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Research article

Decay properties for evolution-parabolic coupled systems related to thermoelastic plate equations

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Abstract: In this paper, we consider the Cauchy problem for a family of evolution-parabolic coupled systems, which are related to the classical thermoelastic plate equations containing non-local operators. By using diagonalization procedure and WKB analysis, we derive representation of solutions in the phase space. Then, sharp decay properties in a framework of $L^p - L^q$ are investigated via these representations. Particularly, some thresholds for the regularity-loss type decay properties are found.

Keywords: thermoelastic plate equations; non-local operator; decay properties; regularity-loss; Wentzel-Kramers-Brillouin (WKB) analysis; fractional power operators
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1. Introduction

In the present paper, the following Cauchy problem for a family of evolution-parabolic coupled systems:

$$\begin{cases} u_{tt} + \Delta^2 u - (-\Delta)^\sigma \theta = 0, & x \in \mathbb{R}^n, \ t > 0, \\ \theta_t - \Delta\theta + (-\Delta)^\sigma u_t = 0, & x \in \mathbb{R}^n, \ t > 0, \\ (u, u_t, \theta)(0, x) = (u_0, u_1, \theta_0)(x), & x \in \mathbb{R}^n, \end{cases} \tag{1.1}$$

is considered, where $u = u(t, x) \in \mathbb{R}$ and $\theta = \theta(t, x) \in \mathbb{R}$ describe the elongation of a plate and the temperature difference to the equilibrium state respectively. Moreover, we assume the real parameter $\sigma \in [0, \infty)$ in the fractional power operators $(-\Delta)^\sigma$, which can be defined by using the pseudo-differential operators with its symbol $|\xi|$. To be specific, we denote

$$(-\Delta)^\sigma f := \mathcal{F}^{-1}\left(|\xi|^{2\sigma}\mathcal{F}(f)\right),$$

where $\mathcal{F}$ and $\mathcal{F}^{-1}$ stand for the Fourier transform and its inverse transform, respectively.

The abstract version of the coupled systems (1.1) has been considered recently by Dell'Oro-Muñoz River-Pata [10]. The authors derived the decay properties of the exponential type or the polynomial type from the point of view of the semigroup. However, the question of sharp decay properties of the corresponding Cauchy problem in a framework of $L^p - L^q$ is unknown. The purpose of this paper is to answer this question by finding the thresholds to distinguish the regularity-loss type decay properties.

To begin with our paper, let us recall some results related to our model (1.1). In recent years, the Cauchy problem for thermoelastic plate equations has attracted a lot of attentions. Let us consider the classical thermoelastic plate equations, i.e. taking $\sigma = 1$ in the coupled systems (1.1), namely,

$$\begin{cases} u_{tt} + \Delta^2 u + \Delta\theta = 0, & x \in \mathbb{R}^n,\ t > 0, \\ \theta_t - \Delta\theta - \Delta u_t = 0, & x \in \mathbb{R}^n,\ t > 0, \\ (u, u_t, \theta)(0, x) = (u_0, u_1, \theta_0)(x), & x \in \mathbb{R}^n. \end{cases} \tag{1.2}$$

The Cauchy problem was first considered by Said-Houari [35] and later by Racke-Ueda [31]. By using energy method in the phase space, Said-Houari [35] proved energy estimates with additional L^1 regularity or even weighted L^1 regularity for the Cauchy problem (1.2). In 2016, the L^2 estimate for thermoelastic plate equations was derived by Racke-Ueda [31]. To understand the optimality of the derived L^2 estimate, they calculated asymptotic expansion of eigenvalues for the Cauchy problem (1.2). We have to remark that the decay estimates of the classical thermoelastic plate Eq (1.2) are polynomial decay for small frequencies and exponential decay for large frequencies, which means that it does not have regularity-loss decay properties. Other studies on the classical thermoelastic plate equations can be found in [1, 4, 11–13, 18–22, 24, 27, 28, 32, 39] and reference therein.

On the other hand, we found that the evolution-parabolic coupled systems (1.1) is a special case of the famous $\alpha - \beta$ coupled systems

$$\begin{cases} u_{tt} + \mathcal{A}u - \gamma_1 \mathcal{A}^{\alpha}\theta = 0, & x \in \mathbb{R}^n,\ t > 0, \\ \theta_t + \gamma_2 \mathcal{A}^{\beta}\theta + \gamma_1 \mathcal{A}^{\alpha} u_t = 0, & x \in \mathbb{R}^n,\ t > 0, \\ (u, u_t, \theta)(0, x) = (u_0, u_1, \theta_0)(x), & x \in \mathbb{R}^n, \end{cases} \tag{1.3}$$

providing that we take $\mathcal{A} = (-\Delta)^2$, $\alpha = \sigma/2, \beta = 1/2$ and $\gamma_1 = \gamma_2 = 1$ in the systems (1.3). Therefore, this paper also will partly answer the regularity-loss threshold in the Cauchy problem for $\alpha - \beta$ coupled systems. Concerning the regularity analysis for $\alpha - \beta$ coupled systems, we refer the interested readers to [7–9, 11, 14–17, 25, 28] and the references therein. Let us turn to the main purpose of this paper. By considering the fractional power operators in the thermoelastic plate equations, one may not expect in general exponential stability. Therefore, it is interesting to discover some thresholds for regularity-loss decay properties. In order to derive qualitative properties of solutions, we should firstly obtain representation of solutions in the Fourier space. However, it is challenging to get the explicit representation of solution due to the coupled structure of thermoelastic plate equation with non-local operator $(-\Delta)^{\sigma}$ for $\sigma \in [0, \infty)$. To overcome this difficulty, we employ the so-called multi-step diagonalization procedure, which has been developed in [26, 33, 34, 36, 38] even for the model with the non-local operator [3, 5, 6, 23]. Recently, A few researchers have employed fractional power nonlocal operators to describe the response of nonlocal plates. These practical applications should be found in [2, 29, 30].

Then, we apply WKB analysis to study the qualitative properties for solutions localized in small frequencies zone and the large frequencies zone, respectively. We should underline that in different range of parameter σ, we may observe the solutions have different behaviors in the $L^p - L^q$ frame. Here, p and q are not necessary on the conjugate line. Precisely, the regularity-loss decay properties appear only when $\sigma \in [0, 1/2) \cup (3/2, \infty)$. In other words, to estimate the solutions in the L^q norm with $q \geqslant 2$, we need to require higher regular data, which means that the thresholds for regularity-loss decay properties are $\sigma = 1/2$ and $\sigma = 3/2$.

