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Research article

Plenty of wave solutions to the ill-posed Boussinesq dynamic wave equation under shallow water beneath gravity

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Abstract: This paper applies two computational techniques for constructing novel solitary wave solutions of the ill-posed Boussinesq dynamic wave (IPB) equation. Jacques Hadamard has formulated this model for studying the dynamic behavior of waves in shallow water under gravity. Extended simple equation (ESE) method and novel Riccati expansion (NRE) method have been applied to the investigated model's converted nonlinear ordinary differential equation through the wave transformation. As a result of this research, many solitary wave solutions have been obtained and represented in different figures in two-dimensional, three-dimensional, and density plots. The explanation of the methods used shows their dynamics and effectiveness in dealing with certain nonlinear evolution equations.

Keywords: ill-posed Boussinesq dynamical wave; extended simple equation (ESE) method; novel Riccati expansion (NRE) method; nonlinear soliton lattice wave solutions
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1. Introduction

Beginning from the mid of 18th century, Euler, Cauchy, d'Alembert, Hamilton, Jacobi, Lagrange, Laplace, Monge, and many others have started formulating some complex nonlinear phenomena through nonlinear partial differential equations [1, 2]. Many complex phenomena, such as solid-state physics, mechanics, fluid mechanics, surface plasma physics, quantum mechanics, civic engineering, population climate, neural networks, epidemiology for infectious disorders, plasma wave, thermodynamics, physics of condensed matter, nonlinear optics, etc., are extracted in distinct mathematical formulas [3–5]. Many computational schemes have been formulated for constructing novel solitary wave solutions that describe the physical and dynamical behavior of these phenomena [6–8]. In 1834 John Russell pointed out the lonely solutions of moving waves by adding a particular parameter value in the closed shape [9,10]. There are several common types of solitary wave solutions. Such solitary wave solutions explain much about the physical properties of these models [11–13].

Here, this article studies the well-known nonlinear IPB model through the ESE and NRE methods for constructing novel lattice soliton solutions and investigating the physical characterizes of the along wave in shallow water beneath gravity [14, 15]. The mathematical model of the considered model is given by [16, 17]

$$\mathcal{U}_{tt} = \mathcal{U}_{xx} + \left(\mathcal{U}^2\right)_{xx} + \mathcal{U}_{xxx}, \tag{1.1}$$

where $\mathcal{U} = \mathcal{U}(x,t)$ describes promulgation of small amplitude long waves (long compared to the amplitude of the wave) in sundry physical contexts inclusive shallow water under gravity. Implementing the next wave transformation $\mathcal{U} = \mathcal{V}(\mathfrak{Z})$, $\mathfrak{Z} = x - c\,t$, where c is arbitrary constant to be evaluated later, converts the nonlinear partial differential equation into the following the ordinary differential equation

$$\left(c^2 - 1\right)\mathcal{V} - \mathcal{V}^2 - \mathcal{V}'' = 0. \tag{1.2}$$

Applying the homogeneous balance principles to the highest order derivative term and nonlinear term of Eq (1.2), get $\mathcal{N} = 2$. Using the general formula of the suggested computational schemes [3–7], get the traveling solutions of the IPB model in the next formula

$$\mathcal{V}(\mathfrak{Z}) = \begin{cases} \sum\limits_{j=-\mathcal{N}}^{\mathcal{N}} a_j\,\phi^j(\mathfrak{Z}) = \frac{a_{-2}}{\phi^2(\mathfrak{Z})} + \frac{a_{-1}}{\phi(\mathfrak{Z})} + a_0 + a_1\,\phi(\mathfrak{Z}) + a_2\,\phi^2(\mathfrak{Z}), \\[2ex] \sum\limits_{j=-\mathcal{N}}^{\mathcal{N}} a_j\,(d+\psi(\mathfrak{Z}))^j = \frac{a_{-2}}{(d+\psi(\mathfrak{Z}))^2} + \frac{a_{-1}}{(d+\psi(\mathfrak{Z}))} + a_0 + a_1\,(d+\psi(\mathfrak{Z})) + a_2\,(d+\psi(\mathfrak{Z}))^2, \end{cases} \tag{1.3}$$

where $a_j, j = -2, \cdots, 2$, are arbitrary constants to be determined later. While $\phi(\mathfrak{Z})$, $\psi(\mathfrak{Z})$ satisfy the following auxiliary equations

$$\begin{cases} \phi'(\mathfrak{Z}) = \mathcal{G}_1 + \mathcal{G}_2\,\phi(\mathfrak{Z}) + \mathcal{G}_3\,\phi(\mathfrak{Z})^2, \\[2ex] \psi'(\mathfrak{Z}) = \left(d^2\,\mathcal{E}_3 - d^2 - d\,\mathcal{E}_2 + \mathcal{E}_1\right) + (-2\,d\,\mathcal{E}_3 + 2d + \mathcal{E}_2)\,\psi(\mathfrak{Z}) + (\mathcal{E}_3 - 1)\,\psi(\mathfrak{Z})^2, \end{cases} \tag{1.4}$$

where d, $\mathcal{G}_i$, $\mathcal{E}_i$, $(i = 1,2,3)$ are arbitrary constants to be determined through the suggested analytical schemes' framework.

The paper's rest sections are organized as follows: Section 2 handles the considered model through the ESE and NRE methods [18–24]. Additionally, the solitary wave solutions are explained through two, three-dimensional, and density plots to illustrate the dynamical behavior of shallow water waves beneath gravity. Section 4 discusses the obtained solitary wave solutions and their novelty. Section 5 gives the conclusion of the whole paper.

2. Nonlinear soliton lattice wave solutions

Here, we use the suggested computational schemes' framework to determine the above-shown parameters.

2.1. ESE method's solutions

Handling the considered model along with the ESE method gives the following sets of parameters' value.

Set A

$$\mathcal{G}_3 = \frac{1-c^2}{16\,\mathcal{G}_1},\ \mathcal{G}_2 = 0,\ a_{-2} = -6\,\mathcal{G}_1^2,\ a_{-1} = a_1 = 0,\ a_0 = \frac{3}{4}\left(c^2-1\right),\ a_2 = \frac{-3\left(1-2\,c^2+c^4\right)}{128\,\mathcal{G}_1^2}.$$

Thus, the model's soliton solutions are given by

When $\mathcal{G}_2 = 0$, we acquire:

Case 1. When $\mathcal{G}_1\,\mathcal{G}_3 > 0$

$$\begin{aligned}\mathcal{U}(\mathfrak{Z}) = &-6\,\mathcal{G}_1\,\mathcal{G}_3\ cot^2\left(\sqrt{\mathcal{G}_1\,\mathcal{G}_3}\,(\mathfrak{Z}+C)\right) + \frac{3\,\left(c^2-1\right)}{4} - \frac{3\left(1-2\,c^2+c^4\right)}{128\,\mathcal{G}_1\,\mathcal{G}_3}\\ &\times tan^2\left(\sqrt{\mathcal{G}_1\,\mathcal{G}_3}\,(\mathfrak{Z}+C)\right),\end{aligned} \tag{2.1}$$

$$\begin{aligned}\mathcal{U}(\mathfrak{Z}) = &-6\,\mathcal{G}_1\,\mathcal{G}_3\ tan^2\left(\sqrt{\mathcal{G}_1\,\mathcal{G}_3}\,(\mathfrak{Z}+C)\right) + \frac{3\,\left(c^2-1\right)}{4} - \frac{3\left(1-2\,c^2+c^4\right)}{128\,\mathcal{G}_1\,\mathcal{G}_3}\\ &\times cot^2\left(\sqrt{\mathcal{G}_1\,\mathcal{G}_3}\,(\mathfrak{Z}+C)\right).\end{aligned} \tag{2.2}$$

Case 2. When $\mathcal{G}_1\,\mathcal{G}_3 < 0$

$$\begin{aligned}\mathcal{U}(\mathfrak{Z}) = &6\,\mathcal{G}_1\,\mathcal{G}_3\ coth^2\left(\sqrt{-\mathcal{G}_1\,\mathcal{G}_3}\,(\mathfrak{Z}+C)\right) + \frac{3\,\left(c^2-1\right)}{4} + \frac{3\left(1-2\,c^2+c^4\right)}{128\,\mathcal{G}_1\,\mathcal{G}_3}\\ &\times tanh^2\left(\sqrt{-\mathcal{G}_1\,\mathcal{G}_3}\,(\mathfrak{Z}+C)\right),\end{aligned} \tag{2.3}$$

$$\begin{aligned}\mathcal{U}(\mathfrak{Z}) = &6\,\mathcal{G}_1\,\mathcal{G}_3\ tanh^2\left(\sqrt{-\mathcal{G}_1\,\mathcal{G}_3}\,(\mathfrak{Z}+C)\right) + \frac{3\,\left(c^2-1\right)}{4} + \frac{3\left(1-2\,c^2+c^4\right)}{128\,\mathcal{G}_1\,\mathcal{G}_3}\\ &\times coth^2\left(\sqrt{-\mathcal{G}_1\,\mathcal{G}_3}\,(\mathfrak{Z}+C)\right).\end{aligned} \tag{2.4}$$

Set B

$$\mathcal{G}_3 = \frac{c^2-1}{16\,\mathcal{G}_1},\ \mathcal{G}_2 = 0,\ a_{-2} = -6\,\mathcal{G}_1^2,\ a_{-1} = a_1 = 0,\ a_0 = \frac{1}{4}\left(c^2-1\right),\ a_2 = \frac{-3\left(1-2\,c^2+c^4\right)}{128\,\mathcal{G}_1^2}. \tag{2.5}$$

Thus, the model's soliton solutions are given by
Case 1. When $\mathcal{G}_1\,\mathcal{G}_3 > 0$

$$\mathcal{U}(\mathfrak{Z}) = -6\,\mathcal{G}_1\,\mathcal{G}_3\,cot^2\left(\sqrt{\mathcal{G}_1\,\mathcal{G}_3}\,(\mathfrak{Z}+C)\right) + \frac{1\left(c^2-1\right)}{4} - \frac{3\left(1-2\,c^2+c^4\right)}{128\,\mathcal{G}_1\,\mathcal{G}_3}\,tan^2\left(\sqrt{\mathcal{G}_1\,\mathcal{G}_3}\,(\mathfrak{Z}+C)\right), \tag{2.6}$$

$$\mathcal{U}(\mathfrak{Z}) = -6\,\mathcal{G}_1\,\mathcal{G}_3\,tan^2\left(\sqrt{\mathcal{G}_1\,\mathcal{G}_3}\,(\mathfrak{Z}+C)\right) + \frac{1\left(c^2-1\right)}{4} - \frac{3\left(1-2\,c^2+c^4\right)}{128\,\mathcal{G}_1\,\mathcal{G}_3}\,cot^2\left(\sqrt{\mathcal{G}_1\,\mathcal{G}_3}\,(\mathfrak{Z}+C)\right). \tag{2.7}$$

Case 2. When $\mathcal{G}_1\,\mathcal{G}_3 < 0$

$$\begin{aligned}\mathcal{U}(\mathfrak{Z}) =& 6\,\mathcal{G}_1\,\mathcal{G}_3\,coth^2\left(\sqrt{-\mathcal{G}_1\,\mathcal{G}_3}\,\mathfrak{Z} \mp \frac{ln(C)}{2}\right) + \frac{1\left(c^2-1\right)}{4} + \frac{3\left(1-2\,c^2+c^4\right)}{128\,\mathcal{G}_1\,\mathcal{G}_3}\\ &\times tanh^2\left(\sqrt{-\mathcal{G}_1\,\mathcal{G}_3}\,\mathfrak{Z} \mp \frac{ln(C)}{2}\right),\end{aligned} \tag{2.8}$$

$$\begin{aligned}\mathcal{U}(\mathfrak{Z}) =& 6\,\mathcal{G}_1\,\mathcal{G}_3\,tanh^2\left(\sqrt{-\mathcal{G}_1\,\mathcal{G}_3}\,\mathfrak{Z} \mp \frac{ln(C)}{2}\right) + \frac{1\left(c^2-1\right)}{4} + \frac{3\left(1-2\,c^2+c^4\right)}{128\,\mathcal{G}_1\,\mathcal{G}_3}\\ &\times coth^2\left(\sqrt{-\mathcal{G}_1\,\mathcal{G}_3}\,\mathfrak{Z} \mp \frac{ln(C)}{2}\right).\end{aligned} \tag{2.9}$$

Set C

$$\mathcal{G}_3 = \frac{c^2-1+\mathcal{G}_2^2}{4\,\mathcal{G}_1},\ a_{-2} = -6\,\mathcal{G}_1^2,\ a_{-1} = -6\,\mathcal{G}_1\,\mathcal{G}_2,\ a_0 = -\frac{1}{2}\left(3\,\mathcal{G}_2^2 - 1 + c^2\right),\ a_1 = a_2 = 0.$$

Thus, the model's soliton solutions are given by
When $\mathcal{G}_2 = 0$, we acquire:
Case 1. When $\mathcal{G}_1\,\mathcal{G}_3 > 0$

$$\mathcal{U}(\mathfrak{Z}) = -6\,\mathcal{G}_1\,\mathcal{G}_3\,cot^2\left(\sqrt{\mathcal{G}_1\,\mathcal{G}_3}\,(\mathfrak{Z}+C)\right) - \frac{1}{2}\left(-1+c^2\right), \tag{2.10}$$

$$\mathcal{U}(\mathfrak{Z}) = -6\,\mathcal{G}_1\,\mathcal{G}_3\,tan^2\left(\sqrt{\mathcal{G}_1\,\mathcal{G}_3}\,(\mathfrak{Z}+C)\right) - \frac{1}{2}\left(-1+c^2\right). \tag{2.11}$$

Case 2. When $\mathcal{G}_1\,\mathcal{G}_3 < 0$

$$\mathcal{U}(\mathfrak{Z}) = 6\,\mathcal{G}_1\,\mathcal{G}_3\,coth^2\left(\sqrt{-\mathcal{G}_1\,\mathcal{G}_3}\,\mathfrak{Z} \mp \frac{ln(C)}{2}\right) - \frac{1}{2}\left(-1+c^2\right), \tag{2.12}$$

$$\mathcal{U}(\mathfrak{z}) = 6\,\mathcal{G}_1\,\mathcal{G}_3\,tanh^2\left(\sqrt{-\mathcal{G}_1\,\mathcal{G}_3}\,\mathfrak{z} \mp \frac{ln(C)}{2}\right) - \frac{1}{2}\left(-1 + c^2\right). \tag{2.13}$$

Additionally, the general soliton solutions are given by
Case 1. When $4\,\mathcal{G}_1\,\mathcal{G}_3 > \mathcal{G}_2^2$ and $\mathcal{G}_3 > 0$

$$\begin{aligned}\mathcal{U}(\mathfrak{z}) =& \frac{-24\,\mathcal{G}_1^2\,\mathcal{G}_3^2}{\left(\sqrt{4\,\mathcal{G}_1\,\mathcal{G}_3 - \mathcal{G}_2^2}\,tan\left(\frac{\sqrt{4\,\mathcal{G}_1\,\mathcal{G}_3 - \mathcal{G}_2^2}}{2}\,(\mathfrak{z} + C)\right) - \mathcal{G}_2\right)^2}\\ &- \frac{12\,\mathcal{G}_1\,\mathcal{G}_2\,\mathcal{G}_3}{\sqrt{4\,\mathcal{G}_1\,\mathcal{G}_3 - \mathcal{G}_2^2}\,tan\left(\frac{\sqrt{4\,\mathcal{G}_1\,\mathcal{G}_3 - \mathcal{G}_2^2}}{2}\,(\mathfrak{z} + C)\right) - \mathcal{G}_2} - \frac{1}{2}\left(3\,\mathcal{G}_2^2 - 1 + c^2\right),\end{aligned} \tag{2.14}$$

$$\begin{aligned}\mathcal{U}(\mathfrak{z}) =& \frac{-24\,\mathcal{G}_1^2\,\mathcal{G}_3^2}{\left(\sqrt{4\,\mathcal{G}_1\,\mathcal{G}_3 - \mathcal{G}_2^2}\,cot\left(\frac{\sqrt{4\,\mathcal{G}_1\,\mathcal{G}_3 - \mathcal{G}_2^2}}{2}\,(\mathfrak{z} + C)\right) - \mathcal{G}_2\right)^2}\\ &- \frac{12\,\mathcal{G}_1\,\mathcal{G}_2\,\mathcal{G}_3}{\sqrt{4\,\mathcal{G}_1\,\mathcal{G}_3 - \mathcal{G}_2^2}\,cot\left(\frac{\sqrt{4\,\mathcal{G}_1\,\mathcal{G}_3 - \mathcal{G}_2^2}}{2}\,(\mathfrak{z} + C)\right) - \mathcal{G}_2} - \frac{1}{2}\left(3\,\mathcal{G}_2^2 - 1 + c^2\right).\end{aligned} \tag{2.15}$$

Case 2. When $4\,\mathcal{G}_1\,\mathcal{G}_3 > \mathcal{G}_2^2$ and $\mathcal{G}_3 < 0$

$$\begin{aligned}\mathcal{U}(\mathfrak{z}) =& \frac{-24\,\mathcal{G}_1^2\,\mathcal{G}_3^2}{\left(\sqrt{4\,\mathcal{G}_1\,\mathcal{G}_3 - \mathcal{G}_2^2}\,tan\left(\frac{\sqrt{4\,\mathcal{G}_1\,\mathcal{G}_3 - \mathcal{G}_2^2}}{2}\,(\mathfrak{z} + C)\right) + \mathcal{G}_2\right)^2}\\ &- \frac{12\,\mathcal{G}_1\,\mathcal{G}_2\,\mathcal{G}_3}{\sqrt{4\,\mathcal{G}_1\,\mathcal{G}_3 - \mathcal{G}_2^2}\,tan\left(\frac{\sqrt{4\,\mathcal{G}_1\,\mathcal{G}_3 - \mathcal{G}_2^2}}{2}\,(\mathfrak{z} + C)\right) + \mathcal{G}_2} - \frac{1}{2}\left(3\,\mathcal{G}_2^2 - 1 + c^2\right),\end{aligned} \tag{2.16}$$

$$\begin{aligned}\mathcal{U}(\mathfrak{z}) =& \frac{-24\,\mathcal{G}_1^2\,\mathcal{G}_3^2}{\left(\sqrt{4\,\mathcal{G}_1\,\mathcal{G}_3 - \mathcal{G}_2^2}\,cot\left(\frac{\sqrt{4\,\mathcal{G}_1\,\mathcal{G}_3 - \mathcal{G}_2^2}}{2}\,(\mathfrak{z} + C)\right) + \mathcal{G}_2\right)^2}\\ &- \frac{12\,\mathcal{G}_1\,\mathcal{G}_2\,\mathcal{G}_3}{\sqrt{4\,\mathcal{G}_1\,\mathcal{G}_3 - \mathcal{G}_2^2}\,cot\left(\frac{\sqrt{4\,\mathcal{G}_1\,\mathcal{G}_3 - \mathcal{G}_2^2}}{2}\,(\mathfrak{z} + C)\right) + \mathcal{G}_2} - \frac{1}{2}\left(3\,\mathcal{G}_2^2 - 1 + c^2\right).\end{aligned} \tag{2.17}$$

Set D

$$\mathcal{G}_3 = -\frac{c^2 - 1 - \mathcal{G}_2^2}{4\,\mathcal{G}_1},\ a_{-2} = -6\,\mathcal{G}_1^2,\ a_{-1} = -6\,\mathcal{G}_1\,\mathcal{G}_2,\ a_0 = -\frac{3}{2}\left(\mathcal{G}_2^2 - 1 + c^2\right),\ a_1 = a_2 = 0.$$

Thus, the model's soliton solutions are given by
When $\mathcal{G}_2 = 0$, we acquire:
Case 1. When $\mathcal{G}_1\,\mathcal{G}_3 > 0$

$$\mathcal{U}(\mathfrak{z}) = -6\,\mathcal{G}_1\,\mathcal{G}_3\,cot^2\left(\sqrt{\mathcal{G}_1\,\mathcal{G}_3}\,(\mathfrak{z} + C)\right) - \frac{3}{2}\left(-1 + c^2\right), \tag{2.18}$$

$$\mathcal{U}(\mathfrak{z}) = -6\,\mathcal{G}_1\,\mathcal{G}_3\,tan^2\left(\sqrt{\mathcal{G}_1\,\mathcal{G}_3}\,(\mathfrak{z} + C)\right) - \frac{3}{2}\left(-1 + c^2\right). \tag{2.19}$$

Case 2. When $\mathcal{G}_1\,\mathcal{G}_3 < 0$

$$\mathcal{U}(\mathfrak{z}) = 6\,\mathcal{G}_1\,\mathcal{G}_3\,coth^2\left(\sqrt{-\mathcal{G}_1\,\mathcal{G}_3}\,\mathfrak{z} \mp \frac{ln(C)}{2}\right) - \frac{3}{2}\left(-1 + c^2\right), \tag{2.20}$$

$$\mathcal{U}(\mathfrak{z}) = 6\,\mathcal{G}_1\,\mathcal{G}_3\,tanh^2\left(\sqrt{-\mathcal{G}_1\,\mathcal{G}_3}\,\mathfrak{z} \mp \frac{ln(C)}{2}\right) - \frac{3}{2}\left(-1 + c^2\right). \tag{2.21}$$

