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Research article

Further study on the conformable fractional Gauss hypergeometric function

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Abstract: This article presents an exhaustive study on the conformable fractional Gauss hypergeometric function (CFGHF). We start by solving the conformable fractional Gauss hypergeometric differential equation (CFGHDE) about the fractional regular singular points $x = 1$ and $x = \infty$. Next, various generating functions of the CFGHF are established. We also develop some differential forms for the CFGHF. Subsequently, differential operators and contiguous relations are reported. Furthermore, we introduce the conformable fractional integral representation and the fractional Laplace transform of CFGHF. As an application, and after making a suitable change of the independent variable, we provide general solutions of some known conformable fractional differential equations, which could be written by means of the CFGHF.

Keywords: conformable fractional derivatives; Gauss hypergeometric functions; fractional differential equations; differential operators; contiguous relations
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1. Introduction

Many research efforts has been devoted to generalize the classical analysis of special functions to either the fractional calculus or the higher dimensional setting [1–3]. Fractional calculus has recently attracted considerable attention. It is defined as a generalization of differentiation and integration to an arbitrary order. It has become a fascinating branch of applied mathematics, which has recently stimulated mathematicians and physicists. Indeed, it represents a powerful tool to study a myriad of problems from different fields of science, such as statistical mechanics, control theory, signal and image processing, thermodynamics and quantum mechanics (see [4–9]).

There is a significant value in exploring conformable derivatives, which is obvious from the enormous number of their meaningful and successful applications in many fields of science in the last few years. We mention, but not limited to, some efforts which proved the vital role of conformable derivatives. For instance, the authors in [10] studied the dynamics of the traveling wave with fractional conformable nonlinear evaluation equations arising in nonlinear wave mechanics. In [11], the authors formulated the exact solutions of the time-fractional Dodd-Bullough-Mikhailov equation, Sinh-Gordon equation, and Liouville equation by utilizing simplest equation method in the conformable fractional derivative sense. Recently, in [12], Rabha et al. introduced different vitalization of the growth of COVID-19 by using controller terms based on the concept of conformable calculus. For more on the conformable and the fractal derivative and their applications to real world problems, we refer the reader to see [13–21] and the references therein. Some growing progress in the fractional differential equations describing real life phenomena has been discussed in [22–24].

Over the last four decades, several interesting and useful extensions of many of the familiar special functions, such as the Gamma, Beta, and Gauss hypergeometric functions have been considered by various authors [25–29]. Functions of hypergeometric type constitute an important class of special functions. The hypergeometric function ${}_2F_1(\mu, \nu; c; x)$ plays a significant role in mathematical analysis and its applications. This function allows one to solve many interesting mathematical and physical problems, such as conformal mapping of triangular domains bounded by line segments or circular arcs and various problems of quantum mechanics. Most of the functions that occur in the analysis are classified as special cases of the hypergeometric functions. Gauss first introduced and studied hypergeometric series, paying particular attention to the cases when a series converges to an elementary function which leads to the study of the hypergeometric series. Eventually, elementary functions and several other important functions in mathematics can be expressed in terms of hypergeometric functions. Hypergeometric functions can also be described as the solutions of special second-order linear differential equations, which are the hypergeometric differential equations. Riemann was the first to exploit this idea and introduced a special symbol to classify hypergeometric functions by singularities and exponents of differential equations. The hypergeometric function is a solution of the following Euler's hypergeometric differential equation

$$x(1-x)\frac{d^2y}{dx^2} + [c - (\mu + \nu + 1)x]\frac{dy}{dx} - \mu\nu y = 0, \tag{1.1}$$

which has three regular singular points 0, 1, and ∞ and the parameters μ, ν and c. The generalization of this equation to three arbitrary regular singular points is given by Riemann's differential equation. Any second order differential equation with three regular singular points can be converted to the hypergeometric differential equation by changing of variables.

The solution of the hypergeometric differential equation includes many of the most interesting special functions of mathematical physics and engineering, for instance, the Jacobi, Gegenbaure, Legendre, and Laguerre polynomials can be expressed in terms of the Gauss hypergeometric functions and other related hypergeometric functions. Every ordinary differential equation of second order with at most three regular singular points can be brought to the hypergeometric differential equation by means of a suitable change of variable. Recently, as a conformable fractional derivative introduced in [30], the authors in [31] used the new concept of fractional regular singular points with the technique of fractional power series to solve the CFGHDE about $x = 0$. They also introduced the form of the conformable fractional derivative and the integral representation of the fractional Gaussian function. Besides, the solution of the fractional $k-$hypergeometric differential equation was introduced in [32]. As the Gauss hypergeometric differential equation appears in many problems of physics, engineering, applied science, as well as finance and many other important problems, it largely motivates us to conduct the present study.

Motivated by the above discussion, we intend to continue the work of Abu Hammad et al. [31] by finding the solutions of the Gauss hypergeometric differential equation via conformable calculus about the fractional regular singular points $x = 1$ and $x = \infty$. Afterward, we give a wide study on the CFGHF includes deriving various generating functions involving expansions and generalizations of the CFGHF and some of the transformation formulas and differential forms. In order to deduce several of contiguous relations, we define a conformable fractional operator. Furthermore, integral representations and Laplace transform of CFGHF in the context of conformable calculus are established. As an application, we give general solutions of a class of conformable fractional differential equations (CFDE), which can be written in terms of the CFGHF.

The structure of this paper is formulated as follows. In section 2, we provide some basic concepts and notations which are essential in the sequel . Section 3 is devoted to the solutions of the conformable fractional Gauss hypergeometric differential equation. Various generating functions of CFGHF are established in section 4. In section 5, we present some of transformation formulas and differential forms. We define a differential operator, then use it to establish several of contiguous relations in section 6. Conformable fractional integral representations are derived in section 7. Various recursion formulas are obtained in section 8. In section 9, we establish the fractional Laplace transform of CFGHF and some useful related identities. General solutions of some interesting CFDEs are obtained by means of CFGHF in section 10. We append our study by pointing out some general remarks and conclusions in section 11.

2. Preliminaries and basic concepts

Various definitions of fractional derivatives are obtained and compared. The most commonly used definitions in literature are due to Riemann-Liouvellie and Caputo. The Riemann-Liouville and Caputo fractional derivatives are non-local operators represented by convolution integrals with weakly singular kernels. Although the non-local fractional derivatives give natural memory and genetic effects in the physical system, the fractional derivatives obtained in this kind of calculus seem very complicated and lose some basic properties of general derivatives, such as product rule and chain rule and others, see for instance [33]. Accordingly, Khalil et al. in [30] introduced a definition of a local kind derivative called from the authors "conformable fractional derivative". The significance of this definition lies on the fact that their derivative stratifies almost all the well-known properties of the integer-order one. Abdeljawad [34] made extensive research on the newly introduced conformable fractional calculus. Also, Martynyuk [35] presented a physical interpretation of such conformable derivative. Afterwords, more than hundred published articles appeared relying this derivative see [36] and references therein.

In [37], Anderson and Ulness made a remark, that since the derivative is local, they suggested the name to be "conformable derivative" instead as introduced "conformable fractional derivative". Some authors [35, 38–40] use the name "fractional-like" instead "conformable" derivative. Since we still have the derivative $f^{(\alpha)}(x)$, for $0 < \alpha \leq 1$, we shall keep the original name "conformable fractional derivative".

The authors in [36] studied the relationship between the conformable derivatives of different order. They obtained a surprising result that a function has a conformable derivative at a point if and only if it has a first-order derivative at the same point, and that holds for all points except the lower terminal. They also answered the question, "what happens in the lower terminal?" From this point of view, the same authors [36] concluded that "not all types of initial value problems involving conformable derivatives are transformed to well studied initial value problems with integer-order derivatives." This phenomenon was supported via an example given in [33].

We briefly state the main advantages of using the conformable derivatives as follows:

(1) It simulates all the concepts and properties of an ordinary derivative such as quotient, product, and chain rules while the other fractional definitions fail to satisfy these rules.
(2) Non-differentiable functions can be differentiated in the conformable sense.
(3) It generalizes the well-known transforms such as Laplace and Sumudu transforms and is used as a tool for solving some singular fractional differential equations.
(4) It paves the way for new comparisons and applications.
(5) It can be extended to solve PDEs exactly and numerically as it was given in the literature.

For the sake of clarity and avoiding ambiguity, we recall the definitions of both the conformable derivative and conformable integral introduced in [30] as well as some of their properties.

Definition 2.1. *Let $f : \Omega \subseteq (0, \infty) \to \mathbb{R}$ and $x \in \Omega$. The conformable fractional derivative of order $\alpha \in (0, 1]$ for f at x is defined as*

$$D^{\alpha} f(x) = \lim_{h \to 0} \frac{f\left(x + hx^{1-\alpha}\right) - f(x)}{h},$$

whenever the limit exists. The function f is called $\alpha-$ conformable fractional differentiable at x. For $x = 0$, $D^{\alpha} f(0) = \lim_{h \to 0^+} D^{\alpha} f(x)$ if such a limit exists.

This definition carries very important and natural properties. Let D^{α} denote the conformable fractional derivative (CFD) operator of order α. We recall from [30, 34, 41] some of its general properties as follows.

Let f and g be $\alpha-$differentiable. Then, we have

(1) Linearity: $D^{\alpha}(af + bg)(t) = aD^{\alpha} f(t) + bD^{\alpha} g(t)$, for all $a, b \in \mathbb{R}$.
(2) Product rule: $D^{\alpha}(fg)(t) = f(t) D^{\alpha} g(t) + g(t) D^{\alpha} f(t)$.
(3) Quotient rule: $D^{\alpha}\left(\frac{f}{g}\right)(t) = \frac{g(t)D^{\alpha} f(t) - f(t)D^{\alpha} g(t)}{g^2(t)}$, where $g(t) \neq 0$.
(4) Chain rule: $D^{\alpha}(f \circ g)(t) = D^{\alpha} f(g(t)) D^{\alpha} g(t) g(t)^{\alpha-1}$.

Notice that for $\alpha = 1$ in the $\alpha-$ conformable fractional derivative, we get the corresponding classical limit definition of the derivative. Also, a function could be $\alpha-$ conformable differentiable at a point but not differentiable in the ordinary sense. For more details, we refer to [30, 34, 42].

