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Research article

α-Admissible mapping in C^*-algebra-valued b-metric spaces and fixed point theorems

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Abstract: In the present paper, for a unital C^*-algebra A, we introduce a version of α_A-admissible on C^*-algebra-valued b-metric space, we proved some Banach and common fixed point theorems using α_A-admissible. Also, we give some non-trivial examples and an application to illustrate our results.

Keywords: C^*-algebra-valued b-metric space; fixed point theorem; α-admissible
Mathematics Subject Classification: 47H10, 46L07

1. Introduction

In 2014, Ma et al [18] introduced the concept of C^*-algebra-valued metric spaces by replacing the range of $\mathbb{R}$ with an unital C^*-algebra. Later in 2015, Ma et al [19] introduced the nation of C^*-algebra-valued metric spaces as a generalization of C^*-algebra-valued metric space. They proved some Banach fixed point theorems. Several research are obtained some results in Banach and common fixed point theorems in C^*-algebra-valued metric spaces (see [2,3,7,10,13,14,25,26,31,34]. The notion of C^*-algebra-valued partial metric space and C^*-algebra-valued partial b-metric spaces are introduced in [8,22] and proved fixed point results as analogous of Banach contraction principle.

In [27] introduced the study of fixed point for the α-admissibility of mappings and generalized several known results of metric spaces see also [28]. Later on, many authors proved α-admissible mappings theorems with various contraction condition see [1,5,9,12,17,20,21,30,32,33,35,36]. The aim of this paper is generalizing some results of metric spaces and C^*-algebra b-valued metric spaces.

We start with some definition and results about C^*-algebra b-valued metric spaces. Suppose that A is a unital C^*-algebra with a unit I. Set $A_h = \{x \in A : x = x^*\}$. An element $x \in A$ is a positive element, if $x = x^*$ and $\sigma(x) \subset \mathbb{R}^+$ is the spectrum of x. We define a partial ordering $\preceq$ on A_h as $x \preceq y$ if $0_A \preceq y - x$, where 0_A means the zero element in A and we let A^+ denote the $\{x \in A : x \succeq 0_A\}$ and $|x| = (x^*x)^{\frac{1}{2}}$.

On the other hand, [27] introduced the study of fixed point for the α-admissibility of mappings and generalized several know results of metric spaces.

Throughout this paper, we use the concept of α-admissibility of mappings defined on C^*-algebra b-valued metric spaces and we defined the generalized Lipschitz contractions on such spaces. The aim of this paper is generalizing some results of metric space and C^*-algebra b-valued metric spaces.

Lemma 1.1. *Suppose that A is a unital C^*-algebra with unit I_A. The following are holds.*
(1) If $a \in A$, with $\|a\| < \frac{1}{2}$, then $1 - a$ is invertible and $\|a(1-a)^{-1}\| < 1$.
(2) For any $x \in A$ and $a, b \in A^+$, such that $a \preceq b$, we have x^*ax and x^*bx are positive element and $x^*ax \preceq x^*bx$.
(3) If $0_A \preceq a \preceq b$ then $\|a\| \leq \|b\|$.
(4) If $a, b \in A^+$ and $ab = ba$, then $a.b \succeq 0_A$.
(5) Let A' denote the set $\{a \in A : ab = ba \ \forall\ b \in A\}$ and let $a \in A'$, if $b, c \in A$ with $b \succeq c \succeq 0_A$ and $1 - a \in (A')^+$ is an invertible element, then $(I_A - a)^{-1}b \prec (I_A - a)^{-1}c$.
We refer [24] for more C^*algebra details.

Definition 1.2. *Let X be a non-empty set and $b \succeq I_A$, $b \in A'$,suppose the mapping*
$d_A : X \times X \to A$, satisfies:
(1) $d_A(x, y) \succeq 0_A$ for all $x, y \in X$ and $d_A(x, y) = 0_A \Leftrightarrow x = y$.
(2) $d_A(x, y) = d_A(y, x)$ for all $x, y \in X$.
(3) $d_A(x, z) \preceq b[d_A(x, y) + d_A(y, z)]$ for all $x, y, z \in X$, where 0_A is zero-element in A and I_A is the unit element in A. Then d_A is called a C^*-algebra valued b-metric on X and (X, A, d_A) is called C^*-algebra-valued b-metric space.

Example 1.3. *Let X be a Banach space, $d_A : X \times X \to A$ given by $d_A(x, y) = \|x - y\|^p \cdot a$, for all $x, y \in X$, $a \in A^+$, $a \succeq 0$ and $p > 1$.*
Its easy to variety that (X, A, d_A) is a C^*-algebra -valued b-metric space.
Using the inequality $(a + b)^p \leq 2^p(a^p + b^p)$ for all $a, b \geq 0$, $p > 1$, we have

$$\|x - z\|^p \leq 2^p(\|x - y\|^p + \|y - z\|^p)$$

for $x, y, z \in X$, which implies that

$$d_A(x, z) \preceq 2^p(d_A(x, y) + d_A(y, z))$$

In the next we give a counter example , show that in general, a C^*-algebra valued b-metric space in not necessary a C^*-algebra valued metric space.