Additionally, the general soliton solutions are given by
Case 1. When $4\,\mathcal{G}_1\,\mathcal{G}_3 > \mathcal{G}_2^2$ and $\mathcal{G}_3 > 0$

$$\begin{aligned}\mathcal{U}(\mathfrak{z}) =& \frac{-24\,\mathcal{G}_1^2\,\mathcal{G}_3^2}{\left(\sqrt{4\,\mathcal{G}_1\,\mathcal{G}_3 - \mathcal{G}_2^2}\,tan\left(\frac{\sqrt{4\,\mathcal{G}_1\,\mathcal{G}_3-\mathcal{G}_2^2}}{2}\,(\mathfrak{z} + C)\right) - \mathcal{G}_2\right)^2} - \frac{12\,\mathcal{G}_1\,\mathcal{G}_2\,\mathcal{G}_3}{\frac{\sqrt{4\,\mathcal{G}_1\,\mathcal{G}_3-\mathcal{G}_2^2}\,tan\left(\frac{\sqrt{4\,\mathcal{G}_1\,\mathcal{G}_3-\mathcal{G}_2^2}}{2}(\mathfrak{z}+C)\right)-\mathcal{G}_2}{2\,\mathcal{G}_3}}\\ &- \frac{3}{2}\left(\mathcal{G}_2^2 - 1 + c^2\right),\end{aligned} \tag{2.22}$$

$$\begin{aligned}\mathcal{U}(\mathfrak{z}) =& \frac{-24\,\mathcal{G}_1^2\,\mathcal{G}_3^2}{\left(\sqrt{4\,\mathcal{G}_1\,\mathcal{G}_3 - \mathcal{G}_2^2}\,cot\left(\frac{\sqrt{4\,\mathcal{G}_1\,\mathcal{G}_3-\mathcal{G}_2^2}}{2}\,(\mathfrak{z} + C)\right) - \mathcal{G}_2\right)^2} - \frac{12\,\mathcal{G}_1\,\mathcal{G}_2\,\mathcal{G}_3}{\frac{\sqrt{4\,\mathcal{G}_1\,\mathcal{G}_3-\mathcal{G}_2^2}\,cot\left(\frac{\sqrt{4\,\mathcal{G}_1\,\mathcal{G}_3-\mathcal{G}_2^2}}{2}(\mathfrak{z}+C)\right)-\mathcal{G}_2}{2\,\mathcal{G}_3}}\\ &- \frac{3}{2}\left(\mathcal{G}_2^2 - 1 + c^2\right).\end{aligned} \tag{2.23}$$

Case 2. When $4\,\mathcal{G}_1\,\mathcal{G}_3 > \mathcal{G}_2^2$ and $\mathcal{G}_3 < 0$

$$\begin{aligned}\mathcal{U}(\mathfrak{z}) =& \frac{-24\,\mathcal{G}_1^2\,\mathcal{G}_3^2}{\left(\sqrt{4\,\mathcal{G}_1\,\mathcal{G}_3 - \mathcal{G}_2^2}\,tan\left(\frac{\sqrt{4\,\mathcal{G}_1\,\mathcal{G}_3-\mathcal{G}_2^2}}{2}\,(\mathfrak{z} + C)\right) + \mathcal{G}_2\right)^2}\\ &- \frac{12\,\mathcal{G}_1\,\mathcal{G}_2\,\mathcal{G}_3}{\sqrt{4\,\mathcal{G}_1\,\mathcal{G}_3 - \mathcal{G}_2^2}\,tan\left(\frac{\sqrt{4\,\mathcal{G}_1\,\mathcal{G}_3-\mathcal{G}_2^2}}{2}\,(\mathfrak{z} + C)\right) + \mathcal{G}_2} - \frac{3}{2}\left(\mathcal{G}_2^2 - 1 + c^2\right),\end{aligned} \tag{2.24}$$

$$\begin{aligned}\mathcal{U}(\mathfrak{z}) =& \frac{-24\,\mathcal{G}_1^2\,\mathcal{G}_3^2}{\left(\sqrt{4\,\mathcal{G}_1\,\mathcal{G}_3 - \mathcal{G}_2^2}\,cot\left(\frac{\sqrt{4\,\mathcal{G}_1\,\mathcal{G}_3-\mathcal{G}_2^2}}{2}\,(\mathfrak{z} + C)\right) + \mathcal{G}_2\right)^2}\\ &- \frac{12\,\mathcal{G}_1\,\mathcal{G}_2\,\mathcal{G}_3}{\sqrt{4\,\mathcal{G}_1\,\mathcal{G}_3 - \mathcal{G}_2^2}\,cot\left(\frac{\sqrt{4\,\mathcal{G}_1\,\mathcal{G}_3-\mathcal{G}_2^2}}{2}\,(\mathfrak{z} + C)\right) + \mathcal{G}_2} - \frac{3}{2}\left(\mathcal{G}_2^2 - 1 + c^2\right),\end{aligned} \tag{2.25}$$

Set E

$$\mathcal{G}_1 = \frac{-36\,\mathcal{G}_3^2 + 36\,c^2\,\mathcal{G}_3^2 + a_1^2}{144\,\mathcal{G}_3^3},\ \mathcal{G}_2 = \frac{-a_1}{6\,\mathcal{G}_3},\ a_{-2} = a_{-1} = 0,\ a_0 = -\frac{a_1^2 - 12\,\mathcal{G}_3^2 + 12\,c^2\,\mathcal{G}_3^2}{24\,\mathcal{G}_3^2},\ a_2 = -6\,\mathcal{G}_3^2.$$

Thus, the model's soliton solutions are given by
When $\mathcal{G}_2 = 0$, we acquire:
Case 1. When $\mathcal{G}_1\,\mathcal{G}_3 > 0$

$$\mathcal{U}(\mathfrak{Z}) = -\frac{a_1^2 - 12\,\mathcal{G}_3^2 + 12\,c^2\,\mathcal{G}_3^2}{24\,\mathcal{G}_3^2} - 6\,\mathcal{G}_3\,\mathcal{G}_1\,tan^2\left(\sqrt{\mathcal{G}_1\,\mathcal{G}_3}\,(\mathfrak{Z} + C)\right), \tag{2.26}$$

$$\mathcal{U}(\mathfrak{Z}) = -\frac{a_1^2 - 12\,\mathcal{G}_3^2 + 12\,c^2\,\mathcal{G}_3^2}{24\,\mathcal{G}_3^2} - 6\,\mathcal{G}_3\,\mathcal{G}_1\,cot^2\left(\sqrt{\mathcal{G}_1\,\mathcal{G}_3}\,(\mathfrak{Z} + C)\right). \tag{2.27}$$

Case 2. When $\mathcal{G}_1\,\mathcal{G}_3 < 0$

$$\mathcal{U}(\mathfrak{Z}) = -\frac{a_1^2 - 12\,\mathcal{G}_3^2 + 12\,c^2\,\mathcal{G}_3^2}{24\,\mathcal{G}_3^2} + 6\,\mathcal{G}_3\,\mathcal{G}_1\,tanh^2\left(\sqrt{-\mathcal{G}_1\,\mathcal{G}_3}\,\mathfrak{Z} \mp \frac{ln(C)}{2}\right), \tag{2.28}$$

$$\mathcal{U}(\mathfrak{Z}) = -\frac{a_1^2 - 12\,\mathcal{G}_3^2 + 12\,c^2\,\mathcal{G}_3^2}{24\,\mathcal{G}_3^2} + 6\,\mathcal{G}_3\,\mathcal{G}_1\,coth^2\left(\sqrt{-\mathcal{G}_1\,\mathcal{G}_3}\,\mathfrak{Z} \mp \frac{ln(C)}{2}\right). \tag{2.29}$$

When $\mathcal{G}_1 = 0$, we acquire
Case 1. When $\mathcal{G}_2 > 0$

$$\mathcal{U}(\mathfrak{Z}) = -\frac{a_1^2 - 12\,\mathcal{G}_3^2 + 12\,c^2\,\mathcal{G}_3^2}{24\,\mathcal{G}_3^2} + \frac{a_1\,\mathcal{G}_2\,e^{\mathcal{G}_2(\mathfrak{Z}+C)}}{1 - \mathcal{G}_3\,e^{\mathcal{G}_2(\mathfrak{Z}+C)}} - 6\,\mathcal{G}_3^2\left(\frac{\mathcal{G}_2\,e^{\mathcal{G}_2(\mathfrak{Z}+C)}}{1 - \mathcal{G}_3\,e^{\mathcal{G}_2(\mathfrak{Z}+C)}}\right)^2, \tag{2.30}$$

Case 2. When $\mathcal{G}_2 < 0$

$$\mathcal{U}(\mathfrak{Z}) = -\frac{a_1^2 - 12\,\mathcal{G}_3^2 + 12\,c^2\,\mathcal{G}_3^2}{24\,\mathcal{G}_3^2} - \frac{a_1\,\mathcal{G}_3\,e^{\mathcal{G}_2(\mathfrak{Z}+C)}}{1 + \mathcal{G}_3\,e^{\mathcal{G}_2(\mathfrak{Z}+C)}} - 6\,\mathcal{G}_3^2\left(\frac{\mathcal{G}_3\,e^{\mathcal{G}_2(\mathfrak{Z}+C)}}{1 + \mathcal{G}_3\,e^{\mathcal{G}_2(\mathfrak{Z}+C)}}\right)^2. \tag{2.31}$$

Additionally, the general soliton solutions are given by
Case 1. When $4\,\mathcal{G}_1\,\mathcal{G}_3 > \mathcal{G}_2^2$ and $\mathcal{G}_3 > 0$

$$\begin{aligned}\mathcal{U}(\mathfrak{Z}) = &-\frac{a_1^2 - 12\,\mathcal{G}_3^2 + 12\,c^2\,\mathcal{G}_3^2}{24\,\mathcal{G}_3^2} + \frac{a_1}{2\,\mathcal{G}_3}\left(\sqrt{4\,\mathcal{G}_1\,\mathcal{G}_3 - \mathcal{G}_2^2}\,tan\left(\frac{\sqrt{4\,\mathcal{G}_1\,\mathcal{G}_3 - \mathcal{G}_2^2}}{2}\,(\mathfrak{Z} + C)\right) - \mathcal{G}_2\right)\\ &- \frac{3}{2}\left(\sqrt{4\,\mathcal{G}_1\,\mathcal{G}_3 - \mathcal{G}_2^2}\,tan\left(\frac{\sqrt{4\,\mathcal{G}_1\,\mathcal{G}_3 - \mathcal{G}_2^2}}{2}\,(\mathfrak{Z} + C)\right) - \mathcal{G}_2\right)^2,\end{aligned} \tag{2.32}$$

$$\mathcal{U}(\mathfrak{Z}) = -\frac{a_1^2 - 12\,\mathcal{G}_3^2 + 12\,c^2\,\mathcal{G}_3^2}{24\,\mathcal{G}_3^2} + \frac{a_1}{2\,\mathcal{G}_3}\left(\sqrt{4\,\mathcal{G}_1\,\mathcal{G}_3 - \mathcal{G}_2^2}\,cot\left(\frac{\sqrt{4\,\mathcal{G}_1\,\mathcal{G}_3 - \mathcal{G}_2^2}}{2}(\mathfrak{Z} + C)\right) - \mathcal{G}_2\right) - \frac{3}{2}\left(\sqrt{4\,\mathcal{G}_1\,\mathcal{G}_3 - \mathcal{G}_2^2}\,cot\left(\frac{\sqrt{4\,\mathcal{G}_1\,\mathcal{G}_3 - \mathcal{G}_2^2}}{2}(\mathfrak{Z} + C)\right) - \mathcal{G}_2\right)^2. \tag{2.33}$$

Case 2. When $4\,\mathcal{G}_1\,\mathcal{G}_3 > \mathcal{G}_2^2$ and $\mathcal{G}_3 < 0$

$$\mathcal{U}(\mathfrak{Z}) = -\frac{a_1^2 - 12\,\mathcal{G}_3^2 + 12\,c^2\,\mathcal{G}_3^2}{24\,\mathcal{G}_3^2} + \frac{a_1}{2\,\mathcal{G}_3}\left(\sqrt{4\,\mathcal{G}_1\,\mathcal{G}_3 - \mathcal{G}_2^2}\,tan\left(\frac{\sqrt{4\,\mathcal{G}_1\,\mathcal{G}_3 - \mathcal{G}_2^2}}{2}(\mathfrak{Z} + C)\right) + \mathcal{G}_2\right) - \frac{3}{2}\left(\sqrt{4\,\mathcal{G}_1\,\mathcal{G}_3 - \mathcal{G}_2^2}\,tan\left(\frac{\sqrt{4\,\mathcal{G}_1\,\mathcal{G}_3 - \mathcal{G}_2^2}}{2}(\mathfrak{Z} + C)\right) + \mathcal{G}_2\right)^2, \tag{2.34}$$

$$\mathcal{U}(\mathfrak{Z}) = -\frac{a_1^2 - 12\,\mathcal{G}_3^2 + 12\,c^2\,\mathcal{G}_3^2}{24\,\mathcal{G}_3^2} + \frac{a_1}{2\,\mathcal{G}_3}\left(\sqrt{4\,\mathcal{G}_1\,\mathcal{G}_3 - \mathcal{G}_2^2}\,cot\left(\frac{\sqrt{4\,\mathcal{G}_1\,\mathcal{G}_3 - \mathcal{G}_2^2}}{2}(\mathfrak{Z} + C)\right) + \mathcal{G}_2\right) - \frac{3}{2}\left(\sqrt{4\,\mathcal{G}_1\,\mathcal{G}_3 - \mathcal{G}_2^2}\,cot\left(\frac{\sqrt{4\,\mathcal{G}_1\,\mathcal{G}_3 - \mathcal{G}_2^2}}{2}(\mathfrak{Z} + C)\right) + \mathcal{G}_2\right)^2. \tag{2.35}$$

Set F

$$\mathcal{G}_1 = -\frac{36\,\mathcal{G}_3^2 + 36\,c^2\,\mathcal{G}_3^2 - a_1^2}{144\,\mathcal{G}_3^3},\ \mathcal{G}_2 = \frac{-a_1}{6\,\mathcal{G}_3},\ a_{-2} = a_{-1} = 0,\ a_0 = -\frac{a_1^2 + 36\,\mathcal{G}_3^2 - 36\,c^2\,\mathcal{G}_3^2}{24\,\mathcal{G}_3^2},\ a_2 = -6\,\mathcal{G}_3^2.$$

Thus, the model's soliton solutions are given by
When $\mathcal{G}_2 = 0$, we acquire:
Case 1. When $\mathcal{G}_1\,\mathcal{G}_3 > 0$

$$\mathcal{U}(\mathfrak{Z}) = -\frac{a_1^2 + 36\,\mathcal{G}_3^2 - 36\,c^2\,\mathcal{G}_3^2}{24\,\mathcal{G}_3^2} - 6\,\mathcal{G}_3\,\mathcal{G}_1\,tan^2\left(\sqrt{\mathcal{G}_1\,\mathcal{G}_3}\,(\mathfrak{Z} + C)\right), \tag{2.36}$$

$$\mathcal{U}(\mathfrak{Z}) = -\frac{a_1^2 + 36\,\mathcal{G}_3^2 - 36\,c^2\,\mathcal{G}_3^2}{24\,\mathcal{G}_3^2} - 6\,\mathcal{G}_3\,\mathcal{G}_1\,cot^2\left(\sqrt{\mathcal{G}_1\,\mathcal{G}_3}\,(\mathfrak{Z} + C)\right). \tag{2.37}$$

Case 2. When $\mathcal{G}_1\,\mathcal{G}_3 < 0$

$$\mathcal{U}(\mathfrak{Z}) = -\frac{a_1^2 + 36\,\mathcal{G}_3^2 - 36\,c^2\,\mathcal{G}_3^2}{24\,\mathcal{G}_3^2} + 6\,\mathcal{G}_3\,\mathcal{G}_1\,tanh^2\left(\sqrt{-\mathcal{G}_1\,\mathcal{G}_3}\,\mathfrak{Z} \mp \frac{ln(C)}{2}\right), \tag{2.38}$$

$$\mathcal{U}(\mathfrak{z}) = -\frac{a_1^2 + 36\,\mathcal{G}_3^2 - 36\,c^2\,\mathcal{G}_3^2}{24\,\mathcal{G}_3^2} + 6\,\mathcal{G}_3\,\mathcal{G}_1\,coth^2\left(\sqrt{-\mathcal{G}_1\,\mathcal{G}_3}\,\mathfrak{z} \mp \frac{ln(C)}{2}\right). \tag{2.39}$$

When $\mathcal{G}_1 = 0$, we acquire
Case 1. When $\mathcal{G}_2 > 0$

$$\mathcal{U}(\mathfrak{z}) = -\frac{a_1^2 + 36\,\mathcal{G}_3^2 - 36\,c^2\,\mathcal{G}_3^2}{24\,\mathcal{G}_3^2} + \frac{a_1\,\mathcal{G}_2\,e^{\mathcal{G}_2(\mathfrak{z}+C)}}{1 - \mathcal{G}_3\,e^{\mathcal{G}_2(\mathfrak{z}+C)}} - 6\,\mathcal{G}_3^2\left(\frac{\mathcal{G}_2\,e^{\mathcal{G}_2(\mathfrak{z}+C)}}{1 - \mathcal{G}_3\,e^{\mathcal{G}_2(\mathfrak{z}+C)}}\right)^2, \tag{2.40}$$

Case 2. When $\mathcal{G}_2 < 0$

$$\mathcal{U}(\mathfrak{z}) = -\frac{a_1^2 + 36\,\mathcal{G}_3^2 - 36\,c^2\,\mathcal{G}_3^2}{24\,\mathcal{G}_3^2} -, \frac{a_1\,\mathcal{G}_3\,e^{\mathcal{G}_2(\mathfrak{z}+C)}}{1 + \mathcal{G}_3\,e^{\mathcal{G}_2(\mathfrak{z}+C)}} - 6\,\mathcal{G}_3^2\left(\frac{\mathcal{G}_3\,e^{\mathcal{G}_2(\mathfrak{z}+C)}}{1 + \mathcal{G}_3\,e^{\mathcal{G}_2(\mathfrak{z}+C)}}\right)^2. \tag{2.41}$$

Additionally, the general soliton solutions are given by
Case 1. When $4\,\mathcal{G}_1\,\mathcal{G}_3 > \mathcal{G}_2^2$ and $\mathcal{G}_3 > 0$

$$\begin{aligned}\mathcal{U}(\mathfrak{z}) =& -\frac{a_1^2 + 36\,\mathcal{G}_3^2 - 36\,c^2\,\mathcal{G}_3^2}{24\,\mathcal{G}_3^2} + \frac{a_1}{2\,\mathcal{G}_3}\left(\sqrt{4\,\mathcal{G}_1\,\mathcal{G}_3 - \mathcal{G}_2^2}\,tan\left(\frac{\sqrt{4\,\mathcal{G}_1\,\mathcal{G}_3 - \mathcal{G}_2^2}}{2}\,(\mathfrak{z} + C)\right) - \mathcal{G}_2\right)\\ &- \frac{3}{2}\left(\sqrt{4\,\mathcal{G}_1\,\mathcal{G}_3 - \mathcal{G}_2^2}\,tan\left(\frac{\sqrt{4\,\mathcal{G}_1\,\mathcal{G}_3 - \mathcal{G}_2^2}}{2}\,(\mathfrak{z} + C)\right) - \mathcal{G}_2\right)^2,\end{aligned} \tag{2.42}$$

$$\begin{aligned}\mathcal{U}(\mathfrak{z}) =& -\frac{a_1^2 + 36\,\mathcal{G}_3^2 - 36\,c^2\,\mathcal{G}_3^2}{24\,\mathcal{G}_3^2} + \frac{a_1}{2\,\mathcal{G}_3}\left(\sqrt{4\,\mathcal{G}_1\,\mathcal{G}_3 - \mathcal{G}_2^2}\,cot\left(\frac{\sqrt{4\,\mathcal{G}_1\,\mathcal{G}_3 - \mathcal{G}_2^2}}{2}\,(\mathfrak{z} + C)\right) - \mathcal{G}_2\right)\\ &- \frac{3}{2}\left(\sqrt{4\,\mathcal{G}_1\,\mathcal{G}_3 - \mathcal{G}_2^2}\,cot\left(\frac{\sqrt{4\,\mathcal{G}_1\,\mathcal{G}_3 - \mathcal{G}_2^2}}{2}\,(\mathfrak{z} + C)\right) - \mathcal{G}_2\right)^2.\end{aligned} \tag{2.43}$$

Case 2. When $4\,\mathcal{G}_1\,\mathcal{G}_3 > \mathcal{G}_2^2$ and $\mathcal{G}_3 < 0$

$$\begin{aligned}\mathcal{U}(\mathfrak{z}) =& -\frac{a_1^2 + 36\,\mathcal{G}_3^2 - 36\,c^2\,\mathcal{G}_3^2}{24\,\mathcal{G}_3^2} + \frac{a_1}{2\,\mathcal{G}_3}\left(\sqrt{4\,\mathcal{G}_1\,\mathcal{G}_3 - \mathcal{G}_2^2}\,tan\left(\frac{\sqrt{4\,\mathcal{G}_1\,\mathcal{G}_3 - \mathcal{G}_2^2}}{2}\,(\mathfrak{z} + C)\right) + \mathcal{G}_2\right)\\ &\frac{3}{2}\left(\sqrt{4\,\mathcal{G}_1\,\mathcal{G}_3 - \mathcal{G}_2^2}\,tan\left(\frac{\sqrt{4\,\mathcal{G}_1\,\mathcal{G}_3 - \mathcal{G}_2^2}}{2}\,(\mathfrak{z} + C)\right) + \mathcal{G}_2\right)^2,\end{aligned} \tag{2.44}$$

$$\mathcal{U}(\mathfrak{Z}) = -\frac{a_1^2 + 36\,\mathcal{G}_3^2 - 36\,c^2\,\mathcal{G}_3^2}{24\,\mathcal{G}_3^2} + \frac{a_1}{2\,\mathcal{G}_3}\left(\sqrt{4\,\mathcal{G}_1\,\mathcal{G}_3 - \mathcal{G}_2^2}\,cot\left(\frac{\sqrt{4\,\mathcal{G}_1\,\mathcal{G}_3 - \mathcal{G}_2^2}}{2}(\mathfrak{Z} + C)\right) + \mathcal{G}_2\right)$$
$$\frac{3}{2}\left(\sqrt{4\,\mathcal{G}_1\,\mathcal{G}_3 - \mathcal{G}_2^2}\,cot\left(\frac{\sqrt{4\,\mathcal{G}_1\,\mathcal{G}_3 - \mathcal{G}_2^2}}{2}(\mathfrak{Z} + C)\right) + \mathcal{G}_2\right)^2. \tag{2.45}$$

2.2. NRE method's solutions

Handling the considered model along with the NRE method gives the following sets of parameters' value.

