X$ of the previous Eq (2.10) such that

$$d_\theta(w^*, x^*) \leq \psi(\varepsilon).$$

The fixed point Eq (2.10) is said to be Ulam-Hyers stable if there exists $c > 0$ such that $\psi(t) = c \cdot t$, for each $t \in \mathbb{R}_+$.

For Ulam-Hyers stability results in the case of fixed point problems see Bota-Boriceanu and Petruşel [9], Lazăr [28], Rus [42, 43].

Regarding the Ulam-Hyers stability problem, the ideas given in Petru, Petruşel and Yao [39] allow us to give the following result.

Theorem 2.4. *Let (X, d_θ) be a complete extended b-metric space. Suppose that all hypotheses of Theorem 2.3 hold and, in addition, there exists a real number $M > 1$ such that $\theta(x, y) < M$, for all $(x, y) \in X \times X$ and a function $\beta : \mathbb{R}_+ \to \mathbb{R}_+$, $\beta(r) := r - M\psi(r)$ strictly increasing and onto. Then*

(a) the fixed point Eq (2.10) *is generalized Ulam-Hyers stable.*
(b) $FixT = \{x^*\}$ and if $x_n \in X$, $n \in \mathbb{N}$ are such that $d_\theta(x_n, Tx_n) \to 0$, as $n \to +\infty$, then $x_n \to x^*$, as $n \to +\infty$, i.e., the fixed point Eq (2.10) *is well posed.*
(c) If $G : X \to X$ is such that there exists $\eta \in [0, +\infty)$ with

$$d_\theta(G(x), T(x)) \le \eta, \text{ for all } x \in X,$$

then

$$y^* \in FixG \text{ implies } d_\theta(x^*, y^*) \le \beta^{-1}(M \cdot \eta).$$

Proof. (a) Since $T : X \to X$ is a Picard operator, we have $FixT = \{x^*\}$.

Let $\varepsilon > 0$ and $w^* \in X$ be an approximate solution of (2.11), i.e.,

$$d_\theta(w^*, Tw^*) \le \varepsilon.$$

Since T is an α-ψ-contractive mapping and since $x^* \in FixT$, there exists $w^* \in X$ such that $\alpha(x^*, w^*) \ge 1$, we obtain:

$$d_\theta(x^*, w^*) = d_\theta(Tx^*, w^*) \le \theta(x^*, w^*)[d_\theta(Tx^*, Tw^*) + d_\theta(Tw^*, w^*)]$$

$$\le \theta(x^*, w^*)[\alpha(x^*, w^*)d_\theta(Tx^*, Tw^*) + \varepsilon] \le \theta(x^*, w^*)[\psi(d_\theta(x^*, w^*)) + \varepsilon].$$

Hence:

$$d_\theta(x^*, w^*) - \theta(x^*, w^*) \cdot \psi(d_\theta(x^*, w^*)) \le \theta(x^*, w^*) \cdot \varepsilon.$$

Therefore, taking into account that $\theta(x, y) < M$ for all $x, y \in X$ we define the function β as follows:

$$\beta(d_\theta(x^*, w^*)) := d_\theta(x^*, w^*) - M\psi(d_\theta(x^*, w^*)) \le M \cdot \varepsilon \text{ implies } d_\theta(x^*, w^*) \le \beta^{-1}(M \cdot \varepsilon).$$

Consequently, the fixed point Eq (2.10) is β^{-1}-generalized Ulam-Hyers stable.

(b) Since T is an α-ψ-contractive mapping and since $x^* \in FixT$, from (H) there exists $x_n \in X$ such that $\alpha(x^*, x_n) \ge 1$, we obtain:

$$d_\theta(x_n, x^*) \le \theta(x_n, x^*)[d_\theta(x_n, Tx_n) + d_\theta(Tx_n, x^*)] = \theta(x_n, x^*)[d_\theta(x_n, Tx_n) + d_\theta(Tx_n, Tx^*)]$$

$$\le \theta(x_n, x^*)[d_\theta(x_n, Tx_n) + \alpha(x_n, x^*)d_\theta(Tx_n, Tx^*)] \le \theta(x_n, x^*)[d_\theta(x_n, Tx_n) + \psi(d_\theta(x_n, x^*))].$$

Therefore, from the hypothesis that $\theta(x, y) < M$ for all $x, y \in X$ we obtain

$$\beta(d_\theta(x_n, x^*)) := d_\theta(x_n, x^*) - M\psi(d_\theta(x_n, x^*)) \le Md_\theta(x_n, Tx_n) \to 0 \text{ as } n \to +\infty$$

$$\text{implies } d_\theta(x_n, x^*) \to 0 \text{ as } n \to +\infty \text{ implies } x_n \to x^*, \text{ as } n \to +\infty.$$

So, the fixed point Eq (2.10) is well posed.