Example 1.4. *Let $X = \mathbb{R}$ and $A = M_2(\mathbb{R})$. Define*

$$d_A(x, y) = \begin{pmatrix} |x - y|^2 & 0 \\ 0 & k|x - y|^2 \end{pmatrix}$$

$x, y \in \mathbb{R}$, $k > 0$, it is clear that (X, A, d_A) is a C^*-algebra valued b-metric space by using the same argument in example 1.3 when $p = 2$

Now, $d_A(0,1) = \begin{pmatrix} 1 & 0 \\ 0 & k.1 \end{pmatrix}$, $d_A(1,2) = \begin{pmatrix} 1 & 0 \\ 0 & k.1 \end{pmatrix}$, $d_A(0,2) = \begin{pmatrix} 4 & 0 \\ 0 & k.4 \end{pmatrix}$.
Its obvious that $d_A(0,2) \geq d_A(0,1) + d_A(1,2)$.
So (X, A, d_A) *is not a* C^**-algebra valued metric space*

Definition 1.5. *Let* (X, A, d_A) *be a* C^**-algebra- valued b-metric space,* $x \in X$*, and* $\{x_n\}_{n=1}^{\infty}$ *be a sequence in X, then*
(i) $\{x_n\}_{n=1}^{\infty}$ *convergent to x whenever, for every* $c \in A$ *with* $c \succ 0_A$*, there is a natural number* $N \in \mathbb{N}$ *such that*

$$d_A(x_n, x) \prec c,$$

for all $n > N$*. We denote this by* $\lim_{n\to\infty} x_n = x$ *or* $x_n \to x$ *as* $n \to +\infty$*.*
(ii) $\{x_n\}_{n=1}^{\infty}$ *is said to be a Cauchy sequence whenever, for every* $c \in A$ *with* $c \succ 0_A$*, there is a natural number* $N \in \mathbb{N}$ *such that*

$$d_A(x_n, x_m) \prec c,$$

for all $n, m > N$*.*

Lemma 1.6. *(i)* $\{x_n\}_{n=1}^{\infty}$ *is a convergence in X. If for any element* $\epsilon > 0$ *there is* $N \in \mathbb{N}$ *such that for all* $n > N$, $\|d(x_n, x)\| \leq \epsilon$*.*
(ii) $\{x_n\}_{n=1}^{\infty}$ *is a Cauchy sequence in X, for any* $\epsilon > 0$ *there* $N \in \mathbb{N}$ *such that*
$\|d_A(x_n, x_m)\| \leq \epsilon$*, for all* $n, m > N$*. We say that* (X, A, d_A) *is a complete* C^**-algebra- valued b-metric space if every Cauchy sequence is convergent with respect to A.*

Example 1.7. *Let* $X = \mathbb{R}$ *and* $A = M_n(\mathbb{R})$ *the set of all* $n \times n$*-matrices with entries in* $\mathbb{R}$*. Define*

$$d_A(a,b) = \begin{pmatrix} \lambda_{1|a_{ii}-b_{ii}|^p} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_{n|a_{nn}-b_{nn}|^p} \end{pmatrix}$$

where $a = (a_{ij})_{i,j=1}^n$ *,* $b = (b_{ij})_{i,j=1}^n$ *are two* $n \times n$*-matrices ,* $a_{ij}, b_{ij} \in \mathbb{R}$ *for all* $i, j = 1, ..., n$*,* $\lambda_i \geq 0$ *for* $i = 1, ..., n$ *are positive real numbers.*
One can define a partial ordering on ($\preceq_{M_n(\mathbb{R})}$*) on* $M_n(\mathbb{R})$ *as following* $a \preceq_{M_n(\mathbb{R})} b$ *if and only if* $a_{ij} \leq b_{ij} \forall i, j = 1, ..., n$*. And an element* $a \succeq_{M_n(\mathbb{R})} 0$ *is positive in* $M_n(\mathbb{R})$ *if and only if* $a_{ij} \geq 0$ *for all* $i, j = 1, ..., n$*.* $(X, M_n(\mathbb{R}), d_{M_n(\mathbb{R})})$ *is* C^**-algebra- valued b-metric space.*
One can prove that

$$d_A(a,c) \preceq_{M_n(\mathbb{R})} 2^p(d_A(a,b) + d_A(b,c)),$$

for all $a, b, c \in M_n(\mathbb{R})$*.*
We need only to use the following inequality in $\mathbb{R}$

$$|x - z|^p \leq 2^p(|x - y|^p + |y - z|^p,$$

where $b = 2^p I_{M_n(\mathbb{R})} \geq I_{M_n(\mathbb{R})}$ $\forall p \geq 1$*, where* $I_{M_n(\mathbb{R})}$*is the unit element in* $M_n(\mathbb{R})$*.*

Remark 1.8. *In the above example the inequality* $|x-z|^p \leq |x-y|^p + |y-z|^p$ *it is impossible for* $x > y > z$*. Then the* (X, A, d) *is not a* C^**-algebra valued metric space.*

It is useful to discuss the relation between C^*-algebra valued metric spaces and lattices-valued metric spaces. To classify C^*-algebra-valued-metric spaces and its relation with lattices, we have to discuss the concept of quantale which introduced by Mulvey [23]. A quantale Q is a complete lattice together with an associative multiplication $\circledast : Q \times Q \to Q$ such that $a \circledast (\vee_i b_i) = \vee_i (a \circledast b_i)$, and $\vee_i (a_i) \circledast b = \vee_i (a_i \circledast b)$ for all $a_i, b_i, a, b \in Q$, $i \in I$, I is an index set the quantale is said to be unit, if it has a unital 'e' satisfy $a \circledast e = e \circledast a$ for all $a \in Q$. And Q is called an involuative quantale with relation $* : Q \to Q$ satisfy $(a^*)^* = a$, $(a \circledast b)^* = b^* \circledast a^*$ and $(\vee_i a_i)^* = \vee_i a_i^*$ for all $a_i, b_i, a \in Q$.

The top element of Q is denoted by 1 and the bottom element denoted by 0. A typical example of quantale is given by $End(S)$, the set of all sublattices of Endomorphisms of the complete lattices S is a unital quantale with join calculated by point wise $(\vee_i f_i)(x) = \vee_i f_i(x)$ and multiplication as composition $(f \circledast g)(x) = (f \circ y)(x)$.