Any linear homogeneous differential equations of order two with three regular singularities can be reduced to (1.1). The hypergeometric function is known as a solution to the hypergeometric equation (1.1). One of the solutions of the hypergeometric equation is given by the following Gauss hypergeometric series in the form

$$ {}_2F_1(\mu, \nu; c; x) = \sum_{n=0}^{\infty} \frac{(\mu)_n (\nu)_n}{(c)_n} \frac{x^n}{n!} \qquad (|x| < 1) \tag{2.1}$$

where $(b)_n$ stands for the usual Pochhammer symbol defined by

$$(b)_n = b(b+1)(b+2)...(b+n-1) = \frac{\Gamma(b+n)}{\Gamma(b)}, \ \ n \in \mathbb{N} \text{ and } (b)_0 = 1.$$

Choosing the values of the parameters μ, ν, and c in an appropriately, one can obtain many elementary and special functions as particular cases of the Gauss hypergeometric series. For instance, the complete elliptic integrals of the first and the second kinds, the Legendre associated functions, ultra-spherical polynomials, and many others are special cases of the function ${}_2F_1(\mu, \nu; c; x)$.

Definition 2.2. *[43] Two hypergeometric functions are said to be contiguous if their parameters μ, ν, and c differ by integers. The relations made by contiguous functions are said to be contiguous function relations.*

Definition 2.3. *The point $x = a$ is called an $\alpha-$regular singular point for the equation*

$$D^{\alpha}D^{\alpha}y + P(x)D^{\alpha}y + Q(x)y = 0, \tag{2.2}$$

if $\lim\limits_{x\to a}(x^{\alpha} - a)P(x)$ and $\lim\limits_{x\to a}(x^{\alpha} - a)^2 Q(x)$ exist.

Definition 2.4. *[34] A series $\sum\limits_{n=0}^{\infty} a_n x^{\alpha n}$ is called a fractional Maclaurin power series.*

Remark 2.1. *We will use $D^{n\alpha}$ to denote $\underbrace{D^{\alpha}D^{\alpha}...D^{\alpha}}_{n-times}$. If $D^{n\alpha}f$ exists for all n in some interval $[0, \lambda]$ then one can write f in the form of a fractional power series*

Definition 2.5. *[30] Suppose that $f : (0, \infty) \to \mathbb{R}$ is α-differentiable, $\alpha \in (0, 1]$, then the α-fractional integral of f is defined by*

$$I_{\alpha}^{a}f(t) = I_1^a\left(t^{\alpha-1}f\right) = \int\limits_a^t \frac{f(x)}{x^{1-\alpha}}dx, \ t \geq 0.$$

For the infinite double series, we have the following useful Lemma (see [44]), which will be used in the sequel.

Lemma 2.1.

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} a_{k,n} = \sum_{m=0}^{\infty}\sum_{j=0}^{m} a_{j,m-j} = \sum_{n=0}^{\infty}\sum_{k=0}^{n} a_{k,n-k}, \tag{2.3}$$

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} b_{k,n} = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} b_{k,n+k}. \tag{2.4}$$

3. Solutions of the conformable fractional Gauss hypergeometric differential equation

In our current study, we are interested to consider a generalization of the differential equation (1.1) to fractional Gauss hypergeometric differential equation, where the involving derivative is CFD. More precisely, we study the equation in the form

$$x^{\alpha}(1 - x^{\alpha})D^{\alpha}D^{\alpha}y + \alpha\left[c - (\mu + \nu + 1)x^{\alpha}\right]D^{\alpha}y - \alpha^2\mu\nu y = 0, \tag{3.1}$$

where $\alpha \in (0, 1]$ and μ, ν and c are reals such that $c \neq 0, -1, -2, \ldots$.

The new concept of fractional regular singular point together with the technique of fractional power series are used to solve the CFGHDE (3.1).

Dividing (3.1) by $x^{\alpha}(1 - x^{\alpha})$, we get

$$D^{\alpha}D^{\alpha}y + \frac{\alpha\{c - (\mu + \nu + 1)x^{\alpha}\}}{x^{\alpha}(1 - x^{\alpha})}D^{\alpha}y - \frac{\alpha^2\mu\nu}{x^{\alpha}(1 - x^{\alpha})}y = 0. \tag{3.2}$$

Comparing (3.2) with (2.2), we have

$$P(x) = \frac{\alpha\{c - (\mu + \nu + 1)x^{\alpha}\}}{x^{\alpha}(1 - x^{\alpha})} \text{ and } Q(x) = \frac{-\alpha^2\mu\nu}{x^{\alpha}(1 - x^{\alpha})}.$$

Clearly $x = 0,\ x = 1$ and $x = \infty$ are $\alpha-$regular singular points for (3.1).

Recently, in [31], the authors used the technique of fractional power series to obtain the general solution of (3.1) about $x = 0$ as

$$y = A\ {}_2F_1(\mu, \nu; c; x^{\alpha}) + B\ x^{\alpha(1-c)}\ {}_2F_1(1 - c + \mu, 1 - c + \nu; 2 - c; x^{\alpha}),$$

where A and B are arbitrary constants and ${}_2F_1(\mu, \nu; c; x^{\alpha})$ is CFGHF defined by

$${}_2F_1(\mu, \nu; c; x^{\alpha}) = \sum_{n=0}^{\infty}\frac{(\mu)_n(\nu)_n}{(c)_n\, n!}x^{\alpha n}; \quad |x^{\alpha}| < 1. \tag{3.3}$$

In this section, we will use a similar technique of [31] to solve the Eq (3.1) about the two $\alpha-$regular singular points $x = 1$ and $x = \infty$.

3.1. Solution of the CFGHE about $x = 1$

As $x = 1$ is an $\alpha-$regular singular point of (3.1), therefore, the solution of (3.1) can be obtained in a series of powers of $(x^{\alpha} - 1)$ as follows:

Taking $x^{\alpha} = 1 - t^{\alpha}$, this transfers the point $x = 1$ to the point $t = 0$ and therefore, we obtain the series solution of the following transformed CFDE in terms of the series of powers of t^{α}:

$$t^{\alpha}(1 - t^{\alpha})D^{\alpha}D^{\alpha}y(t) + \alpha\{[\mu + \nu + 1 - c] - (\mu + \nu + 1)t^{\alpha}\}D^{\alpha}y(t) - \alpha^2\mu\nu y(t) = 0. \tag{3.4}$$

Putting $c' = \mu + \nu + 1 - c$, in (3.4), we get

$$t^{\alpha}(1 - t^{\alpha})D^{\alpha}D^{\alpha}y(t) + \alpha\{c' - (\mu + \nu + 1)t^{\alpha}\}D^{\alpha}y(t) - \alpha^2\mu\nu y(t) = 0. \tag{3.5}$$

This conformable fractional differential equation is similar to CFGHE (3.1). So, the two linearly independent solutions of (3.5) can be stated in the form

$$y_1 = {}_2F_1(\mu, \nu; c'; t^{\alpha}) \text{ and } y_2 = t^{\alpha(1-c)}\ {}_2F_1(1 - c' + \mu, 1 - c' + \nu; 2 - c'; t^{\alpha}). \tag{3.6}$$

Now, replacing c' by $(\mu + \nu + 1 - c)$ and t^{α} by $(1 - x^{\alpha})$ in (3.6), we get

$$y_1 = {}_2F_1(\mu, \nu; \mu + \nu + 1 - c; t^{\alpha})$$

and

$$y_2 = (1 - x^{\alpha})^{(c-\mu-\nu)} \ {}_2F_1 \left(c - \nu, c - \mu; c - \mu - \nu + 1; 1 - x^{\alpha}\right).$$

Thus, the general solution of Eq (3.1) about $x = 1$ is given by

$$\begin{aligned} y =& A \ {}_2F_1 \left(\mu, \nu; \mu + \nu + 1 - c; t^{\alpha}\right) \\ &+ B \ (1 - x^{\alpha})^{(c-\mu-\nu)} \ {}_2F_1 \left(c - \nu, c - \mu; c - \mu - \nu + 1; 1 - x^{\alpha}\right), \end{aligned} \tag{3.7}$$

where A and B are arbitrary constants.

3.2. Solution of the CFGHE about $x = \infty$

As $x = \infty$ is an $\alpha-$regular singular point of (3.1), thus, the solution of (3.1) can be obtained in a series about $x = \infty$ by putting $x^{\alpha} = \frac{1}{\zeta^{\alpha}}$ in (3.1). Therefore,

$$D_x^{\alpha} \ y = -\zeta^{2\alpha} D_{\zeta}^{\alpha} \ y \ \text{ and } D_x^{\alpha} D_x^{\alpha} \ y = \left[2\alpha\zeta^{3\alpha} D_{\zeta}^{\alpha} \ y + \zeta^{4\alpha} \ D_{\zeta}^{\alpha} D_{\zeta}^{\alpha} \ y\right]. \tag{3.8}$$

In view of (3.1), we get

$$\zeta^{2\alpha} \left(1 - \zeta^{\alpha}\right) D_{\zeta}^{\alpha} D_{\zeta}^{\alpha} \ y + \alpha \left\{2\zeta^{\alpha} \left(1 - \zeta^{\alpha}\right) + c\zeta^{2\alpha} - \zeta^{\alpha} \left(\mu + \nu + 1\right)\right\} D_{\zeta}^{\alpha} \ y \ + \alpha^2 \mu\nu y = 0. \tag{3.9}$$

Now, to find the solution, we proceed as follows. Let $y = \sum\limits_{n=0}^{\infty} a_n \zeta^{\alpha(s+n)}; \ a_0 \neq 0$ be the series solution of Eq (3.9) about $\zeta = 0$. Then from the basic properties of the CFD, we get

$$D_{\zeta}^{\alpha} y = \sum_{n=0}^{\infty} a_n \ \alpha \left(s + n\right) \zeta^{\alpha(s+n-1)} \text{ and } D_{\zeta}^{\alpha} D_{\zeta}^{\alpha} y = \sum_{n=0}^{\infty} a_n \ \alpha^2 \left(s + n\right) \left(s + n - 1\right) \zeta^{\alpha(s+n-2)}$$

Thus, owing to (3.9), we have

$$\begin{aligned} \zeta^{2\alpha} \left(1 - \zeta^{\alpha}\right) \sum_{n=0}^{\infty} a_n \ \alpha^2 \left(s + n\right) \left(s + n - 1\right) \zeta^{\alpha(s+n-2)} + \alpha \left\{2\zeta^{\alpha} \left(1 - \zeta^{\alpha}\right) + c\zeta^{2\alpha} - \zeta^{\alpha} \left(\mu + \nu + 1\right)\right\} \\ \times \sum_{n=0}^{\infty} a_n \ \alpha \left(s + n\right) \zeta^{\alpha(s+n-1)} + \alpha^2 \mu\nu \sum_{n=0}^{\infty} a_n \zeta^{\alpha(s+n)} = 0. \end{aligned}$$