The main contribution of this paper is to find two non-trivial thresholds ($\sigma = 1/2$ and $\sigma = 3/2$) for the regularity-loss decay properties to this evolution-parabolic coupled system. To the best of authors' knowledge, these crucial thresholds were not found in evolution-parabolic coupled system (1.1). This phenomenon indicates that even coupling with parabolic-like equation (in some sense, general Fourier's law in physic), the exponential stability does not hold anymore if $\sigma \in [0, 1/2) \cup (3/2, \infty)$. Namely, this coupling is not sufficient to provide exponential decay for large frequencies.

To end this section, we give some notations to be used in this paper.

Notation 1. We denote the identity matrix of dimension 3×3 by I, i.e., $I := \text{diag}(1, 1, 1)$.

Notation 2. $f \lesssim g$ means that there exists a positive constant C such that $f \leqslant Cg$.

Notation 3. $H^s_p(\mathbb{R}^n)$ and $\dot{H}^s_p(\mathbb{R}^n)$ with $s \geqslant 0$ and $1 \leqslant p < \infty$, denote Bessel and Riesz potential spaces based on $L^p(\mathbb{R}^n)$, respectively. Here $\langle D\rangle^s$ and $|D|^s$ stand for the pseudo-differential operators with symbols $\langle\xi\rangle^s$ and $|\xi|^s$.

Notation 4. We divide the Fourier space into three parts

$$\begin{aligned} Z_{\text{int}}(\varepsilon) &:= \{\xi \in \mathbb{R}^n : |\xi| < \varepsilon \ll 1\}, \\ Z_{\text{mid}}(\varepsilon, N) &:= \{\xi \in \mathbb{R}^n : \varepsilon \leqslant |\xi| \leqslant N\}, \\ Z_{\text{ext}}(N) &:= \{\xi \in \mathbb{R}^n : |\xi| > N \gg 1\}, \end{aligned} \tag{1.4}$$

for small, bounded and large frequencies. Furthermore, let us define $\chi_{\text{int}}(\xi), \chi_{\text{mid}}(\xi), \chi_{\text{ext}}(\xi) \in C^\infty(\mathbb{R}^n)$ having their supports in $Z_{\text{int}}(\varepsilon)$, $Z_{\text{mid}}(\varepsilon/2, 2N)$ and $Z_{\text{ext}}(N)$, respectively, fulfilling

$$\chi_{\text{mid}}(\xi) = 1 - \chi_{\text{int}}(\xi) - \chi_{\text{ext}}(\xi). \tag{1.5}$$

2. Treatment by using multi-step diagonalization procedure

In this section, we will prepare representation of solutions in the Fourier space by applying multi-step diagonalization procedure. Due to the fact that

$$\mathcal{F}\left((-\Delta)^\sigma f\right)(\xi) = |\xi|^{2\sigma} \hat{f}(\xi), \tag{2.1}$$

it give us an opportunity to discuss the non-local operator by considering $|\xi|$-dependent systems. Moreover, because the arbitrary number σ appears in the systems, it seems that the approach of asymptotic expansions of eigenvalues and corresponding eigenprojections does not work well.

To begin with, we apply the partial Fourier transformation with respect to spatial variables to (1.1) to arrive at

$$\begin{cases} \hat{u}_{tt} + |\xi|^4 \hat{u} - |\xi|^{2\sigma} \hat{\theta} = 0, & \xi \in \mathbb{R}^n,\ t > 0, \\ \hat{\theta}_t + |\xi|^2 \hat{\theta} + |\xi|^{2\sigma} \hat{u}_t = 0, & \xi \in \mathbb{R}^n,\ t > 0, \\ (\hat{u}, \hat{u}_t, \hat{\theta})(0, \xi) = (\hat{u}_0, \hat{u}_1, \hat{\theta}_0)(\xi), & \xi \in \mathbb{R}^n. \end{cases} \tag{2.2}$$

Let us introduce the quantity $\hat{v} = \hat{v}(t,\xi)$ such that

$$\hat{v} := \left(\hat{u}_t + |\xi|^2\hat{u}, \hat{u}_t - |\xi|^2, \hat{\theta}\right)^{\mathrm{T}}, \tag{2.3}$$

which is the solution to the next first-order coupled system:

$$\begin{cases} \hat{v}_t + \left(A_0|\xi|^2 + A_1|\xi|^{2\sigma}\right)\hat{v} = 0, & \xi \in \mathbb{R}^n,\ t > 0, \\ \hat{v}(0,\xi) = \hat{v}_0(\xi), & \xi \in \mathbb{R}^n, \end{cases} \tag{2.4}$$

where the coefficient matrices are given by

$$A_0 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad A_1 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & -1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}. \tag{2.5}$$

We would like to remark that the choice of the quantity (2.3) is quite important to our approach, which gives us a suitable structure of the coefficient matrices A_0 and A_1.

With the aim of studying the dominant part in the first step of multi-step diagonalization procedure, we need to clarify between the next cases with respect to the value of parameter $|\xi|$:

- Case 2.1: We consider $\sigma \in [0,1)$ with $\xi \in Z_{\mathrm{ext}}(N)$, or $\sigma \in (1,\infty)$ with $\xi \in Z_{\mathrm{int}}(\varepsilon)$.
- Case 2.2: We consider $\sigma \in (1,\infty)$ with $\xi \in Z_{\mathrm{ext}}(N)$, or $\sigma \in [0,1)$ with $\xi \in Z_{\mathrm{int}}(\varepsilon)$.
- Case 2.3: We consider $\sigma = 1$ for all frequencies.
- Case 2.4: We consider $\sigma \neq 1$ with $\xi \in Z_{\mathrm{mid}}(\varepsilon, N)$.

More precisely, we would like to employ multi-step diagonalization procedure to derive asymptotic expansion of eigenvalues in Cases 2.1 and 2.2; the usual diagonalization to get explicit eigenvalues in Case 2.3; contradiction argument to investigate exponential stability of eigenvalues in the last case. Particularly, the case when $\sigma = 1$ stands for the classical thermoelastic plate equations.