And it is unit identity is Id_S. An element $a \in Q$ is said to be right-sided if $a \circledast 1 \leq a$, for all $a \in Q$, denote by $R(Q)$ the set of all right-sided elements. Similarly, an element $a \in Q$ is said to be left- side if $1 \circledast a \leq a$ for all $a \in Q$, $L(Q)$ denote the set of left-sided elements.if $a \in Q$ is right-sided elements and left-sided elements it is said to be 2-sided elements and the set of 2-sided-elements denoted by $I(Q)$. Any two sided-elements a is distributive in the sense that $a \wedge \vee_i b_i = \vee_i (a \wedge b_i)$.

A quantale is commutative if it is commutative under the multiplication. If the quntale is commutative then $Q \cong I(Q)$. If A is a C^*-algebra and $R(A)$ is the lattice of all closed right ideals of A, then $R(A)$ is a quantale and the multiplication of closed right ideals obtained by taking the topological closure of the usual product of ideals, simply, $I \circledast J = \overline{IJ}$ for any two ideals $I, J \in R(A)$.
By Gelfand duality theorem [11]. Any commutative C^*-algebra is isomorphic to the set of all continuous functions of the compact Hausdorf topological space. So, in this case $R(A)$ is isomorphic to the lattice $O(\widehat{A})$ of all open sublattices of $\widehat{A}$, where $\widehat{A}$ is the topological space determined by A, the spectrum of A. Therefore, commutative C^*-algebra classify by commutative quantales as given in [6]. A is a commutative C^*-algebra if and only if $R(A)$ is commutative quantale.

On the other hand 'Sherman [29]' show that if A_{sa} is the space of self-.adjoint elements of a C^*-algebra A with the canonical order $\leq$ given by $a \leq b$ if and only if $b - a \geq 0$ is positive. Then A_{sa} is a lattice ordered if and only if A is commutative. Therefore, the C^*-algebra valued metric space in commutative case coinside with the commutative quantale -valued-metric space, with a suitable metric For a non-commutative C^*-algebra A with unit, by $MaxA$ is meant. The set of all subspace of A together with the multiplication defined by $M \circledast N = \overline{MN}$ to be the closure of product liner subspace, for each $M, N \in MaxA$, and the join "$\vee''$" defined by $\vee_i M_i = \overline{\sum_i M_i}$, and the involution $M^* = \{a^* : a \in M\}$ and the unit of $MaxA$ is given by the identity. So, $MaxA$ is defined A unital involutive quantale. In the case the non-commutative C^*-algebra is classify by $MaxA$, following [15,16]. If A and B are two unital C^*-algebras. Then A and B are isomorphic if and only if $MaxA$ and $MaxB$ are isomorphic as a unital involutive quantale . So, C^*-algebra valued metric spaces are classify by the unital involutive quantale-valued metric space.

2. Main results

In 2012 Samet et al [27], introduced the concept of α-admissible mapping as follows.

Definition 2.1. *Let $T : X \to X$ be self map and $\alpha : X \times X \to [0, +\infty)$. Then T is called α-admissible if for all $x, y \in X$ with $\alpha(x, y) \geq 1$ implies $\alpha(Tx, Ty) \geq 1$.*

Next, we introduced an analogue definition of α-admissible for a unital C^*-algebra.

Definition 2.2. *Let X be a non-empty set and $\alpha_A : X \times X \to (A')^+$ be a function, we say that the self map T is α_A - admissible if $(x, y) \in X \times X$, $\alpha_A(x, y) \succeq I_A \Rightarrow \alpha_A(Tx, Ty) \succeq I_A$, where I_A the unity of A.*

Definition 2.3. *Let (X, A, d) be a complete C^*-algebra- valued b-metric space, the mapping $T : X \to X$ is said to be generalised Lipschitz condition if there exist $a \in A$ such that $\|a\| < 1$ and*

$$d_A(Tx, Ty) \preceq a^* d_A(x, y)a, \tag{2.1}$$

for all $x, y \in X$ with $\alpha_A(x, y) \succeq I_A$.

Example 2.4. *Let $X = \mathbb{R}$ and $A = M_n(\mathbb{R})$ as given in example (1.7), define $T : X \to X$, by $Tx = \dfrac{x}{2}$, and $\alpha_{M_n(\mathbb{R})} : X \times X \to M_n(\mathbb{R})^+$, given by $\alpha_{M_n(\mathbb{R})}(x, y) = I_{M_n(\mathbb{R})}$ and $\alpha_{M_n(\mathbb{R})}(Tx, Ty) = \alpha_{M_n(\mathbb{R})}(\frac{x}{2}, \frac{y}{2}) = I_{M_n(\mathbb{R})}$ thus T is $\alpha_{M_n(\mathbb{R})}-$admissible, where $M_n(\mathbb{R})^+$ is the set of all positive elements*

$$\alpha_{M_n(\mathbb{R})}(x, y) d_{M_n(\mathbb{R})}(Tx, Ty) \preceq_{M_n(\mathbb{R})} I_{M_n(\mathbb{R})}.\begin{pmatrix} \lambda_{1|\frac{x}{2} - \frac{y}{2}|^p} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_{n|\frac{x}{2} - \frac{y}{2}|^p} \end{pmatrix} \preceq_{M_n(\mathbb{R})} \frac{I_{M_n(\mathbb{R})}}{(2)^p}.d_{M_n(\mathbb{R})}(x, y),$$

and $a = \dfrac{I_{M_n(\mathbb{R})}}{(\sqrt{2})^p}$, $a^* = \dfrac{I_{M_n(\mathbb{R})}}{(\sqrt{2})^p}$, so T satisfy the generalised Lipschitz condition.