Set A

$c = \sqrt{4\,\mathcal{E}_1\,\mathcal{E}_3 - 4\,\mathcal{E}_1 + 1 - \mathcal{E}_2^2}$, $a_{-2} = a_{-1} = 0$, $a_0 = -(2\,\mathcal{E}_3 - 2)\,\mathcal{E}_1 - {\mathcal{E}_2}^2 - (-6\,\mathcal{E}_3\,d + 6\,d)\,\mathcal{E}_2 - 6\,{\mathcal{E}_3}^2 d^2 + 12\,\mathcal{E}_3\,d^2 - 6\,d^2$, $a_1 = -\mathcal{E}_2\,(6\,\mathcal{E}_3 - 6) + 12\,{\mathcal{E}_3}^2 d - 24\,\mathcal{E}_3\,d + 12\,d$, $a_2 = -6\,{\mathcal{E}_3}^2 - 6 + 12\,\mathcal{E}_3$.

Thus, the model's soliton solutions are given by

When $\left(\mathfrak{C} = \mathcal{E}_2^2 - 4\,\mathcal{E}_2\,\mathcal{E}_1 + 4\,\mathcal{E}_1 > 0\right)$ **and** $(\mathcal{E}_2(\mathcal{E}_3 - 1) \neq 0)$ **or** $(\mathcal{E}_1(\mathcal{E}_3 - 1) \neq 0)$**:**

$$\mathcal{U}(\mathfrak{Z}) = -(2\,\mathcal{E}_3 - 2)\,\mathcal{E}_1 - {\mathcal{E}_2}^2 - (-6\,\mathcal{E}_3\,d + 6\,d)\,\mathcal{E}_2 - 6\,{\mathcal{E}_3}^2 d^2 + 12\,\mathcal{E}_3\,d^2 - 6\,d^2 + (-\mathcal{E}_2 \times (6\,\mathcal{E}_3 - 6) + 12\,{\mathcal{E}_3}^2 d - 24\,\mathcal{E}_3\,d + 12\,d)(d - \frac{1}{2(\mathcal{E}_3 - 1)}(\mathcal{E}_2 + \sqrt{\mathfrak{C}}\,tanh(\frac{\sqrt{\mathfrak{C}}}{2}\mathfrak{Z}))) + \left(-6\,{\mathcal{E}_3}^2 - 6 + 12\,\mathcal{E}_3\right)\left(d - \frac{1}{2(\mathcal{E}_3 - 1)}(\mathcal{E}_2 + \sqrt{\mathfrak{C}}\,tanh(\frac{\sqrt{\mathfrak{C}}}{2}\mathfrak{Z}))\right)^2, \tag{2.46}$$

$$\mathcal{U}(\mathfrak{Z}) = -(2\,\mathcal{E}_3 - 2)\,\mathcal{E}_1 - {\mathcal{E}_2}^2 - (-6\,\mathcal{E}_3\,d + 6\,d)\,\mathcal{E}_2 - 6\,{\mathcal{E}_3}^2 d^2 + 12\,\mathcal{E}_3\,d^2 - 6\,d^2 + (-\mathcal{E}_2\,(6\,\mathcal{E}_3 - 6) + 12\,{\mathcal{E}_3}^2 d - 24\,\mathcal{E}_3\,d + 12\,d)(d - \frac{1}{2(\mathcal{E}_3 - 1)}(\mathcal{E}_2 + \sqrt{\mathfrak{C}}\,coth(\frac{\sqrt{\mathfrak{C}}}{2}\mathfrak{Z}))) + (-6\,{\mathcal{E}_3}^2 - 6 + 12\,\mathcal{E}_3)\left(d - \frac{1}{2(\mathcal{E}_3 - 1)}(\mathcal{E}_2 + \sqrt{\mathfrak{C}}\,coth(\frac{\sqrt{\mathfrak{C}}}{2}\mathfrak{Z}))\right)^2, \tag{2.47}$$

$$\mathcal{U}(\mathfrak{Z}) = -(2\,\mathcal{E}_3 - 2)\,\mathcal{E}_1 - {\mathcal{E}_2}^2 - (-6\,\mathcal{E}_3\,d + 6\,d)\,\mathcal{E}_2 - 6\,{\mathcal{E}_3}^2 d^2 + 12\,\mathcal{E}_3\,d^2 - 6\,d^2 + (-\mathcal{E}_2\,(6\,\mathcal{E}_3 - 6) + 12\,{\mathcal{E}_3}^2 d - 24\,\mathcal{E}_3\,d + 12\,d)(d - \frac{1}{2(\mathcal{E}_3 - 1)}(\mathcal{E}_2 + \sqrt{\mathfrak{C}}\,(tanh\left(\sqrt{\mathfrak{C}}\,\mathfrak{Z}\right) \pm i\,sech(\sqrt{\mathfrak{C}}\,\mathfrak{Z})))) + \left(-6\,{\mathcal{E}_3}^2 - 6 + 12\,\mathcal{E}_3\right)\left(d - \frac{1}{2(\mathcal{E}_3 - 1)}(\mathcal{E}_2 + \sqrt{\mathfrak{C}}\,(tanh(\sqrt{\mathfrak{C}}\,\mathfrak{Z}) \pm i\,sech(\sqrt{\mathfrak{C}}\,\mathfrak{Z})))\right)^2, \tag{2.48}$$

$$\mathcal{U}(\mathfrak{Z}) = -(2\,\mathcal{E}_3 - 2)\,\mathcal{E}_1 - {\mathcal{E}_2}^2 - (-6\,\mathcal{E}_3\,d + 6\,d)\,\mathcal{E}_2 - 6\,{\mathcal{E}_3}^2 d^2 + 12\,\mathcal{E}_3\,d^2 - 6\,d^2 + \left(-\mathcal{E}_2\,(6\,\mathcal{E}_3 - 6) + 12\,{\mathcal{E}_3}^2 d - 24\,\mathcal{E}_3\,d + 12\,d\right)(d - \frac{1}{2(\mathcal{E}_3 - 1)}(\mathcal{E}_2 + \sqrt{\mathfrak{C}}\,(coth(\sqrt{\mathfrak{C}}\,\mathfrak{Z}) \pm csch(\sqrt{\mathfrak{C}}\,\mathfrak{Z})))) + \left(-6\,{\mathcal{E}_3}^2 - 6 + 12\,\mathcal{E}_3\right)\left(d - \frac{1}{2(\mathcal{E}_3 - 1)}(\mathcal{E}_2 + \sqrt{\mathfrak{C}}\,(coth(\sqrt{\mathfrak{C}}\,\mathfrak{Z}) \pm csch(\sqrt{\mathfrak{C}}\,\mathfrak{Z})))\right)^2. \tag{2.49}$$

$$\begin{aligned}\mathcal{U}(\mathfrak{z}) = &-(2\mathcal{E}_3-2)\mathcal{E}_1-\mathcal{E}_2^2-(-6\mathcal{E}_3 d+6d)\mathcal{E}_2-6\mathcal{E}_3^2d^2+12\mathcal{E}_3 d^2-6d^2+\Big(-\mathcal{E}_2\,(6\mathcal{E}_3-6)\\ &+12\mathcal{E}_3^2d-24\mathcal{E}_3 d+12d\Big)(d-\frac{1}{4(\mathcal{E}_3-1)}(2\mathcal{E}_2+\sqrt{\mathfrak{C}}(tanh\left(\frac{\sqrt{\mathfrak{C}}}{4}\mathfrak{z}\right)\pm coth(\frac{\sqrt{\mathfrak{C}}}{4}\mathfrak{z}\Big))))\\ &+(-6\mathcal{E}_3^2-6+12\mathcal{E}_3)(d-\frac{1}{4(\mathcal{E}_3-1)}(2\mathcal{E}_2+\sqrt{\mathfrak{C}}(tanh\left(\frac{\sqrt{\mathfrak{C}}}{4}\mathfrak{z}\right)\pm coth(\frac{\sqrt{\mathfrak{C}}}{4}\mathfrak{z}\Big))))^2,\end{aligned} \tag{2.50}$$

$$\begin{aligned}\mathcal{U}(\mathfrak{z}) = &-(2\mathcal{E}_3-2)\mathcal{E}_1-\mathcal{E}_2^2-(-6\mathcal{E}_3 d+6d)\mathcal{E}_2-6\mathcal{E}_3^2d^2+12\mathcal{E}_3 d^2-6d^2+(-\mathcal{E}_2\,(6\mathcal{E}_3-6)\\ &+12\mathcal{E}_3^2d-24\mathcal{E}_3 d+12d)(d+\frac{1}{2(\mathcal{E}_3-1)}(-\mathcal{E}_2+\frac{\pm\sqrt{\mathfrak{C}\,(A^2+B^2)}-A\ \sqrt{\mathfrak{C}}\,cosh(\sqrt{\mathfrak{C}}\,\mathfrak{z})}{A\,sinh(\sqrt{\mathfrak{C}}\,\mathfrak{z})+B}))\\ &+\Big(-6\mathcal{E}_3^2-6+12\mathcal{E}_3\Big)\left(d+\frac{1}{2(\mathcal{E}_3-1)}(-\mathcal{E}_2+\frac{\pm\sqrt{\mathfrak{C}\,(A^2+B^2)}-A\ \sqrt{\mathfrak{C}}\,cosh(\sqrt{\mathfrak{C}}\,\mathfrak{z})}{A\,sinh(\sqrt{\mathfrak{C}}\,\mathfrak{z})+B})\right)^2,\end{aligned} \tag{2.51}$$

$$\begin{aligned}\mathcal{U}(\mathfrak{z}) = &-(2\mathcal{E}_3-2)\mathcal{E}_1-\mathcal{E}_2^2-(-6\mathcal{E}_3 d+6d)\mathcal{E}_2-6\mathcal{E}_3^2d^2+12\mathcal{E}_3 d^2-6d^2+(-\mathcal{E}_2\,(6\mathcal{E}_3-6)\\ &+12\mathcal{E}_3^2d-24\mathcal{E}_3 d+12d)(d+\frac{1}{2(\mathcal{E}_3-1)}(-\mathcal{E}_2+\frac{\pm\sqrt{\mathfrak{C}\,(A^2+B^2)}+A\ \sqrt{\mathfrak{C}}\,cosh(\sqrt{\mathfrak{C}}\,\mathfrak{z})}{A\,sinh(\sqrt{\mathfrak{C}}\,\mathfrak{z})+B}))\\ &+(-6\mathcal{E}_3^2-6+12\mathcal{E}_3)\left(d+\frac{1}{2(\mathcal{E}_3-1)}\left(-\mathcal{E}_2+\frac{\pm\sqrt{\mathfrak{C}\,(A^2+B^2)}+A\ \sqrt{\mathfrak{C}}\,cosh(\sqrt{\mathfrak{C}}\,\mathfrak{z})}{A\,sinh(\sqrt{\mathfrak{C}}\,\mathfrak{z})+B}\right)\right)^2,\end{aligned} \tag{2.52}$$

$$\begin{aligned}\mathcal{U}(\mathfrak{z}) = &-(2\mathcal{E}_3-2)\mathcal{E}_1-\mathcal{E}_2^2-(-6\mathcal{E}_3 d+6d)\mathcal{E}_2-6\mathcal{E}_3^2d^2+12\mathcal{E}_3 d^2-6d^2+(-\mathcal{E}_2\,(6\mathcal{E}_3-6)+12\mathcal{E}_3^2d\\ &-24\mathcal{E}_3 d+12d)\left(d+\frac{2\mathcal{E}_1\,cosh(\frac{\sqrt{\mathfrak{C}}}{2}\mathfrak{z})}{\sqrt{\mathfrak{C}}\,sinh(\frac{\sqrt{\mathfrak{C}}}{2}\mathfrak{z})-\mathcal{E}_2\,cosh(\frac{\sqrt{\mathfrak{C}}}{2}\mathfrak{z})}\right)+\Big(-6\mathcal{E}_3^2-6+12\mathcal{E}_3\Big)\Big(d\\ &+\frac{2\mathcal{E}_1\,cosh(\frac{\sqrt{\mathfrak{C}}}{2}\mathfrak{z})}{\sqrt{\mathfrak{C}}\,sinh(\frac{\sqrt{\mathfrak{C}}}{2}\mathfrak{z})-\mathcal{E}_2\,cosh(\frac{\sqrt{\mathfrak{C}}}{2}\mathfrak{z})}\Big)^2,\end{aligned} \tag{2.53}$$

$$\begin{aligned}\mathcal{U}(\mathfrak{z}) = &-(2\mathcal{E}_3-2)\mathcal{E}_1-\mathcal{E}_2^2-(-6\mathcal{E}_3 d+6d)\mathcal{E}_2-6\mathcal{E}_3^2d^2+12\mathcal{E}_3 d^2-6d^2+(-\mathcal{E}_2\,(6\mathcal{E}_3-6)+12\mathcal{E}_3^2d\\ &-24\mathcal{E}_3 d+12d)(d+\frac{2\mathcal{E}_1\,sinh(\frac{\sqrt{\mathfrak{C}}}{2}\mathfrak{z})}{\sqrt{\mathfrak{C}}\,cosh(\frac{\sqrt{\mathfrak{C}}}{2}\mathfrak{z})-\mathcal{E}_2\,sinh(\frac{\sqrt{\mathfrak{C}}}{2}\mathfrak{z})})+\Big(-6\mathcal{E}_3^2-6+12\mathcal{E}_3\Big)\Big(d\\ &+\frac{2\mathcal{E}_1\,sinh(\frac{\sqrt{\mathfrak{C}}}{2}\mathfrak{z})}{\sqrt{\mathfrak{C}}\,cosh(\frac{\sqrt{\mathfrak{C}}}{2}\mathfrak{z})-\mathcal{E}_2\,sinh(\frac{\sqrt{\mathfrak{C}}}{2}\mathfrak{z})}\Big)^2,\end{aligned} \tag{2.54}$$

$$\begin{aligned}\mathcal{U}(\mathfrak{z}) = &-(2\mathcal{E}_3-2)\mathcal{E}_1-\mathcal{E}_2^2-(-6\mathcal{E}_3 d+6d)\mathcal{E}_2-6\mathcal{E}_3^2d^2+12\mathcal{E}_3 d^2-6d^2+(-\mathcal{E}_2\,(6\mathcal{E}_3-6)+12\mathcal{E}_3^2d\\ &-24\mathcal{E}_3 d+12d)(d+\frac{2\mathcal{E}_1\,cosh(\sqrt{\mathfrak{C}}\,\mathfrak{z})}{\sqrt{\mathfrak{C}}\,sinh(\sqrt{\mathfrak{C}}\,\mathfrak{z})-\mathcal{E}_2\,cosh(\sqrt{\mathfrak{C}}\,\mathfrak{z})\pm i\,\sqrt{\mathfrak{C}}})+\Big(-6\mathcal{E}_3^2-6+12\mathcal{E}_3\Big)\,(d\\ &+\frac{2\mathcal{E}_1\,cosh(\sqrt{\mathfrak{C}}\,\mathfrak{z})}{\sqrt{\mathfrak{C}}\,sinh(\sqrt{\mathfrak{C}}\,\mathfrak{z})-\mathcal{E}_2\,cosh(\sqrt{\mathfrak{C}}\,\mathfrak{z})\pm i\,\sqrt{\mathfrak{C}}})^2,\end{aligned} \tag{2.55}$$

$$\mathcal{U}(\mathfrak{Z}) = -(2\mathcal{E}_3-2)\mathcal{E}_1-\mathcal{E}_2{}^2-(-6\mathcal{E}_3\,d+6\,d)\mathcal{E}_2-6\mathcal{E}_3{}^2d^2+12\mathcal{E}_3\,d^2-6\,d^2+(-\mathcal{E}_2\,(6\mathcal{E}_3-6)+12\mathcal{E}_3{}^2d$$
$$-24\mathcal{E}_3\,d+12\,d)(d+\frac{2\mathcal{E}_1\,sinh(\sqrt{\mathfrak{C}}\mathfrak{Z})}{\sqrt{\mathfrak{C}}\,cosh(\sqrt{\mathfrak{C}}\mathfrak{Z})-\mathcal{E}_2\,sinh(\sqrt{\mathfrak{C}}\mathfrak{Z})\pm i\,\sqrt{\mathfrak{C}}})+\left(-6\mathcal{E}_3{}^2-6+12\mathcal{E}_3\right)(d \tag{2.56}$$
$$+\frac{2\mathcal{E}_1\,sinh(\sqrt{\mathfrak{C}}\mathfrak{Z})}{\sqrt{\mathfrak{C}}\,cosh(\sqrt{\mathfrak{C}}\mathfrak{Z})-\mathcal{E}_2\,sinh(\sqrt{\mathfrak{C}}\mathfrak{Z})\pm i\,\sqrt{\mathfrak{C}}})^2,$$

While A, B are arbitrary real constants and $A^2+B^2>0$.
When $\left(\mathfrak{C}=\mathcal{E}_2^2-4\mathcal{E}_2\mathcal{E}_1+4\mathcal{E}_1<0\right)$ **and** $(\mathcal{E}_2(\mathcal{E}_3-1)\neq 0)$ **or** $(\mathcal{E}_1(\mathcal{E}_3-1)\neq 0)$**:**

$$\mathcal{U}(\mathfrak{Z}) = -(2\mathcal{E}_3-2)\mathcal{E}_1-\mathcal{E}_2{}^2-(-6\mathcal{E}_3\,d+6\,d)\mathcal{E}_2-6\mathcal{E}_3{}^2d^2+12\mathcal{E}_3\,d^2-6\,d^2+(-\mathcal{E}_2\,(6\mathcal{E}_3-6)$$
$$+12\mathcal{E}_3{}^2d-24\mathcal{E}_3\,d+12\,d)(d+\frac{1}{2(\mathcal{E}_3-1)}(-\mathcal{E}_2+\sqrt{-\mathfrak{C}}\,tanh(\frac{\sqrt{-\mathfrak{C}}}{2}\mathfrak{Z}))) \tag{2.57}$$
$$+\left(-6\mathcal{E}_3{}^2-6+12\mathcal{E}_3\right)(d+\frac{1}{2(\mathcal{E}_3-1)}(-\mathcal{E}_2+\sqrt{-\mathfrak{C}}\,tanh(\frac{\sqrt{-\mathfrak{C}}}{2}\mathfrak{Z})))^2,$$

$$\mathcal{U}(\mathfrak{Z}) = -(2\mathcal{E}_3-2)\mathcal{E}_1-\mathcal{E}_2{}^2-(-6\mathcal{E}_3\,d+6\,d)\mathcal{E}_2-6\mathcal{E}_3{}^2d^2+12\mathcal{E}_3\,d^2-6\,d^2+(-\mathcal{E}_2\,(6\mathcal{E}_3-6)$$
$$+12\mathcal{E}_3{}^2d-24\mathcal{E}_3\,d+12\,d)(d-\frac{1}{2(\mathcal{E}_3-1)}(\mathcal{E}_2+\sqrt{-\mathfrak{C}}\,coth(\frac{\sqrt{-\mathfrak{C}}}{2}\mathfrak{Z}))) \tag{2.58}$$
$$+\left(-6\mathcal{E}_3{}^2-6+12\mathcal{E}_3\right)(d-\frac{1}{2(\mathcal{E}_3-1)}(\mathcal{E}_2+\sqrt{-\mathfrak{C}}\,coth(\frac{\sqrt{-\mathfrak{C}}}{2}\mathfrak{Z})))^2,$$

$$\mathcal{U}(\mathfrak{Z}) = -(2\mathcal{E}_3-2)\mathcal{E}_1-\mathcal{E}_2{}^2-(-6\mathcal{E}_3\,d+6\,d)\mathcal{E}_2-6\mathcal{E}_3{}^2d^2+12\mathcal{E}_3\,d^2-6\,d^2+(-\mathcal{E}_2\,(6\mathcal{E}_3-6)$$
$$+12\mathcal{E}_3{}^2d-24\mathcal{E}_3\,d+12\,d)(d+\frac{1}{2(\mathcal{E}_3-1)}(-\mathcal{E}_2+\sqrt{-\mathfrak{C}}\,(tan(\sqrt{-\mathfrak{C}}\mathfrak{Z})\pm\,sec(\sqrt{-\mathfrak{C}}\mathfrak{Z})))) \tag{2.59}$$
$$+\left(-6\mathcal{E}_3{}^2-6+12\mathcal{E}_3\right)(d+\frac{1}{2(\mathcal{E}_3-1)}(-\mathcal{E}_2+\sqrt{-\mathfrak{C}}\,(tan(\sqrt{-\mathfrak{C}}\mathfrak{Z})\pm\,sec(\sqrt{-\mathfrak{C}}\mathfrak{Z}))))^2,$$

$$\mathcal{U}(\mathfrak{Z}) = -(2\mathcal{E}_3-2)\mathcal{E}_1-\mathcal{E}_2{}^2-(-6\mathcal{E}_3\,d+6\,d)\mathcal{E}_2-6\mathcal{E}_3{}^2d^2+12\mathcal{E}_3\,d^2-6\,d^2+(-\mathcal{E}_2\,(6\mathcal{E}_3-6)$$
$$+12\mathcal{E}_3{}^2d-24\mathcal{E}_3\,d+12\,d)(d+\frac{1}{2(\mathcal{E}_3-1)}(-\mathcal{E}_2+\sqrt{-\mathfrak{C}}\,(tan(\sqrt{-\mathfrak{C}}\mathfrak{Z})\pm\,sec(\sqrt{-\mathfrak{C}}\mathfrak{Z})))) \tag{2.60}$$
$$+\left(-6\mathcal{E}_3{}^2-6+12\mathcal{E}_3\right)(d+\frac{1}{2(\mathcal{E}_3-1)}(-\mathcal{E}_2+\sqrt{-\mathfrak{C}}\,(tan(\sqrt{-\mathfrak{C}}\mathfrak{Z})\pm\,sec(\sqrt{-\mathfrak{C}}\mathfrak{Z}))))^2,$$