(c) Since $x^* \in FixT$, from (H) there exists $x \in X$ such that $\alpha(x^*, x) \geq 1$, we obtain:

$$d_\theta(x, x^*) \leq \theta(x, x^*)[d_\theta(x, Tx) + d_\theta(Tx, x^*)] = \theta(x, x^*)[d_\theta(x, Tx) + d_\theta(Tx, Tx^*)] \leq$$

$$\leq \theta(x, x^*)[d_\theta(x, Tx) + \alpha(x, x^*)d_\theta(Tx, Tx^*)] \leq \theta(x, x^*)[d_\theta(x, Tx) + \psi(d_\theta(x, x^*))].$$

Therefore, using the hypothesis that $\theta(x, y) < M$ for all $x, y \in X$ we obtain

$$\beta(d_\theta(x, x^*)) := d_\theta(x, x^*) - M\psi(d_\theta(x, x^*)) \leq M \cdot d_\theta(x, Tx).$$

So, we have the following estimation

$$d_\theta(x, x^*) \leq \beta^{-1}(M \cdot d_\theta(x, Tx)). \tag{2.12}$$

Rewriting (2.12) for $x := y^*$ we get:

$$d_\theta(x^*, y^*) \leq \beta^{-1}(M \cdot d_\theta(y^*, Ty^*)) = \beta^{-1}(M \cdot d_\theta(Gy^*, Ty^*)) \text{ implies } d_\theta(x^*, y^*) \leq \beta^{-1}(M \cdot \eta).$$

□

3. Applications to integral equations and nonlinear differential fractional equations

Further we shall establish the existence of a solution to the following integral equation

$$x(t) = p(t) + \int_0^t P(t, u)g(u, x(u))du, \quad t \in [0, 1], \tag{3.1}$$

where $g : [0, 1]\times\mathbb{R} \to \mathbb{R}$, $p : [0, 1] \to \mathbb{R}$ are two bounded continuous functions and $P : [0, 1]\times[0, 1] \to [0, +\infty)$ is a function such that $P(t, \cdot) \in L^1([0, 1])$ for all $t \in [0, 1]$.

Consider the operator $T : Y \to Y$, where $Y = \bigcup_{i=1}^{p} A_i \subseteq X$, given by

$$T(x)(t) = p(t) + \int_0^t P(t, u)g(u, x(u))du. \tag{3.2}$$

Observe that each fixed point of T is a solution of integral Eq (3.1). Also, T is well defined since g and p are two bounded continuous functions.

Then let us give the following theorem on the existence of a fixed point for (3.2), which in turn reduces to the result for the existence of a solution to (3.1).

Theorem 3.1. *Let $T : Y \to Y$, where $Y = \bigcup_{i=1}^{3} A_i \subseteq X$, be a cyclic integral operator given by (3.2). Suppose that the following conditions hold:*

(i) *for* $x, y \in Y$ *and for every* $u \in [0,1]$ *we have*

$$0 \leq T(u, x(u)) - T(u, y(u)) \leq \frac{1}{2}\sqrt{e^{|x(u)-y(u)|^2}}.$$

(ii) *for every* $u \in [0,1]$ *we have*

$$\|\int_0^1 P(t,u)du\|_{+\infty} < 1.$$

Then T *has a fixed point.*

Proof. Consider the space $X = C([0,1], \mathbb{R})$ of all continuous real valued functions defined on $[0,1]$. Then (X, d_θ) is a complete extended b-metric space with respect to

$$d_\theta(x,y) = \|x-y\|_{+\infty} = \sup_{t\in[ab]} |x(t) - y(t)|^2,$$

where $\theta : X \times X \to [1, +\infty)$ is defined by

$$\theta(x,y) = |x(t)| + |y(t)| + 1.$$

Let $A_1 = A_2 = A_3 = X = C([0,1], \mathbb{R})$ are non empty subsets of X. It is obvious A_1, A_2, A_3 are closed subsets of (X, d_θ). Clearly $T(A_1) \subset A_2, T(A_2) \subset A_3$ and $T(A_3) \subset A_1$. Then T is a cyclic operator on $\bigcup_{i=1}^{3} A_i$.