Theorem 2.5. *Let (X, A, d_A) be a complete C^*-algebra- valued b-metric space, with $b \succeq I_A$, $b \in A', \|b\| \|a\|^2 < 1$ suppose that $T : X \to X$, be a generalised Lipschitz contraction satisfies the following conditions:*
(i) T is α_A-admissible.
(ii) There exists $x_0 \in X$ such that $\alpha_A(x_0, Tx_0) \succeq I_A$.
(iii) T is continuous.
Then T has a fixed point.

Proof: let $x_0 \in X$ such that $\alpha_A(x_0, Tx_0) \succeq I_A$ and define a sequence $\{x_n\}_{n=0}^{\infty} \subseteq X$ such that $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$. If $x_n = x_{n+1}$ for some $n \in \mathbb{N}$, then $x = x_n$ is a fixed point for T.
Assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$, since T is α_A-admissible, we have

$$\begin{aligned} \alpha_A(x_0, x_1) &= \alpha_A(x_0, Tx_0) \succeq I_A \Rightarrow \\ \alpha_A(Tx_0, T^2x_0) &= \alpha(x_1, x_2) \succeq I_A. \end{aligned}$$

By induction we get

$$\alpha_A(x_n, x_{n+1}) \succeq I_A. \tag{2.2}$$

Since T is generalised Lipschitz condition , then

$$\begin{aligned} d_A(x_n, x_{n+1}) &= d_A(Tx_{n-1}, Tx_n) \preceq a^* d_A(x_{n-1}, x_n)a \\ &\preceq (a^*)^2 d_A(x_{n-2}, x_{n-1})a^2 \end{aligned}$$

$$
\begin{aligned}
&\cdot\\
&\cdot\\
&\cdot\\
&\preceq\ (a^*)^n d_A(x_o, x_1) a^n\\
&\preceq\ (a^*)^n d_0 a^n.
\end{aligned}
$$

Denote that $d_0 := d_A(x_0, x_1)$ in A, notice that in C^*-algebra, if $a, b \in A^+$ and $0_A \preceq a \preceq b$, then for any $x \in A$ both x^*ax and x^*bx are positive elements and

$$0_A \preceq x^*ax \preceq x^*bx.$$

Now, for $m \geq 1$, $p \geq 1$ it following that

$$
\begin{aligned}
d_A(x_m, x_{m+p}) &\preceq b[d_A(x_m, x_{m+1}) + d_A(x_{m+1}, x_{m+p})]\\
&\preceq bd_A(x_m, x_{m+1}) + b^2 d_A(x_{m+1}, x_{m+2}) + ...\\
&+ b^{p-1} d_A(x_{m+p-2}, x_{m+p-1}) + b^{p-1} d_A(x_{m+p-1}, x_{m+p})\\
&\preceq b((a^*)^m d_0 a^m) + b^2((a^*)^{m+1} d_0 a^{m+1}) + ...\\
&+ b^{p-1}((a^*)^{m+p-2} d_0 a^{m+p-2}) + b^{p-1}((a^*)^{m+p-1} d_0 a^{m+p-1})\\
&= \sum_{k=1}^{p-1} b^k((a^*)^{m+k-1}\ d_0\ a^{m+k-1}) + b^{p-1}(a^*)^{m+p-1} d_0\ a^{m+p-1}\\
&= \sum_{k=1}^{p-1} b^k((a^*)^{m+k-1}\ d_0^{\frac{1}{2}} d_0^{\frac{1}{2}}\ a^{m+k-1}) + b^{p-1}(a^*)^{m+p-1} d_0\ a^{m+p-1}\\
&= \sum_{k=1}^{p-1}((a^*)^{m+k-1}\ b^{\frac{k}{2}} d_0^{\frac{1}{2}})(d_0^{\frac{1}{2}}\ b^{\frac{k}{2}}\ a^{m+k-1}) + (b^{\frac{p-1}{2}}(a^*)^{m+p-1} d_0^{\frac{1}{2}}\ d_0^{\frac{1}{2}} b^{\frac{p-1}{2}}(a^*)^{m+p-1})\\
&= \sum_{k=1}^{p-1}(d_0^{\frac{1}{2}}\ b^{\frac{k}{2}} a^{m+k-1})^*(d_0^{\frac{1}{2}}\ b^{\frac{k}{2}}\ a^{m+k-1}) + (d_0^{\frac{1}{2}} b^{\frac{p-1}{2}} a^{m+p-1})^*(d_0^{\frac{1}{2}} b^{\frac{p-1}{2}} a^{m+p-1})\\
&= \sum_{k=1}^{p-1} |d_0^{\frac{1}{2}}\ b^{\frac{k}{2}} a^{m+k-1}|^2 + |d_0^{\frac{1}{2}} b^{\frac{p-1}{2}} a^{m+p-1}|^2\\
&\preceq \sum_{k=1}^{p-1} ||d_0^{\frac{1}{2}}\ b^{\frac{k}{2}} a^{m+k-1}||^2.I_A + ||d_0^{\frac{1}{2}} b^{\frac{p-1}{2}} a^{m+p-1}||^2.I_A\\
&\preceq ||d_0^{\frac{1}{2}}||^2 \sum_{k=1}^{p-1} ||\ a||^{2(m+k-1)}||b||^k.I_A + ||d_0^{\frac{1}{2}}||^2||b||^{p-1}\ ||a||^{2(m+p-1)}.I_A\\
&= ||d_0||^{\cdot}[||b||||a||^{2m}(\frac{1-(||b||||a||^2)^{p-1}}{1-||b||||a||^2})].I_A + ||d_0||||b||^{p-1}.||a||^{2(m+p-1)}.I_A\\
&= ||d_0||[||b||||a||^{2m}(\frac{(||b||||a||^2)^{p-1}-1}{||b||||a||^2-1})].I_A + ||d_0||||b||^{p-1}.||a||^{2(m+p-1)} \to 0_A,
\end{aligned}
$$

with the condition $||b||||a||^2 < 1$ and at $m \to +\infty$.