$$\mathcal{U}(\mathfrak{Z}) = -(2\mathcal{E}_3-2)\mathcal{E}_1-\mathcal{E}_2{}^2-(-6\mathcal{E}_3\,d+6\,d)\mathcal{E}_2-6\mathcal{E}_3{}^2d^2+12\mathcal{E}_3\,d^2-6\,d^2+(-\mathcal{E}_2\,(6\mathcal{E}_3-6)$$
$$+12\mathcal{E}_3{}^2d-24\mathcal{E}_3\,d+12\,d)(d-\frac{1}{2(\mathcal{E}_3-1)}(\mathcal{E}_2+\sqrt{-\mathfrak{C}}\,(cot(\sqrt{-\mathfrak{C}}\mathfrak{Z})\pm\,csc(\sqrt{-\mathfrak{C}}\mathfrak{Z})))) \tag{2.61}$$
$$+\left(-6\mathcal{E}_3{}^2-6+12\mathcal{E}_3\right)\left(d-\frac{1}{2(\mathcal{E}_3-1)}(\mathcal{E}_2+\sqrt{-\mathfrak{C}}\,(cot(\sqrt{-\mathfrak{C}}\mathfrak{Z})\pm\,csc(\sqrt{-\mathfrak{C}}\mathfrak{Z})))\right)^2,$$

$$
\begin{aligned}
\mathcal{U}(\mathfrak{Z}) = &-(2\mathcal{E}_3-2)\mathcal{E}_1-\mathcal{E}_2^{\,2}-(-6\mathcal{E}_3\,d+6\,d)\mathcal{E}_2-6\mathcal{E}_3^{\,2}d^2+12\mathcal{E}_3\,d^2-6\,d^2+(-\mathcal{E}_2\;(6\mathcal{E}_3-6)\\
&+12\mathcal{E}_3^{\,2}d-24\mathcal{E}_3\,d+12\,d)\Big(d+\frac{1}{4(\mathcal{E}_3-1)}(-2\mathcal{E}_2+\sqrt{-\mathfrak{C}}\,(tan(\frac{\sqrt{-\mathfrak{C}}}{4}\mathfrak{Z})-\\
&cot(\frac{\sqrt{-\mathfrak{C}}}{4}\mathfrak{Z})))\Big)+(-6\mathcal{E}_3^{\,2}-6+12\mathcal{E}_3)\Big(d+\frac{1}{4(\mathcal{E}_3-1)}\Big(-2\mathcal{E}_2+\sqrt{-\mathfrak{C}}\\
&\times\Big(tan(\frac{\sqrt{-\mathfrak{C}}}{4}\mathfrak{Z})-cot(\frac{\sqrt{-\mathfrak{C}}}{4}\mathfrak{Z})\Big)\Big)\Big)^2,
\end{aligned}
\tag{2.62}
$$

$$
\begin{aligned}
\mathcal{U}(\mathfrak{Z}) = &-(2\mathcal{E}_3-2)\mathcal{E}_1-\mathcal{E}_2^{\,2}-(-6\mathcal{E}_3\,d+6\,d)\mathcal{E}_2-6\mathcal{E}_3^{\,2}d^2+12\mathcal{E}_3\,d^2-6\,d^2+\Big(-\mathcal{E}_2\;(6\mathcal{E}_3-6)+12\mathcal{E}_3^{\,2}d\\
&-24\mathcal{E}_3\,d+12\,d\Big)\left(d+\frac{1}{2(\mathcal{E}_3-1)}\left(-\mathcal{E}_2+\frac{\pm\sqrt{-\mathfrak{C}\,(A^2-B^2)}-A\sqrt{-\mathfrak{C}}\,cos(\sqrt{-\mathfrak{C}}\,\mathfrak{Z})}{A\,sin(\sqrt{-\mathfrak{C}}\,\mathfrak{Z})+B}\right)\right)\\
&+\Big(-6\mathcal{E}_3^{\,2}-6+12\mathcal{E}_3\Big)\left(d+\frac{1}{2(\mathcal{E}_3-1)}\left(-\mathcal{E}_2+\frac{\pm\sqrt{-\mathfrak{C}\,(A^2-B^2)}-A\sqrt{-\mathfrak{C}}\,cos(\sqrt{-\mathfrak{C}}\,\mathfrak{Z})}{A\,sin(\sqrt{-\mathfrak{C}}\,\mathfrak{Z})+B}\right)\right)^2,
\end{aligned}
\tag{2.63}
$$

$$
\begin{aligned}
\mathcal{U}(\mathfrak{Z}) = &-(2\mathcal{E}_3-2)\mathcal{E}_1-\mathcal{E}_2^{\,2}-(-6\mathcal{E}_3\,d+6\,d)\mathcal{E}_2-6\mathcal{E}_3^{\,2}d^2+12\mathcal{E}_3\,d^2-6\,d^2+(-\mathcal{E}_2\;(6\mathcal{E}_3-6)+12\mathcal{E}_3^{\,2}d\\
&-24\mathcal{E}_3\,d+12\,d)\left(d+\frac{1}{2(\mathcal{E}_3-1)}(-\mathcal{E}_2+\frac{\pm\sqrt{-\mathfrak{C}\,(A^2-B^2)}+A\sqrt{-\mathfrak{C}}\,cos(\sqrt{-\mathfrak{C}}\,\mathfrak{Z})}{A\,sin(\sqrt{-\mathfrak{C}}\,\mathfrak{Z})+B})\right)+(-6\mathcal{E}_3^{\,2}\\
&-6+12\mathcal{E}_3)\left(d+\frac{1}{2(\mathcal{E}_3-1)}\left(-\mathcal{E}_2+\frac{\pm\sqrt{-\mathfrak{C}\,(A^2-B^2)}+A\sqrt{-\mathfrak{C}}\,cos(\sqrt{-\mathfrak{C}}\,\mathfrak{Z})}{A\,sin(\sqrt{-\mathfrak{C}}\,\mathfrak{Z})+B}\right)\right)^2,
\end{aligned}
\tag{2.64}
$$

$$
\begin{aligned}
\mathcal{U}(\mathfrak{Z}) = &-(2\mathcal{E}_3-2)\mathcal{E}_1-\mathcal{E}_2^{\,2}-(-6\mathcal{E}_3\,d+6\,d)\mathcal{E}_2-6\mathcal{E}_3^{\,2}d^2+12\mathcal{E}_3\,d^2-6\,d^2+(-\mathcal{E}_2\;(6\mathcal{E}_3-6)+12\mathcal{E}_3^{\,2}d\\
&-24\mathcal{E}_3\,d+12\,d)(d-\frac{2\mathcal{E}_1\,cos(\frac{\sqrt{-\mathfrak{C}}}{2}\mathfrak{Z})}{\sqrt{-\mathfrak{C}}\,sin(\frac{\sqrt{-\mathfrak{C}}}{2}\mathfrak{Z})+\mathcal{E}_2\,cos(\frac{\sqrt{-\mathfrak{C}}}{2}\mathfrak{Z})})+\Big(-6\mathcal{E}_3^{\,2}-6+12\mathcal{E}_3\Big)\\
&\times\left(d-\frac{2\mathcal{E}_1\,cos(\frac{\sqrt{-\mathfrak{C}}}{2}\mathfrak{Z})}{\sqrt{-\mathfrak{C}}\,sin(\frac{\sqrt{-\mathfrak{C}}}{2}\mathfrak{Z})+\mathcal{E}_2\,cos(\frac{\sqrt{-\mathfrak{C}}}{2}\mathfrak{Z})}\right)^2,
\end{aligned}
\tag{2.65}
$$

$$
\begin{aligned}
\mathcal{U}(\mathfrak{Z}) = &-(2\mathcal{E}_3-2)\mathcal{E}_1-\mathcal{E}_2^{\,2}-(-6\mathcal{E}_3\,d+6\,d)\mathcal{E}_2-6\mathcal{E}_3^{\,2}d^2+12\mathcal{E}_3\,d^2-6\,d^2+(-\mathcal{E}_2\;(6\mathcal{E}_3-6)+12\mathcal{E}_3^{\,2}d\\
&-24\mathcal{E}_3\,d+12\,d)(d+\frac{2\mathcal{E}_1\,sin(\frac{\sqrt{-\mathfrak{C}}}{2}\mathfrak{Z})}{\sqrt{-\mathfrak{C}}\,cos(\sqrt{-\mathfrak{C}}2\,\mathfrak{Z})-\mathcal{E}_2\,sin(\frac{\sqrt{-\mathfrak{C}}}{2}\mathfrak{Z})})+\Big(-6\mathcal{E}_3^{\,2}-6+12\mathcal{E}_3\Big)\\
&\times\left(d+\frac{2\mathcal{E}_1\,sin(\frac{\sqrt{-\mathfrak{C}}}{2}\mathfrak{Z})}{\sqrt{-\mathfrak{C}}\,cos(\frac{\sqrt{-\mathfrak{C}}}{2}\mathfrak{Z})-\mathcal{E}_2\,sin(\frac{\sqrt{-\mathfrak{C}}}{2}\mathfrak{Z})}\right)^2,
\end{aligned}
\tag{2.66}
$$

$$
\begin{aligned}
\mathcal{U}(\mathfrak{Z}) = &-(2\mathcal{E}_3-2)\mathcal{E}_1-\mathcal{E}_2^{\,2}-(-6\mathcal{E}_3\,d+6\,d)\mathcal{E}_2-6\mathcal{E}_3^{\,2}d^2+12\mathcal{E}_3\,d^2-6\,d^2+(-\mathcal{E}_2\;(6\mathcal{E}_3-6)+12\mathcal{E}_3^{\,2}d\\
&-24\mathcal{E}_3\,d+12\,d)\left(d+\frac{2\mathcal{E}_1\,sin(\sqrt{-\mathfrak{C}}\,\mathfrak{Z})}{\sqrt{-\mathfrak{C}}\,cos(\sqrt{-\mathfrak{C}}\,\mathfrak{Z})-\mathcal{E}_2\,sin(\sqrt{-\mathfrak{C}}\,\mathfrak{Z})\pm\sqrt{-\mathfrak{C}}}\right)+\Big(-6\mathcal{E}_3^{\,2}-6+12\mathcal{E}_3\Big)\\
&\times\left(d+\frac{2\mathcal{E}_1\,sin(\sqrt{-\mathfrak{C}}\,\mathfrak{Z})}{\sqrt{-\mathfrak{C}}\,cos(\sqrt{-\mathfrak{C}}\,\mathfrak{Z})-\mathcal{E}_2\,sin(\sqrt{-\mathfrak{C}}\,\mathfrak{Z})\pm\sqrt{-\mathfrak{C}}}\right)^2,
\end{aligned}
\tag{2.67}
$$

While A, B are arbitrary real constants and $A^2 - B^2 > 0$.
When $\mathcal{E}_1 = 0$ and $\mathcal{E}_2(\mathcal{E}_3 - 1) \neq 0$, we have:

$$\begin{aligned}\mathcal{U}(\mathfrak{Z}) = &-(2\mathcal{E}_3 - 2)\mathcal{E}_1 - \mathcal{E}_2{}^2 - (-6\mathcal{E}_3 d + 6d)\mathcal{E}_2 - 6\mathcal{E}_3{}^2 d^2 + 12\mathcal{E}_3 d^2 - 6d^2 + (-\mathcal{E}_2\,(6\mathcal{E}_3 - 6) + 12\mathcal{E}_3{}^2 d \\ &- 24\mathcal{E}_3 d + 12d)(d - \frac{\mathcal{E}_2 k}{(\mathcal{E}_3 - 1)(k + cosh(\mathcal{E}_2\mathfrak{Z}) - sinh(\mathcal{E}_2\mathfrak{Z}))}) + \left(-6\mathcal{E}_3{}^2 - 6 + 12\mathcal{E}_3\right) \\ &\times\left(d - \frac{\mathcal{E}_2 k}{(\mathcal{E}_3 - 1)(k + cosh(\mathcal{E}_2\mathfrak{Z}) - sinh(\mathcal{E}_2\mathfrak{Z}))}\right)^2,\end{aligned} \tag{2.68}$$

$$\begin{aligned}\mathcal{U}(\mathfrak{Z}) = &-(2\mathcal{E}_3 - 2)\mathcal{E}_1 - \mathcal{E}_2{}^2 - (-6\mathcal{E}_3 d + 6d)\mathcal{E}_2 - 6\mathcal{E}_3{}^2 d^2 + 12\mathcal{E}_3 d^2 - 6d^2 + (-\mathcal{E}_2\,(6\mathcal{E}_3 - 6) + 12\mathcal{E}_3{}^2 d \\ &- 24\mathcal{E}_3 d + 12d)(d - \frac{\mathcal{E}_2\,(cosh(\mathcal{E}_2\mathfrak{Z}) + sinh(\mathcal{E}_2\mathfrak{Z}))}{(\mathcal{E}_3 - 1)(k + cosh(\mathcal{E}_2\mathfrak{Z}) + sinh(\mathcal{E}_2\mathfrak{Z}))}) + \left(-6\mathcal{E}_3{}^2 - 6 + 12\mathcal{E}_3\right) \\ &\times\left(d - \frac{\mathcal{E}_2\,(cosh(\mathcal{E}_2\mathfrak{Z}) + sinh(\mathcal{E}_2\mathfrak{Z}))}{(\mathcal{E}_3 - 1)(k + cosh(\mathcal{E}_2\mathfrak{Z}) + sinh(\mathcal{E}_2\mathfrak{Z}))}\right)^2,\end{aligned} \tag{2.69}$$

$$\begin{aligned}\mathcal{U}(\mathfrak{Z}) = &-(2\mathcal{E}_3 - 2)\mathcal{E}_1 - \mathcal{E}_2{}^2 - (-6\mathcal{E}_3 d + 6d)\mathcal{E}_2 - 6\mathcal{E}_3{}^2 d^2 + 12\mathcal{E}_3 d^2 - 6d^2 + (-\mathcal{E}_2\,(6\mathcal{E}_3 - 6) + 12\mathcal{E}_3{}^2 d \\ &- 24\mathcal{E}_3 d + 12d)(d - \frac{1}{(\mathcal{E}_3 - 1)\mathfrak{Z} + C}) + \left(-6\mathcal{E}_3{}^2 - 6 + 12\mathcal{E}_3\right) \\ &\times\left(d - \frac{1}{(\mathcal{E}_3 - 1)\mathfrak{Z} + C}\right)^2.\end{aligned} \tag{2.70}$$

Set B

$$\begin{aligned}&c = \sqrt{-4\mathcal{E}_1\mathcal{E}_3 + 4\mathcal{E}_1 + 1 - \mathcal{E}_2^2},\ a_{-2} = a_{-1} = 0,\ a_0 = -(6\mathcal{E}_3 - 6)\mathcal{E}_1 - \mathcal{E}_2{}^2 - (-6\mathcal{E}_3 d + 6d)\mathcal{E}_2 - 6\mathcal{E}_3{}^2 d^2 \\ &+ 12\mathcal{E}_3 d^2 - 6d^2, a_1 = -\mathcal{E}_2\,(6\mathcal{E}_3 - 6) + 12\mathcal{E}_3{}^2 d - 24\mathcal{E}_3 d + 12d,\ a_2 = -6\mathcal{E}_3{}^2 - 6 + 12\mathcal{E}_3.\end{aligned} \tag{2.71}$$

Thus, the model's soliton solutions are given by
When $\left(\mathfrak{C} = \mathcal{E}_2^2 - 4\mathcal{E}_2\mathcal{E}_1 + 4\mathcal{E}_1 > 0\right)$ and $(\mathcal{E}_2(\mathcal{E}_3 - 1) \neq 0)$ or $(\mathcal{E}_1(\mathcal{E}_3 - 1) \neq 0)$:

$$\begin{aligned}\mathcal{U}(\mathfrak{Z}) = &-(6\mathcal{E}_3 - 6)\mathcal{E}_1 - \mathcal{E}_2{}^2 - (-6\mathcal{E}_3 d + 6d)\mathcal{E}_2 - 6\mathcal{E}_3{}^2 d^2 + 12\mathcal{E}_3 d^2 - 6d^2 + (-\mathcal{E}_2\,(6\mathcal{E}_3 - 6) + 12\mathcal{E}_3{}^2 d \\ &- 24\mathcal{E}_3 d + 12d)(d - \frac{1}{2(\mathcal{E}_3 - 1)}(\mathcal{E}_2 + \sqrt{\mathfrak{C}}\,tanh(\frac{\sqrt{\mathfrak{C}}}{2}\mathfrak{Z}))) + \left(-6\mathcal{E}_3{}^2 - 6 + 12\mathcal{E}_3\right) \\ &\times(d - \frac{1}{2(\mathcal{E}_3 - 1)}(\mathcal{E}_2 + \sqrt{\mathfrak{C}}\,tanh(\frac{\sqrt{\mathfrak{C}}}{2}\mathfrak{Z})))^2,\end{aligned} \tag{2.72}$$

$$\begin{aligned}\mathcal{U}(\mathfrak{Z}) = &-(6\mathcal{E}_3 - 6)\mathcal{E}_1 - \mathcal{E}_2{}^2 - (-6\mathcal{E}_3 d + 6d)\mathcal{E}_2 - 6\mathcal{E}_3{}^2 d^2 + 12\mathcal{E}_3 d^2 - 6d^2 + (-\mathcal{E}_2\,(6\mathcal{E}_3 - 6) + 12\mathcal{E}_3{}^2 d \\ &- 24\mathcal{E}_3 d + 12d)(d - \frac{1}{2(\mathcal{E}_3 - 1)}(\mathcal{E}_2 + \sqrt{\mathfrak{C}}\,coth(\frac{\sqrt{\mathfrak{C}}}{2}\mathfrak{Z}))) + \left(-6\mathcal{E}_3{}^2 - 6 + 12\mathcal{E}_3\right) \\ &\times(d - \frac{1}{2(\mathcal{E}_3 - 1)}(\mathcal{E}_2 + \sqrt{\mathfrak{C}}\,coth(\frac{\sqrt{\mathfrak{C}}}{2}\mathfrak{Z})))^2,\end{aligned} \tag{2.73}$$

$$\begin{aligned}\mathcal{U}(\mathfrak{Z}) =& -(6\,\mathcal{E}_3-6)\,\mathcal{E}_1-{\mathcal{E}_2}^2-(-6\,\mathcal{E}_3\,d+6\,d)\,\mathcal{E}_2-6\,{\mathcal{E}_3}^2d^2+12\,\mathcal{E}_3\,d^2-6\,d^2+(-\mathcal{E}_2\,(6\,\mathcal{E}_3-6)+12\,{\mathcal{E}_3}^2d\\ &-24\,\mathcal{E}_3\,d+12\,d)(d-\frac{1}{2(\mathcal{E}_3-1)}(\mathcal{E}_2+\sqrt{\mathfrak{C}}\,(tanh\left(\sqrt{\mathfrak{C}}\,\mathfrak{Z}\right)\pm i\,sech(\sqrt{\mathfrak{C}}\,\mathfrak{Z})))))+\left(-6\,{\mathcal{E}_3}^2-6+12\,\mathcal{E}_3\right)\\ &\times(d-\frac{1}{2(\mathcal{E}_3-1)}(\mathcal{E}_2+\sqrt{\mathfrak{C}}\,(tanh(\sqrt{\mathfrak{C}}\,\mathfrak{Z})\pm i\,sech(\sqrt{\mathfrak{C}}\,\mathfrak{Z}))))^2,\end{aligned}\tag{2.74}$$

$$\begin{aligned}\mathcal{U}(\mathfrak{Z}) =& -(6\,\mathcal{E}_3-6)\,\mathcal{E}_1-{\mathcal{E}_2}^2-(-6\,\mathcal{E}_3\,d+6\,d)\,\mathcal{E}_2-6\,{\mathcal{E}_3}^2d^2+12\,\mathcal{E}_3\,d^2-6\,d^2+(-\mathcal{E}_2\,(6\,\mathcal{E}_3-6)+12\,{\mathcal{E}_3}^2d\\ &-24\,\mathcal{E}_3\,d+12\,d)(d-\frac{1}{2(\mathcal{E}_3-1)}(\mathcal{E}_2+\sqrt{\mathfrak{C}}\,(coth(\sqrt{\mathfrak{C}}\,\mathfrak{Z})\pm\,csch(\sqrt{\mathfrak{C}}\,\mathfrak{Z}))))+\left(-6\,{\mathcal{E}_3}^2-6+12\,\mathcal{E}_3\right)\\ &\times(d-\frac{1}{2(\mathcal{E}_3-1)}(\mathcal{E}_2+\sqrt{\mathfrak{C}}\,(coth(\sqrt{\mathfrak{C}}\,\mathfrak{Z})\pm\,csch(\sqrt{\mathfrak{C}}\,\mathfrak{Z}))))^2.\end{aligned}\tag{2.75}$$

$$\begin{aligned}\mathcal{U}(\mathfrak{Z}) =& -(6\,\mathcal{E}_3-6)\,\mathcal{E}_1-{\mathcal{E}_2}^2-(-6\,\mathcal{E}_3\,d+6\,d)\,\mathcal{E}_2-6\,{\mathcal{E}_3}^2d^2+12\,\mathcal{E}_3\,d^2-6\,d^2+(-\mathcal{E}_2\,(6\,\mathcal{E}_3-6)+12\,{\mathcal{E}_3}^2d\\ &-24\,\mathcal{E}_3\,d+12\,d)(d-\frac{1}{4(\mathcal{E}_3-1)}(2\,\mathcal{E}_2+\sqrt{\mathfrak{C}}\,(tanh\left(\frac{\sqrt{\mathfrak{C}}}{4}\,\mathfrak{Z})\pm\,coth(\frac{\sqrt{\mathfrak{C}}}{4}\,\mathfrak{Z}\right))))+(-6\,{\mathcal{E}_3}^2-6\\ &+12\,\mathcal{E}_3)(d-\frac{1}{4(\mathcal{E}_3-1)}(2\,\mathcal{E}_2+\sqrt{\mathfrak{C}}\,(tanh(\frac{\sqrt{\mathfrak{C}}}{4}\,\mathfrak{Z})\pm\,coth(\frac{\sqrt{\mathfrak{C}}}{4}\,\mathfrak{Z}))))^2,\end{aligned}\tag{2.76}$$