2.1. Treatment for Case 2.1

In Case 2.1, we immediately find that the matrices $A_0|\xi|^2$ has a dominant influence in the coefficient matrix $A(|\xi|;\sigma) = A_0|\xi|^2 + A_1|\xi|^{2\sigma}$. So, we now have to start the diagonalization procedure with the matrix $A_0|\xi|^2$.

By introducing a new variable $\hat{v}^{(1)} := L_1^{-1}\hat{v}$ with

$$L_1 := \begin{pmatrix} 0 & 1 & 1 \\ 0 & -i & i \\ 1 & 0 & 0 \end{pmatrix}, \tag{2.6}$$

we may get

$$\hat{v}_t^{(1)} + \Lambda_0\hat{v}^{(1)} + A_1^{(0)}|\xi|^{2\sigma}\hat{v}^{(1)} = 0, \tag{2.7}$$

where the diagonal matrix is given by

$$\Lambda_0 = |\xi|^2\mathrm{diag}(1, i, -i) \tag{2.8}$$

and the matrix $A_1^{(0)}$ is defined by

$$A_1^{(0)} := L_1^{-1}A_1L_1 = \begin{pmatrix} 0 & \frac{1-i}{2} & \frac{1+i}{2} \\ \frac{-1-i}{2} & 0 & 0 \\ \frac{-1+i}{2} & 0 & 0 \end{pmatrix}. \tag{2.9}$$

Next, we construct a helpful matrix

$$L_2 := |\xi|^{2\sigma-2}\begin{pmatrix} 0 & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{i}{2} & 0 & 0 \\ \frac{i}{2} & 0 & 0 \end{pmatrix}. \tag{2.10}$$

Then, by introducing a new variable $\hat{v}^{(2)} := (I + L_2(|\xi|))^{-1}\hat{v}^{(1)}$, we derive

$$\hat{v}_t^{(2)} + \Lambda_0\hat{v}^{(2)} + (I + L_2)^{-1}\left(A_1^{(0)}|\xi|^{2\sigma} - [L_2, \Lambda_0]\right)\hat{v}^{(2)} + (I + L_2)^{-1}A_1^{(0)}L_2|\xi|^{2\sigma}\hat{v}^{(2)} = 0. \tag{2.11}$$

The choice of matrix L_2 contributes to the next equality. Due to the fact that

$$A_1^0|\xi|^{2\sigma} - [L_2, \Lambda_0] = 0, \tag{2.12}$$

the following first-order system holds:

$$\hat{v}_t^{(2)} + \Lambda_0\hat{v}^{(2)} + A_1^0L_2|\xi|^{2\sigma}\hat{v}^{(2)} + R\hat{v}^{(2)} = 0, \tag{2.13}$$

where the remainder term was denoted by

$$R = (I + L_2)^{-1}L_2A_1^{(0)}L_2(|\xi|)|\xi|^{2\sigma}. \tag{2.14}$$

Similarly, we introduce $\hat{v}^{(3)} := (I + L_3)^{-1}\hat{v}^{(2)}$ with

$$L_3 := |\xi|^{4\sigma-4}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{-1+i}{8} \\ 0 & \frac{-1-i}{8} & 0 \end{pmatrix} \tag{2.15}$$

to obtain directly

$$\begin{aligned} &\hat{v}_t^{(3)} + \Lambda_0\hat{v}^{(3)} + (I + L_3)^{-1}\left(A_1^{(0)}L_2|\xi|^{2\sigma} + [L_3, \Lambda_0]\right)\hat{v}^{(3)} \\ &\quad + (I + L_3)^{-1}A_1^{(0)}L_2L_3|\xi|^{2\sigma}\hat{v}^{(3)} + (I + L_3)^{-1}R(I + L_3)\hat{v}^{(3)} = 0. \end{aligned} \tag{2.16}$$

According to the simple calculation that

$$\Lambda_1 = |\xi|^{4\sigma-2}\,\mathrm{diag}\left(-\frac{1}{2}, \frac{1+i}{4}, \frac{1-i}{4}\right), \tag{2.17}$$

we may have the final first-order coupled system

$$\hat{v}_t^{(3)} + (\Lambda_0 + \Lambda_1)\hat{v}^{(3)} + \tilde{R}\hat{v}^{(3)} = 0, \tag{2.18}$$

where the new remainder should be of the form

$$\tilde{R} = (I + L_3)^{-1}\left(A_1^{(0)}L_2L_3|\xi|^{2\sigma} - L_3\Lambda_1\right) + (I + L_3)^{-1}R(I + L_3). \tag{2.19}$$

Summarizing the above diagonalization procedures, we obtain non-zero pairwise distinct eigenvalues, where their first real values are positive. Thus, the following proposition for the asymptotic behavior of eigenvalues and representation of solutions can be concluded.

Proposition 2.1. *When $\sigma \in [0,1)$ with $\xi \in Z_{\text{ext}}(N)$, or $\sigma \in (1,\infty)$ with $\xi \in Z_{\text{int}}(\varepsilon)$, we have the results that the eigenvalues $\lambda_j = \lambda_j(|\xi|)$ of the coefficient matrix $A(|\xi|;\sigma)$ from the Cauchy problem* (2.2) *can be written as:*

$$\begin{aligned}\lambda_1(|\xi|) &= |\xi|^2 - \frac{1}{2}|\xi|^{4\sigma-2} + O\left(|\xi|^{6\sigma-4}\right),\\ \lambda_2(|\xi|) &= i|\xi|^2 + \frac{1+i}{4}|\xi|^{4\sigma-2} + O\left(|\xi|^{6\sigma-4}\right),\\ \lambda_3(|\xi|) &= -i|\xi|^2 + \frac{1-i}{4}|\xi|^{4\sigma-2} + O\left(|\xi|^{6\sigma-4}\right).\end{aligned} \tag{2.20}$$

Furthermore, the solution in the Fourier space has the following representations:

- *When $\sigma \in [0,1)$ with $\xi \in Z_{\text{ext}}(N)$, the solution to the Cauchy problem* (2.2) *is*

$$\hat{v}(t,\xi) = T_{\text{ext}}(\xi)\,\text{diag}\left(\mathrm{e}^{-\lambda_1(|\xi|)t}, \mathrm{e}^{-\lambda_2(|\xi|)t}, \mathrm{e}^{-\lambda_3(|\xi|)t}\right) T_{\text{ext}}^{-1}(\xi)\hat{v}_0(\xi); \tag{2.21}$$