By condition (ii) we get for $x \in Y = \bigcup_{i=1}^{3} A_i$

$$\begin{aligned}
|T(x)(t) - T(y)(t)|^2 &= |\int_0^t P(t,u)[g(u,x(u)) - g(u,y(u))]du|^2 \\
&\leq \int_0^t |P(t,u)|^2 |g(u,x(u)) - g(u,y(u)|^2 du \\
&\leq \frac{1}{4}\int_0^t |P(t,u)|^2 e^{|x(u)-y(u)|^2} du \\
&\leq \frac{1}{4} e^{\|x(u)-y(u)\|_{+\infty}}.
\end{aligned}$$

Then we get

$$\|Tx - Ty\|_{+\infty} \leq \frac{1}{4} e^{\|x(u)-y(u)\|_{+\infty}}.$$

Hence $d_\theta(Tx, Ty) \leq \lambda\varphi(d_\theta(x,y))$ where $\varphi(t) = \frac{1}{4}e^t$ is a comparison function. For $x \in Y$, $\lim_{n,m\to+\infty} \theta(x_n, x_m) = 1 < 2$. Then, by Lemma 1.1, we get that $\lambda\varphi$ is an extended b-comparison function, with $\lambda = \frac{1}{4}$.

Thus all the conditions of Theorem 2.2 are satisfied. Then the cyclic integral operator T has a fixed point. □

Theorem 3.2. *Let $T : Y \to Y$, where $Y = \bigcup\limits_{i=1}^{p} A_i \subseteq X$, be a cyclic integral operator given by*

$$T(x)(t) = p(t) + \int\limits_0^t \frac{(t-u)^{\alpha-1}}{\Gamma(\alpha)} T(u, x(u))du, \quad with \;\; t \in [0,1] \;\; and \;\; \alpha \in (0,1)$$

where Γ is the Euler gamma function given by

$$\Gamma(\alpha) = \int\limits_0^{+\infty} t^{\alpha-1} e^{-t} du.$$

Suppose that for $x \in \bigcup\limits_{i=1}^{3} A_i$ we have

$$0 \le T(u, x(u)) - T(u, y(u)) \le \frac{\Gamma(\alpha+1)}{4} \sqrt{e^{|x(u)-y(u)|^2}} \;\; for \;\; every \;\; u \in [0,1].$$

Then T has a fixed point.

The next point of this section is to give an application of our new fixed point results in the framework of fractional differential equations. Let us recall some notions concerning Atangana-Băleanu fractional operator, known in the related literature as the fractional operator with Mittag-Leffler kernel (see [1]).

Let $x \in H^1(a,b)$, $a < b$ and $q \in [0,1]$. The Caputo Atangana-Băleanu fractional derivative of x of order q is defined by

$$({}_a^{ABC}D^q x)(t) = \frac{B(q)}{1-q} \int_t^a x'(\beta) E_q \left[-q \frac{(t-\beta)^q}{1-q} \right] d\beta, \tag{3.3}$$

where E_q is Mittag-Leffler function defined by $E_q(z) = \sum\limits_{n=0}^{+\infty} \frac{z^n}{\Gamma(nq+1)}$ [21, 51] and $B(q)$ is a normalizing positive function satisfying $B(0) = B(1) = 1$.

The associated fractional integral is defined by

$$({}_a^{AB}I^q x)(t) = \frac{1-q}{B(q)} x(t) + \frac{q}{B(q)} ({}_aI^q x)(t), \tag{3.4}$$

where ${}_aI^q$ is the left Riemann-Liouville fractional integral given as

$$({}_aI^q x)(t) = \frac{1}{\Gamma(q)} \int_a^t (t-\beta)^{q-1} x(\beta) d\beta. \tag{3.5}$$

We consider the next differential equation

$$({}_0^{ABC}D^q x)(t) = f(t, x(t)), \tag{3.6}$$

with an initial condition

$$x(0) = x_0, \tag{3.7}$$

where $f \in C[0,1]$ such that $f(0, x(0)) = 0$, $q \in (0,1]$, $t \in [0,1]$ and x_0 is a constant.