$$\begin{aligned}\mathcal{U}(\mathfrak{Z}) =& -(6\,\mathcal{E}_3-6)\,\mathcal{E}_1-{\mathcal{E}_2}^2-(-6\,\mathcal{E}_3\,d+6\,d)\,\mathcal{E}_2-6\,{\mathcal{E}_3}^2d^2+12\,\mathcal{E}_3\,d^2-6\,d^2+(-\mathcal{E}_2\,(6\,\mathcal{E}_3-6)+12\,{\mathcal{E}_3}^2d\\ &-24\,\mathcal{E}_3\,d+12\,d)(d+\frac{1}{2(\mathcal{E}_3-1)}(-\mathcal{E}_2+\frac{\pm\sqrt{\mathfrak{C}\,(A^2+B^2)}-A\,\sqrt{\mathfrak{C}}\,cosh(\sqrt{\mathfrak{C}}\,\mathfrak{Z})}{A\,sinh(\sqrt{\mathfrak{C}}\,\mathfrak{Z})+B}))+(-6\,{\mathcal{E}_3}^2-6\\ &+12\,\mathcal{E}_3)(d+\frac{1}{2(\mathcal{E}_3-1)}(-\mathcal{E}_2+\frac{\pm\sqrt{\mathfrak{C}\,(A^2+B^2)}-A\,\sqrt{\mathfrak{C}}\,cosh(\sqrt{\mathfrak{C}}\,\mathfrak{Z})}{A\,sinh(\sqrt{\mathfrak{C}}\,\mathfrak{Z})+B}))^2,\end{aligned}\tag{2.77}$$

$$\begin{aligned}\mathcal{U}(\mathfrak{Z}) =& -(6\,\mathcal{E}_3-6)\,\mathcal{E}_1-{\mathcal{E}_2}^2-(-6\,\mathcal{E}_3\,d+6\,d)\,\mathcal{E}_2-6\,{\mathcal{E}_3}^2d^2+12\,\mathcal{E}_3\,d^2-6\,d^2+(-\mathcal{E}_2\,(6\,\mathcal{E}_3-6)+12\,{\mathcal{E}_3}^2d\\ &-24\,\mathcal{E}_3\,d+12\,d)(d+\frac{1}{2(\mathcal{E}_3-1)}(-\mathcal{E}_2+\frac{\pm\sqrt{\mathfrak{C}\,(A^2+B^2)}+A\,\sqrt{\mathfrak{C}}\,cosh(\sqrt{\mathfrak{C}}\,\mathfrak{Z})}{A\,sinh(\sqrt{\mathfrak{C}}\,\mathfrak{Z})+B}))+(-6\,{\mathcal{E}_3}^2-6\\ &+12\,\mathcal{E}_3)\left(d+\frac{1}{2(\mathcal{E}_3-1)}(-\mathcal{E}_2+\frac{\pm\sqrt{\mathfrak{C}\,(A^2+B^2)}+A\,\sqrt{\mathfrak{C}}\,cosh(\sqrt{\mathfrak{C}}\,\mathfrak{Z})}{A\,sinh(\sqrt{\mathfrak{C}}\,\mathfrak{Z})+B})\right)^2,\end{aligned}\tag{2.78}$$

$$\begin{aligned}\mathcal{U}(\mathfrak{Z}) =& -(6\,\mathcal{E}_3-6)\,\mathcal{E}_1-{\mathcal{E}_2}^2-(-6\,\mathcal{E}_3\,d+6\,d)\,\mathcal{E}_2-6\,{\mathcal{E}_3}^2d^2+12\,\mathcal{E}_3\,d^2-6\,d^2+(-\mathcal{E}_2\,(6\,\mathcal{E}_3-6)+12\,{\mathcal{E}_3}^2d\\ &-24\,\mathcal{E}_3\,d+12\,d)\left(d+\frac{2\,\mathcal{E}_1\,cosh(\frac{\sqrt{\mathfrak{C}}}{2}\,\mathfrak{Z})}{\sqrt{\mathfrak{C}}\,sinh(\frac{\sqrt{\mathfrak{C}}}{2}\,\mathfrak{Z})-\mathcal{E}_2\,cosh(\frac{\sqrt{\mathfrak{C}}}{2}\,\mathfrak{Z})}\right)+\left(-6\,{\mathcal{E}_3}^2-6+12\,\mathcal{E}_3\right)\\ &\times\left(d+\frac{2\,\mathcal{E}_1\,cosh(\frac{\sqrt{\mathfrak{C}}}{2}\,\mathfrak{Z})}{\sqrt{\mathfrak{C}}\,sinh(\frac{\sqrt{\mathfrak{C}}}{2}\,\mathfrak{Z})-\mathcal{E}_2\,cosh(\frac{\sqrt{\mathfrak{C}}}{2}\,\mathfrak{Z})}\right)^2,\end{aligned}\tag{2.79}$$

$$\mathcal{U}(\mathfrak{Z}) = -(6\,\mathcal{E}_3 - 6)\,\mathcal{E}_1 - {\mathcal{E}_2}^2 - (-6\,\mathcal{E}_3\,d + 6\,d)\,\mathcal{E}_2 - 6\,{\mathcal{E}_3}^2 d^2 + 12\,\mathcal{E}_3\,d^2 - 6\,d^2 + (-\mathcal{E}_2\,(6\,\mathcal{E}_3 - 6) + 12\,{\mathcal{E}_3}^2 d - 24\,\mathcal{E}_3\,d + 12\,d)\left(d + \frac{2\,\mathcal{E}_1\,sinh(\frac{\sqrt{\mathfrak{C}}}{2}\,\mathfrak{Z})}{\sqrt{\mathfrak{C}}\,cosh(\frac{\sqrt{\mathfrak{C}}}{2}\,\mathfrak{Z}) - \mathcal{E}_2\,sinh(\frac{\sqrt{\mathfrak{C}}}{2}\,\mathfrak{Z})}\right) + \left(-6\,{\mathcal{E}_3}^2 - 6 + 12\,\mathcal{E}_3\right) \times \left(d + \frac{2\,\mathcal{E}_1\,sinh(\frac{\sqrt{\mathfrak{C}}}{2}\,\mathfrak{Z})}{\sqrt{\mathfrak{C}}\,cosh(\frac{\sqrt{\mathfrak{C}}}{2}\,\mathfrak{Z}) - \mathcal{E}_2\,sinh(\frac{\sqrt{\mathfrak{C}}}{2}\,\mathfrak{Z})}\right)^2, \tag{2.80}$$

$$\mathcal{U}(\mathfrak{Z}) = -(6\,\mathcal{E}_3 - 6)\,\mathcal{E}_1 - {\mathcal{E}_2}^2 - (-6\,\mathcal{E}_3\,d + 6\,d)\,\mathcal{E}_2 - 6\,{\mathcal{E}_3}^2 d^2 + 12\,\mathcal{E}_3\,d^2 - 6\,d^2 + (-\mathcal{E}_2\,(6\,\mathcal{E}_3 - 6) + 12\,{\mathcal{E}_3}^2 d - 24\,\mathcal{E}_3\,d + 12\,d)\left(d + \frac{2\,\mathcal{E}_1\,cosh(\sqrt{\mathfrak{C}}\,\mathfrak{Z})}{\sqrt{\mathfrak{C}}\,sinh(\sqrt{\mathfrak{C}}\,\mathfrak{Z}) - \mathcal{E}_2\,cosh(\sqrt{\mathfrak{C}}\,\mathfrak{Z}) \pm i\,\sqrt{\mathfrak{C}}}\right) + \left(-6\,{\mathcal{E}_3}^2 - 6 + 12\,\mathcal{E}_3\right)(d + \frac{2\,\mathcal{E}_1\,cosh(\sqrt{\mathfrak{C}}\,\mathfrak{Z})}{\sqrt{\mathfrak{C}}\,sinh(\sqrt{\mathfrak{C}}\,\mathfrak{Z}) - \mathcal{E}_2\,cosh(\sqrt{\mathfrak{C}}\,\mathfrak{Z}) \pm i\,\sqrt{\mathfrak{C}}})^2, \tag{2.81}$$

$$\mathcal{U}(\mathfrak{Z}) = -(6\,\mathcal{E}_3 - 6)\,\mathcal{E}_1 - {\mathcal{E}_2}^2 - (-6\,\mathcal{E}_3\,d + 6\,d)\,\mathcal{E}_2 - 6\,{\mathcal{E}_3}^2 d^2 + 12\,\mathcal{E}_3\,d^2 - 6\,d^2 + (-\mathcal{E}_2\,(6\,\mathcal{E}_3 - 6) + 12\,{\mathcal{E}_3}^2 d - 24\,\mathcal{E}_3\,d + 12\,d)\left(d + \frac{2\,\mathcal{E}_1\,sinh(\sqrt{\mathfrak{C}}\,\mathfrak{Z})}{\sqrt{\mathfrak{C}}\,cosh(\sqrt{\mathfrak{C}}\,\mathfrak{Z}) - \mathcal{E}_2\,sinh(\sqrt{\mathfrak{C}}\,\mathfrak{Z}) \pm i\,\sqrt{\mathfrak{C}}}\right) + \left(-6\,{\mathcal{E}_3}^2 - 6 + 12\,\mathcal{E}_3\right)(d + \frac{2\,\mathcal{E}_1\,sinh(\sqrt{\mathfrak{C}}\,\mathfrak{Z})}{\sqrt{\mathfrak{C}}\,cosh(\sqrt{\mathfrak{C}}\,\mathfrak{Z}) - \mathcal{E}_2\,sinh(\sqrt{\mathfrak{C}}\,\mathfrak{Z}) \pm i\,\sqrt{\mathfrak{C}}})^2, \tag{2.82}$$

While A, B are arbitrary real constants and $A^2 + B^2 > 0$.
When $\left(\mathfrak{C} = \mathcal{E}_2^2 - 4\,\mathcal{E}_2\,\mathcal{E}_1 + 4\,\mathcal{E}_1 < 0\right)$ **and** $(\mathcal{E}_2(\mathcal{E}_3 - 1) \neq 0)$ **or** $(\mathcal{E}_1(\mathcal{E}_3 - 1) \neq 0)$**:**

$$\mathcal{U}(\mathfrak{Z}) = -(6\,\mathcal{E}_3 - 6)\,\mathcal{E}_1 - {\mathcal{E}_2}^2 - (-6\,\mathcal{E}_3\,d + 6\,d)\,\mathcal{E}_2 - 6\,{\mathcal{E}_3}^2 d^2 + 12\,\mathcal{E}_3\,d^2 - 6\,d^2 + (-\mathcal{E}_2\,(6\,\mathcal{E}_3 - 6) + 12\,{\mathcal{E}_3}^2 d - 24\,\mathcal{E}_3\,d + 12\,d)(d + \frac{1}{2(\mathcal{E}_3 - 1)}(-\mathcal{E}_2 + \sqrt{-\mathfrak{C}}\,tanh(\frac{\sqrt{-\mathfrak{C}}}{2}\,\mathfrak{Z}))) + \left(-6\,{\mathcal{E}_3}^2 - 6 + 12\,\mathcal{E}_3\right)\left(d + \frac{1}{2(\mathcal{E}_3 - 1)}(-\mathcal{E}_2 + \sqrt{-\mathfrak{C}}\,tanh(\frac{\sqrt{-\mathfrak{C}}}{2}\,\mathfrak{Z}))\right)^2, \tag{2.83}$$

$$\mathcal{U}(\mathfrak{Z}) = -(6\,\mathcal{E}_3 - 6)\,\mathcal{E}_1 - {\mathcal{E}_2}^2 - (-6\,\mathcal{E}_3\,d + 6\,d)\,\mathcal{E}_2 - 6\,{\mathcal{E}_3}^2 d^2 + 12\,\mathcal{E}_3\,d^2 - 6\,d^2 + (-\mathcal{E}_2\,(6\,\mathcal{E}_3 - 6) + 12\,{\mathcal{E}_3}^2 d - 24\,\mathcal{E}_3\,d + 12\,d)(d - \frac{1}{2(\mathcal{E}_3 - 1)}(\mathcal{E}_2 + \sqrt{-\mathfrak{C}}\,coth(\frac{\sqrt{-\mathfrak{C}}}{2}\,\mathfrak{Z}))) + \left(-6\,{\mathcal{E}_3}^2 - 6 + 12\,\mathcal{E}_3\right) \times \left(d - \frac{1}{2(\mathcal{E}_3 - 1)}(\mathcal{E}_2 + \sqrt{-\mathfrak{C}}\,coth(\frac{\sqrt{-\mathfrak{C}}}{2}\,\mathfrak{Z}))\right)^2, \tag{2.84}$$

$$\mathcal{U}(\mathfrak{Z}) = -(6\mathcal{E}_3-6)\mathcal{E}_1-\mathcal{E}_2{}^2-(-6\mathcal{E}_3 d+6d)\mathcal{E}_2-6\mathcal{E}_3{}^2d^2+12\mathcal{E}_3 d^2-6d^2+(-\mathcal{E}_2\,(6\mathcal{E}_3-6)+12\mathcal{E}_3{}^2 d$$
$$-24\mathcal{E}_3 d+12d)(d+\frac{1}{2(\mathcal{E}_3-1)}(-\mathcal{E}_2+\sqrt{-\mathfrak{C}}\,(tan(\sqrt{-\mathfrak{C}}\,\mathfrak{Z})\pm sec(\sqrt{-\mathfrak{C}}\,\mathfrak{Z}))))+(-6\mathcal{E}_3{}^2-6$$
$$+12\mathcal{E}_3)\left(d+\frac{1}{2(\mathcal{E}_3-1)}(-\mathcal{E}_2+\sqrt{-\mathfrak{C}}\,(tan(\sqrt{-\mathfrak{C}}\,\mathfrak{Z})\pm sec(\sqrt{-\mathfrak{C}}\,\mathfrak{Z})))\right)^2, \tag{2.85}$$

$$\mathcal{U}(\mathfrak{Z}) = -(6\mathcal{E}_3-6)\mathcal{E}_1-\mathcal{E}_2{}^2-(-6\mathcal{E}_3 d+6d)\mathcal{E}_2-6\mathcal{E}_3{}^2d^2+12\mathcal{E}_3 d^2-6d^2+(-\mathcal{E}_2\,(6\mathcal{E}_3-6)+12\mathcal{E}_3{}^2 d$$
$$-24\mathcal{E}_3 d+12d)(d+\frac{1}{2(\mathcal{E}_3-1)}(-\mathcal{E}_2+\sqrt{-\mathfrak{C}}\,(tan(\sqrt{-\mathfrak{C}}\,\mathfrak{Z})\pm sec(\sqrt{-\mathfrak{C}}\,\mathfrak{Z}))))+(-6\mathcal{E}_3{}^2$$
$$-6+12\mathcal{E}_3)\left(d+\frac{1}{2(\mathcal{E}_3-1)}(-\mathcal{E}_2+\sqrt{-\mathfrak{C}}\,(tan(\sqrt{-\mathfrak{C}}\,\mathfrak{Z})\pm sec(\sqrt{-\mathfrak{C}}\,\mathfrak{Z})))\right)^2, \tag{2.86}$$

$$\mathcal{U}(\mathfrak{Z}) = -(6\mathcal{E}_3-6)\mathcal{E}_1-\mathcal{E}_2{}^2-(-6\mathcal{E}_3 d+6d)\mathcal{E}_2-6\mathcal{E}_3{}^2d^2+12\mathcal{E}_3 d^2-6d^2+(-\mathcal{E}_2\,(6\mathcal{E}_3-6)+12\mathcal{E}_3{}^2 d$$
$$-24\mathcal{E}_3 d+12d)(d-\frac{1}{2(\mathcal{E}_3-1)}(\mathcal{E}_2+\sqrt{-\mathfrak{C}}\,(cot(\sqrt{-\mathfrak{C}}\,\mathfrak{Z})\pm csc(\sqrt{-\mathfrak{C}}\,\mathfrak{Z}))))+(-6\mathcal{E}_3{}^2$$
$$-6+12\mathcal{E}_3)\left(d-\frac{1}{2(\mathcal{E}_3-1)}(\mathcal{E}_2+\sqrt{-\mathfrak{C}}\,(cot(\sqrt{-\mathfrak{C}}\,\mathfrak{Z})\pm csc(\sqrt{-\mathfrak{C}}\,\mathfrak{Z})))\right)^2, \tag{2.87}$$

$$\mathcal{U}(\mathfrak{Z}) = -(6\mathcal{E}_3-6)\mathcal{E}_1-\mathcal{E}_2{}^2-(-6\mathcal{E}_3 d+6d)\mathcal{E}_2-6\mathcal{E}_3{}^2d^2+12\mathcal{E}_3 d^2-6d^2+(-\mathcal{E}_2\,(6\mathcal{E}_3-6)+12\mathcal{E}_3{}^2 d$$
$$-24\mathcal{E}_3 d+12d)(d+\frac{1}{4(\mathcal{E}_3-1)}(-2\mathcal{E}_2+\sqrt{-\mathfrak{C}}\,(tan(\frac{\sqrt{-\mathfrak{C}}}{4}\mathfrak{Z})-cot(\frac{\sqrt{-\mathfrak{C}}}{4}\mathfrak{Z}))))+(-6\mathcal{E}_3{}^2$$
$$-6+12\mathcal{E}_3)\left(d+\frac{1}{4(\mathcal{E}_3-1)}(-2\mathcal{E}_2+\sqrt{-\mathfrak{C}}\,(tan(\frac{\sqrt{-\mathfrak{C}}}{4}\mathfrak{Z})-cot(\frac{\sqrt{-\mathfrak{C}}}{4}\mathfrak{Z})))\right)^2, \tag{2.88}$$

$$\mathcal{U}(\mathfrak{Z}) = -(6\mathcal{E}_3-6)\mathcal{E}_1-\mathcal{E}_2{}^2-(-6\mathcal{E}_3 d+6d)\mathcal{E}_2-6\mathcal{E}_3{}^2d^2+12\mathcal{E}_3 d^2-6d^2+(-\mathcal{E}_2\,(6\mathcal{E}_3-6)+12\mathcal{E}_3{}^2 d$$
$$-24\mathcal{E}_3 d+12d)(d+\frac{1}{2(\mathcal{E}_3-1)}(-\mathcal{E}_2+\frac{\pm\sqrt{-\mathfrak{C}\,(A^2-B^2)}-A\sqrt{-\mathfrak{C}}\,cos(\sqrt{-\mathfrak{C}}\,\mathfrak{Z})}{A\,sin(\sqrt{-\mathfrak{C}}\,\mathfrak{Z})+B}))$$
$$+\left(-6\mathcal{E}_3{}^2-6+12\mathcal{E}_3\right)\left(d+\frac{1}{2(\mathcal{E}_3-1)}(-\mathcal{E}_2+\frac{\pm\sqrt{-\mathfrak{C}\,(A^2-B^2)}-A\sqrt{-\mathfrak{C}}\,cos(\sqrt{-\mathfrak{C}}\,\mathfrak{Z})}{A\,sin(\sqrt{-\mathfrak{C}}\,\mathfrak{Z})+B})\right)^2, \tag{2.89}$$

$$\mathcal{U}(\mathfrak{Z}) = -(6\mathcal{E}_3-6)\mathcal{E}_1-\mathcal{E}_2{}^2-(-6\mathcal{E}_3 d+6d)\mathcal{E}_2-6\mathcal{E}_3{}^2d^2+12\mathcal{E}_3 d^2-6d^2+(-\mathcal{E}_2\,(6\mathcal{E}_3-6)+12\mathcal{E}_3{}^2 d$$
$$-24\mathcal{E}_3 d+12d)(d-\frac{2\mathcal{E}_1\,cos(\frac{\sqrt{-\mathfrak{C}}}{2}\mathfrak{Z})}{\sqrt{-\mathfrak{C}}\,sin(\frac{\sqrt{-\mathfrak{C}}}{2}\mathfrak{Z})+\mathcal{E}_2\,cos(\frac{\sqrt{-\mathfrak{C}}}{2}\mathfrak{Z})})+\left(-6\mathcal{E}_3{}^2-6+12\mathcal{E}_3\right)$$
$$\times(d-\frac{2\mathcal{E}_1\,cos(\frac{\sqrt{-\mathfrak{C}}}{2}\mathfrak{Z})}{\sqrt{-\mathfrak{C}}\,sin(\frac{\sqrt{-\mathfrak{C}}}{2}\mathfrak{Z})+\mathcal{E}_2\,cos(\frac{\sqrt{-\mathfrak{C}}}{2}\mathfrak{Z})})^2, \tag{2.90}$$

$$
\begin{aligned}
\mathcal{U}(\mathfrak{z}) = &-(6\,\mathcal{E}_3-6)\,\mathcal{E}_1-\mathcal{E}_2{}^2-(-6\,\mathcal{E}_3\,d+6\,d)\,\mathcal{E}_2-6\,\mathcal{E}_3{}^2d^2+12\,\mathcal{E}_3\,d^2-6\,d^2+(-\mathcal{E}_2\,(6\,\mathcal{E}_3-6)+12\,\mathcal{E}_3{}^2d\\
&-24\,\mathcal{E}_3\,d+12\,d)(d+\frac{2\,\mathcal{E}_1\,sin(\frac{\sqrt{-\mathfrak{C}}}{2}\,\mathfrak{z})}{\sqrt{-\mathfrak{C}}\,cos(\sqrt{-\mathfrak{C}}2\,\mathfrak{z})-\mathcal{E}_2\,sin(\frac{\sqrt{-\mathfrak{C}}}{2}\,\mathfrak{z})})+\left(-6\,\mathcal{E}_3{}^2-6+12\,\mathcal{E}_3\right)\\
&\times\left(d+\frac{2\,\mathcal{E}_1\,sin(\frac{\sqrt{-\mathfrak{C}}}{2}\,\mathfrak{z})}{\sqrt{-\mathfrak{C}}\,cos(\frac{\sqrt{-\mathfrak{C}}}{2}\,\mathfrak{z})-\mathcal{E}_2\,sin(\frac{\sqrt{-\mathfrak{C}}}{2}\,\mathfrak{z})}\right)^2,
\end{aligned}
\tag{2.91}
$$

$$
\begin{aligned}
\mathcal{U}(\mathfrak{z}) = &-(6\,\mathcal{E}_3-6)\,\mathcal{E}_1-\mathcal{E}_2{}^2-(-6\,\mathcal{E}_3\,d+6\,d)\,\mathcal{E}_2-6\,\mathcal{E}_3{}^2d^2+12\,\mathcal{E}_3\,d^2-6\,d^2+(-\mathcal{E}_2\,(6\,\mathcal{E}_3-6)+12\,\mathcal{E}_3{}^2d\\
&-24\,\mathcal{E}_3\,d+12\,d)(d+\frac{2\,\mathcal{E}_1\,sin(\sqrt{-\mathfrak{C}}\,\mathfrak{z})}{\sqrt{-\mathfrak{C}}\,cos(\sqrt{-\mathfrak{C}}\,\mathfrak{z})-\mathcal{E}_2\,sin(\sqrt{-\mathfrak{C}}\,\mathfrak{z})\pm\sqrt{-\mathfrak{C}}})+\left(-6\,\mathcal{E}_3{}^2-6+12\,\mathcal{E}_3\right)\\
&\times(d+\frac{2\,\mathcal{E}_1\,sin(\sqrt{-\mathfrak{C}}\,\mathfrak{z})}{\sqrt{-\mathfrak{C}}\,cos(\sqrt{-\mathfrak{C}}\,\mathfrak{z})-\mathcal{E}_2\,sin(\sqrt{-\mathfrak{C}}\,\mathfrak{z})\pm\sqrt{-\mathfrak{C}}})^2,
\end{aligned}
\tag{2.92}
$$