- *When $\sigma \in (1,\infty)$ with $\xi \in Z_{\text{int}}(\varepsilon)$, the solution to the Cauchy problem* (2.2) *is*

$$\hat{v}(t,\xi) = T_{\text{int}}(\xi)\,\text{diag}\left(\mathrm{e}^{-\lambda_1(|\xi|)t}, \mathrm{e}^{-\lambda_2(|\xi|)t}, \mathrm{e}^{-\lambda_3(|\xi|)t}\right) T_{\text{int}}^{-1}(\xi)\hat{v}_0(\xi); \tag{2.22}$$

where $T_{\text{int}}(\xi) = L_1(I+L_2)(I+L_3)$ when $\sigma \in (1,\infty)$ with $\xi \in Z_{\text{int}}(\varepsilon)$, and $T_{\text{ext}}(\xi) = L_1(I+L_2)(I+L_3)$ when $\sigma \in [0,1)$ with $\xi \in Z_{\text{ext}}(N)$.

2.2. Treatment for Case 2.2

In this case, we observe that comparing the matrices $A_1|\xi|^{2\sigma}$ with $A_0|\xi|^2$, the matrix $A_1|\xi|^{2\sigma}$ has a dominant influence. Thus, we now begin diagonalization procedure with the matrix $A_1|\xi|^{2\sigma}$.

Similarly, defining a new variable $\hat{v}^{(1)} := L_4^{-1}\hat{v}$ with a matrix

$$L_4 := \begin{pmatrix} 1 & i & -i \\ -1 & i & -i \\ 0 & 1 & 1 \end{pmatrix}, \tag{2.23}$$

we may transfer our system to the following one:

$$\hat{v}_t^{(1)} + \Lambda_1\hat{v}^{(1)} + A_0^0|\xi|^2\hat{v}^{(1)} = 0 \tag{2.24}$$

with the new coefficient matrices

$$\Lambda_1 = |\xi|^{2\sigma}\,\text{diag}(0,i,-i) \quad \text{and} \quad A_0^0 = L_4^{-1}A_0L_4 = \begin{pmatrix} 0 & -i & i \\ -\frac{i}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{i}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix}. \tag{2.25}$$

By introducing $\hat{v}^{(2)} := (I+L_5)^{-1}\hat{v}^{(1)}$ and a suitable matrix

$$L_5 = |\xi|^{2-2\sigma}\begin{pmatrix} 0 & -1 & -1 \\ \frac{1}{2} & 0 & \frac{i}{4} \\ \frac{1}{2} & -\frac{i}{4} & 0 \end{pmatrix}, \tag{2.26}$$

we immediately obtain the first-order coupled system

$$\hat{v}_t^{(2)} + \Lambda_1 \hat{v}^{(2)} + (I + L_5)^{-1} \left(A_0^0 |\xi|^2 - [L_5, \Lambda_1] \right) \hat{v}^{(2)} + (I + L_5)^{-1} A_0^0 L_5 |\xi|^2 \hat{v}^{(2)} = 0. \tag{2.27}$$

In other words, making use of the structure of the suitable matrix L_5, it leads to

$$\hat{v}_t^{(2)} + \Lambda_1 \hat{v}^{(2)} + \Lambda_2 \hat{v}^{(2)} - (I + L_5)^{-1} L_5 \Lambda_2 \hat{v}^{(2)} + (I + L_5)^{-1} A_0^0 L_5 |\xi|^2 \hat{v}^{(2)} = 0 \tag{2.28}$$

carrying the diagonal matrix

$$\Lambda_2 = |\xi|^2 \operatorname{diag}\left(0, \frac{1}{2}, \frac{1}{2}\right) \tag{2.29}$$

or, equivalently,

$$\hat{v}_t^{(2)} + \Lambda_1 \hat{v}^{(2)} + \Lambda_2 \hat{v}^{(2)} + A_0^1 \hat{v}^{(2)} + R_1 \hat{v}^{(2)} = 0. \tag{2.30}$$

In the above, we define

$$A_0^1 = A_0^0 L_5 |\xi|^2 - L_5 \Lambda_2 = |\xi|^{4-2\sigma} \begin{pmatrix} 0 & \frac{3}{4} & \frac{3}{4} \\ \frac{1}{2} & \frac{3i}{8} & \frac{i}{2} \\ \frac{1}{2} & -\frac{i}{2} & -\frac{3i}{8} \end{pmatrix} \tag{2.31}$$

and the remainder in this case is

$$R_1 = -(I + L_5)^{-1} L_5 A_0^1. \tag{2.32}$$

Analogously, denoting $\hat{v}^{(3)} := (I + L_6)^{-1} \hat{v}^{(2)}$ with

$$L_6 = |\xi|^{4-4\sigma} \begin{pmatrix} 0 & -\frac{3i}{4} & \frac{3i}{4} \\ \frac{i}{2} & 0 & -\frac{1}{4} \\ -\frac{i}{2} & -\frac{1}{4} & 0 \end{pmatrix}, \tag{2.33}$$

we may immediately derive

$$\begin{aligned} &\hat{v}_t^{(3)} + \Lambda_1 \hat{v}^{(3)} + \Lambda_2 \hat{v}^{(3)} + (I + L_6)^{-1} \left(A_0^1 - [L_6, \Lambda_1] \right) \hat{v}^{(3)} \\ &\quad + (I + L_6)^{-1} \left(A_0^1 L_6 - [L_6, \Lambda_2] \right) \hat{v}^{(3)} + (I + L_6)^{-1} R_1 (I + L_6) \hat{v}^{(3)} = 0. \end{aligned} \tag{2.34}$$

It is obvious that the diagonal matrix is given by

$$\Lambda_3 = A_0^1 - [L_6, \Lambda_1] = |\xi|^{4-2\sigma} \operatorname{diag}\left(0, \frac{3i}{8}, -\frac{3i}{8}\right). \tag{2.35}$$