Therefore,

$$\begin{aligned} &\sum_{n=0}^{\infty} \alpha^2 a_n \left[(s + n) (s + n - 1) + 2 \ (s + n) - (\mu + \nu + 1) (s + n) + \mu\nu\right] \zeta^{\alpha(s+n)} \\ &- \sum_{n=0}^{\infty} \alpha^2 a_n \left[(s + n) (s + n - 1) + 2 \ (s + n) - c \ (s + n)\right] \zeta^{\alpha(s+n+1)} = 0. \end{aligned}$$

Then we have

$$\begin{aligned} &\alpha^2 a_0 \left[s (s - 1) + 2 \ s - s (\mu + \nu + 1) + \mu\nu\right] \zeta^{\alpha s} \\ &+ \sum_{n=1}^{\infty} \alpha^2 a_n \left[(s + n) (s + n - 1) + 2 \ (s + n) - (\mu + \nu + 1) (s + n) + \mu\nu\right] \zeta^{\alpha(s+n)} \\ &- \sum_{n=0}^{\infty} \alpha^2 a_n \left[(s + n) (s + n - 1) + 2 \ (s + n) - c \ (s + n)\right] \zeta^{\alpha(s+n+1)} = 0. \end{aligned}$$

A shift of index yields

$$\begin{aligned}
&\alpha^2 a_0\left[s(s-1)+2\ s-s(\mu+\nu+1)+\mu\nu\right]\zeta^{\alpha s}\\
&+\alpha^2\sum_{n=0}^{\infty}a_{n+1}\left[(s+n+1)(s+n)+2\ (s+n+1)-(\mu+\nu+1)(s+n+1)+\mu\nu\right]\\
&-a_n\left[(s+n)(s+n-1)+2\ (s+n)-c\ (s+n)\right]\zeta^{\alpha(s+n+1)}=0.
\end{aligned} \tag{3.10}$$

Equating the coefficients of $\zeta^{\alpha s}$ to zero in (3.10), we get the following indicial equation

$$s^2-s(\mu+\nu)+\mu\nu=0 \tag{3.11}$$

This Eq (3.11) has two indicial roots $s=s_1=\mu$ and $s=s_2=\nu$.
Again, equating to zero the coefficient of $\zeta^{\alpha(s+n+1)}$ in (3.10), yields the recursion relation for a_n

$$a_{n+1}=\frac{\left[(s+n)(s+n-1)+2\ (s+n)-c\ (s+n)\right]}{\left[(s+n+1)(s+n)+2\ (s+n+1)-(\mu+\nu+1)(s+n+1)+\mu\nu\right]}a_n, \tag{3.12}$$

or

$$a_{n+1}=\frac{(s+n)(s+n+1-c)}{(s+n+1)\left[(s+n+1)-\mu-\nu\right]+\mu\nu}a_n \tag{3.13}$$

To find the first solution of (3.9), we put $s=\mu$ in (3.13) to get

$$a_{n+1}=\frac{(\mu+n)(\mu+n+1-c)}{(\mu+n+1)\left[(n+1)-\nu\right]+\mu\nu}a_n=\frac{(\mu+n)(\mu+n+1-c)}{(n+1)\left[(n+1)+\mu-\nu\right]}a_n.$$

Note that if $n=0$, one can see

$$a_1=\frac{(\mu)(\mu-c+1)}{[\mu-\nu+1]}a_0.$$

and for $n=1$, we obtain

$$a_2=\frac{(\mu+1)(\mu-c+2)}{[\mu-\nu+2]}a_1=\frac{(\mu)(\mu+1)(\mu-c+1)(\mu-c+2)}{2\,[\mu-\nu+1]\,[\mu-\nu+2]}a_0.$$

Using the Pochhammer symbol, we have

$$a_2=\frac{(\mu)_2(\mu-c+1)_2}{2!\,(\mu-\nu+1)_2}a_0.$$

In general, we may write

$$a_n=\frac{(\mu)_n(\mu-c+1)_n}{n!\,(\mu-\nu+1)_n}a_0. \tag{3.14}$$

Letting $a_0=A$, the first solution y_1 is given by

$$\begin{aligned}
y_1=&A\sum_{n=0}^{\infty}\frac{(\mu)_n(\mu-c+1)_n}{(\mu-\nu+1)_n}\frac{\zeta^{\alpha(\mu+n)}}{n!}=A\zeta^{\alpha\mu}\ {}_2F_1\left(\mu,\mu-c+1;\mu-\nu+1;\zeta^{\alpha}\right)\\
=&Ax^{-\alpha\mu}\ {}_2F_1\left(\mu,\mu-c+1;\mu-\nu+1;\frac{1}{x^{\alpha}}\right)
\end{aligned}$$

To find the second solution of (3.9), putting $s = \nu$ in (3.13), we have

$$a_{n+1} = \frac{(\nu+n)(\nu+n+1-c)}{(\nu+n+1)[(n+1)-\mu]+\mu\nu}a_n = \frac{(\nu+n)(\nu+n+1-c)}{(n+1)[(n+1)+\nu-\mu]}a_n,$$

from which we get

$$a_1 = \frac{(\nu)(\nu-c+1)}{[\nu-\mu+1]}a_0.$$

Thus

$$a_2 = \frac{(\nu+1)(\nu-c+2)}{[\nu-\mu+2]}a_1 = \frac{(\nu)(\nu+1)(\nu-c+1)(\nu-c+2)}{2[\nu-\mu+1][\nu-\mu+2]}a_0.$$

Again by Pochhammer symbol yields

$$a_2 = \frac{(\nu)_2(\nu-c+1)_2}{2!(\nu-\mu+1)_2}a_0$$

and in general

$$a_n = \frac{(\nu)_n(\nu-c+1)_n}{n!(\nu-\mu+1)_n}a_0 \tag{3.15}$$

Putting $a_0 = B$, the second solution y_2 is given by

$$\begin{aligned} y_2 =& B\sum_{n=0}^{\infty}\frac{(\nu)_n(\nu-c+1)_n}{(\nu-\mu+1)_n}\frac{\zeta^{\alpha(\nu+n)}}{n!} = B\zeta^{\alpha\nu}\ {}_2F_1(\nu,\nu-c+1;\nu-\mu+1;\zeta^\alpha) \\ =& Bx^{-\alpha\nu}\ {}_2F_1\left(\nu,\nu-c+1;\nu-\mu+1;\frac{1}{x^\alpha}\right). \end{aligned}$$

Therefore, the general solution of (3.1) about $x = \infty$ is

$$y = Ax^{-\alpha\mu}\ {}_2F_1\left(\mu,\mu-c+1;\mu-\nu+1;\frac{1}{x^\alpha}\right) + Bx^{-\alpha\nu}\ {}_2F_1\left(\nu,\nu-c+1;\nu-\mu+1;\frac{1}{x^\alpha}\right),$$

where A and B are arbitrary constants.

Remark 3.1. *It is worth mentioning that the presented CFDE* (3.9) *is distinct to that one which was treated in [31, 32]. In fact* (3.9) *extended the Gauss hypergeometric differential equation given in [45] to the conformable fractional context.*

4. Generating functions

Generating functions provide an important way to transform formal power series into functions and to analyze asymptotic properties of sequences. In what follows, we characterize the CFGHF by means of various generating functions.

Theorem 4.1. *For $\alpha \in (0, 1]$, the following generating function holds true*

$$\sum_{m=0}^{\infty}(\mu)_m\ {}_2F_1(\mu+m,\nu;c;x^\alpha)\,.\frac{t^{\alpha m}}{m!} = (1-t^\alpha)^{-\mu}\ {}_2F_1\left(\mu,\nu;c;\frac{x^\alpha}{1-t^\alpha}\right), \tag{4.1}$$

where $|x^\alpha| < 1$, and $|t^\alpha| < 1$.

Proof. For convenience, let $\Im$ denote the left-hand side of (4.1). In view of (3.3), it follows that

$$\Im = \sum_{m=0}^{\infty} (\mu)_m \left\{ \sum_{n=0}^{\infty} \frac{(\mu+m)_n (\nu)_n}{(c)_n} \frac{x^{\alpha n}}{n!} \right\} . \frac{t^{\alpha m}}{m!}. \tag{4.2}$$

Changing the order of summations in (4.2) and make use of identity $(\mu)_m (\mu+m)_n = (\mu)_{m+n} = (\mu)_n (\mu+n)_m$, yields

$$\Im = \sum_{n=0}^{\infty} \frac{(\mu)_n (\nu)_n}{(c)_n} \frac{x^{\alpha n}}{n!} . \sum_{m=0}^{\infty} \frac{(\mu+n)_m}{m!} t^{\alpha m} .$$

Using the equality $\sum_{m=0}^{\infty} \frac{(\mu+n)_m}{m!} t^{\alpha m} = (1-t^{\alpha})^{-(\mu+n)}$; $|t^{\alpha}| < 1$ and the definition (3.3) immediately leads to the required result. □

Theorem 4.2. *For* $\alpha \in (0,1]$, *we have the following relation*

$$\sum_{m=0}^{\infty} (\mu)_m \ {}_2F_1 (-m, \nu; c; x^{\alpha}) . \frac{t^{\alpha m}}{m!} = (1-t^{\alpha})^{-\mu} \ {}_2F_1 \left(\mu, \nu; c; \frac{-x^{\alpha} t^{\alpha}}{1-t^{\alpha}} \right), \tag{4.3}$$

where $|x^{\alpha}| < 1$, *and* $|t^{\alpha}| < 1$

Proof. For short, set $\Im$ to denote the left-hand side of (4.3). Using (3.3), one gets

$$\Im = \sum_{m=0}^{\infty} (\mu)_m \left\{ \sum_{n=0}^{\infty} \frac{(-m)_n (\nu)_n}{(c)_n} \frac{x^{\alpha n}}{n!} \right\} . \frac{t^{\alpha m}}{m!} \tag{4.4}$$

Since $(-m)_n = 0$ if $n > m$, then we may write

$$\Im = \sum_{m=0}^{\infty} \sum_{n=0}^{m} \frac{(\mu)_m}{m!} \frac{(-m)_n (\nu)_n}{(c)_n \, n!} x^{\alpha n} t^{\alpha m}$$