While A, B are arbitrary real constants and $A^2-B^2>0$.
When $\mathcal{E}_1=0$ and $\mathcal{E}_2(\mathcal{E}_3-1)\neq 0$, we have:

$$
\begin{aligned}
\mathcal{U}(\mathfrak{z}) = &-(6\,\mathcal{E}_3-6)\,\mathcal{E}_1-\mathcal{E}_2{}^2-(-6\,\mathcal{E}_3\,d+6\,d)\,\mathcal{E}_2-6\,\mathcal{E}_3{}^2d^2+12\,\mathcal{E}_3\,d^2-6\,d^2+(-\mathcal{E}_2\,(6\,\mathcal{E}_3-6)+12\,\mathcal{E}_3{}^2d\\
&-24\,\mathcal{E}_3\,d+12\,d)(d-\frac{\mathcal{E}_2\,k}{(\mathcal{E}_3-1)\,(k+cosh(\mathcal{E}_2\,\mathfrak{z})-sinh(\mathcal{E}_2\,\mathfrak{z}))})+\left(-6\,\mathcal{E}_3{}^2-6+12\,\mathcal{E}_3\right)\\
&\times\left(d-\frac{\mathcal{E}_2\,k}{(\mathcal{E}_3-1)\,(k+cosh(\mathcal{E}_2\,\mathfrak{z})-sinh(\mathcal{E}_2\,\mathfrak{z}))}\right)^2,
\end{aligned}
\tag{2.93}
$$

$$
\begin{aligned}
\mathcal{U}(\mathfrak{z}) = &-(6\,\mathcal{E}_3-6)\,\mathcal{E}_1-\mathcal{E}_2{}^2-(-6\,\mathcal{E}_3\,d+6\,d)\,\mathcal{E}_2-6\,\mathcal{E}_3{}^2d^2+12\,\mathcal{E}_3\,d^2-6\,d^2+(-\mathcal{E}_2\,(6\,\mathcal{E}_3-6)+12\,\mathcal{E}_3{}^2d\\
&-24\,\mathcal{E}_3\,d+12\,d)(d-\frac{\mathcal{E}_2\,(cosh(\mathcal{E}_2\,\mathfrak{z})+sinh(\mathcal{E}_2\,\mathfrak{z}))}{(\mathcal{E}_3-1)\,(k+cosh(\mathcal{E}_2\,\mathfrak{z})+sinh(\mathcal{E}_2\,\mathfrak{z}))})+\left(-6\,\mathcal{E}_3{}^2-6+12\,\mathcal{E}_3\right)\\
&\times\left(d-\frac{\mathcal{E}_2\,(cosh(\mathcal{E}_2\,\mathfrak{z})+sinh(\mathcal{E}_2\,\mathfrak{z}))}{(\mathcal{E}_3-1)\,(k+cosh(\mathcal{E}_2\,\mathfrak{z})+sinh(\mathcal{E}_2\,\mathfrak{z}))}\right)^2,
\end{aligned}
\tag{2.94}
$$

$$
\begin{aligned}
\mathcal{U}(\mathfrak{z}) = &-(6\,\mathcal{E}_3-6)\,\mathcal{E}_1-\mathcal{E}_2{}^2-(-6\,\mathcal{E}_3\,d+6\,d)\,\mathcal{E}_2-6\,\mathcal{E}_3{}^2d^2+12\,\mathcal{E}_3\,d^2-6\,d^2+(-\mathcal{E}_2\,(6\,\mathcal{E}_3-6)+12\,\mathcal{E}_3{}^2d\\
&-24\,\mathcal{E}_3\,d+12\,d)(d-\frac{1}{(\mathcal{E}_3-1)\,\mathfrak{z}+C})+\left(-6\,\mathcal{E}_3{}^2-6+12\,\mathcal{E}_3\right)\left(d-\frac{1}{(\mathcal{E}_3-1)\,\mathfrak{z}+C}\right)^2,
\end{aligned}
\tag{2.95}
$$

Set C

$c=\sqrt{4\,\mathcal{E}_1\,\mathcal{E}_3-4\,\mathcal{E}_1+1-\mathcal{E}_2^2}$, $a_{-2}=-6\,\mathcal{E}_1{}^2-\left(-12\,\mathcal{E}_2\,d-12\,d^2+12\,\mathcal{E}_3\,d^2\right)\mathcal{E}_1-6\,\mathcal{E}_2{}^2d^2-(-12\,\mathcal{E}_3\,d^3+12\,d^3)\mathcal{E}_2-6\,d^4-6\,\mathcal{E}_3{}^2d^4+12\,\mathcal{E}_3d^4$, $a_{-1}=-(-12\,\mathcal{E}_3\,d+12\,d+6\,\mathcal{E}_2)\,\mathcal{E}_1+6\,\mathcal{E}_2{}^2d-(18\,\mathcal{E}_3\,d^2-18\,d^2)\mathcal{E}_2+12\,\mathcal{E}_3{}^2d^3-24\,\mathcal{E}_3d^3+12\,d^3$, $a_1=a_2=0$, $a_0=-(2\,\mathcal{E}_3-2)\,\mathcal{E}_1-\mathcal{E}_2{}^2-(-6\,\mathcal{E}_3\,d+6\,d)\,\mathcal{E}_2-6\,\mathcal{E}_3{}^2d^2+12\,\mathcal{E}_3\,d^2-6\,d^2$.

Thus, the model's soliton solutions are given by
When $\left(\mathfrak{C} = \mathcal{E}_2^2 - 4\,\mathcal{E}_2\,\mathcal{E}_1 + 4\,\mathcal{E}_1 > 0\right)$ **and** $(\mathcal{E}_2(\mathcal{E}_3 - 1) \neq 0)$ **or** $(\mathcal{E}_1(\mathcal{E}_3 - 1) \neq 0)$**:**

$$\begin{aligned}\mathcal{U}(\mathfrak{z}) =&(-6\,\mathcal{E}_1{}^2 - (-12\,\mathcal{E}_2\,d - 12\,d^2 + 12\,\mathcal{E}_3\,d^2)\mathcal{E}_1 - 6\,\mathcal{E}_2{}^2d^2 - (-12\,\mathcal{E}_3\,d^3 + 12\,d^3)\mathcal{E}_2 - 6\,d^4 - 6\,\mathcal{E}_3{}^2d^4\\ &+ 12\,\mathcal{E}_3d^4)(d - \frac{1}{2(\mathcal{E}_3-1)}(\mathcal{E}_2 + \sqrt{\mathfrak{C}}\,tanh(\frac{\sqrt{\mathfrak{C}}}{2}\,\mathfrak{z})))^{-2} + (-(-12\,\mathcal{E}_3\,d + 12\,d + 6\,\mathcal{E}_2)\mathcal{E}_1 + 6\,\mathcal{E}_2{}^2d\\ &- (18\,\mathcal{E}_3\,d^2 - 18\,d^2)\mathcal{E}_2 + 12\,\mathcal{E}_3{}^2d^3 - 24\,\mathcal{E}_3d^3 + 12\,d^3)\,(d - \frac{1}{2(\mathcal{E}_3-1)}(\mathcal{E}_2 + \sqrt{\mathfrak{C}}\,tanh(\frac{\sqrt{\mathfrak{C}}}{2}\,\mathfrak{z})))^{-1}\\ &- (2\,\mathcal{E}_3 - 2)\mathcal{E}_1 - \mathcal{E}_2{}^2 - (-6\,\mathcal{E}_3\,d + 6\,d)\mathcal{E}_2 - 6\,\mathcal{E}_3{}^2d^2 + 12\,\mathcal{E}_3\,d^2 - 6\,d^2,\end{aligned} \tag{2.96}$$

$$\begin{aligned}\mathcal{U}(\mathfrak{z}) =&(-6\,\mathcal{E}_1{}^2 - (-12\,\mathcal{E}_2\,d - 12\,d^2 + 12\,\mathcal{E}_3\,d^2)\mathcal{E}_1 - 6\,\mathcal{E}_2{}^2d^2 - (-12\,\mathcal{E}_3\,d^3 + 12\,d^3)\mathcal{E}_2 - 6\,d^4 - 6\,\mathcal{E}_3{}^2d^4\\ &+ 12\,\mathcal{E}_3d^4)(d - \frac{1}{2(\mathcal{E}_3-1)}(\mathcal{E}_2 + \sqrt{\mathfrak{C}}\,coth(\frac{\sqrt{\mathfrak{C}}}{2}\,\mathfrak{z})))^{-2} + (-(-12\,\mathcal{E}_3\,d + 12\,d + 6\,\mathcal{E}_2)\mathcal{E}_1 + 6\,\mathcal{E}_2{}^2d\\ &- (18\,\mathcal{E}_3\,d^2 - 18\,d^2)\mathcal{E}_2 + 12\,\mathcal{E}_3{}^2d^3 - 24\,\mathcal{E}_3d^3 + 12\,d^3)\,(d - \frac{1}{2(\mathcal{E}_3-1)}(\mathcal{E}_2 + \sqrt{\mathfrak{C}}\,coth(\frac{\sqrt{\mathfrak{C}}}{2}\,\mathfrak{z})))^{-1}\\ &- (2\,\mathcal{E}_3 - 2)\mathcal{E}_1 - \mathcal{E}_2{}^2 - (-6\,\mathcal{E}_3\,d + 6\,d)\mathcal{E}_2 - 6\,\mathcal{E}_3{}^2d^2 + 12\,\mathcal{E}_3\,d^2 - 6\,d^2,\end{aligned} \tag{2.97}$$

$$\begin{aligned}\mathcal{U}(\mathfrak{z}) =&(-6\,\mathcal{E}_1{}^2 - (-12\,\mathcal{E}_2\,d - 12\,d^2 + 12\,\mathcal{E}_3\,d^2)\mathcal{E}_1 - 6\,\mathcal{E}_2{}^2d^2 - (-12\,\mathcal{E}_3\,d^3 + 12\,d^3)\mathcal{E}_2 - 6\,d^4 - 6\,\mathcal{E}_3{}^2d^4\\ &+ 12\,\mathcal{E}_3d^4)(d - \frac{1}{2(\mathcal{E}_3-1)}(\mathcal{E}_2 + \sqrt{\mathfrak{C}}\,(tanh(\sqrt{\mathfrak{C}}\,\mathfrak{z}) \pm i\,sech(\sqrt{\mathfrak{C}}\,\mathfrak{z}))))^{-2} + (-(-12\,\mathcal{E}_3\,d + 12\,d + 6\,\mathcal{E}_2)\\ &\times \mathcal{E}_1 + 6\,\mathcal{E}_2{}^2d - (18\,\mathcal{E}_3\,d^2 - 18\,d^2)\mathcal{E}_2 + 12\,\mathcal{E}_3{}^2d^3 - 24\,\mathcal{E}_3d^3 + 12\,d^3)\,(d - \frac{1}{2(\mathcal{E}_3-1)}(\mathcal{E}_2 + \sqrt{\mathfrak{C}}\\ &\times (tanh(\sqrt{\mathfrak{C}}\,\mathfrak{z}) \pm i\,sech(\sqrt{\mathfrak{C}}\,\mathfrak{z}))))^{-1} - (2\,\mathcal{E}_3 - 2)\mathcal{E}_1 - \mathcal{E}_2{}^2 - (-6\,\mathcal{E}_3\,d + 6\,d)\mathcal{E}_2 - 6\,\mathcal{E}_3{}^2d^2\\ &+ 12\,\mathcal{E}_3\,d^2 - 6\,d^2,\end{aligned} \tag{2.98}$$

$$\begin{aligned}\mathcal{U}(\mathfrak{z}) =&(-6\,\mathcal{E}_1{}^2 - (-12\,\mathcal{E}_2\,d - 12\,d^2 + 12\,\mathcal{E}_3\,d^2)\mathcal{E}_1 - 6\,\mathcal{E}_2{}^2d^2 - (-12\,\mathcal{E}_3\,d^3 + 12\,d^3)\mathcal{E}_2 - 6\,d^4\\ &- 6\,\mathcal{E}_3{}^2d^4 + 12\,\mathcal{E}_3d^4)(d - \frac{1}{2(\mathcal{E}_3-1)}(\mathcal{E}_2 + \sqrt{\mathfrak{C}}\,(coth(\sqrt{\mathfrak{C}}\,\mathfrak{z}) \pm csch(\sqrt{\mathfrak{C}}\,\mathfrak{z}))))^{-2}\\ &+ (-(-12\,\mathcal{E}_3\,d + 12\,d + 6\,\mathcal{E}_2)\mathcal{E}_1 + 6\,\mathcal{E}_2{}^2d - (18\,\mathcal{E}_3\,d^2 - 18\,d^2)\mathcal{E}_2 + 12\,\mathcal{E}_3{}^2d^3 - 24\,\mathcal{E}_3d^3\\ &+ 12\,d^3)(d - \frac{1}{2(\mathcal{E}_3-1)}(\mathcal{E}_2 + \sqrt{\mathfrak{C}}(coth(\sqrt{\mathfrak{C}}\,\mathfrak{z}) \pm csch(\sqrt{\mathfrak{C}}\,\mathfrak{z}))))^{-1} - (2\,\mathcal{E}_3 - 2)\mathcal{E}_1 - \mathcal{E}_2{}^2\\ &- (-6\,\mathcal{E}_3\,d + 6\,d)\mathcal{E}_2 - 6\,\mathcal{E}_3{}^2d^2 + 12\,\mathcal{E}_3\,d^2 - 6\,d^2,\end{aligned} \tag{2.99}$$

$$\begin{aligned}\mathcal{U}(\mathfrak{z}) =&(-6\,\mathcal{E}_1{}^2 - (-12\,\mathcal{E}_2\,d - 12\,d^2 + 12\,\mathcal{E}_3\,d^2)\mathcal{E}_1 - 6\,\mathcal{E}_2{}^2d^2 - (-12\,\mathcal{E}_3\,d^3 + 12\,d^3)\mathcal{E}_2 - 6\,d^4\\ &- 6\,\mathcal{E}_3{}^2d^4 + 12\,\mathcal{E}_3d^4)(d - \frac{1}{4(\mathcal{E}_3-1)}(2\,\mathcal{E}_2 + \sqrt{\mathfrak{C}}\,(tanh(\frac{\sqrt{\mathfrak{C}}}{4}\,\mathfrak{z}) \pm coth(\frac{\sqrt{\mathfrak{C}}}{4}\,\mathfrak{z}))))^{-2}\\ &+ (-(-12\,\mathcal{E}_3\,d + 12\,d + 6\,\mathcal{E}_2)\mathcal{E}_1 + 6\,\mathcal{E}_2{}^2d - (18\,\mathcal{E}_3\,d^2 - 18\,d^2)\mathcal{E}_2 + 12\,\mathcal{E}_3{}^2d^3\\ &- 24\,\mathcal{E}_3d^3 + 12\,d^3)\,(d - \frac{1}{4(\mathcal{E}_3-1)}(2\,\mathcal{E}_2 + \sqrt{\mathfrak{C}}(tanh(\frac{\sqrt{\mathfrak{C}}}{4}\,\mathfrak{z}) \pm coth(\frac{\sqrt{\mathfrak{C}}}{4}\,\mathfrak{z}))))^{-1}\\ &- (2\,\mathcal{E}_3 - 2)\mathcal{E}_1 - \mathcal{E}_2{}^2 - (-6\,\mathcal{E}_3\,d + 6\,d)\mathcal{E}_2 - 6\,\mathcal{E}_3{}^2d^2 + 12\,\mathcal{E}_3\,d^2 - 6\,d^2,\end{aligned} \tag{2.100}$$

$$\begin{aligned}\mathcal{U}(\mathfrak{z}) =&(-6\,\mathcal{E}_1{}^2-(-12\,\mathcal{E}_2\,d-12\,d^2+12\,\mathcal{E}_3\,d^2)\mathcal{E}_1-6\,\mathcal{E}_2{}^2d^2-(-12\,\mathcal{E}_3\,d^3+12\,d^3)\mathcal{E}_2-6\,d^4\\&-6\,\mathcal{E}_3{}^2d^4+12\,\mathcal{E}_3d^4)(d+\frac{1}{2(\mathcal{E}_3-1)}(-\mathcal{E}_2+\frac{\pm\sqrt{\mathfrak{C}\,(A^2+B^2)}-A\,\sqrt{\mathfrak{C}}\,cosh(\sqrt{\mathfrak{C}}\,\mathfrak{z})}{A\,sinh(\sqrt{\mathfrak{C}}\,\mathfrak{z})+B}))^{-2}\\&+(-(-12\,\mathcal{E}_3\,d+12\,d+6\,\mathcal{E}_2)\mathcal{E}_1+6\,\mathcal{E}_2{}^2d-(18\,\mathcal{E}_3\,d^2-18\,d^2)\mathcal{E}_2+12\,\mathcal{E}_3{}^2d^3\\&-24\,\mathcal{E}_3d^3+12\,d^3)(d+\frac{1}{2(\mathcal{E}_3-1)}(-\mathcal{E}_2+\frac{\pm\sqrt{\mathfrak{C}\,(A^2+B^2)}-A\,\sqrt{\mathfrak{C}}\,cosh(\sqrt{\mathfrak{C}}\,\mathfrak{z})}{A\,sinh(\sqrt{\mathfrak{C}}\,\mathfrak{z})+B}))^{-1}\\&-(2\,\mathcal{E}_3-2)\mathcal{E}_1-\mathcal{E}_2{}^2-(-6\,\mathcal{E}_3\,d+6\,d)\mathcal{E}_2-6\,\mathcal{E}_3{}^2d^2+12\,\mathcal{E}_3\,d^2-6\,d^2,\end{aligned} \tag{2.101}$$

$$\begin{aligned}\mathcal{U}(\mathfrak{z}) =&(-6\,\mathcal{E}_1{}^2-(-12\,\mathcal{E}_2\,d-12\,d^2+12\,\mathcal{E}_3\,d^2)\mathcal{E}_1-6\,\mathcal{E}_2{}^2d^2-(-12\,\mathcal{E}_3\,d^3+12\,d^3)\mathcal{E}_2-6\,d^4\\&-6\,\mathcal{E}_3{}^2d^4+12\,\mathcal{E}_3d^4)(d+\frac{1}{2(\mathcal{E}_3-1)}(-\mathcal{E}_2+\frac{\pm\sqrt{\mathfrak{C}\,(A^2+B^2)}+A\,\sqrt{\mathfrak{C}}\,cosh(\sqrt{\mathfrak{C}}\,\mathfrak{z})}{A\,sinh(\sqrt{\mathfrak{C}}\,\mathfrak{z})+B}))^{-2}\\&+(-(-12\,\mathcal{E}_3\,d+12\,d+6\,\mathcal{E}_2)\mathcal{E}_1+6\,\mathcal{E}_2{}^2d-(18\,\mathcal{E}_3\,d^2-18\,d^2)\mathcal{E}_2+12\,\mathcal{E}_3{}^2d^3\\&-24\,\mathcal{E}_3d^3+12\,d^3)\,(d+\frac{1}{2(\mathcal{E}_3-1)}(-\mathcal{E}_2+\frac{\pm\sqrt{\mathfrak{C}\,(A^2+B^2)}+A\,\sqrt{\mathfrak{C}}\,cosh(\sqrt{\mathfrak{C}}\,\mathfrak{z})}{A\,sinh(\sqrt{\mathfrak{C}}\,\mathfrak{z})+B}))^{-1}\\&-(2\,\mathcal{E}_3-2)\mathcal{E}_1-\mathcal{E}_2{}^2-(-6\,\mathcal{E}_3\,d+6\,d)\mathcal{E}_2-6\,\mathcal{E}_3{}^2d^2+12\,\mathcal{E}_3\,d^2-6\,d^2,\end{aligned} \tag{2.102}$$

$$\begin{aligned}\mathcal{U}(\mathfrak{z}) =&(-6\,\mathcal{E}_1{}^2-(-12\,\mathcal{E}_2\,d-12\,d^2+12\,\mathcal{E}_3\,d^2)\mathcal{E}_1-6\,\mathcal{E}_2{}^2d^2-(-12\,\mathcal{E}_3\,d^3+12\,d^3)\mathcal{E}_2-6\,d^4-6\,\mathcal{E}_3{}^2d^4\\&+12\,\mathcal{E}_3d^4)(d+\frac{2\,\mathcal{E}_1\,cosh(\frac{\sqrt{\mathfrak{C}}}{2}\,\mathfrak{z})}{\sqrt{\mathfrak{C}}\,sinh(\frac{\sqrt{\mathfrak{C}}}{2}\,\mathfrak{z})-\mathcal{E}_2\,cosh(\frac{\sqrt{\mathfrak{C}}}{2}\,\mathfrak{z})})^{-2}+(-(-12\,\mathcal{E}_3\,d+12\,d+6\,\mathcal{E}_2)\mathcal{E}_1+6\,\mathcal{E}_2{}^2d\\&-(18\,\mathcal{E}_3\,d^2-18\,d^2)\mathcal{E}_2+12\,\mathcal{E}_3{}^2d^3-24\,\mathcal{E}_3d^3+12\,d^3)\,(d+\frac{2\,\mathcal{E}_1\,cosh(\frac{\sqrt{\mathfrak{C}}}{2}\,\mathfrak{z})}{\sqrt{\mathfrak{C}}\,sinh(\frac{\sqrt{\mathfrak{C}}}{2}\,\mathfrak{z})-\mathcal{E}_2\,cosh(\frac{\sqrt{\mathfrak{C}}}{2}\,\mathfrak{z})})^{-1}\\&-(2\,\mathcal{E}_3-2)\mathcal{E}_1-\mathcal{E}_2{}^2-(-6\,\mathcal{E}_3\,d+6\,d)\mathcal{E}_2-6\,\mathcal{E}_3{}^2d^2+12\,\mathcal{E}_3\,d^2-6\,d^2,\end{aligned} \tag{2.103}$$