Therefore, we may obtain the next system:

$$\begin{aligned} &\hat{v}_t^{(3)} + \Lambda_1 \hat{v}^{(3)} + \Lambda_2 \hat{v}^{(3)} + \Lambda_3 \hat{v}^{(3)} + (I + L_6)^{-1} L_6 \left([L_6, \Lambda_2] - \Lambda_3 \right) \hat{v}^{(3)} \\ &\quad + (I + L_6)^{-1} \left(A_0^1 L_6 - [R_1, L_6] \right) \hat{v}^{(3)} + (I + L_5)^{-1} L_5 L_5 A_0^1 \hat{v}^{(3)} - \left(L_5 A_0^1 + [L_6, \Lambda_2] \right) \hat{v}^{(3)} = 0, \end{aligned} \tag{2.36}$$

which can be rewritten by

$$\hat{v}_t^{(3)} + \Lambda_1 \hat{v}^{(3)} + \Lambda_2 \hat{v}^{(3)} + \Lambda_3 \hat{v}^{(3)} + A_0^2 \hat{v}^{(3)} + R_2 \hat{v}^{(3)} = 0, \tag{2.37}$$

where we denoted the coefficient matrices by

$$A_0^2 = -\left(L_5A_0^1 + [L_6,\Lambda_2]\right) = |\xi|^{6-4\sigma}\begin{pmatrix} 1 & \frac{i}{4} & -\frac{i}{4} \\ \frac{i}{8} & -\frac{1}{2} & -\frac{15}{32} \\ -\frac{i}{8} & -\frac{15}{32} & -\frac{1}{2} \end{pmatrix} \tag{2.38}$$

and

$$R_2 = (I+L_6)^{-1}L_6\left([L_6,\Lambda_2]-\Lambda_3\right) + (I+L_6)^{-1}\left(A_0^1L_6 - [R_1,L_6]\right) + (I+L_5)^{-1}L_5L_5A_0^1. \tag{2.39}$$

However, we now need to do one step more due to the zero value in the first position of the diagonal element. Finally, we introduce $\hat{v}^{(4)} := (I+L_7)^{-1}\hat{v}^{(3)}$ with

$$L_7 = |\xi|^{6-6\sigma}\begin{pmatrix} 0 & \frac{1}{4} & \frac{1}{4} \\ -\frac{1}{8} & 0 & -\frac{15i}{64} \\ \frac{1}{8} & \frac{15i}{64} & 0 \end{pmatrix}. \tag{2.40}$$

In this way, the next system is derived:

$$\hat{v}_t^{(4)} + \Lambda_1\hat{v}^{(4)} + \Lambda_2\hat{v}^{(4)} + \Lambda_3\hat{v}^{(4)} + \Lambda_4\hat{v}^{(4)} + R_3\hat{v}^{(4)} = 0, \tag{2.41}$$

where

$$\Lambda_4 = A_0^2 - [L_7,\Lambda_1] = |\xi|^{6-4\sigma}\operatorname{diag}\left(1,-\frac{1}{2},-\frac{1}{2}\right), \tag{2.42}$$

$$\begin{aligned} R_3 &= (I+L_7)^{-1}R_2(I+L_7) + (I+L_7)^{-1}\left(A_0^2L_7 - L_7\Lambda_4\right) + (I+L_7)^{-1}[\Lambda_2,L_7] \\ &\quad + (I+L_7)^{-1}[\Lambda_3,L_7]. \end{aligned} \tag{2.43}$$

Considering all steps of diagonalization procedure in the above, we obtain non-zero pairwise distinct eigenvalues, where their first real values are positive. Hence, the following proposition for the asymptotic behavior of eigenvalues and representation of solutions can be concluded.

Proposition 2.2. *When $\sigma \in (1,\infty)$ with $\xi \in Z_{\text{ext}}(N)$, or $\sigma \in [0,1)$ with $\xi \in Z_{\text{int}}(\varepsilon)$, the eigenvalues $\lambda_j = \lambda_j(|\xi|)$ of the coefficient matrix $A(|\xi|;\sigma)$ from the Cauchy problem* (2.2) *can be written as:*

$$\begin{aligned} \lambda_1(|\xi|) &= |\xi|^{6-4\sigma} + O\left(|\xi|^{8-6\sigma}\right), \\ \lambda_2(|\xi|) &= i|\xi|^{2\sigma} + \frac{1}{2}|\xi|^2 + \frac{3i}{8}|\xi|^{4-2\sigma} - \frac{1}{2}|\xi|^{6-4\sigma} + O\left(|\xi|^{8-6\sigma}\right), \\ \lambda_3(|\xi|) &= -i|\xi|^{2\sigma} + \frac{1}{2}|\xi|^2 - \frac{3i}{8}|\xi|^{4-2\sigma} - \frac{1}{2}|\xi|^{6-4\sigma} + O\left(|\xi|^{8-6\sigma}\right). \end{aligned} \tag{2.44}$$

Furthermore, the solution in the Fourier space has the following representations:

- *When $\sigma > 1$ with $\xi \in Z_{\text{ext}}(N)$, the solution to the Cauchy problem* (2.2) *is*

$$\hat{v}(t,\xi) = T_{\text{ext}}(\xi)\operatorname{diag}\left(e^{-\lambda_1(|\xi|)t}, e^{-\lambda_2(|\xi|)t}, e^{-\lambda_3(|\xi|)t}\right)T_{\text{ext}}^{-1}(\xi)\hat{v}_0(\xi); \tag{2.45}$$

- *When $\sigma \in [0,1)$ with $\xi \in Z_{\text{int}}(\varepsilon)$, the solution to the Cauchy problem* (2.2) *is*

$$\hat{v}(t,\xi) = T_{\text{int}}(\xi)\operatorname{diag}\left(e^{-\lambda_1(|\xi|)t}, e^{-\lambda_2(|\xi|)t}, e^{-\lambda_3(|\xi|)t}\right)T_{\text{int}}^{-1}(\xi)\hat{v}_0(\xi); \tag{2.46}$$

where $T_{\text{ext}}(\xi) = L_4(I+L_5)(I+L_6)(I+L_7)$ when $\sigma \in (1,\infty)$ with $\xi \in Z_{\text{ext}}(N)$, and $T_{\text{int}}(\xi) = L_4(I+L_5)(I+L_6)(I+L_7)$ when $\sigma \in [0,1)$ with $\xi \in Z_{\text{int}}(\varepsilon)$.

2.3. Treatment for Case 2.3

In this case, we only need to diagonalize the matrix $(A_0 + A_1)|\xi|^2$ as a whole due to the fact that the matrices $A_0|\xi|^2$ and $A_1|\xi|^{2\sigma}$ have the same influence while $\sigma = 1$. Then, we have the following result.