Using the congruence relation $(-m)_n = \frac{(-1)^n m!}{(m-n)!}$, we get

$$\Im = \sum_{m=0}^{\infty} \sum_{n=0}^{m} \frac{(-1)^n (\mu)_m (\nu)_n}{(c)_n (m-n)! n!} x^{\alpha n} t^{\alpha m} \tag{4.5}$$

Using lemma 2.1, Eq (4.5) becomes

$$\Im = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^n (\mu)_{n+m} (\nu)_n}{(c)_n \, m! n!} x^{\alpha n} t^{\alpha(n+m)} \tag{4.6}$$

Changing the order of summations in (4.6) and make use of identity $(\mu)_{n+m} = (\mu)_n (\mu+n)_m$, we get

$$\Im = \sum_{n=0}^{\infty} \frac{(\mu)_n (\nu)_n}{(c)_n \, n!} (-x^{\alpha} t^{\alpha})^n . \sum_{m=0}^{\infty} \frac{(\mu+n)_m}{m!} t^{\alpha m}$$

Using the binomial relation $\sum\limits_{m=0}^{\infty} \frac{(\mu+n)_m}{m!} t^{\alpha m} = (1-t^{\alpha})^{-(\mu+n)}$, $(|t^{\alpha}| < 1)$, it follows that

$$\begin{aligned}\mathfrak{I} &= \sum_{n=0}^{\infty} \frac{(\mu)_n (\nu)_n}{(c)_n \, n!} (-x^{\alpha} t^{\alpha})^n . (1-t^{\alpha})^{-(\mu+n)} = (1-t^{\alpha})^{-\mu} \sum_{n=0}^{\infty} \frac{(\mu)_n (\nu)_n}{(c)_n \, n!} \left(\frac{-x^{\alpha} t^{\alpha}}{1-t^{\alpha}} \right)^n \\ &= (1-t^{\alpha})^{-\mu} \ {}_2F_1\left(\mu, \nu; c; \frac{-x^{\alpha} t^{\alpha}}{1-t^{\alpha}}\right).\end{aligned}$$

□

Theorem 4.3. *For* $\alpha \in (0,1]$, *the following generating relation is valid*

$$\sum_{m=0}^{\infty} \frac{(\mu)_m (\nu)_m}{(c)_m} \, {}_2F_1(\mu+m, \nu+m; c+m; x^{\alpha}) . \frac{t^{\alpha m}}{m!} = {}_2F_1(\mu, \nu; c; x^{\alpha}+t^{\alpha}), \tag{4.7}$$

where $|x^{\alpha}| < 1$, $|t^{\alpha}| < 1$, *and* $|x^{\alpha}+t^{\alpha}| < 1$.

Proof. Let $\mathfrak{I}$ denote the left-hand side of (4.7), then using (3.3), we obtain

$$\mathfrak{I} = \sum_{m=0}^{\infty} \frac{(\mu)_m (\nu)_m}{(c)_m} \left\{ \sum_{n=0}^{\infty} \frac{(\mu+m)_n (\nu+m)_n}{(c+m)_n} \frac{x^{\alpha n}}{n!} \right\} . \frac{t^{\alpha m}}{m!}$$

With the help of $(\mu)_{n+m} = (\mu)_m (\mu+m)_n$, we get

$$\mathfrak{I} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\mu)_{m+n} (\nu)_{m+n}}{(c)_{m+n} \, n! m!} x^{\alpha n} t^{\alpha m}.$$

Using lemma 2.1, we have

$$\begin{aligned}\mathfrak{I} &= \sum_{m=0}^{\infty} \sum_{n=0}^{m} \frac{(\mu)_m (\nu)_m}{(c)_m \, n! \, (m-n)!} x^{\alpha n} t^{\alpha(m-n)} \\ &= \sum_{m=0}^{\infty} \frac{(\mu)_m (\nu)_m}{(c)_m \, m!} \sum_{n=0}^{m} \frac{m!}{n! \, (m-n)!} x^{\alpha n} t^{\alpha(m-n)}.\end{aligned}$$

The binomial theorem immediately gives

$$\begin{aligned}\mathfrak{I} &= \sum_{m=0}^{\infty} \frac{(\mu)_m (\nu)_m}{(c)_m \, m!} (x^{\alpha}+t^{\alpha})^m \\ &= {}_2F_1(\mu, \nu; c; x^{\alpha}+t^{\alpha})\end{aligned}$$

as required. □

5. Transmutation formulas and differential forms

5.1. Transmutation formulas

Theorem 5.1. *For $|x^\alpha| < 1$ and $\left|\frac{x^\alpha}{1-x^\alpha}\right| < 1$, the following identity holds true*

$$ {}_2F_1(\mu, \nu; c; x^\alpha) = (1-x^\alpha)^{-\mu}\ {}_2F_1\left(\mu, c-\nu; c; \frac{-x^\alpha}{1-x^\alpha}\right). \tag{5.1}$$

Proof. Consider

$$(1-x^\alpha)^{-\mu}\ {}_2F_1\left(\mu, c-\nu; c; \frac{-x^\alpha}{1-x^\alpha}\right) = \sum_{k=0}^{\infty} \frac{(-1)^k (\mu)_k (c-\nu)_k}{(c)_k\, k!} x^{\alpha k} (1-x^\alpha)^{-(k+\mu)}.$$

In view of the expansion $(1-x^\alpha)^{-\mu} = \sum\limits_{n=0}^{\infty} \frac{(\mu)_n}{n!} x^{\alpha n}$; $|x^\alpha| < 1$, we may write

$$(1-x^\alpha)^{-\mu}\ {}_2F_1\left(\mu, c-\nu; c; \frac{-x^\alpha}{1-x^\alpha}\right) = \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} \frac{(-1)^k (\mu)_k (c-\nu)_k (\mu+k)_n}{(c)_k\, k! n!} x^{\alpha(k+n)}.$$

Using the identity $(\mu+k)_n = (\mu)_{k+n}$, it is obvious

$$(1-x^\alpha)^{-\mu}\ {}_2F_1\left(\mu, c-\nu; c; \frac{-x^\alpha}{1-x^\alpha}\right) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^k (\mu)_{k+n} (c-\nu)_k}{(c)_k\, k! n!} x^{\alpha(k+n)}. \tag{5.2}$$

In virtue of lemma 2.1 and using the fact $(-n)_k = \frac{(-1)^k n!}{(n-k)!}$, one easily gets

$$(1-x^\alpha)^{-\mu}\ {}_2F_1\left(\mu, c-\nu; c; \frac{-x^\alpha}{1-x^\alpha}\right) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{(-n)_k (c-\nu)_k}{(c)_k\, k!} \frac{(\mu)_n x^{\alpha n}}{n!}. \tag{5.3}$$

Since $(-n)_k = 0$ if $k > n$, then (5.3) becomes

$$(1-x^\alpha)^{-\mu}\ {}_2F_1\left(\mu, c-\nu; c; \frac{-x^\alpha}{1-x^\alpha}\right) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-n)_k (c-\nu)_k}{(c)_k\, k!} \frac{(\mu)_n x^{\alpha n}}{n!}. \tag{5.4}$$

Since the inner sum on the right of (5.4) is a terminating hypergeometric series, then

$$(1-x^\alpha)^{-\mu}\ {}_2F_1\left(\mu, c-\nu; c; \frac{-x^\alpha}{1-x^\alpha}\right) = \sum_{n=0}^{\infty} {}_2F_1(-n, c-\nu; c; 1) \frac{(\mu)_n x^{\alpha n}}{n!}. \tag{5.5}$$

Due to ${}_2F_1(-n, c-\nu; c; 1) = \frac{(\nu)_n}{(c)_n}$, the proof is therefore completed. □

Theorem 5.2. *For $|x^\alpha| < 1$, the following identity is true*

$$ {}_2F_1(\mu, \nu; c; x^\alpha) = (1-x^\alpha)^{c-\mu-\nu}\ {}_2F_1(c-\mu, c-\nu; c; x^\alpha). \tag{5.6}$$

Proof. By using assertion of theorem 5.1 and assuming that $y^\alpha = \frac{-x^\alpha}{1-x^\alpha}$, it follows that

$$ {}_2F_1(\mu, c-\nu; c; y^\alpha) = (1-y^\alpha)^{-(c-\nu)}\ {}_2F_1\left(c-\mu, c-\nu; c; \frac{-y^\alpha}{1-y^\alpha}\right). \tag{5.7} $$

From the assumption, we have $x^\alpha = \frac{-y^\alpha}{1-y^\alpha}$ which gives

$$ {}_2F_1\left(\mu, c-\nu; c; \frac{-x^\alpha}{1-x^\alpha}\right) = (1-x^\alpha)^{(c-\nu)}\ {}_2F_1(c-\mu, c-\nu; c; x^\alpha). \tag{5.8} $$

With a combinations of (5.8) and (5.1), the result follows. □

5.2. Some differential forms

According to the notation $D^{\alpha n}$, and due to the fact $D^\alpha x^p = px^{p-\alpha}$ with $\alpha \in (0,1]$ and $|x^\alpha| < 1$, we state some interesting conformable fractional differential formulas for ${}_2F_1(\mu, \nu; c; x^\alpha)$ as follows

$$ D^\alpha\ {}_2F_1(\mu, \nu; c; x^\alpha) = \frac{\alpha\mu\nu}{c}\ {}_2F_1(\mu+1, \nu+1; c+1; x^\alpha), \tag{5.9} $$

$$ D^{n\alpha}\ {}_2F_1(\mu, \nu; c; x^\alpha) = \frac{\alpha^n(\mu)_n(\nu)_n}{(c)_n}\ {}_2F_1(\mu+n, \nu+n; c+n; x^\alpha), \tag{5.10} $$

$$ D^{n\alpha}\ \left\{x^{\alpha(\mu+n-1)}\ {}_2F_1(\mu, \nu; c; x^\alpha)\right\} = \alpha^n(\mu)_n\ x^{\alpha(\mu-1)}\ {}_2F_1(\mu+n, \nu; c; x^\alpha), \tag{5.11} $$

$$ D^{n\alpha}\ \left\{x^{\alpha(c-1)}\ {}_2F_1(\mu, \nu; c; x^\alpha)\right\} = \alpha^n(c-n)_n\ x^{\alpha(c-n-1)}\ {}_2F_1(\mu, \nu; c-n; x^\alpha), \tag{5.12} $$