In [7] we find the following result (Proposition 3.5.).

Remark 3.1. *For $0 < q < 1$, we conclude that*

$$({}^{AB}I_b^q\,{}^{ABC}D_b^q)f(x) = f(x) - f(b). \tag{3.8}$$

Let us consider $X = C([0,1],\mathbb{R})$ be the set of all continuous real functions on $[0,1]$. Then (X, d_θ) is a complete extended b-metric space with respect to

$$d_\theta(x,y) = (\|x\|_\infty + \|y\|_\infty)^2,$$

where $\|x\|_\infty = \sup_{t\in[0,1]} |x(t)|$ and $\theta : X \times X \to [1,+\infty)$ is defined by $\theta(x,y) = |x(t)| + 2|y(t)| + 1$.

Theorem 3.3. *Suppose that $a > \sqrt{2}$ such that*

$$(|\, f(t,x(\beta))\,| + |\, f(t,y(\beta))\,|) \le \frac{B(q)\Gamma(q)}{2a\,((1-q)\Gamma(q)+1)\left[1 + \sup_{\alpha\in[0,1]}|x(\alpha)| + \sup_{\alpha\in[0,1]}|y(\alpha)|\right]} \times (|x(\beta)| + |y(\beta)|),$$

for all $0 \le t \le 1$ and $x, y \in C([0,1],\mathbb{R})$. Then the initial value problem (3.6) and (3.7) has a unique solution $x(t) \in C([0,1],\mathbb{R})$.

Proof. Applying Atangana-Băleanu integral to both sides of relation (3.6), by Remark 3.1 and relation (3.7) we obtain

$$x(t) = x_0 + {}_0^{AB}I^q f(t,x(t)).$$

Let $A_1 = A_2 = X = C([0,1],\mathbb{R})$ are non empty subsets of X. It is obvious A_1, A_2 are closed subsets of (X, d_θ). Let us define the operator $T : Y \to Y$ where $Y = \bigcup_{i=1}^{2} A_i \subseteq X$ as $(Tx)(t) = x_0 + {}_0^{AB}I^q f(t,x(t))$. Clearly $T(A_1) \subset A_2$ and $T(A_2) \subset A_1$. Then T is a cyclic operator on $\bigcup_{i=1}^{2} A_i$.

Clearly, if $x^* \in \bigcap_{i=1}^{2} A_i \subset C([0,1],\mathbb{R})$ is a fixed point of T then x^* is a solution of Eqs (3.6) and (3.7). We shall prove there exists a fixed point of T if all hypothesis of Theorem 2.2 are satisfied.

For $x, y \in Y \subseteq C([0,1],\mathbb{R})$ we get

$$\begin{aligned}
(|(Tx)(t)| + |(Ty)(t)|)^2 &= \left({}_0^{AB}I^q\,[|f(\beta,x(\beta))| + |f(\beta,y(\beta))|]\right)^2 \\
&= \left\{\frac{1-q}{B(q)}\,[|\, f(t,x(t))\,| + |\, f(t,y(t))\,|] + \frac{q}{B(q)}\,{}_0I^q\,[|\, f(\beta,x(\beta))\,| + |\, f(\beta,y(\beta))\,|]\right\}^2 \\
&\le \left\{\frac{B(q)\Gamma(q)}{2a\,((1-q)\Gamma(q)+1)\left[1 + \sup_{\alpha\in[0,1]}|x(\alpha)| + \sup_{\alpha\in[0,1]}|y(\alpha)|\right]}\right\}^2 \\
&\quad \times \left\{\frac{1-q}{B(q)} + \frac{q}{B(q)}\,{}_0I^q(1)\right\}(|x(\beta)| + |y(\beta)|)^2
\end{aligned}$$

$$= \left\{ \frac{B(q)\Gamma(q)}{2a\left((1-q)\Gamma(q)+1\right)\left[1+\sup_{\alpha\in[0,1]}|x(\alpha)|+\sup_{\alpha\in[0,1]}|y(\alpha)|\right]} \right\}^2$$