Proposition 2.3. *After one step of the diagonalization procedure, the starting Cauchy problem* (2.4) *can be transformed to*

$$\begin{cases} \hat{v}_t^{(1)} + (A_0 + A_1)|\xi|^2 \hat{v}^{(1)} = 0, & \xi \in \mathbb{R}^n,\ t > 0, \\ \hat{v}^{(1)}(0,\xi) = \hat{v}_0^{(1)}(\xi), & \xi \in \mathbb{R}^n. \end{cases} \tag{2.47}$$

Proof. We directly calculate the eigenvalues of the matrix $(A_0 + A_1)|\xi|^2$ to get

$$0 = \det\Big((A_0 + A_1)|\xi|^2 - \lambda I\Big) = \begin{vmatrix} -\lambda & -|\xi|^2 & -|\xi|^2 \\ |\xi|^2 & -\lambda & -|\xi|^2 \\ \frac{|\xi|^2}{2} & \frac{|\xi|^2}{2} & |\xi|^2 - \lambda \end{vmatrix} = -\lambda^3 + \lambda^2|\xi|^2 - 2|\xi|^4\lambda + |\xi|^6. \tag{2.48}$$

Then, the solutions of the above cubic equation are

$$\lambda_j(|\xi|) = |\xi|^2 z_j \quad \text{with} \quad j = 1,2,3, \tag{2.49}$$

where the elements are

$$\begin{aligned} z_1 &= \frac{1}{3}\left(1 + \sqrt[3]{\frac{11 + 3\sqrt{69}}{2}} + \sqrt[3]{\frac{11 - 3\sqrt{69}}{2}}\right), \\ z_2 &= \frac{1}{3}\left(1 + \frac{-1 + \sqrt{3}i}{2}\sqrt[3]{\frac{11 + 3\sqrt{69}}{2}} - \frac{1 + \sqrt{3}i}{2}\sqrt[3]{\frac{11 - 3\sqrt{69}}{2}}\right), \\ z_3 &= \frac{1}{3}\left(1 - \frac{1 + \sqrt{3}i}{2}\sqrt[3]{\frac{11 + 3\sqrt{69}}{2}} + \frac{-1 + \sqrt{3}i}{2}\sqrt[3]{\frac{11 - 3\sqrt{69}}{2}}\right). \end{aligned} \tag{2.50}$$

We point out that the constants $z_1 \neq z_2 \neq z_3$ and $\mathrm{Re}\, z_j < 0$ for all $j = 1,2,3$, which means the pairwise distinct eigenvalues with negative real parts are obtained. By introducing $\hat{v}^{(1)} = T^{-1}\hat{v}$, which satisfies

$$\hat{v}_t^{(1)} + \Lambda_1 \hat{v}^{(1)} = 0 \tag{2.51}$$

with the diagonal matrix

$$\Lambda_1 = |\xi|^2 T^{-1}(A_0 + A_1)T = |\xi|^2 \operatorname{diag}(z_1, z_2, z_3), \tag{2.52}$$

where the right matrix is denoted by

$$T = \begin{pmatrix} 2|\xi|^4 - 3z_1|\xi|^2 & 2|\xi|^4 - 3z_2|\xi|^2 & 2|\xi|^4 - 3z_3|\xi|^2 \\ 2|\xi|^2 - 3z_1|\xi|^2 & 2|\xi|^2 - 3z_2|\xi|^2 & 2|\xi|^2 - 3z_3|\xi|^2 \\ z_1^2 + z_1|\xi|^2 & z_2^2 + z_2|\xi|^2 & z_3^2 + z_3|\xi|^2 \end{pmatrix}. \tag{2.53}$$

As a consequence, we complete the proof immediately. □

2.4. Treatment for Case 2.4

Finally, in Case 2.4, with the aim of guaranteeing the exponential stabilities of eigenvalues, we need to derive the exponential decay results by obtaining a priori estimates for eigenvalues. We will give the proof by using contraction argument.

Let us assume that there is a purely imaginary eigenvalue $\lambda = ia$ with $a \in \mathbb{R} \neq 0$ of the matrix $A_0|\xi|^2 + A_1|\xi|^\sigma$ for $\xi \in Z_{\mathrm{mid}}(\varepsilon, N)$. The non-zero real number a fulfills the cubic equation

$$\begin{aligned} 0 = \det\left(A_0|\xi|^2 + A_1|\xi|^{2\sigma} - \lambda I\right) &= \begin{vmatrix} -\lambda & -|\xi|^2 & -|\xi|^{2\sigma} \\ |\xi|^2 & -\lambda & -|\xi|^{2\sigma} \\ \frac{|\xi|^{2\sigma}}{2} & \frac{|\xi|^{2\sigma}}{2} & |\xi|^2 - \lambda \end{vmatrix} \\ &= -\lambda^3 + \lambda^2|\xi|^2 - \left(|\xi|^{4\sigma} + |\xi|^4\right)\lambda + |\xi|^6 \\ &= ia^3 - |\xi|^2 a^2 - \left(|\xi|^{4\sigma} + |\xi|^4\right) ia + |\xi|^6, \end{aligned} \tag{2.54}$$

which implies the non-zero constant a should be the solution of the following equations:

$$\begin{cases} -|\xi|^2 a^2 + |\xi|^6 = 0, \\ ia^3 - \left(|\xi|^{4\sigma} + |\xi|^4\right) ia = 0. \end{cases} \tag{2.55}$$

Then, we may obtain the solution of a^2 such that $a^2 = |\xi|^4$ and $a^2 = |\xi|^2 + |\xi|^{4\sigma}$. We can conclude a contradiction immediately. Recalling Proposition 2.1 and Proposition 2.2, we find that the dominant real parts of all eigenvalues are positive. In other words, according to the compactness of the zone, for $\xi \in Z_{\mathrm{mid}}(\varepsilon, N)$ the next estimates hold:

$$|\hat{v}(t,\xi)| \lesssim \mathrm{e}^{-ct}|\hat{v}_0(\xi)| \tag{2.56}$$

for $t \geqslant 0$ and $c > 0$.

3. Decay properties in the $L^p - L^q$ frame

In this section, we will study $L^p - L^q$ estimates away of the conjugate line, i.e. $1 \leqslant p \leqslant 2 \leqslant q \leqslant \infty$. To do this, we first introduce a useful lemma. One may see, for example, [3, 37] recently by using Hölder's inequality and the Hausdorff-Young inequality.