$$ \begin{aligned} &D^{n\alpha}\ \left\{x^{\alpha(c-\mu+n-1)}\ (1-x^\alpha)^{\mu+\nu-c}\ {}_2F_1(\mu, \nu; c; x^\alpha)\right\}, \\ &= \alpha^n(c-\mu)_n\ x^{\alpha(c-\mu-1)}(1-x^\alpha)^{\mu+\nu-c-n}\ {}_2F_1(\mu-n, \nu; c; x^\alpha), \end{aligned} \tag{5.13} $$

$$ \begin{aligned} &D^{n\alpha}\ \{\ (1-x^\alpha)^{\mu+\nu-c}\ {}_2F_1(\mu, \nu; c; x^\alpha)\} \\ &= \frac{\alpha^n(c-\mu)_n(c-\gamma)_n}{(c)_n}\ (1-x^\alpha)^{\mu+\nu-c-n}\ {}_2F_1(\mu, \nu; c+n; x^\alpha), \end{aligned} \tag{5.14} $$

$$ \begin{aligned} &D^{n\alpha}\ \left\{x^{\alpha(c-1)}\ (1-x^\alpha)^{\mu+\nu-c}\ {}_2F_1(\mu, \nu; c; x^\alpha)\right\} \\ &= \alpha^n(c-n)_n\ x^{\alpha(c-n-1)}(1-x^\alpha)^{\mu+\nu-c-n}\ {}_2F_1(\mu-n, \nu-n; c-n; x^\alpha), \end{aligned} \tag{5.15} $$

$$ \begin{aligned} &D^{n\alpha}\ \left\{x^{\alpha(n+c-1)}\ (1-x^\alpha)^{n+\mu+\nu-c}\ {}_2F_1(\mu+n, \nu+n; c+n; x^\alpha)\right\} \\ &= \alpha^n(c)_n\ x^{\alpha(c-1)}(1-x^\alpha)^{\mu+\nu-c}\ {}_2F_1(\mu, \nu; c; x^\alpha). \end{aligned} \tag{5.16} $$

Such formulas can be proved using the series expansions of ${}_2F_1(\mu, \nu; c; x^\alpha)$ as given in (3.3). However, we are going to prove the validity of (5.11) and (5.14), while the other formulas can be proved similarly. First, note that

$$ D^{n\alpha}\ \left\{x^{\alpha(\mu+n-1)}\ {}_2F_1(\mu, \nu; c; x^\alpha)\right\} = \sum_{k=0}^{\infty} \frac{(\mu)_k(\nu)_k}{(c)_k\, k!} D^{n\alpha}\left\{x^{\alpha(\mu+n+k-1)}\right\}. $$

The action of the conformable derivative gives

$$
\begin{aligned}
D^{n\alpha}\left\{x^{\alpha(\mu+n-1)}\ {}_2F_1\left(\mu,\nu;c;x^{\alpha}\right)\right\} &= \sum_{k=0}^{\infty}\frac{(\mu)_k\,(\nu)_k}{(c)_k\,k!}\alpha^n\frac{\Gamma(\mu+n+k)}{\Gamma(\mu+k)}x^{\alpha(\mu+k-1)} \\
&= \alpha^n\sum_{k=0}^{\infty}\frac{(\mu)_{n+k}\,(\nu)_k}{(c)_k\,k!}x^{\alpha(\mu+k-1)}.
\end{aligned}
$$

Knowing that $(\mu)_{n+k} = (\mu)_n\,(\mu+n)_k$, it can be seen

$$
\begin{aligned}
D^{n\alpha}\left\{x^{\alpha(\mu+n-1)}\ {}_2F_1\left(\mu,\nu;c;x^{\alpha}\right)\right\} &= \alpha^n\,(\mu)_n\,x^{\alpha(\mu-1)}\sum_{k=0}^{\infty}\frac{(\mu+n)_k\,(\nu)_k}{(c)_k\,k!}x^{\alpha k} \\
&= \alpha^n\,(\mu)_n\ x^{\alpha(\mu-1)}\ {}_2F_1\left(\mu+n,\nu;c;x^{\alpha}\right)
\end{aligned}
$$

as required.

In view of (5.6), we have

$$
D^{n\alpha}\left\{(1-x^{\alpha})^{\mu+\nu-c}\ {}_2F_1\left(\mu,\nu;c;x^{\alpha}\right)\right\} = D^{n\alpha}\ {}_2F_1\left(c-\mu,c-\nu;c;x^{\alpha}\right) \tag{5.17}
$$

Using (5.10), we get

$$
D^{n\alpha}\left\{(1-x^{\alpha})^{\mu+\nu-c}\ {}_2F_1\left(\mu,\nu;c;x^{\alpha}\right)\right\} = \frac{\alpha^n\,(c-\mu)_n\,(c-\nu)_n}{(c)_n}\ {}_2F_1\left(c-\mu+n,c-\nu+n;c+n;x^{\alpha}\right) \tag{5.18}
$$

Return to (5.6), we obtain

$$
\begin{aligned}
D^{n\alpha}\left\{\ (1-x^{\alpha})^{\mu+\nu-c}\ {}_2F_1\left(\mu,\nu;c;x^{\alpha}\right)\right\} =&\frac{\alpha^n\,(c-\mu)_n\,(c-\gamma)_n}{(c)_n}\ (1-x^{\alpha})^{\mu+\nu-c-n} \\
&\times\ {}_2F_1\left(\mu,\nu;c+n;x^{\alpha}\right).
\end{aligned} \tag{5.19}
$$

Remark 5.1. *In case of* $\mu = -n$ *in* (5.16)*, we obtain*

$$
D^{n\alpha}\left\{x^{\alpha(n+c-1)}\ (1-x^{\alpha})^{\nu-c}\ \right\} = \alpha^n\,(c)_n\ x^{\alpha(c-1)}\,(1-x^{\alpha})^{\nu-c-n}\ {}_2F_1\left(-n,\nu;c;x^{\alpha}\right). \tag{5.20}
$$

6. A differential operator and contiguous relations of the CFGHF

Following [44], define the conformable fractional operator θ^{α} in the form

$$
\theta^{\alpha} = \frac{1}{\alpha}x^{\alpha}D^{\alpha}. \tag{6.1}
$$

This operator has the particularly pleasant property that $\theta^{\alpha}x^{n\alpha} = nx^{n\alpha}$, which makes it handy to be used on power series. In this section, relying on definition 2.2, we establish several results concerning contiguous relations for the CFGHF. To achieve that, we have to prove the following lemma.

Lemma 6.1. *Let* $\alpha \in (0,1]$, *then the CFGHF* ${}_2F_1\left(\mu,\nu;c;x^{\alpha}\right)$ *satisfies the following*

$$
(\theta^{\alpha}+\mu)\ {}_2F_1\left(\mu,\nu;c;x^{\alpha}\right) = \mu\,{}_2F_1\left(\mu+1,\nu;c;x^{\alpha}\right) \tag{6.2}
$$

$$
(\theta^{\alpha}+\nu)\ {}_2F_1\left(\mu,\nu;c;x^{\alpha}\right) = \nu\,{}_2F_1\left(\mu,\nu+1;c;x^{\alpha}\right) \tag{6.3}
$$

$$
(\theta^{\alpha}+c-1)\ {}_2F_1\left(\mu,\nu;c;x^{\alpha}\right) = (c-1)\ {}_2F_1\left(\mu,\nu;c-1;x^{\alpha}\right) \tag{6.4}
$$

Proof. Using (3.3) and (6.1), it follows that

$$\begin{aligned}(\theta^{\alpha}+\mu)\ {}_2F_1(\mu,\nu;c;x^{\alpha}) &= \sum_{n=0}^{\infty}\frac{(\mu)_n(\nu)_n}{(c)_n\,n!}(\theta^{\alpha}+\mu)\,x^{\alpha n}\\ &= \sum_{n=0}^{\infty}\frac{(\mu)_n(\nu)_n}{(c)_n\,n!}(n+\mu)\,x^{\alpha n} = \sum_{n=0}^{\infty}\frac{(\mu)_{n+1}(\nu)_n}{(c)_n\,n!}x^{\alpha n}\\ &= \sum_{n=0}^{\infty}\frac{\mu(\mu+1)_n(\nu)_n}{(c)_n\,n!}x^{\alpha n} = \mu\,{}_2F_1(\mu+1,\nu;c;x^{\alpha}).\end{aligned}$$

Similarly, we have

$$(\theta^{\alpha}+\nu)\ {}_2F_1(\mu,\nu;c;x^{\alpha}) = \nu\,{}_2F_1(\mu,\nu+1;c;x^{\alpha}).$$

Analogously, we obtain

$$(\theta^{\alpha}+c-1)\ {}_2F_1(\mu,\nu;c;x^{\alpha}) = \sum_{n=0}^{\infty}\frac{(\mu)_n(\nu)_n}{(c)_n\,n!}(\theta^{\alpha}+c-1)\,x^{\alpha n} = \sum_{n=0}^{\infty}\frac{(\mu)_n(\nu)_n}{(c)_n\,n!}(n+c-1)\,x^{\alpha n}.$$

Therefore,

$$\begin{aligned}(\theta^{\alpha}+c-1)\ {}_2F_1(\mu,\nu;c;x^{\alpha}) &= \sum_{n=0}^{\infty}\frac{(\mu)_n(\nu)_n}{(c)_{n-1}\,n!}x^{\alpha n}\\ &= \sum_{n=0}^{\infty}\frac{(c-1)(\mu)_n(\nu)_n}{(c-1)_n\,n!}x^{\alpha n} = (c-1)\ {}_2F_1(\mu,\nu;c-1;x^{\alpha}).\end{aligned}$$

□

The following result is an immediate consequence of Lemma 6.1.