It implies that $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence. By completeness of X that exists $x \in X$ such that $x_n \to x$ as $n \to +\infty$.
Since T is continuous and the lim is unique it follows $x_{n+1} = Tx_n \to Tx$ as $n \to +\infty$ such that $x = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} Tx_n = Tx$, so, $Tx = x$ is a fixed point for T.
Now, we replace the assumption of continuoity of T in the above theorem by another condition.

Theorem 2.6. *Let (X, A, d_A) be a complete C^*-algebra- valued b-metric space, with $b \geq I_A$.*
Let $T : X \to X$ be generalized Lipschitz condition as in (2.5) and the following conditions are satisfies:
(i) T is α_A-admissible.
(ii) There exists $x_0 \in X$ such that $\alpha_A(x_0, Tx_0) \geq I_A$.
(iii) If $\{x_n\}_{n=0}^{\infty}$ is a sequence in X such that $\alpha_A(x_n, x_{n+1} \geq I_A$ for all $n \in \mathbb{N}$ and $x_n \to x \in X$, as $n \to +\infty$, then $\alpha_A(x_n, x) \geq I_A$ for all $n \in \mathbb{N}$. Then T has a fixed point in X.

Proof: From theorem 2.5, we Know that $\{x_n\}_{n=0}^{\infty}$ is a Couchy sequence in (X, A, d_A), then there exists $x \in X$ such that $x_n \to x$ as $n \to +\infty$.
On the other hand from equation(3.1) and by hypothesis (iii) , we have $d_A(x_n, x) \geq I_A$, for all $n \in \mathbb{N}$, since T is generalized Lipschitz Contraction using 2.2 we get

$$\begin{aligned} d_A(x, Tx) &\preceq b[d_A(x, x_{n+1}) + d_A(x_{n+1}, Tx) \\ &= b[d_A(x, x_{n+1}) + d_A(Tx_n, Tx) \\ &\preceq b[d_A(x, x_{n+1}) + a^*(d(x_n, x)a] \\ &\to 0_A \ \ as \ \ n \to +\infty. \end{aligned}$$

$$d_A(x, Tx) = 0_A \Rightarrow Tx = x.$$

To prove the uniqueness of the fixed point of generalized Lipschitz mapping we have to consider the following property.
(H): For all $x, y \in X$, there exists $z \in X$ such that $d_A(x, z) \geq I_A$ and $d_A(y, z) \geq I_A$.

Theorem 2.7. *Adding condition (H) to the hypothesis of theorem (2.5) we obtain the uniqueness of the fixed point of T.*

Proof: Suppose that x and y are two fixed points of T from (H), there exists $z \in X$ such that

$$\alpha_A(x, z) \geq I_A \ \ and \ \ \alpha_A(y, z) \geq I_A. \tag{2.3}$$

Since T is α_A-admissible, from (2.2) we have

$$\alpha_A(x, T^n z)) \geq I_A \ and \ \alpha_A(y, T^n z)) \geq I_A. \tag{2.4}$$

Since T is generalized Lipschitz contraction, so by using (2.4), we have

$$\begin{aligned} d_A(x, T^n z) &= d_A(Tx, T(T^{n-1}z)) \\ &\preceq a^* d_A(x, T^{n-1}z)a \\ &\;\;. \\ &\;\;. \end{aligned}$$

$$
\begin{aligned}
& \cdot \\
& \leq (a^*)^n d_A(x,z)a \quad for \;\; all \;\; n \in \mathbb{N} \\
\|d_A(x,T^n z)\| & \leq \|a\|^{2n}\|d_A(x,z)\|.
\end{aligned}
$$

Since $\|b\|\|a\|^2 < 1$, $\|a\| < 1$, we have $\|a\|^{2n} \to 0_A$ as $n \to +\infty$ and $d_A(x,T^n z) \to 0_A$, $T^n z = x$ as $n \to +\infty$.

Similarly we get $T^n z = y$ as $n \to +\infty$, there for by uniqueness of the limit, we obtain $x = y$. This complete the proof.

3. Common fixed point theorems

Now, we give a common fixed point theorems for two mappings satisfy a common α_A-admissible.

Definition 3.1. *let $(T,S) : X \to X$ be a continuous self mappings on X. $\alpha_A : X \times X \to A^+$. (T,S) are said to be common α_A-admissible if for any $x_0 \in X$,*

$$\alpha_A(x_0,y) \geq I_A \Rightarrow \alpha_A(Tx_0,Sy) \geq I_A \Rightarrow \alpha_A(T^2x_0,S^2y) \geq I_A.$$

Theorem 3.2. *Let (X,A,d_A) be complete C^*-algebra- valued b-metric space and $T,S : X \to X$, such that*

$$\alpha_A(x,y)d_A(Tx,Sy) \leq a^* d_A(x,y)a, \tag{3.1}$$

and $\|a\| < 1$, $\|b\|.\|a\|^2 < 1$ and the following conditions are satisfies:
(i) (T,S) are common α_A-admissible.
(ii) The exists $x_0 \in X$ such that

$$\alpha_A(x_0,y) \geq I_A \Rightarrow \alpha_A(Tx_0,Sy) \geq I_A.$$

(iii) T and S are continuous and have a common fixed point in X.