Lemma 3.1. *Let $f \in \mathcal{S}(\mathbb{R}^n)$ and $\kappa_1 > 0$, $\kappa_2 \in \mathbb{R}$, $s \geqslant 0$. Then, the next estimates hold:*

$$\left\|\mathcal{F}^{-1}_{\xi\to x}\left(\chi_{\mathrm{int}}(\xi)|\xi|^s \mathrm{e}^{-c|\xi|^{\kappa_1}t}\hat{f}(\xi)\right)\right\|_{L^q(\mathbb{R}^n)} \lesssim (1+t)^{-\frac{s}{\kappa_1}-\frac{n}{\kappa_1}\left(\frac{1}{p}-\frac{1}{q}\right)}\|f\|_{L^p(\mathbb{R}^n)}, \tag{3.1}$$

$$\left\|\mathcal{F}^{-1}_{\xi\to x}\left(\chi_{\mathrm{ext}}(\xi)|\xi|^s \mathrm{e}^{-c|\xi|^{\kappa_2}t}\hat{f}(\xi)\right)\right\|_{L^q(\mathbb{R}^n)} \lesssim \begin{cases} \mathrm{e}^{-ct}\left\|\langle D\rangle^{s+\ell}f\right\|_{L^p(\mathbb{R}^n)} & \text{if } \kappa_2 \geqslant 0, \\ (1+t)^{\frac{\ell}{\kappa_2}-\frac{n}{\kappa_2}\left(\frac{1}{r}-\frac{1}{q}\right)}\left\|\langle D\rangle^{s+\ell}f\right\|_{L^r(\mathbb{R}^n)} & \text{if } \kappa_2 < 0, \end{cases} \tag{3.2}$$

where $c > 0$, $1 \leqslant p, r \leqslant 2 \leqslant q \leqslant \infty$ and $\ell > n(1/p - 1/q)$.

It is well-known that the pointwise estimates in the Fourier space are useful for us to describe the decay properties of solutions. Hence, summarizing the result in the last section, we may derive the following pointwise estimates in the Fourier space, which is sharp since the application of diagonalization procedure.

Proposition 3.1. *The solution $\hat{v} = \hat{v}(t, \xi)$ to the Cauchy problem* (2.2) *for $\sigma \in [0, \infty)$ satisfies the next pointwise estimates for any $\xi \in \mathbb{R}^n$ and $t \geqslant 0$:*

$$|\hat{v}(t, \xi)| \lesssim \exp\left(-c\rho(|\xi|)t\right) |\hat{v}_0(\xi)|, \tag{3.3}$$

where the function $\rho(|\xi|)$ characterizing the decay properties can be represented by

$$\rho(|\xi|) = \begin{cases} \dfrac{|\xi|^{6-4\sigma}}{1 + |\xi|^{8-8\sigma}} & \text{if } \ \sigma \in [0, 1], \\ \dfrac{|\xi|^{4\sigma-2}}{1 + |\xi|^{8\sigma-8}} & \text{if } \ \sigma \in (1, \infty). \end{cases} \tag{3.4}$$

Let us now analyze the decay properties according to the pointwise estimates.

- We may observe that the decay property for small frequencies is changed from $\sigma \in [0, 1]$ to $\sigma \in (1, \infty)$, which implies the first threshold $\sigma = 1$. It will lead to the decay rate for L^p data changing. We will show more detail in the forthcoming part.
- Concerning the decay property for large frequencies, we find that the solutions localized for large frequencies decay exponentially if and only if $\sigma \in [1/2, 3/2]$. In other words, $\rho(|\xi|) \approx |\xi|^{2(2\sigma-1)}$ for $\sigma \in [1/2, 1]$ and $\rho(|\xi|) \approx |\xi|^{2(3-2\sigma)}$ for $\sigma \in (1, 3/2]$. In the remaining case $\sigma \in [0, 1/2) \cup (3/2, \infty)$, the decay property is regularity-loss type. By this way, one may expect the second and third threshold for decay property are $\sigma = 1/2$ and $\sigma = 3/2$, respectively.

Let us state our main theorem on the decay estimates of solutions.

Theorem 3.1. *Let us assume $|D|^2 u_0, u_1, \theta_0 \in H^{s+\ell}_p(\mathbb{R}^n)$ with $s \geqslant 0$ and $\ell > n(1/p - 1/q)$. Then, the following estimates for the solutions to the Cauchy problem* (1.1) *hold:*

$$\left\|\left(|D|^2 u, u_t, \theta\right)(t, \cdot)\right\|_{(\dot{H}^s_q(\mathbb{R}^n))^3} \lesssim \begin{cases} (1+t)^{\max\left\{-\frac{s}{6-4\sigma} - \frac{n}{6-4\sigma}\left(\frac{1}{p} - \frac{1}{q}\right), \frac{\ell}{4\sigma-2} - \frac{n}{4\sigma-2}\left(\frac{1}{p} - \frac{1}{q}\right)\right\}} \left\|\left(|D|^2 u_0, u_1, \theta_0\right)\right\|_{H^{s+\ell}_p(\mathbb{R}^n)} \\ \text{if } \ \sigma \in [0, 1/2), \\ (1+t)^{-\frac{s}{6-4\sigma} - \frac{n}{6-4\sigma}\left(\frac{1}{p} - \frac{1}{q}\right)} \left\|\left(|D|^2 u_0, u_1, \theta_0\right)\right\|_{(H^{s+\ell}_p(\mathbb{R}^n))^3} \\ \text{if } \ \sigma \in [1/2, 1], \\ (1+t)^{-\frac{s}{4\sigma-2} - \frac{n}{4\sigma-2}\left(\frac{1}{p} - \frac{1}{q}\right)} \left\|\left(|D|^2 u_0, u_1, \theta_0\right)\right\|_{(H^{s+\ell}_p(\mathbb{R}^n))^3} \\ \text{if } \ \sigma \in (1, 3/2), \\ (1+t)^{\max\left\{-\frac{s}{4\sigma-2} - \frac{n}{4\sigma-2}\left(\frac{1}{p} - \frac{1}{q}\right), \frac{\ell}{6-4\sigma} - \frac{n}{6-4\sigma}\left(\frac{1}{p} - \frac{1}{q}\right)\right\}} \left\|\left(|D|^2 u_0, u_1, \theta_0\right)\right\|_{(H^{s+\ell}_p(\mathbb{R}^n))^3} \\ \text{if } \ \sigma \in [3/2, \infty), \end{cases} \tag{3.5}$$

for any $t \geqslant 0$ and $1 \leqslant p \leqslant 2 \leqslant q \leqslant \infty$.