Theorem 6.1. *Let* $\alpha\in(0,1]$, *then the CFGHF* ${}_2F_1(\mu,\nu;c;x^{\alpha})$ *satisfies the following contiguous relations*

$$(\mu-\nu)\ {}_2F_1(\mu,\nu;c;x^{\alpha}) = \mu\,{}_2F_1(\mu+1,\nu;c;x^{\alpha}) - \nu\,{}_2F_1(\mu,\nu+1;c;x^{\alpha}) \tag{6.5}$$

and

$$(\mu+c-1)\ {}_2F_1(\mu,\nu;c;x^{\alpha}) = \mu\,{}_2F_1(\mu+1,\nu;c;x^{\alpha}) - (c-1)\ {}_2F_1(\mu,\nu;c-1;x^{\alpha}) \tag{6.6}$$

Proof. Using (6.2) and (6.3) immediately give (6.5) and similarly (6.2) and (6.4) assert (6.6). □

Theorem 6.2. *Let* $\alpha\in(0,1]$, *then the CFGHF,* ${}_2F_1(\mu,\nu;c;x^{\alpha})$ *satisfies the following contiguous relation*

$$\begin{aligned}\left[\mu+(\nu-c)\,x^{\alpha}\right]\ {}_2F_1(\mu,\nu;c;x^{\alpha}) =&\mu(1-x^{\alpha})\ {}_2F_1(\mu+1,\nu;c;x^{\alpha})\\ &- c^{-1}(c-\mu)(c-\nu)\,x^{\alpha}\ {}_2F_1(\mu,\nu;c+1;x^{\alpha}).\end{aligned} \tag{6.7}$$

Proof. Consider

$$\theta^{\alpha}{}_2F_1(\mu,\nu;c;x^{\alpha}) = \sum_{n=1}^{\infty}\frac{(\mu)_n(\nu)_n}{(c)_n\,n!}nx^{\alpha n}.$$

A shift of index gives

$$\theta^{\alpha}{}_2F_1(\mu,\nu;c;x^{\alpha}) = \sum_{n=0}^{\infty}\frac{(\mu)_{n+1}(\nu)_{n+1}}{(c)_{n+1}\,n!}x^{\alpha(n+1)} = x^{\alpha}\sum_{n=0}^{\infty}\frac{(\mu+n)(\nu+n)}{(c+n)}\frac{(\mu)_n(\nu)_n}{(c)_n\,n!}x^{\alpha n}. \tag{6.8}$$

Since

$$\frac{(\mu+n)(\nu+n)}{(c+n)} = n + (\mu+\nu-c) + \frac{(c-\mu)(c-\nu)}{c+n},$$

then Eq (6.8) yields

$$\begin{aligned}\theta^{\alpha}{}_2F_1(\mu,\nu;c;x^{\alpha}) =& x^{\alpha}\sum_{n=0}^{\infty}\frac{(\mu)_n(\nu)_n}{(c)_n\,n!}nx^{\alpha n} + (\mu+\nu-c)\,x^{\alpha}\sum_{n=0}^{\infty}\frac{(\mu)_n(\nu)_n}{(c)_n\,n!}x^{\alpha n}\\ &+ x^{\alpha}\frac{(c-\mu)(c-\nu)}{c}\sum_{n=0}^{\infty}\frac{c}{c+n}\frac{(\mu)_n(\nu)_n}{(c)_n\,n!}x^{\alpha n}\\ =& x^{\alpha}\theta^{\alpha}{}_2F_1(\mu,\nu;c;x^{\alpha}) + (\mu+\nu-c)\,x^{\alpha}\,{}_2F_1(\mu,\nu;c;x^{\alpha})\\ &+ \frac{(c-\mu)(c-\nu)}{c}x^{\alpha}\sum_{n=0}^{\infty}\frac{(\mu)_n(\nu)_n}{(c+1)_n\,n!}x^{\alpha n}.\end{aligned}$$

Hence, we can write

$$\begin{aligned}(1-x^{\alpha})\,\theta^{\alpha}{}_2F_1(\mu,\nu;c;x^{\alpha}) =& (\mu+\nu-c)\,x^{\alpha}\,{}_2F_1(\mu,\nu;c;x^{\alpha})\\ &+ c^{-1}(c-\mu)(c-\nu)\,x^{\alpha}\,{}_2F_1(\mu,\nu;c+1;x^{\alpha})\end{aligned} \tag{6.9}$$

From (6.2), we obtain

$$\begin{aligned}(1-x^{\alpha})\,\theta^{\alpha}{}_2F_1(\mu,\nu;c;x^{\alpha}) =& -\mu\,(1-x^{\alpha})\;{}_2F_1(\mu,\nu;c;x^{\alpha})\\ &+ \mu\,(1-x^{\alpha})\;{}_2F_1(\mu+1,\nu;c;x^{\alpha})\end{aligned}$$

which implies together with (6.9) the required relation. □

Theorem 6.3. *For $\alpha \in (0,1]$, then the CFGHF, ${}_2F_1(\mu,\nu;c;x^{\alpha})$ satisfies the following contiguous relation*

$$(1-x^{\alpha})\;{}_2F_1(\mu,\nu;c;x^{\alpha}) = {}_2F_1(\mu-1,\nu;c;x^{\alpha}) - c^{-1}(c-\nu)\,x^{\alpha}\;{}_2F_1(\mu,\nu;c+1;x^{\alpha}), \tag{6.10}$$

$$(1-x^{\alpha})\;{}_2F_1(\mu,\nu;c;x^{\alpha}) = {}_2F_1(\mu,\nu-1;c;x^{\alpha}) - c^{-1}(c-\mu)\,x^{\alpha}\;{}_2F_1(\mu,\nu;c+1;x^{\alpha}). \tag{6.11}$$

Proof. By operating $\theta^{\alpha}\;{}_2F_1(\mu-1,\nu;c;x^{\alpha})$, we obtain

$$\theta^{\alpha}\;{}_2F_1(\mu-1,\nu;c;x^{\alpha}) = \theta^{\alpha}\;\sum_{n=0}^{\infty}\frac{(\mu-1)_n(\nu)_n}{(c)_n\,n!}x^{\alpha n} = \sum_{n=1}^{\infty}\frac{(\mu-1)_n(\nu)_n}{(c)_n\,n!}nx^{\alpha n}.$$

A shift of index yields

$$\begin{aligned}\theta^{\alpha}\ {}_2F_1\left(\mu-1,\nu;c;x^{\alpha}\right) &= \sum_{n=0}^{\infty}\frac{(\mu-1)_{n+1}\,(\nu)_{n+1}}{(c)_{n+1}\,n!}x^{\alpha(n+1)} \\ &= (\mu-1)\,x^{\alpha}\sum_{n=0}^{\infty}\frac{(\nu+n)}{(c+n)}\frac{(\mu)_n\,(\nu)_n}{(c)_n\,n!}x^{\alpha n}\end{aligned} \tag{6.12}$$

But $\frac{(\nu+n)}{(c+n)} = 1-\frac{c-\nu}{c+n}$, thus (6.12) becomes

$$\begin{aligned}\theta^{\alpha}\ {}_2F_1\left(\mu-1,\nu;c;x^{\alpha}\right) &= (\mu-1)\,x^{\alpha}\left[\sum_{n=0}^{\infty}\frac{(\mu)_n\,(\nu)_n}{(c)_n\,n!}x^{\alpha n}+\frac{c-\nu}{c}\sum_{n=0}^{\infty}\frac{c}{c+n}\frac{(\mu)_n\,(\nu)_n}{(c)_n\,n!}x^{\alpha n}\right] \\ &= (\mu-1)\,x^{\alpha}\left[{}_2F_1\left(\mu,\nu;c;x^{\alpha}\right)-\frac{c-\nu}{c}\ {}_2F_1\left(\mu,\nu;c+1;x^{\alpha}\right)\right]\end{aligned}$$

which yield

$$\begin{aligned}\theta^{\alpha}\ {}_2F_1\left(\mu-1,\nu;c;x^{\alpha}\right) = (\mu-1)\,x^{\alpha}\ {}_2F_1\left(\mu,\nu;c;x^{\alpha}\right) \\ -c^{-1}\,(c-\nu)\,(\mu-1)\,x^{\alpha}\ \ {}_2F_1\left(\mu,\nu;c+1;x^{\alpha}\right).\end{aligned} \tag{6.13}$$

Now, replacing μ by $(\mu-1)$ in (6.2) implies that

$$\theta^{\alpha}\ {}_2F_1\left(\mu-1,\nu;c;x^{\alpha}\right) = -(\mu-1)\ \ {}_2F_1\left(\mu-1,\nu;c;x^{\alpha}\right)+(\mu-1)\ \ {}_2F_1\left(\mu,\nu;c;x^{\alpha}\right). \tag{6.14}$$

From (6.13) and (6.14), the relation (6.10) is verified. Similarly, since μ and ν can be interchanged without affecting the hypergeometric series, (6.11) yields. □

Observe that from the contiguous relations we just derived in Theorems 6.1, 6.2, and 6.3, we can obtain further relations by performing some suitable eliminations as follows.

From (6.7) and (6.10), we get

$$\begin{aligned}\left[2\mu-c+(\nu-\mu)\,x^{\alpha}\right]\ {}_2F_1\left(\mu,\nu;c;x^{\alpha}\right) =& \mu\,(1-x^{\alpha})\ \ {}_2F_1\left(\mu+1,\nu;c;x^{\alpha}\right) \\ &-(c-\mu)\ \ {}_2F_1\left(\mu-1,\nu;c;x^{\alpha}\right).\end{aligned} \tag{6.15}$$

A combination of (6.7) and (6.11) gives

$$\begin{aligned}\left[\mu+\gamma-c\right]\ {}_2F_1\left(\mu,\nu;c;x^{\alpha}\right) =& \mu\,(1-x^{\alpha})\ \ {}_2F_1\left(\mu+1,\nu;c;x^{\alpha}\right) \\ &-(c-\gamma)\ \ {}_2F_1\left(\mu,\nu-1;c;x^{\alpha}\right).\end{aligned} \tag{6.16}$$

Inserting (6.5) in (6.15) implies

$$\begin{aligned}\left[c-\mu-\nu\right]_2 F_1\left(\mu,\nu;c;x^{\alpha}\right) =& (c-\mu)\ \ {}_2F_1\left(\mu-1,\nu;c;x^{\alpha}\right) \\ &-\nu\,(1-x^{\alpha})\ \ {}_2F_1\left(\mu,\nu+1;c;x^{\alpha}\right).\end{aligned} \tag{6.17}$$

Moreover, from (6.15) and (6.16), we get

$$(\nu-\mu)\,(1-x^{\alpha})\ \ {}_2F_1\left(\mu,\nu;c;x^{\alpha}\right) = (c-\mu)\ \ {}_2F_1\left(\mu-1,\nu;c;x^{\alpha}\right)-(c-\nu)\ \ {}_2F_1\left(\mu,\nu-1;c;x^{\alpha}\right). \tag{6.18}$$

Use (6.6) and (6.16) to obtain

$$\begin{aligned}\left[1-\mu+(c-\nu-1)\,x^{\alpha}\right]\;{}_2F_1\left(\mu,\nu;c;x^{\alpha}\right)=&(c-\mu)\;{}_2F_1\left(\mu-1,\nu;c;x^{\alpha}\right)\\&-(c-1)\left(1-x^{\alpha}\right)\;{}_2F_1\left(\mu,\nu;c-1;x^{\alpha}\right).\end{aligned}\tag{6.19}$$

By interchanging μ and ν in (6.15), we have

$$\begin{aligned}\left[2\nu-c+(\mu-\nu)\,x^{\alpha}\right]\;{}_2F_1\left(\mu,\nu;c;x^{\alpha}\right)=&\nu\left(1-x^{\alpha}\right)\;{}_2F_1\left(\mu,\nu+1;c;x^{\alpha}\right)\\&-(c-\nu)\;{}_2F_1\left(\mu,\nu-1;c;x^{\alpha}\right).\end{aligned}\tag{6.20}$$

We append this section by driving the CFGHE. The conformable fractional operator (6.1) can be employed to derive a conformable fractional differential equation characterized by (3.3).