Proof: Let $x_0 \in X$ and construct a sequence $\{x_n\} \subseteq X$ such that $Tx_{2n} = x_{2n+1}$, $Sx_{2n+1} = x_{2n+2}$ form (3.1), we get

$$
\begin{aligned}
\alpha_A(x_0,x_1) &= \alpha_A(Tx_0,Sx_1) \geq I_A \\
&\Rightarrow \alpha_A(T^2x_0,S^2x_1) \geq I_A \\
&\Rightarrow \alpha_A(x_2,x_3) \geq I_A,
\end{aligned}
$$

by induction, we have $\alpha_A(x_{2n},x_{2n+1}) \geq I_A$, for all $n \in \mathbb{N}$.

$$
\begin{aligned}
d_A(x_{2n+1},x_{2n+2}) &= d_A(Tx_{2n},Sx_{2n+1}) \\
&\leq \alpha_A(x_{2n},x_{2n+1})d_A(Tx_{2n},Sx_{2n+1}) \\
&\leq a^* d_A(x_{2n},x_{2n+1})a,
\end{aligned}
$$

by induction, we obtain

$$d_A(x_{2n+1},x_{2n+2}) \leq (a^*)^{2n+1} d_A(x_0,x_1)a^{2n+1}.$$

Similarly,

$$d_A(x_{2n}, x_{2n+1}) \preceq (a^*)^{2n} d_A(x_0, x_1) a^{2n}.$$

Now, we can obtain for any $n \in \mathbb{N}$

$$d_A(x_n, x_{n+1}) \preceq (a^*)^n d_A(x_0, x_1) a^n.$$

Then for $p \in \mathbb{N}$, $p \geq 1$, $m \geq 1$, and applying the triangle inequality,
we have

$$\begin{aligned} d_A(x_m, x_{m+p}) &\preceq b[d_A(x_m, x_{m+1}) + b^2 d_A(x_{m+1}, x_{m+2}) + ... \\ &+ b^{p-2} d_A(x_{m+p-2}, x_{m+p-1}) + b^{p-1} d_A(x_{m+p-1}, x_{m+p}) \\ &\preceq \sum_{k=1}^{p-1} b^k((a^*)^{m+k-1} d_0 a^{m+k-1}) + b^{p-1}(a^*)^{m+p-1} d_0 a^{m+p-1}, \end{aligned}$$

by similar calculation as theorem (2.5), we get

$$d_A(x_m, x_{m+p}) \preceq \|d_0\|^{\cdot}[\|b\|\|a\|^{2m}(\frac{1-(\|b\|\|a\|^2)^{p-1}}{1-\|b\|\|a\|^2})]I_A + \|d_0\|\|b\|^{p-1}\|a\|^{2(m+p-1)} I_A \to 0_A,$$

as $n \to +\infty$, where I_A is the unitary in A, $d_0 := d_A(x_0, x_1)$, $b \in (A^+)'$.
So, $\{x_n\}$ is a Cauchy sequence in X.
The completion of (X, A, d_A) implies that there exists $x \in X$ such that $\lim_{n\to\infty} x_n = x$
Now, we using triangle inequality and (3.1), we set

$$\begin{aligned} d_A(x, Sx) &\preceq b[d_A(x, x_{2n+1}) + d_A(x_{2n+1}, Sx)] \\ &\preceq b[d_A(x, x_{2n+1}) + d_A(Tx_{2n}, Sx)] \\ &\textit{and} \quad \alpha_A(x_{2n}, x) \succeq I_A, \; \textit{we get} \\ d_A(x, Sx) &\preceq b[d_A(x, x_{2n+1}) + a^* d_A(x_{2n}, x)a] \\ \|d_A(x, Sx)\| &\leq \|b\|\|d_A(x, x_{2n+1})\| + \|b\|\|a\|^2\|d_A(x_{2n}, x)\| \\ \|d_A(x, Sx)\| &\leq \|d_A(x, x_{2n+1})\|(\|b\| + \|b\|\|a\|^2). \end{aligned}$$

Since $\|a\| < 1$, we have a contradiction $\Rightarrow d_A(x, Sx) = 0_A \Rightarrow Sx = x$, similarly, we get $Tx = x$, so, S and T have a common fixed point .

In the following, we will show that the uniquely of common fixed point in X, for that assume that is another fixed point $y \in X$ such that $Ty = y = y$.
Since x satisfy property H, and (T, S) are α_A -admissible, we have

$$\begin{aligned} d_A(x, S^n z) &= d_A(Tx, S^n z) \\ &\preceq a^* d_A(Tx, S^{n-1} z)a \\ &\;. \\ &\;. \\ &\;. \\ &\preceq (a^*)^n d_A(x, z)a^n \end{aligned}$$

$$\|d_A(x, S^n z)\| \leq \|a\|^{2n}\|d_A(x, z)\| \to 0 \ \ as \ \ n \to +\infty.$$

So, $d_A(x, S^n z) = 0_A$ this implies that $S^n z = x$.
Similarly, we get $S^n z = y$ Thus x is a unique common fixed point.

Theorem 3.3. *Let (X, A, d_A) be a complete C^*-algebra- valued b-metric space, suppose that two mappings $T, S : X \to X$, satisfy*

$$\alpha(x, y)d_A(Tx, Ty) \preceq a^* d_A(Sx, Sy)a \ \ for \ any \ x, y \in X, \tag{3.2}$$

where $a \in A$, with $\|b\|\|a\|^2 < 1$ and $\|a\| < 1$.
If $R(T) \subseteq R(S)$ and $R(S)$ is complete in X, T and S are weakly compatible, such that the following holds
(i) (T, S) are common α_A-admissible.
(ii) There is $x_0 \in X$ such that $\alpha_A(x_0, y) \succeq I_A \Rightarrow \alpha_A(Tx_0, Sy) \succeq I_A$.
(iii) T and S are continuous.
(iv) X has a property (H), they T and S have a unique common fixed point in X.