Proof. By using the pointwise estimates of solution for small frequencies and Lemma 3.1, we derive

$$\left\|\chi_{\rm int}(D)|D|^s\left(|D|^2u,u_t,\theta\right)(t,\cdot)\right\|_{(L^q(\mathbb{R}^n))^3}\lesssim\left\|\mathcal{F}^{-1}_{\xi\to x}\left(\chi_{\rm int}(\xi)|\xi|^s\mathrm{e}^{-c\rho(|\xi|)t}\hat{v}_0(\xi)\right)(t,\cdot)\right\|_{(L^q(\mathbb{R}^n))^3} \tag{3.6}$$

$$\lesssim\begin{cases}(1+t)^{-\frac{s}{6-4\sigma}-\frac{n}{6-4\sigma}\left(\frac{1}{p}-\frac{1}{q}\right)}\left\|\left(|D|^2u_0,u_1,\theta_0\right)\right\|_{(L^p(\mathbb{R}^n))^3}\\ \text{if}\quad\sigma\in[0,1],\\ (1+t)^{-\frac{s}{4\sigma-2}-\frac{n}{4\sigma-2}\left(\frac{1}{p}-\frac{1}{q}\right)}\left\|\left(|D|^2u_0,u_1,\theta_0\right)\right\|_{(L^p(\mathbb{R}^n))^3}\\ \text{if}\quad\sigma\in(1,\infty),\end{cases} \tag{3.7}$$

for any $s\geqslant 0$ and $1\leqslant p\leqslant 2\leqslant q\leqslant\infty$.

Secondly, we consider the pointwise estimates for large frequencies to get

$$\left\|\chi_{\rm ext}(D)|D|^s\left(|D|^2u,u_t,\theta\right)(t,\cdot)\right\|_{(L^q(\mathbb{R}^n))^3}\lesssim\left\|\mathcal{F}^{-1}_{\xi\to x}\left(\chi_{\rm ext}(\xi)|\xi|^s\mathrm{e}^{-c\rho(|\xi|)t}\hat{v}_0(\xi)\right)(t,\cdot)\right\|_{(L^q(\mathbb{R}^n))^3} \tag{3.8}$$

$$\lesssim\begin{cases}(1+t)^{\frac{\ell}{4\sigma-2}-\frac{n}{4\sigma-2}\left(\frac{1}{p}-\frac{1}{q}\right)}\left\|\left(|D|^2u_0,u_1,\theta_0\right)\right\|_{(H^{s+\ell}_p(\mathbb{R}^n))^3}\\ \text{if}\quad\sigma\in[0,1/2),\\ \mathrm{e}^{-ct}\left\|\left(|D|^2u_0,u_1,\theta_0\right)\right\|_{(H^{s+\ell}_p(\mathbb{R}^n))^3}\\ \text{if}\quad\sigma\in[1/2,3/2],\\ (1+t)^{\frac{\ell}{6-4\sigma}-\frac{n}{6-4\sigma}\left(\frac{1}{p}-\frac{1}{q}\right)}\left\|\left(|D|^2u_0,u_1,\theta_0\right)\right\|_{(H^{s+\ell}_p(\mathbb{R}^n))^3}\\ \text{if}\quad\sigma\in(3/2,\infty),\end{cases} \tag{3.9}$$

where we used Lemma 3.1. Here, we take $s\geqslant 0$, $\ell>n(1/p-1/q)$ and $1\leqslant p\leqslant 2\leqslant q\leqslant\infty$.

Finally, due to the fact the exponential stability comes for $\xi\in Z_{\rm mid}(\varepsilon,N)$, we may derive exponential decay of solutions without asking additional regularity for initial data. The proof is completed. □

Remark 3.1. *One also may derive L^2-L^2 estimates with additional L^m regularities for $m\in[1,2)$ or even weighted L^1 regularities by following the approach of [3]. However, it is still open to derive the estimates in the L^p-L^q frame carrying $1\leqslant p\leqslant q\leqslant 2$. The main difficulty stays at some oscillating integrals in the L^1 space.*

Remark 3.2. *As we mentioned in the previous part, the thresholds for decay properties are described by the numbers $\sigma=1/2$, $\sigma=1$ and $\sigma=3/2$, which are not trivial from the view of the model. This discovery is the core of this paper.*

Remark 3.3. *Let us give some comments by comparing with the known results.*

- *In the last paper Dell'Oro-Muñoz River-Pata [10], the authors investigated two type decay properties by some tools form semigroup. In Theorem 3.1, we derive some new decay estimates of solutions in the whole space $\mathbb{R}^n$. It provides a complement of the study for this evolution-plate equations. Moreover, it also shows large-time behaviors of solutions under different value of σ.*

- *In this work, we couple the plate-like equation with the parabolic-like equation, which shows regularity-loss decay properties. However, Racke-Ueda [31] considered the coupling with classical Fourier's law, which derives exponential stability. In other words, a coupling with parabolic-like equation will destroy the exponential stability that is a new effect from the viewpoint of authors.*
- *The recent paper Chen [3] shows the fractional power operator* $(-\Delta)^{\sigma}$ *in the plate equation will lead to some smoothing effects and change the decay rates. However, his work only presents small changes by the fractional power operator in evolution-parabolic coupled system. In our work, the fractional power operator appears in the parabolic equation, and brings a great change (from exponential stability to polynomial stability). This phenomenon does not occur in Chen [3].*

4. Conclusions

In this work, we investigated decay properties for a evolution-parabolic coupled system in the field of thermoelastic plates by distinguishing regularity-loss type decay properties and no-loss behaviors, which is mainly determined by the parameter σ in the coupling terms. This parameter actually may influence on the type of model. Motivated by the recent researches in this field, we believe the study for asymptotic profiles as $t \to \infty$ is quite interesting. Benefit from our main tools (diagonalization procedure and WKB analysis), we are able to observe the dominant part of the characteristic roots in Propositions 2.1 and 2.2. Therefore, we conjecture that one may obtain asymptotic profiles of solutions by following our approaches.

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Conflict of interest

The authors declare that they have no competing interests.

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