Relation (3.3) with the operator θ^{α} defined by (6.1) gives

$$\theta^{\alpha}\left(\theta^{\alpha}+c-1\right)y=\theta^{\alpha}\sum_{n=0}^{\infty}\frac{(\mu)_n\,(\nu)_n}{(c)_n\,n!}\,(n+c-1)\,x^{\alpha n}=\sum_{n=1}^{\infty}\frac{(\mu)_n\,(\nu)_n}{(c)_n\,n!}n\,(n+c-1)\,x^{\alpha n}.$$

A shift of index yields

$$\begin{aligned}\theta^{\alpha}\left(\theta^{\alpha}+c-1\right)y\;&=\;\sum_{n=0}^{\infty}\frac{(\mu)_{n+1}\,(\nu)_{n+1}}{(c)_{n+1}\,n!}\,(n+c)\,x^{\alpha(n+1)}\\&=\;x^{\alpha}\sum_{n=0}^{\infty}\frac{(\mu)_n\,(\nu)_n}{(c)_n\,n!}\,(n+\mu)\,(n+\nu)\,x^{\alpha n}=x^{\alpha}\left(\theta^{\alpha}+\mu\right)\left(\theta^{\alpha}+\nu\right)y\end{aligned}$$

This shows $y={}_2F_1\left(\mu,\nu;c;x^{\alpha}\right)$ is a solution of the following CFDE

$$\left[\theta^{\alpha}\left(\theta^{\alpha}+c-1\right)-x^{\alpha}\left(\theta^{\alpha}+\mu\right)\left(\theta^{\alpha}+\nu\right)\right]y=0,\qquad\theta^{\alpha}=\frac{1}{\alpha}x^{\alpha}D^{\alpha}\tag{6.21}$$

Owing to $\theta^{\alpha}y=\frac{1}{\alpha}x^{\alpha}D^{\alpha}y$ and $\theta^{\alpha}\theta^{\alpha}y=\frac{1}{\alpha^2}x^{2\alpha}D^{\alpha}D^{\alpha}y+\frac{1}{\alpha}x^{\alpha}D^{\alpha}$, then Eq (6.21) can be written in the form

$$x^{\alpha}\left(1-x^{\alpha}\right)D^{\alpha}D^{\alpha}y+\alpha\left[c-(\mu+\nu+1)\,x^{\alpha}\right]D^{\alpha}y-\alpha^2\mu\nu y=0,$$

which coincide with (3.1).

7. Conformable fractional integral of the CFGHF

Taking into account the α-integral given in Definition 2.5, we provide some forms of fractional integral related to the α-Gauss hypergeometric function. Thus according to Definition 2.5, it follows that

$$I_{\alpha}f(x)=\int_0^x t^{\alpha-1}f(t)\,dt.\tag{7.1}$$

In this regard, we state the following important result given in [34].

Lemma 7.1. *Suppose that $f:[0,\infty)\to\mathbb{R}$ is α-differentiable for $\alpha\in(0,1]$, then for all $x>0$ one can write:*

$$I_{\alpha}D^{\alpha}\left(f(x)\right)=f(x)-f(0)\tag{7.2}$$

With the aid of (7.1) and (7.2), the following result can be deduced.

Theorem 7.1. *For* $\alpha \in (0,1]$*, then the conformable fractional integral* I_α *of CFGHF,* ${}_2F_1(\mu,\nu;c;x^\alpha)$ *can be written as*

$$I_\alpha\, {}_2F_1(\mu,\nu;c;x^\alpha) = \frac{(c-1)}{\alpha(\mu-1)(\nu-1)}\left[{}_2F_1(\mu-1,\nu-1;c-1;x^\alpha) - 1\right] \tag{7.3}$$

Proof. Relation (5.9), gives

$$D^\alpha\, {}_2F_1(\mu-1,\nu-1;c-1;x^\alpha) = \frac{\alpha(\mu-1)(\nu-1)}{(c-1)}\, {}_2F_1(\mu,\nu;c;x^\alpha).$$

Acting by the conformable fractional integral on both sides we obtain

$$I_\alpha D^\alpha\, {}_2F_1(\mu-1,\nu-1;c-1;x^\alpha) = \frac{\alpha(\mu-1)(\nu-1)}{(c-1)} I_\alpha\, {}_2F_1(\mu,\nu;c;x^\alpha).$$

Using (7.2), we have

$${}_2F_1(\mu-1,\nu-1;c-1;x^\alpha) - 1 = \frac{\alpha(\mu-1)(\nu-1)}{(c-1)} I_\alpha\, {}_2F_1(\mu,\nu;c;x^\alpha).$$

Therefore, it follows that

$$I_\alpha\, {}_2F_1(\mu,\nu;c;x^\alpha) = \frac{(c-1)}{\alpha(\mu-1)(\nu-1)}\left[{}_2F_1(\mu-1,\nu-1;c-1;x^\alpha) - 1\right]$$

as required. □

Theorem 7.2. *For* $\alpha \in (0,1]$*, then the CFGHF,* ${}_2F_1(\mu,\nu;c;x^\alpha)$ *has a conformable fractional integral representation in the form*

$${}_2F_1(\mu,\nu;c;x^\alpha) = 1 + \frac{\alpha\mu\nu}{c}\int_0^x {}_2F_1(\mu+1,\nu+1;c+1;t^\alpha)\, d_\alpha t$$

where $d_\alpha t = t^{\alpha-1}dt$

Proof. In view of theorem 7.1, we obtain

$$I_\alpha\, {}_2F_1(\mu+1,\nu+1;c+1;x^\alpha) = \frac{c}{\alpha\mu\nu}\left[{}_2F_1(\mu,\nu;c;x^\alpha) - 1\right]$$

Hence,

$$\begin{aligned} {}_2F_1(\mu,\nu;c;x^\alpha) &= 1 + \frac{\alpha\mu\nu}{c} I_\alpha\left[{}_2F_1(\mu+1,\nu+1;c+1;x^\alpha)\right] \\ &= 1 + \frac{\alpha\mu\nu}{c}\int_0^x {}_2F_1(\mu+1,\nu+1;c+1;t^\alpha)\, d_\alpha t \\ &= 1 + \frac{\alpha\mu\nu}{c}\int_0^x {}_2F_1(\mu+1,\nu+1;c+1;t^\alpha)\; t^{\alpha-1}dt \end{aligned}$$

as required. □

Now, following [44], we state the following result.

Theorem 7.3. *For $\alpha \in (0,1]$ and $c > \nu > 0$, the CFGHF, ${}_2F_1(\mu,\nu;c;x^\alpha)$ has an integral representation*

$$ {}_2F_1(\mu,\nu;c;x^\alpha) = \frac{\Gamma(c)}{\Gamma(\nu)\Gamma(c-\nu)} \int_0^1 \tau^{\nu-1}(1-\tau)^{c-\nu-1}(1-x^\alpha\tau)^{-\mu}\,d\tau \tag{7.4} $$

Proof. From the definition of CFGHF (3.3), we have

$$ \begin{aligned} {}_2F_1(\mu,\nu;c;x^\alpha) &= \sum_{n=0}^{\infty} \frac{(\mu)_n\ \Gamma(\nu+n)\ \Gamma(c)}{\Gamma(\nu)\Gamma(c+n)\,n!} x^{\alpha n} \\ &= \frac{\Gamma(c)}{\Gamma(\nu)\Gamma(c-\nu)} \sum_{n=0}^{\infty} \frac{(\mu)_n\ \Gamma(\nu+n)\ \Gamma(c-\nu)}{\Gamma(c+n)\,n!} x^{\alpha n} \\ &= \frac{\Gamma(c)}{\Gamma(\nu)\Gamma(c-\nu)} \sum_{n=0}^{\infty} \beta(c-\nu,\nu+n)\,\frac{(\mu)_n}{n!} x^{\alpha n}. \end{aligned} $$

Using the integral form of beta function, we get

$$ {}_2F_1(\mu,\nu;c;x^\alpha) = \frac{\Gamma(c)}{\Gamma(\nu)\Gamma(c-\nu)} \int_0^1 \tau^{\nu-1}(1-\tau)^{c-\nu-1}\,.\sum_{n=0}^{\infty} \frac{(\mu)_n}{n!}(x^\alpha\tau)^n\,d\tau. $$

By using the identity $\sum\limits_{n=0}^{\infty} \frac{(\mu)_n}{n!} t^n = (1-t)^{-(\mu)}$, $|t| < 1$, it follows that

$$ {}_2F_1(\mu,\nu;c;x^\alpha) = \frac{\Gamma(c)}{\Gamma(\nu)\Gamma(c-\nu)} \int_0^1 \tau^{\nu-1}(1-\tau)^{c-\nu-1}\,.(1-x^\alpha\tau)^{-\mu}\,d\tau $$

as required. □

8. Recursion formulas for formulas for the CFGHF

Employing the assertion in theorem 7.3, and owing to the results given in [46], we state the following recursion formulas.