Proof: Let $x_0 \in X$, choose $x_1 \in X$, such that $Sx_1 = Sx_0$, which can be done since $R(T) \subseteq R(S)$. Let $x_0 \in X$ such that $Sx_2 = Tx_1$.
Repeating the process, we have a sequence $\{Sx_n\}_{n=1}^{\infty}$ in X satisfying $Sx_n = Tx_{n-1}$.
Then, since (T, S) are α_A-admissible, we get

$$\begin{aligned}
\alpha_A(Sx_1, Sx_2) &= \alpha_A(Tx_0, Tx_1) \succeq I_A \\
&\Rightarrow \alpha_A(T^2x_0, T^2x_1) \succeq I_A \\
&\Rightarrow \alpha_A(Sx_2, Sx_3) \succeq I_A \\
&. \\
&. \\
&. \\
&\Rightarrow \alpha_A(Sx_n, Sx_{n+1}) \succeq I_A.
\end{aligned}$$

Now,

$$\begin{aligned}
d_A(Sx_n, Sx_{n+1}) &= d_A(Tx_{n-1}, Tx_n) \\
&\preceq a^* d_A(Sx_{n-1}, Sx_n)a \\
&. \\
&. \\
&. \\
&\preceq (a^*)^n d_A(Sx_0, Sx_1)a^n.
\end{aligned}$$

For $m \geq 1$, $p \geq 1$.

$$\begin{aligned}
d_A(Sx_m, Sx_{m+p}) &\preceq bd_A(Sx_m, Sx_{m+1}) + b^2 d_A(Sx_{m+1}, Sx_{m+2}) + ... + \\
&+ b^{p-1} d_A(Sx_{m+p-2}, Sx_{m+p+1}) + b^{p-1} d_A(Sx_{m+p-1}, Sx_{m+p})
\end{aligned}$$

$$\leq \sum_{k=1}^{p-1} b^k (a^*)^{m+k-1} d_0 (a)^{m+k-1} + ... + b^{p-1} (a^*)^{m+p-1} d_0 (a)^{m+p-1}.$$

Using similar calculation as in theorem 2.5 , we get

$$\begin{aligned} d_A(Sx_m, Sx_{m+p}) &\leq ||d_0|| \frac{[||b||||a||^{2m}(||b||||a||^2)^{p-1} - 1]}{||b||||a||^2 - 1} I_A \\ &+ ||d_0||||b||^{p-1}||a||^{2(m+p+1)} I_A \to 0_A \ \ as \ \ m \to +\infty. \end{aligned}$$

Where $d_0 = d_A(Sx_0, Sx_1)$.
So, $\{Sx_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $R(S)$ and is complete in X, there exists $x \in X$ such that $\lim_{n \to +\infty} Sx_n = Sx$.
Also,

$$\begin{aligned} d_A(Sx_n, Tx) &= d_A(Tx_{n-1}, Tx) \\ &\leq a^* d_A(Sx_n, x) a \to 0_A, \ \ as \ \ n \to +\infty. \end{aligned}$$

So, $Sx_n \to Tx$ as $n \to +\infty$. Hens $Sx_n = Tx = Sx$, so x is coincidence common fixed point in X.
Morovere of y is another common fixed point such that $Ty = Sy = y$, so

$$d_A(Sx, Sy) = d_A(Tx, Ty) \leq a^* d_A(Sx, Sy) a$$

$$||d_A(Sx, Sy)|| \leq ||a||^2 ||d_A(Sx, Sy)||.$$

Since $||a|| < 1$, so we yet $d_A(Sx, Sy) = 0_A \Rightarrow Sx = Sy$.
So S , T have coincidence fixed point is unique $Sx = Tx = x$.
Since $\{Sx_n\}_{n=1}^{\infty}$ is a sequence in X, convergent to Sx and Sy respectively,
$Sx = \lim_{n \to +\infty} Sx_n = Tx$, since the lim is unique, so $Tx = Sx = x$, so S and T have a common fixed point in X.
Since X has a property (H) and (S, T) are α_A-admissible, we get

$$\begin{aligned} d_A(x, T^n x) &= d_A(Tx_1, T^n z) = d_A(Tx, T^{n-1} z) \\ &\leq a^* d_A(Sx, S(T^{n-1} z_n)) a \\ &. \\ &. \\ &. \\ &\leq (a^*)^n d_A(Sx, Sz) a^n \\ ||d_A(x, T^n z)|| &\leq ||a||^{2n} ||d_A(Sx, Sz)|| \to 0 \ \ as \ \ n \to +\infty. \\ &\Rightarrow d_A(x, T^n z) = 0_A \Rightarrow T^n z = x. \end{aligned}$$

Similarly $T^n z = y$, so $x = y$ and this complete the proof.

4. Application

We introduce a non-trivial example satisfy the theorem 2.5.

Example 4.1. *Let* $X = [0,1]$, $A = M_2(\mathbb{R})$, $p > 1$ *and* $k > 0$ *is a constant, we define* $d_A = X \times X \to A$ *as* $d_A(x,y) = \begin{pmatrix} |x-y|^p & 0 \\ 0 & k|x-y|^p \end{pmatrix}$ *for all* $x, y \in X$. *Then* (X, A, d_A) *is* C^**-algebra valued b-metric space. Define* $T : X \to X$ *as* $Tx = x^2$, *then*

$$\begin{aligned} d_A(Tx,Ty) &= \begin{pmatrix} |x^2-y^2|^p & 0 \\ 0 & k|x^2-y^2|^p \end{pmatrix} = \begin{pmatrix} |x-y|^p|x+y|^p & 0 \\ 0 & k|x-y|^p|x+y|^p \end{pmatrix} \\ &\leq 2^p.I \begin{pmatrix} |x-y|^p & 0 \\ 0 & k|x-y|^p \end{pmatrix} \end{aligned}$$