$$\times \left\{ \frac{1-q}{B(q)} + \frac{q}{B(q)}\frac{1}{q\Gamma(q)} \right\} \left\{ \sqrt{(|x(\beta)|+|y(\beta)|)^2} \right\}^2$$

$$\leq \left\{ \frac{B(q)\Gamma(q)}{2a\left((1-q)\Gamma(q)+1\right)\left[1+\sup_{\alpha\in[0,1]}|x(\alpha)|+\sup_{\alpha\in[0,1]}|y(\alpha)|\right]} \right\}^2$$

$$\times \left\{ \frac{1-q}{B(q)} + \frac{1}{B(q)\Gamma(q)} \right\} \left\{ \sqrt{(|x(\beta)|+|y(\beta)|)^2} \right\}^2 .$$

Applying supremum on both sides and taking into consideration that $\frac{1}{(2+2\sigma)^2} < \frac{1}{(2+\sigma)^2} < \frac{1}{2^2+\sigma^2} < \frac{1}{2+\sigma^2}$, for every $\sigma > 0$, we get

$$\sup_{t\in[0,1]} (|(Tx)(t)|+|(Ty)(t)|)^2 \leq \frac{1}{a^2\left[2+2\left(\sup_{\alpha\in[0,1]}|x(\alpha)|+\sup_{\alpha\in[0,1]}|y(\alpha)|\right)\right]^2} \left\{ \sqrt{\left(\sup_{\beta\in[0,1]}|x(\beta)|+\sup_{\beta\in[0,1]}|y(\beta)|\right)^2} \right\}^2$$

$$\leq \frac{1}{a^2\left[2+\left(\sup_{\alpha\in[0,1]}|x(\alpha)|+\sup_{\alpha\in[0,1]}|y(\alpha)|\right)^2\right]} \left\{ \sqrt{\left(\sup_{\beta\in[0,1]}|x(\beta)|+\sup_{\beta\in[0,1]}|y(\beta)|\right)^2} \right\}^2 .$$

Then we can write

$$(\|(Tx)(t)\|_\infty + \|(Ty)(t)\|_\infty)^2 \leq \frac{1}{a^2\left[2+(\|x(\alpha)\|_\infty+\|y(\alpha)\|_\infty)^2\right]} (\|x(\beta)\|_\infty + \|y(\beta)\|_\infty)^2 .$$

Then we get

$$d_\theta(Tx,Ty) \leq \frac{d_\theta(x,y)}{a^2(2+d_\theta(x,y))} \leq \frac{1}{a^2} d_\theta(x,y).$$

Results $d_\theta(Tx,Ty) \leq \lambda\varphi(d_\theta(x,y))$, where $\varphi(t) = \frac{1}{a^2}t$ is a comparison function. It follows by Lemma 1.1 that $\lambda\varphi$ is an extended b-comparison function with $\lambda = \frac{1}{a^2}$.

Furthermore, for $x \in Y = \bigcup_{i=1}^{2} A_i$, $\lim_{n,m\to+\infty} \theta(x_n,x_m) = 1 < a^2$, where $a > \sqrt{2}$. Thus all the conditions of Theorem 2.2 are satisfied. Then the cyclic integral operator T has a fixed point.

□

4. Conclusions

It is well known that for proving the existence and uniqueness of the solution of different type of equations one can use the fixed point theory technique. One of the most researched areas of mathematics in the last years is the fractional differential calculus because of its utility in modelling real world phenomena.

The aim of this paper is to combine both fields. First, we give some fixed point results for φ and α-φ cyclic-type contractions in an extended b-metric space. The notion of cyclic operator has gained a lot of attention recently, because of its applications in different fields such as: physics, computer science, engineering. The last section, the one devoted to applications, studies the existence and uniqueness of a solution of an integral type equation and of a nonlinear fractional differential equation using the Atangana-Băleanu fractional operator.

Open questions

We can remark the interesting results regarding the notion of cyclic contraction obtained in [44–47]. The authors proved fixed point theorems in different metric spaces. These results can also be given in the case of extended b-metric spaces.

Another open problem is the study of the uniqesness of the fixed point in the case of an extended b-metric space where the functional d_θ is not necessary continuous.

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Conflict of interest

The authors declare that they don't have any conflict of interest.

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