$$\begin{aligned}\mathcal{U}(\mathfrak{z}) =&(-6\,\mathcal{E}_1{}^2-(-12\,\mathcal{E}_2\,d-12\,d^2+12\,\mathcal{E}_3\,d^2)\mathcal{E}_1-6\,\mathcal{E}_2{}^2d^2-(-12\,\mathcal{E}_3\,d^3+12\,d^3)\mathcal{E}_2-6\,d^4-6\,\mathcal{E}_3{}^2d^4\\&+12\,\mathcal{E}_3d^4)(d+\frac{2\,\mathcal{E}_1\,sinh(\frac{\sqrt{\mathfrak{C}}}{2}\,\mathfrak{z})}{\sqrt{\mathfrak{C}}\,cosh(\frac{\sqrt{\mathfrak{C}}}{2}\,\mathfrak{z})-\mathcal{E}_2\,sinh(\frac{\sqrt{\mathfrak{C}}}{2}\,\mathfrak{z})})^{-2}+(-(-12\,\mathcal{E}_3\,d+12\,d+6\,\mathcal{E}_2)\mathcal{E}_1+6\,\mathcal{E}_2{}^2d\\&-(18\,\mathcal{E}_3\,d^2-18\,d^2)\mathcal{E}_2+12\,\mathcal{E}_3{}^2d^3-24\,\mathcal{E}_3d^3+12\,d^3)\,(d+\frac{2\,\mathcal{E}_1\,sinh(\frac{\sqrt{\mathfrak{C}}}{2}\,\mathfrak{z})}{\sqrt{\mathfrak{C}}\,cosh(\frac{\sqrt{\mathfrak{C}}}{2}\,\mathfrak{z})-\mathcal{E}_2\,sinh(\frac{\sqrt{\mathfrak{C}}}{2}\,\mathfrak{z})})^{-1}\\&-(2\,\mathcal{E}_3-2)\mathcal{E}_1-\mathcal{E}_2{}^2-(-6\,\mathcal{E}_3\,d+6\,d)\mathcal{E}_2-6\,\mathcal{E}_3{}^2d^2+12\,\mathcal{E}_3\,d^2-6\,d^2,\end{aligned} \tag{2.104}$$

$$
\begin{aligned}
\mathcal{U}(\mathfrak{Z}) =&(-6\,\mathcal{E}_1{}^2 - (-12\,\mathcal{E}_2\,d - 12\,d^2 + 12\,\mathcal{E}_3\,d^2)\mathcal{E}_1 - 6\,\mathcal{E}_2{}^2 d^2 - (-12\,\mathcal{E}_3\,d^3 + 12\,d^3)\mathcal{E}_2 - 6\,d^4 - 6\,\mathcal{E}_3{}^2 d^4 \\
&+ 12\,\mathcal{E}_3 d^4)(d + \frac{2\,\mathcal{E}_1\,cosh(\sqrt{\mathfrak{C}}\,\mathfrak{Z})}{\sqrt{\mathfrak{C}}\,sinh(\sqrt{\mathfrak{C}}\,\mathfrak{Z}) - \mathcal{E}_2\,cosh(\sqrt{\mathfrak{C}}\,\mathfrak{Z}) \pm i\,\sqrt{\mathfrak{C}}})^{-2} + (-(-12\,\mathcal{E}_3\,d + 12\,d \\
&+ 6\,\mathcal{E}_2)\mathcal{E}_1 + 6\,\mathcal{E}_2{}^2 d - (18\,\mathcal{E}_3\,d^2 - 18\,d^2)\mathcal{E}_2 + 12\,\mathcal{E}_3{}^2 d^3 - 24\,\mathcal{E}_3 d^3 + 12\,d^3) \\
&\times (d + \frac{2\,\mathcal{E}_1\,cosh(\sqrt{\mathfrak{C}}\,\mathfrak{Z})}{\sqrt{\mathfrak{C}}\,sinh(\sqrt{\mathfrak{C}}\,\mathfrak{Z}) - \mathcal{E}_2\,cosh(\sqrt{\mathfrak{C}}\,\mathfrak{Z}) \pm i\,\sqrt{\mathfrak{C}}})^{-1} - (2\,\mathcal{E}_3 - 2)\mathcal{E}_1 - \mathcal{E}_2{}^2 - (-6\,\mathcal{E}_3\,d \\
&+ 6\,d)\mathcal{E}_2 - 6\,\mathcal{E}_3{}^2 d^2 + 12\,\mathcal{E}_3\,d^2 - 6\,d^2,
\end{aligned}
\tag{2.105}
$$

$$
\begin{aligned}
\mathcal{U}(\mathfrak{Z}) =&(-6\,\mathcal{E}_1{}^2 - (-12\,\mathcal{E}_2\,d - 12\,d^2 + 12\,\mathcal{E}_3\,d^2)\mathcal{E}_1 - 6\,\mathcal{E}_2{}^2 d^2 - (-12\,\mathcal{E}_3\,d^3 + 12\,d^3)\mathcal{E}_2 - 6\,d^4 - 6\,\mathcal{E}_3{}^2 d^4 \\
&+ 12\,\mathcal{E}_3 d^4)(d + \frac{2\,\mathcal{E}_1\,sinh(\sqrt{\mathfrak{C}}\,\mathfrak{Z})}{\sqrt{\mathfrak{C}}\,cosh(\sqrt{\mathfrak{C}}\,\mathfrak{Z}) - \mathcal{E}_2\,sinh(\sqrt{\mathfrak{C}}\,\mathfrak{Z}) \pm i\,\sqrt{\mathfrak{C}}})^{-2} + (-(-12\,\mathcal{E}_3\,d + 12\,d + 6\,\mathcal{E}_2)\mathcal{E}_1 \\
&+ 6\,\mathcal{E}_2{}^2 d - (18\,\mathcal{E}_3\,d^2 - 18\,d^2)\mathcal{E}_2 + 12\,\mathcal{E}_3{}^2 d^3 - 24\,\mathcal{E}_3 d^3 + 12\,d^3)\,(d+ \\
&\frac{2\,\mathcal{E}_1\,sinh(\sqrt{\mathfrak{C}}\,\mathfrak{Z})}{\sqrt{\mathfrak{C}}\,cosh(\sqrt{\mathfrak{C}}\,\mathfrak{Z}) - \mathcal{E}_2\,sinh(\sqrt{\mathfrak{C}}\,\mathfrak{Z}) \pm i\,\sqrt{\mathfrak{C}}})^{-1} - (2\,\mathcal{E}_3 - 2)\mathcal{E}_1 - \mathcal{E}_2{}^2 - (-6\,\mathcal{E}_3\,d + 6\,d)\mathcal{E}_2 \\
&- 6\,\mathcal{E}_3{}^2 d^2 + 12\,\mathcal{E}_3\,d^2 - 6\,d^2,
\end{aligned}
\tag{2.106}
$$

While A, B are arbitrary real constants and $A^2 + B^2 > 0$.

When $\left(\mathfrak{C} = \mathcal{E}_2^2 - 4\,\mathcal{E}_2\,\mathcal{E}_1 + 4\,\mathcal{E}_1 < 0\right)$ **and** $(\mathcal{E}_2(\mathcal{E}_3 - 1) \neq 0)$ **or** $(\mathcal{E}_1(\mathcal{E}_3 - 1) \neq 0)$**:**

$$
\begin{aligned}
\mathcal{U}(\mathfrak{Z}) =&(-6\,\mathcal{E}_1{}^2 - (-12\,\mathcal{E}_2\,d - 12\,d^2 + 12\,\mathcal{E}_3\,d^2)\mathcal{E}_1 - 6\,\mathcal{E}_2{}^2 d^2 - (-12\,\mathcal{E}_3\,d^3 + 12\,d^3)\mathcal{E}_2 - 6\,d^4 - 6\,\mathcal{E}_3{}^2 d^4 \\
&+ 12\,\mathcal{E}_3 d^4)(d + \frac{1}{2(\mathcal{E}_3 - 1)}(-\mathcal{E}_2 + \sqrt{-\mathfrak{C}}\,tanh(\frac{\sqrt{-\mathfrak{C}}}{2}\,\mathfrak{Z})))^{-2} + (-(-12\,\mathcal{E}_3\,d + 12\,d + 6\,\mathcal{E}_2)\mathcal{E}_1 \\
&+ 6\,\mathcal{E}_2{}^2 d - (18\,\mathcal{E}_3\,d^2 - 18\,d^2)\mathcal{E}_2 + 12\,\mathcal{E}_3{}^2 d^3 - 24\,\mathcal{E}_3 d^3 + 12\,d^3)\,(d + \frac{1}{2(\mathcal{E}_3 - 1)}(-\mathcal{E}_2 + \sqrt{-\mathfrak{C}} \\
&\times tanh(\frac{\sqrt{-\mathfrak{C}}}{2}\,\mathfrak{Z})))^{-1} - (2\,\mathcal{E}_3 - 2)\mathcal{E}_1 - \mathcal{E}_2{}^2 - (-6\,\mathcal{E}_3\,d + 6\,d)\mathcal{E}_2 - 6\,\mathcal{E}_3{}^2 d^2 + 12\,\mathcal{E}_3\,d^2 - 6\,d^2,
\end{aligned}
\tag{2.107}
$$

$$
\begin{aligned}
\mathcal{U}(\mathfrak{Z}) =&(-6\,\mathcal{E}_1{}^2 - (-12\,\mathcal{E}_2\,d - 12\,d^2 + 12\,\mathcal{E}_3\,d^2)\mathcal{E}_1 - 6\,\mathcal{E}_2{}^2 d^2 - (-12\,\mathcal{E}_3\,d^3 + 12\,d^3)\mathcal{E}_2 - 6\,d^4 - 6\,\mathcal{E}_3{}^2 d^4 \\
&+ 12\,\mathcal{E}_3 d^4)(d - \frac{1}{2(\mathcal{E}_3 - 1)}(\mathcal{E}_2 + \sqrt{-\mathfrak{C}}\,coth(\frac{\sqrt{-\mathfrak{C}}}{2}\,\mathfrak{Z})))^{-2} + (-(-12\,\mathcal{E}_3\,d + 12\,d + 6\,\mathcal{E}_2)\mathcal{E}_1 + 6\,\mathcal{E}_2{}^2 d \\
&- (18\,\mathcal{E}_3\,d^2 - 18\,d^2)\mathcal{E}_2 + 12\,\mathcal{E}_3{}^2 d^3 - 24\,\mathcal{E}_3 d^3 + 12\,d^3)\,(d - \frac{1}{2(\mathcal{E}_3 - 1)}(\mathcal{E}_2 + \sqrt{-\mathfrak{C}}\,coth(\frac{\sqrt{-\mathfrak{C}}}{2}\,\mathfrak{Z})))^{-1} \\
&- (2\,\mathcal{E}_3 - 2)\mathcal{E}_1 - \mathcal{E}_2{}^2 - (-6\,\mathcal{E}_3\,d + 6\,d)\mathcal{E}_2 - 6\,\mathcal{E}_3{}^2 d^2 + 12\,\mathcal{E}_3\,d^2 - 6\,d^2,
\end{aligned}
\tag{2.108}
$$

$$
\begin{aligned}
\mathcal{U}(\mathfrak{Z}) =&(-6\,\mathcal{E}_1{}^2 - (-12\,\mathcal{E}_2\,d - 12\,d^2 + 12\,\mathcal{E}_3\,d^2)\mathcal{E}_1 - 6\,\mathcal{E}_2{}^2 d^2 - (-12\,\mathcal{E}_3\,d^3 + 12\,d^3)\mathcal{E}_2 - 6\,d^4 - 6\,\mathcal{E}_3{}^2 d^4 \\
&+ 12\,\mathcal{E}_3 d^4)(d + \frac{1}{2(\mathcal{E}_3 - 1)}(-\mathcal{E}_2 + \sqrt{-\mathfrak{C}}\,(tan(\sqrt{-\mathfrak{C}}\,\mathfrak{Z}) \pm\, sec(\sqrt{-\mathfrak{C}}\,\mathfrak{Z}))))^{-2} + (-(-12\,\mathcal{E}_3\,d \\
&+ 12\,d + 6\,\mathcal{E}_2)\mathcal{E}_1 + 6\,\mathcal{E}_2{}^2 d - (18\,\mathcal{E}_3\,d^2 - 18\,d^2)\mathcal{E}_2 + 12\,\mathcal{E}_3{}^2 d^3 - 24\,\mathcal{E}_3 d^3 + 12\,d^3)\,(d + \frac{1}{2(\mathcal{E}_3 - 1)} \\
&\times (-\mathcal{E}_2 + \sqrt{-\mathfrak{C}}\,(tan(\sqrt{-\mathfrak{C}}\,\mathfrak{Z}) \pm\, sec(\sqrt{-\mathfrak{C}}\,\mathfrak{Z}))))^{-1} - (2\,\mathcal{E}_3 - 2)\mathcal{E}_1 - \mathcal{E}_2{}^2 \\
&- (-6\,\mathcal{E}_3\,d + 6\,d)\mathcal{E}_2 - 6\,\mathcal{E}_3{}^2 d^2 + 12\,\mathcal{E}_3\,d^2 - 6\,d^2,
\end{aligned}
\tag{2.109}
$$

$$\begin{aligned}\mathcal{U}(\mathfrak{z}) =&(-6\,\mathcal{E}_1{}^2 - (-12\,\mathcal{E}_2\,d - 12\,d^2 + 12\,\mathcal{E}_3\,d^2)\mathcal{E}_1 - 6\,\mathcal{E}_2{}^2 d^2 - (-12\,\mathcal{E}_3\,d^3 + 12\,d^3)\mathcal{E}_2 - 6\,d^4 - 6\,\mathcal{E}_3{}^2 d^4\\ &+ 12\,\mathcal{E}_3 d^4)(d - \frac{1}{2(\mathcal{E}_3 - 1)}(\mathcal{E}_2 + \sqrt{-\mathfrak{C}}\,(cot(\sqrt{-\mathfrak{C}}\,\mathfrak{z}) \pm csc(\sqrt{-\mathfrak{C}}\,\mathfrak{z}))))^{-2} + (-(-12\,\mathcal{E}_3\,d + 12\,d\\ &+ 6\,\mathcal{E}_2)\mathcal{E}_1 + 6\,\mathcal{E}_2{}^2 d - (18\,\mathcal{E}_3\,d^2 - 18\,d^2)\mathcal{E}_2 + 12\,\mathcal{E}_3{}^2 d^3 - 24\,\mathcal{E}_3 d^3 + 12\,d^3)\,(d - \frac{1}{2(\mathcal{E}_3 - 1)}(\mathcal{E}_2\\ &+ \sqrt{-\mathfrak{C}}\,(cot(\sqrt{-\mathfrak{C}}\,\mathfrak{z}) \pm csc(\sqrt{-\mathfrak{C}}\,\mathfrak{z}))))^{-1} - (2\,\mathcal{E}_3 - 2)\mathcal{E}_1 - \mathcal{E}_2{}^2 - (-6\,\mathcal{E}_3\,d + 6\,d)\mathcal{E}_2\\ &- 6\,\mathcal{E}_3{}^2 d^2 + 12\,\mathcal{E}_3\,d^2 - 6\,d^2,\end{aligned} \tag{2.110}$$

$$\begin{aligned}\mathcal{U}(\mathfrak{z}) =&(-6\,\mathcal{E}_1{}^2 - (-12\,\mathcal{E}_2\,d - 12\,d^2 + 12\,\mathcal{E}_3\,d^2)\mathcal{E}_1 - 6\,\mathcal{E}_2{}^2 d^2 - (-12\,\mathcal{E}_3\,d^3 + 12\,d^3)\mathcal{E}_2 - 6\,d^4 - 6\,\mathcal{E}_3{}^2 d^4\\ &+ 12\,\mathcal{E}_3 d^4)(d + \frac{1}{4(\mathcal{E}_3 - 1)}(-2\,\mathcal{E}_2 + \sqrt{-\mathfrak{C}}\,(tan(\frac{\sqrt{-\mathfrak{C}}}{4}\,\mathfrak{z}) - cot(\frac{\sqrt{-\mathfrak{C}}}{4}\,\mathfrak{z}))))^{-2} + (-(-12\,\mathcal{E}_3\,d\\ &+ 12\,d + 6\,\mathcal{E}_2)\mathcal{E}_1 + 6\,\mathcal{E}_2{}^2 d - (18\,\mathcal{E}_3\,d^2 - 18\,d^2)\mathcal{E}_2 + 12\,\mathcal{E}_3{}^2 d^3 - 24\,\mathcal{E}_3 d^3 + 12\,d^3)\,(d + \frac{1}{4(\mathcal{E}_3 - 1)}\\ &\times (-2\,\mathcal{E}_2 + \sqrt{-\mathfrak{C}}\,(tan(\frac{\sqrt{-\mathfrak{C}}}{4}\,\mathfrak{z}) - cot(\frac{\sqrt{-\mathfrak{C}}}{4}\,\mathfrak{z}))))^{-1} - (2\,\mathcal{E}_3 - 2)\mathcal{E}_1 - \mathcal{E}_2{}^2\\ &- (-6\,\mathcal{E}_3\,d + 6\,d)\mathcal{E}_2 - 6\,\mathcal{E}_3{}^2 d^2 + 12\,\mathcal{E}_3\,d^2 - 6\,d^2,\end{aligned} \tag{2.111}$$

$$\begin{aligned}\mathcal{U}(\mathfrak{z}) =&(-6\,\mathcal{E}_1{}^2 - (-12\,\mathcal{E}_2\,d - 12\,d^2 + 12\,\mathcal{E}_3\,d^2)\mathcal{E}_1 - 6\,\mathcal{E}_2{}^2 d^2 - (-12\,\mathcal{E}_3\,d^3 + 12\,d^3)\mathcal{E}_2 - 6\,d^4 - 6\,\mathcal{E}_3{}^2 d^4\\ &+ 12\,\mathcal{E}_3 d^4)(d + \frac{1}{2(\mathcal{E}_3 - 1)}(-\mathcal{E}_2 + \frac{\pm\sqrt{-\mathfrak{C}\,(A^2 - B^2)} - A\,\sqrt{-\mathfrak{C}}\,cos(\sqrt{-\mathfrak{C}}\,\mathfrak{z})}{A\,sin(\sqrt{-\mathfrak{C}}\,\mathfrak{z}) + B}))^{-2}\\ &+ (-(-12\,\mathcal{E}_3\,d + 12\,d + 6\,\mathcal{E}_2)\mathcal{E}_1 + 6\,\mathcal{E}_2{}^2 d - (18\,\mathcal{E}_3\,d^2 - 18\,d^2)\mathcal{E}_2 + 12\,\mathcal{E}_3{}^2 d^3 - 24\,\mathcal{E}_3 d^3 + 12\,d^3)\\ &\times (d + \frac{1}{2(\mathcal{E}_3 - 1)}(-\mathcal{E}_2 + \frac{\pm\sqrt{-\mathfrak{C}\,(A^2 - B^2)} - A\,\sqrt{-\mathfrak{C}}\,cos(\sqrt{-\mathfrak{C}}\,\mathfrak{z})}{A\,sin(\sqrt{-\mathfrak{C}}\,\mathfrak{z}) + B}))^{-1} - (2\,\mathcal{E}_3 - 2)\mathcal{E}_1 - \mathcal{E}_2{}^2\\ &- (-6\,\mathcal{E}_3\,d + 6\,d)\mathcal{E}_2 - 6\,\mathcal{E}_3{}^2 d^2 + 12\,\mathcal{E}_3\,d^2 - 6\,d^2,\end{aligned} \tag{2.112}$$

$$\begin{aligned}\mathcal{U}(\mathfrak{z}) =&(-6\,\mathcal{E}_1{}^2 - (-12\,\mathcal{E}_2\,d - 12\,d^2 + 12\,\mathcal{E}_3\,d^2)\mathcal{E}_1 - 6\,\mathcal{E}_2{}^2 d^2 - (-12\,\mathcal{E}_3\,d^3 + 12\,d^3)\mathcal{E}_2 - 6\,d^4 - 6\,\mathcal{E}_3{}^2 d^4\\ &+ 12\,\mathcal{E}_3 d^4)(d + \frac{1}{2(\mathcal{E}_3 - 1)}(-\mathcal{E}_2 + \frac{\pm\sqrt{-\mathfrak{C}\,(A^2 - B^2)} + A\,\sqrt{-\mathfrak{C}}\,cos(\sqrt{-\mathfrak{C}}\,\mathfrak{z})}{A\,sin(\sqrt{-\mathfrak{C}}\,\mathfrak{z}) + B}))^{-2}\\ &+ (-(-12\,\mathcal{E}_3\,d + 12\,d + 6\,\mathcal{E}_2)\mathcal{E}_1 + 6\,\mathcal{E}_2{}^2 d - (18\,\mathcal{E}_3\,d^2 - 18\,d^2)\mathcal{E}_2 + 12\,\mathcal{E}_3{}^2 d^3 - 24\,\mathcal{E}_3 d^3 + 12\,d^3)\\ &\times (d + \frac{1}{2(\mathcal{E}_3 - 1)}(-\mathcal{E}_2 + \frac{\pm\sqrt{-\mathfrak{C}\,(A^2 - B^2)} + A\,\sqrt{-\mathfrak{C}}\,cos(\sqrt{-\mathfrak{C}}\,\mathfrak{z})}{A\,sin(\sqrt{-\mathfrak{C}}\,\mathfrak{z}) + B}))^{-1} - (2\,\mathcal{E}_3 - 2)\mathcal{E}_1 - \mathcal{E}_2{}^2\\ &- (-6\,\mathcal{E}_3\,d + 6\,d)\mathcal{E}_2 - 6\,\mathcal{E}_3{}^2 d^2 + 12\,\mathcal{E}_3\,d^2 - 6\,d^2,\end{aligned} \tag{2.113}$$