Define, $\alpha_A : X \times X \to A$, *by* $\alpha_A(x,y) = \begin{cases} \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} & \text{if } x = y = 1 \\ 0 & \text{otherwise,} \end{cases}$

it is clear that $\alpha_A(x,y) = \begin{cases} I_{M_2(\mathbb{R})} & \text{if } x = y, \\ 0 & \text{otherwise,} \end{cases}$

$\alpha_A(Tx,Ty) = \begin{cases} \begin{pmatrix} x^2 & 0 \\ 0 & y^2 \end{pmatrix} & \text{if } x = y = 1 \\ 0 & \text{otherwise,} \end{cases}$

$\alpha_A(x,y) = I_{M_2(\mathbb{R})} \Rightarrow \alpha_A(Tx,Ty) = I_{M_2(\mathbb{R})}$

So, $\alpha_A(x,y)d_A(Tx,Ty) \leq ((\sqrt{2})^p)d_A(x,y)(\sqrt{2})^p$. *So, it is satisfy the conditions of theorem 2.5, and then* T *has a fixed point* $0 \in X$.

As an application, we use the C^*-algebra-valued b-metric space to study the existence and uniqueness of the system of matrix equations in [4] by using theorem 2.5.

Example 4.2. *Application: Suppose that* $M_n(\mathbb{C})$ *the set of all* $m \times n$ *matrices with complex entries.* $M_n(\mathbb{C})$ *is a* C^**-algebra with the operator norm. Let* B_1, B_2,...,$B_n \in M_n(\mathbb{C})$ *are diagonal matrices which satisfy* $\sum_{k=1}^{n} |B_k|^2 < 1$.

Let $A = (a_{ij})_{1\leq i,j\leq n} \in M_n(\mathbb{C})$ *and* $C = (c_{ij})_{1\leq i,j\leq n} \in M_n(\mathbb{C})^+$, *where* $M_n(\mathbb{C})^+$ *denote the set of all positive definite matrices "hermitian and the eigenvalues are non-negative". Then the matrix equations*

$$A - \sum_{k=1}^{n} B_k^* A B_k = C, \tag{4.1}$$

has a unique solution.

Proof: *Set* $\alpha = \sum_{k=1}^{n} |B_k|^2$, *clear if* $\alpha = 0$, *then the equations has a unique solution in* $M_n(\mathbb{C})$. *Without loss of generality, suppose that* $\alpha > 0$. *For* $A, D \in M_n(\mathbb{C})$ *and* $p \geq 1$,
define $d_{M_n(\mathbb{C})} : M_n(\mathbb{C}) \times M_n(\mathbb{C}) \to M_n(\mathbb{C})^+$ *as*

$d_{M_n(\mathbb{C})}(A,D) = diag(\lambda_1|a_{11} - d_{11},...,\lambda_n|a_{nn} - d_{nn}|^p)$, $\lambda_1,...,\lambda_n > 0$, then $(M_n(\mathbb{C}), d_{M_n(\mathbb{C})})$ is a C^*-algebra valued b-metric space and is complete since the set $M_n(\mathbb{C})$ is complete (the proof is given in the example 1.7). Consider the map $T = (T_{ii}) : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ defined by
$T_{ii}(a_{ij})_{1\le i,j\le n} = \sum_{k=1}^{n} B_k^*(a_{ii})B_k + c_{ii}$. Define $\alpha_{M_n(\mathbb{C})} : M_n(\mathbb{C}) \times M_n(\mathbb{C}) \to M_n(\mathbb{C})^+$,
$\alpha_{M_n(\mathbb{C})}(A,B) = I_{M_n(\mathbb{C})}$, clear that T is $\alpha_{M_2(\mathbb{C})}$ admissible. Then

$$\begin{aligned} d_{M_n(\mathbb{C})}(TA,TD) &= diag(\lambda_1|(\sum_{k=1}^{n} B_k^* a_{11} B_k + c_{11}) - (\sum_{k=1}^{n} B_k^* d_{11} B_k + c_{11}),...,\lambda_n|(\sum_{k=1}^{n} B_k^* a_{nn} B_k \\ &+ c_{nn}) - (\sum_{k=1}^{n} B_k^* d_{nn} B_k + c_{nn})|^p) \\ &= diag(\lambda_1|(\sum_{k=1}^{n} B_k^*(a_{11} - d_{11})B_k|^p,...,\lambda_n|(\sum_{k=1}^{n} B_k^*(a_{nn} - d_{nn})B_k|^p) \\ &= diag(\lambda_1(\sum_{k=1}^{n} |B_k|^2)^p|a_{11} - d_{11}|^p,...,\lambda_n(\sum_{k=1}^{n} |B_k|^2)^p|a_{nn} - d_{nn}|^p) \\ &= diag(\sum_{k=1}^{n} |B_k|^2)^p(\lambda_1|a_{11} - d_{11}|^p,...,\lambda_n|a_{nn} - d_{nn}|^p) = \alpha^p d_{M_n(\mathbb{C})}(A,D). \end{aligned}$$

Therefore, T satisfy the condition of theorem 2.5 and has a fixed point. So the matrix equations (4.1) has a solution on $M_n(\mathbb{C})$. Moreover $\alpha_{M_n(\mathbb{C})}$ is satisfy the condition (H), so the system of matrix equations have a unique hermitian matrix solution A.

5. Conclusions

In this paper, we define a new version of α_A-admissible in the case of self map $T : A \to A$ and α_A-admissible in two self mappings (T, S). We prove the principal Banach fixed point theorem and two common fixed point theorems in the C^*-algebra- valued b-metric space, which generalized the given results in [18,19,26,27]. Moreover, we introduced an application to show that the useful of C^*-algebra-valued b-metric space to study the existence and unique of system matrix equations.

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Conflict of interest

The authors of this current research declaring that this study has been done without any competing intersts.

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