$$\begin{aligned}\mathcal{U}(\mathfrak{Z}) =&(-6\,\mathcal{E}_1{}^2 - (-12\,\mathcal{E}_2\,d - 12\,d^2 + 12\,\mathcal{E}_3\,d^2)\mathcal{E}_1 - 6\,\mathcal{E}_2{}^2 d^2 - (-12\,\mathcal{E}_3\,d^3 + 12\,d^3)\mathcal{E}_2 - 6\,d^4 - 6\,\mathcal{E}_3{}^2 d^4\\ &+ 12\,\mathcal{E}_3 d^4)(d - \frac{2\,\mathcal{E}_1\,cos(\frac{\sqrt{-\mathfrak{C}}}{2}\,\mathfrak{Z})}{\sqrt{-\mathfrak{C}}\,sin(\frac{\sqrt{-\mathfrak{C}}}{2}\,\mathfrak{Z}) + \mathcal{E}_2\,cos(\frac{\sqrt{-\mathfrak{C}}}{2}\,\mathfrak{Z})})^{-2} + (-(-12\,\mathcal{E}_3\,d + 12\,d + 6\,\mathcal{E}_2)\mathcal{E}_1 + 6\,\mathcal{E}_2{}^2 d\\ &- (18\,\mathcal{E}_3\,d^2 - 18\,d^2)\mathcal{E}_2 + 12\,\mathcal{E}_3{}^2 d^3 - 24\,\mathcal{E}_3 d^3 + 12\,d^3)\,(d - \frac{2\,\mathcal{E}_1\,cos(\frac{\sqrt{-\mathfrak{C}}}{2}\,\mathfrak{Z})}{\sqrt{-\mathfrak{C}}\,sin(\frac{\sqrt{-\mathfrak{C}}}{2}\,\mathfrak{Z}) + \mathcal{E}_2\,cos(\frac{\sqrt{-\mathfrak{C}}}{2}\,\mathfrak{Z})})^{-1}\\ &- (2\,\mathcal{E}_3 - 2)\mathcal{E}_1 - \mathcal{E}_2{}^2 - (-6\,\mathcal{E}_3\,d + 6\,d)\mathcal{E}_2 - 6\,\mathcal{E}_3{}^2 d^2 + 12\,\mathcal{E}_3\,d^2 - 6\,d^2,\end{aligned} \tag{2.114}$$

$$\begin{aligned}\mathcal{U}(\mathfrak{Z}) =&(-6\,\mathcal{E}_1{}^2 - (-12\,\mathcal{E}_2\,d - 12\,d^2 + 12\,\mathcal{E}_3\,d^2)\mathcal{E}_1 - 6\,\mathcal{E}_2{}^2 d^2 - (-12\,\mathcal{E}_3\,d^3 + 12\,d^3)\mathcal{E}_2 - 6\,d^4 - 6\,\mathcal{E}_3{}^2 d^4\\ &+ 12\,\mathcal{E}_3 d^4)(d + \frac{2\,\mathcal{E}_1\,sin(\frac{\sqrt{-\mathfrak{C}}}{2}\,\mathfrak{Z})}{\sqrt{-\mathfrak{C}}\,cos(\sqrt{-\mathfrak{C}}2\,\mathfrak{Z}) - \mathcal{E}_2\,sin(\frac{\sqrt{-\mathfrak{C}}}{2}\,\mathfrak{Z})})^{-2} + (-(-12\,\mathcal{E}_3\,d + 12\,d + 6\,\mathcal{E}_2)\mathcal{E}_1 + 6\,\mathcal{E}_2{}^2 d\\ &- (18\,\mathcal{E}_3\,d^2 - 18\,d^2)\mathcal{E}_2 + 12\,\mathcal{E}_3{}^2 d^3 - 24\,\mathcal{E}_3 d^3 + 12\,d^3)\,(d + \frac{2\,\mathcal{E}_1\,sin(\frac{\sqrt{-\mathfrak{C}}}{2}\,\mathfrak{Z})}{\sqrt{-\mathfrak{C}}\,cos(\sqrt{-\mathfrak{C}}2\,\mathfrak{Z}) - \mathcal{E}_2\,sin(\frac{\sqrt{-\mathfrak{C}}}{2}\,\mathfrak{Z})})^{-1}\\ &- (2\,\mathcal{E}_3 - 2)\mathcal{E}_1 - \mathcal{E}_2{}^2 - (-6\,\mathcal{E}_3\,d + 6\,d)\mathcal{E}_2 - 6\,\mathcal{E}_3{}^2 d^2 + 12\,\mathcal{E}_3\,d^2 - 6\,d^2,\end{aligned} \tag{2.115}$$

$$\begin{aligned}\mathcal{U}(\mathfrak{Z}) =&(-6\,\mathcal{E}_1{}^2 - (-12\,\mathcal{E}_2\,d - 12\,d^2 + 12\,\mathcal{E}_3\,d^2)\mathcal{E}_1 - 6\,\mathcal{E}_2{}^2 d^2 - (-12\,\mathcal{E}_3\,d^3 + 12\,d^3)\mathcal{E}_2 - 6\,d^4 - 6\,\mathcal{E}_3{}^2 d^4\\ &+ 12\,\mathcal{E}_3 d^4)(d + \frac{2\,\mathcal{E}_1\,sin(\sqrt{-\mathfrak{C}}\,\mathfrak{Z})}{\sqrt{-\mathfrak{C}}\,cos(\sqrt{-\mathfrak{C}}\,\mathfrak{Z}) - \mathcal{E}_2\,sin(\sqrt{-\mathfrak{C}}\,\mathfrak{Z}) \pm \sqrt{-\mathfrak{C}}})^{-2} + (-(-12\,\mathcal{E}_3\,d + 12\,d + 6\,\mathcal{E}_2)\mathcal{E}_1\\ &+ 6\,\mathcal{E}_2{}^2 d - (18\,\mathcal{E}_3\,d^2 - 18\,d^2)\mathcal{E}_2 + 12\,\mathcal{E}_3{}^2 d^3 - 24\,\mathcal{E}_3 d^3 + 12\,d^3)\,(d\\ &+ \frac{2\,\mathcal{E}_1\,sin(\sqrt{-\mathfrak{C}}\,\mathfrak{Z})}{\sqrt{-\mathfrak{C}}\,cos(\sqrt{-\mathfrak{C}}\,\mathfrak{Z}) - \mathcal{E}_2\,sin(\sqrt{-\mathfrak{C}}\,\mathfrak{Z}) \pm \sqrt{-\mathfrak{C}}})^{-1} - (2\,\mathcal{E}_3 - 2)\mathcal{E}_1 - \mathcal{E}_2{}^2 - (-6\,\mathcal{E}_3\,d + 6\,d)\mathcal{E}_2\\ &- 6\,\mathcal{E}_3{}^2 d^2 + 12\,\mathcal{E}_3\,d^2 - 6\,d^2,\end{aligned} \tag{2.116}$$

while A, B are arbitrary real constants and $A^2 - B^2 > 0$.
When $\mathcal{E}_1 = 0$ and $\mathcal{E}_2(\mathcal{E}_3 - 1) \neq 0$, we have:

$$\begin{aligned}\mathcal{U}(\mathfrak{Z}) =&(-6\,\mathcal{E}_1{}^2 - (-12\,\mathcal{E}_2\,d - 12\,d^2 + 12\,\mathcal{E}_3\,d^2)\mathcal{E}_1 - 6\,\mathcal{E}_2{}^2 d^2 - (-12\,\mathcal{E}_3\,d^3 + 12\,d^3)\mathcal{E}_2 - 6\,d^4 - 6\,\mathcal{E}_3{}^2 d^4\\ &+ 12\,\mathcal{E}_3 d^4)(d - \frac{\mathcal{E}_2\,k}{(\mathcal{E}_3 - 1)\,(k + cosh(\mathcal{E}_2\,\mathfrak{Z}) - sinh(\mathcal{E}_2\,\mathfrak{Z}))})^{-2} + (-(-12\,\mathcal{E}_3\,d + 12\,d + 6\,\mathcal{E}_2)\mathcal{E}_1\\ &+ 6\,\mathcal{E}_2{}^2 d - (18\,\mathcal{E}_3\,d^2 - 18\,d^2)\mathcal{E}_2 + 12\,\mathcal{E}_3{}^2 d^3 - 24\,\mathcal{E}_3 d^3 + 12\,d^3)\\ &\times (d - \frac{\mathcal{E}_2\,k}{(\mathcal{E}_3 - 1)\,(k + cosh(\mathcal{E}_2\,\mathfrak{Z}) - sinh(\mathcal{E}_2\,\mathfrak{Z}))})^{-1} - (2\,\mathcal{E}_3 - 2)\mathcal{E}_1 - \mathcal{E}_2{}^2 - (-6\,\mathcal{E}_3\,d + 6\,d)\mathcal{E}_2\\ &- 6\,\mathcal{E}_3{}^2 d^2 + 12\,\mathcal{E}_3\,d^2 - 6\,d^2,\end{aligned} \tag{2.117}$$

$$\begin{aligned}\mathcal{U}(\mathfrak{z}) =&(-6\,\mathcal{E}_1{}^2 - (-12\,\mathcal{E}_2\,d - 12\,d^2 + 12\,\mathcal{E}_3\,d^2)\mathcal{E}_1 - 6\,\mathcal{E}_2{}^2 d^2 - (-12\,\mathcal{E}_3\,d^3 + 12\,d^3)\mathcal{E}_2 - 6\,d^4 - 6\,\mathcal{E}_3{}^2 d^4\\ &+ 12\,\mathcal{E}_3 d^4)\Big(d - \frac{\mathcal{E}_2\,(cosh(\mathcal{E}_2\,\mathfrak{z}) + sinh(\mathcal{E}_2\,\mathfrak{z}))}{(\mathcal{E}_3 - 1)\,(k + cosh(\mathcal{E}_2\,\mathfrak{z}) + sinh(\mathcal{E}_2\,\mathfrak{z}))}\Big)^{-2} + (-(-12\,\mathcal{E}_3\,d + 12\,d + 6\,\mathcal{E}_2)\mathcal{E}_1 + 6\,\mathcal{E}_2{}^2 d\\ &- (18\,\mathcal{E}_3\,d^2 - 18\,d^2)\mathcal{E}_2 + 12\,\mathcal{E}_3{}^2 d^3 - 24\,\mathcal{E}_3 d^3 + 12\,d^3)\Big(d - \frac{\mathcal{E}_2\,(cosh(\mathcal{E}_2\,\mathfrak{z}) + sinh(\mathcal{E}_2\,\mathfrak{z}))}{(\mathcal{E}_3 - 1)\,(k + cosh(\mathcal{E}_2\,\mathfrak{z}) + sinh(\mathcal{E}_2\,\mathfrak{z}))}\Big)^{-1}\\ &- (2\,\mathcal{E}_3 - 2)\mathcal{E}_1 - \mathcal{E}_2{}^2 - (-6\,\mathcal{E}_3\,d + 6\,d)\mathcal{E}_2 - 6\,\mathcal{E}_3{}^2 d^2 + 12\,\mathcal{E}_3\,d^2 - 6\,d^2,\end{aligned} \tag{2.118}$$

$$\begin{aligned}\mathcal{U}(\mathfrak{z}) =&(-6\,\mathcal{E}_1{}^2 - (-12\,\mathcal{E}_2\,d - 12\,d^2 + 12\,\mathcal{E}_3\,d^2)\mathcal{E}_1 - 6\,\mathcal{E}_2{}^2 d^2 - (-12\,\mathcal{E}_3\,d^3 + 12\,d^3)\mathcal{E}_2 - 6\,d^4 - 6\,\mathcal{E}_3{}^2 d^4\\ &+ 12\,\mathcal{E}_3 d^4)\Big(d - \frac{1}{(\mathcal{E}_3 - 1)\,\mathfrak{z} + C}\Big)^{-2} + (-(-12\,\mathcal{E}_3\,d + 12\,d + 6\,\mathcal{E}_2)\mathcal{E}_1 + 6\,\mathcal{E}_2{}^2 d - (18\,\mathcal{E}_3\,d^2\\ &- 18\,d^2)\mathcal{E}_2 + 12\,\mathcal{E}_3{}^2 d^3 - 24\,\mathcal{E}_3 d^3 + 12\,d^3)\Big(d - \frac{1}{(\mathcal{E}_3 - 1)\,\mathfrak{z} + C}\Big)^{-1} - (2\,\mathcal{E}_3 - 2)\mathcal{E}_1 - \mathcal{E}_2{}^2\\ &- (-6\,\mathcal{E}_3\,d + 6\,d)\mathcal{E}_2 - 6\,\mathcal{E}_3{}^2 d^2 + 12\,\mathcal{E}_3\,d^2 - 6\,d^2,\end{aligned} \tag{2.119}$$

3. Solutions' demonstrating

This section explains the constructed solutions through some distinct plots in two, three, and contour graphs under some special values of above-mentioned parameters. The following Figures 1–5 show cone, dark, solitary, bright, and singular dark waves respectively.

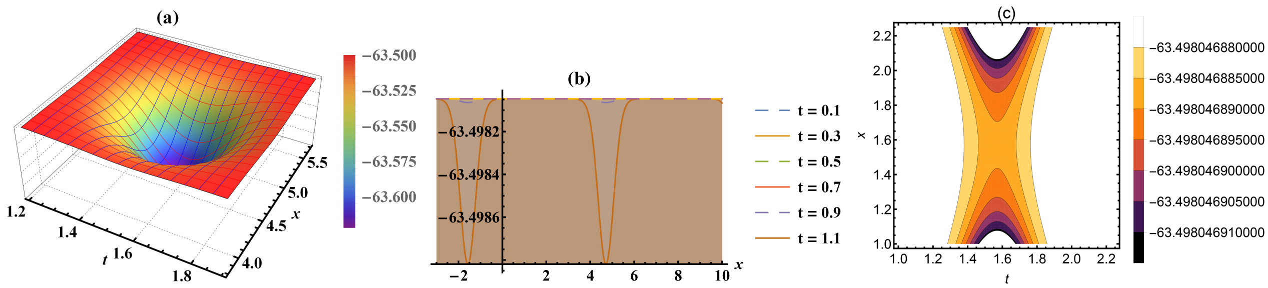

Figure 1. Cone wave graphs of Eq (2.3) represented by three, two-dimensional and contour plot three dimensional.

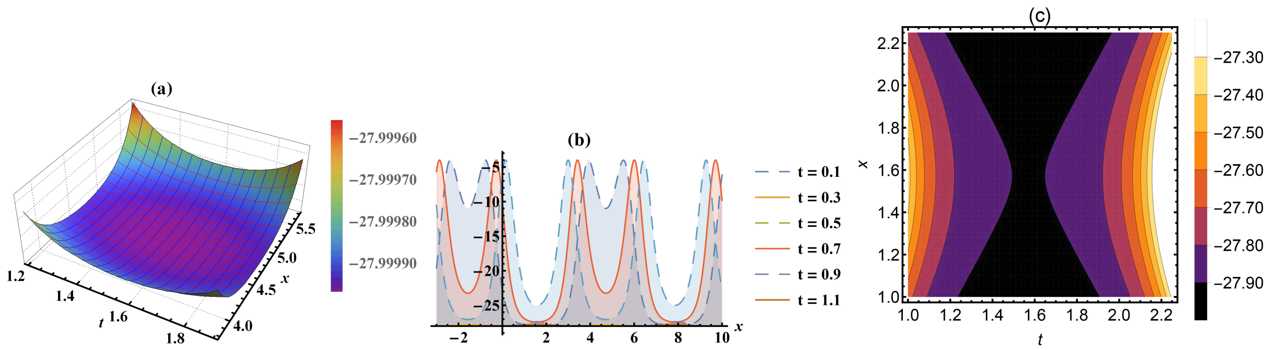

Figure 2. Dark wave graphs of Eq (2.12) represented by three, two-dimensional and contour plot three dimensional.

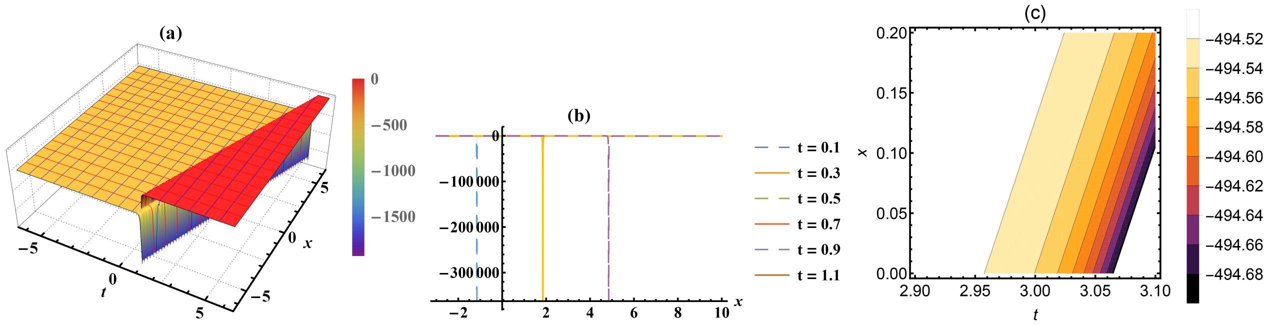

Figure 3. Solitary wave graphs of Eq (2.30) represented by three, two-dimensional and contour plot three dimensional.

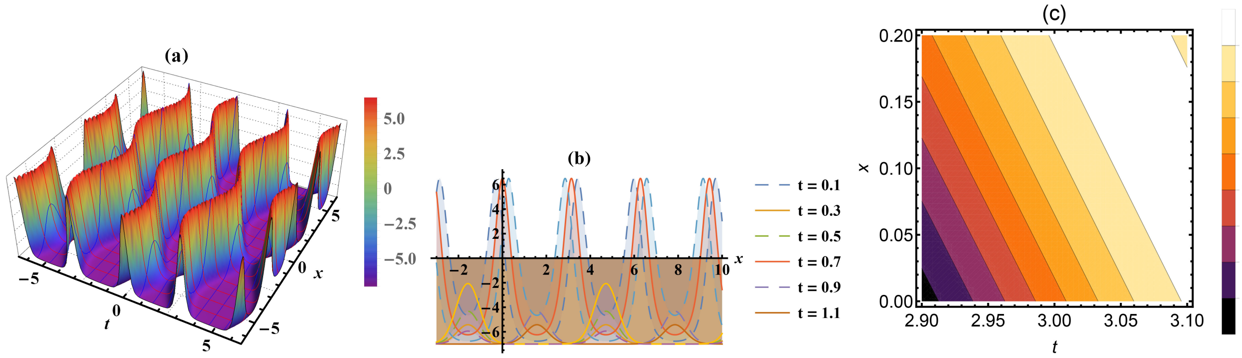

Figure 4. Bright wave graphs of Eq (2.46) represented by three, two-dimensional and contour plot three dimensional.

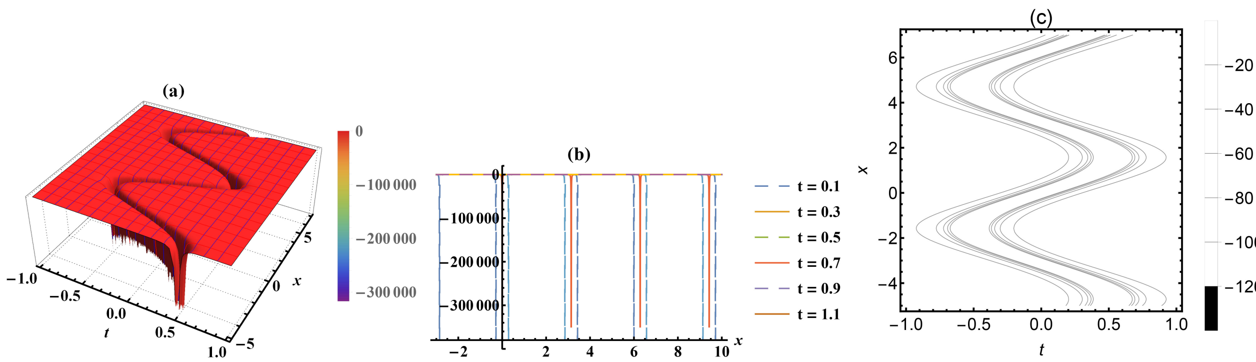

Figure 5. Singular dark wave graphs of Eq (2.50) represented by three, two-dimensional and contour plot three dimensional.

4. Results and discussion

Here, the solutions' physical interpretation and novelty are shown for demonstrating the research paper's contribution. This process is given in the following items:

- **Employed schemes:**
  Two computational schemes (ESE and NRE methods) have been employed in the nonlinear IPB

model for constructing novel nonlinear soliton lattice wave solutions. These methods have not been applied to this model before. Both schemes depend on different forms of well-known Riccati equation that helps formulate their solutions in various forms such as hyperbolic, trigonometric, exponential and rational forms. Additionally, Their obtained solutions show the dynamical and physical characterizations of the shallow water waves under gravity.

- **Obtained results:**
  The obtained schemes have obtained many different forms of solutions that cover many previous published solutions through using some different schemes. All our solutions are completely different and novel that have been obtained in [25–30].
- **Shown figures:**
  Some solutions have been explained through some different figures in various forms such as contour, two, three-dimensional plots. These solutions show many novel properties of the shallow water waves under gravity, such as cone, solitary, dark, bright and periodic features. These figures have been plotted using a particular value of each show-parameters in the solutions.

5. Conclusions

This article has studied the nonlinear IPB model along with two recent analytical schemes. Many novel solutions have been obtained and demonstrated through some magnificent figures to show many undiscovered features of the considered model. The novelty and paper's contribution are investigated. The used schemes' performance shows their effectiveness and the ability to handle many nonlinear evolution equations.

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Conflicts of interest

There is no conflict of interest.

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