Theorem 8.1. *Let $\alpha \in (0,1]$. The following recursion formulas hold for the CFGHF*

$$ {}_2F_1(\mu+n,\nu;c;x^\alpha) = {}_2F_1(\mu,\nu;c;x^\alpha) + \frac{\nu x^\alpha}{c}\sum_{k=1}^{n} {}_2F_1(\mu+n-k+1,\nu+1;c+1;x^\alpha), \tag{8.1} $$

$$ {}_2F_1(\mu-n,\nu;c;x^\alpha) = {}_2F_1(\mu,\nu;c;x^\alpha) - \frac{\nu x^\alpha}{c}\sum_{k=1}^{n} {}_2F_1(\mu-k+1,\nu+1;c+1;x^\alpha), \tag{8.2} $$

where $|x^\alpha| < 1$, $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Proof. By means of (7.4), we have

$$\begin{aligned} {}_2F_1(\mu+n,\nu;c;x^\alpha) =& \frac{\Gamma(c)}{\Gamma(\nu)\Gamma(c-\nu)}\int_0^1 \tau^{\nu-1}(1-\tau)^{c-\nu-1}(1-x^\alpha\tau)^{-\mu-n-1}\,d\tau \\ &- \frac{\Gamma(c)\,x^\alpha}{\Gamma(\nu)\Gamma(c-\nu)}\int_0^1 \tau^{\nu}(1-\tau)^{c-\nu-1}(1-x^\alpha\tau)^{-\mu-n-1}\,d\tau \\ =& \frac{\Gamma(c)}{\Gamma(\nu)\Gamma(c-\nu)}\int_0^1 \tau^{\nu-1}(1-\tau)^{c-\nu-1}(1-x^\alpha\tau)^{-\mu-n-1}\,d\tau \\ &- \frac{x^\alpha}{\alpha(\mu+n)}\left\{\frac{\alpha(\mu+n)\Gamma(c)}{\Gamma(\nu)\Gamma(c-\nu)}\int_0^1 \tau^{\nu}(1-\tau)^{c-\nu-1}(1-x^\alpha\tau)^{-\mu-n-1}\,d\tau\right\}. \end{aligned}$$

In virtue of conformable derivative, we may write

$$\begin{aligned} {}_2F_1(\mu+n,\nu;c;x^\alpha) =& \frac{\Gamma(c)}{\Gamma(\nu)\Gamma(c-\nu)}\int_0^1 \tau^{\nu-1}(1-\tau)^{c-\nu-1}(1-x^\alpha\tau)^{-\mu-n-1}\,d\tau \\ &- \frac{x^\alpha}{\alpha(\mu+n)}D^\alpha\left\{\frac{\Gamma(c)}{\Gamma(\nu)\Gamma(c-\nu)}\int_0^1 \tau^{\nu-1}(1-\tau)^{c-\nu-1}(1-x^\alpha\tau)^{-\mu-n}\,d\tau\right\}. \end{aligned}$$

Again using (7.4), we have

$${}_2F_1(\mu+n,\nu;c;x^\alpha) = {}_2F_1(\mu+n+1,\nu;c;x^\alpha) - \frac{x^\alpha}{\alpha(\mu+n)}D^\alpha\left\{{}_2F_1(\mu+n,\nu;c;x^\alpha)\right\}.$$

Thus,

$${}_2F_1(\mu+n-1,\nu;c;x^\alpha) = {}_2F_1(\mu+n,\nu;c;x^\alpha) - \frac{x^\alpha}{\alpha(\mu+n-1)}D^\alpha\left\{{}_2F_1(\mu+n-1,\nu;c;x^\alpha)\right\},$$

or

$${}_2F_1(\mu+n,\nu;c;x^\alpha) = {}_2F_1(\mu+n-1,\nu;c;x^\alpha) + \frac{x^\alpha}{\alpha(\mu+n-1)}D^\alpha\left\{{}_2F_1(\mu+n-1,\nu;c;x^\alpha)\right\}. \quad (8.3)$$

Applying this last identity (8.3), we get

$$\begin{aligned} {}_2F_1(\mu+n,\nu;c;x^\alpha) &= {}_2F_1(\mu+n-2,\nu;c;x^\alpha) + \frac{x^\alpha}{\alpha(\mu+n-2)}D^\alpha\left\{{}_2F_1(\mu+n-2,\nu;c;x^\alpha)\right\} \\ &\quad + \frac{x^\alpha}{\alpha(\mu+n-1)}D^\alpha\left\{{}_2F_1(\mu+n-1,\nu;c;x^\alpha)\right\} \\ &= {}_2F_1(\mu+n-2,\nu;c;x^\alpha) + \frac{x^\alpha}{\alpha}.\sum_{k=1}^{2}\frac{1}{(\mu+n-k)}D^\alpha\left\{{}_2F_1(\mu+n-k,\nu;c;x^\alpha)\right\}. \end{aligned}$$

Again apply (8.3) recursively n-times, we obtain

$$ {}_2F_1\left(\mu+n,\nu;c;x^{\alpha}\right)= {}_2F_1\left(\mu,\nu;c;x^{\alpha}\right)+\frac{x^{\alpha}}{\alpha}.\sum_{k=1}^{n}\frac{1}{(\mu+n-k)}D^{\alpha}\left\{{}_2F_1\left(\mu+n-k,\nu;c;x^{\alpha}\right)\right\}. \tag{8.4}$$

Using (5.9), we have

$$ {}_2F_1\left(\mu+n,\nu;c;x^{\alpha}\right)= {}_2F_1\left(\mu,\nu;c;x^{\alpha}\right)+\frac{x^{\alpha}\nu}{c}.\sum_{k=1}^{n}\left\{{}_2F_1\left(\mu+n-k+1,\nu+1;c+1;x^{\alpha}\right)\right\}.$$

Furthermore, the assertion of theorem 7.3 gives

$$\begin{aligned} {}_2F_1\left(\mu-n,\nu;c;x^{\alpha}\right)=&\frac{\Gamma(c)}{\Gamma(\nu)\Gamma(c-\nu)}\int_0^1\tau^{\nu-1}(1-\tau)^{c-\nu-1}(1-x^{\alpha}\tau)^{n-\mu-1}\,d\tau\\ &-\frac{\Gamma(c)\,x^{\alpha}}{\Gamma(\nu)\Gamma(c-\nu)}\int_0^1\tau^{\nu}(1-\tau)^{c-\nu-1}(1-x^{\alpha}\tau)^{n-\mu-1}\,d\tau\\ =&\frac{\Gamma(c)}{\Gamma(\nu)\Gamma(c-\nu)}\int_0^1\tau^{\nu-1}(1-\tau)^{c-\nu-1}(1-x^{\alpha}\tau)^{n-\mu-1}\,d\tau\\ &-\frac{x^{\alpha}}{\alpha(\mu-n)}\left\{\frac{\alpha(\mu-n)\Gamma(c)}{\Gamma(\nu)\Gamma(c-\nu)}\int_0^1\tau^{\nu}(1-\tau)^{c-\nu-1}(1-x^{\alpha}\tau)^{n-\mu-1}\,d\tau\right\}\\ =&\frac{\Gamma(c)}{\Gamma(\nu)\Gamma(c-\nu)}\int_0^1\tau^{\nu-1}(1-\tau)^{c-\nu-1}(1-x^{\alpha}\tau)^{n-\mu-1}\,d\tau\\ &-\frac{x^{\alpha}}{\alpha(\mu-n)}D^{\alpha}\left\{\frac{\Gamma(c)}{\Gamma(\nu)\Gamma(c-\nu)}\int_0^1\tau^{\nu-1}(1-\tau)^{c-\nu-1}(1-x^{\alpha}\tau)^{n-\mu}\,d\tau\right\}\end{aligned}$$

Relying on the integral representation (7.4), we have

$$ {}_2F_1\left(\mu-n,\nu;c;x^{\alpha}\right)= {}_2F_1\left(\mu-n+1,\nu;c;x^{\alpha}\right)-\frac{x^{\alpha}}{\alpha(\mu-n)}D^{\alpha}\left\{{}_2F_1\left(\mu-n,\nu;c;x^{\alpha}\right)\right\}.$$

Therefore,

$$ {}_2F_1\left(\mu-n-1,\nu;c;x^{\alpha}\right)= {}_2F_1\left(\mu-n,\nu;c;x^{\alpha}\right)-\frac{x^{\alpha}}{\alpha(\mu-n-1)}D^{\alpha}\left\{{}_2F_1\left(\mu-n-1,\nu;c;x^{\alpha}\right)\right\},$$

or

$$ {}_2F_1\left(\mu-n,\nu;c;x^{\alpha}\right)= {}_2F_1\left(\mu-n-1,\nu;c;x^{\alpha}\right)+\frac{x^{\alpha}}{\alpha(\mu-n-1)}D^{\alpha}\left\{{}_2F_1\left(\mu-n-1,\nu;c;x^{\alpha}\right)\right\}. \tag{8.5}$$

Applying relation (8.5) recursively, we obtain

$$
\begin{aligned}
{}_2F_1(\mu-n,\nu;c;x^\alpha) = & {}_2F_1(\mu-n-2,\nu;c;x^\alpha) + \frac{x^\alpha}{\alpha(\mu-n-2)} D^\alpha\{{}_2F_1(\mu-n-2,\nu;c;x^\alpha)\} \\
& + \frac{x^\alpha}{\alpha(\mu-n-1)} D^\alpha\{{}_2F_1(\mu-n-1,\nu;c;x^\alpha)\} \\
= & {}_2F_1(\mu-n-2,\nu;c;x^\alpha) + \frac{x^\alpha}{\alpha}\sum_{k=1}^{2}\frac{1}{(\mu-n-k)} D^\alpha\{{}_2F_1(\mu-n-k,\nu;c;x^\alpha)\}.
\end{aligned}
$$

Repeating the recurrence relation (8.5) n-times and dappling the derivative formula (5.9), we have

$$
{}_2F_1(\mu-n,\nu;c;x^\alpha) = {}_2F_1(\mu-2n,\nu;c;x^\alpha) + \frac{x^\alpha\nu}{c}.\sum_{k=1}^{n}\{{}_2F_1(\mu-n-k+1,\nu+1;c+1;x^\alpha)\}. \quad (8.6)
$$

The relation (8.2) follows directly from (8.6) by replacing μ by $(\mu+n)$ where $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. □

Theorem 8.2. *For $\alpha \in (0,1]$. The following recursion formulas hold true for the CFGHF, ${}_2F_1(\mu,\nu;c;x^\alpha)$*

$$
{}_2F_1(\mu+n,\nu;c;x^\alpha) = \sum_{k=0}^{n}\binom{n}{k}\frac{(\nu)_k}{(c)_k}x^{\alpha k}\,{}_2F_1(\mu+k,\nu+k;c+k;x^\alpha), \quad (8.7)
$$

and

$$
{}_2F_1(\mu-n,\nu;c;x^\alpha) = \sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{(\nu)_k}{(c)_k}x^{\alpha k}\,{}_2F_1(\mu,\nu+k;c+k;x^\alpha), \quad (8.8)
$$

where $|x^\alpha| < 1,\ n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Proof. Employing (8.1) of theorem 8.1 with $n = 1$, we obtain

$$
{}_2F_1(\mu+1,\nu;c;x^\alpha) = {}_2F_1(\mu,\nu;c;x^\alpha) + \frac{\nu x^\alpha}{c}\,{}_2F_1(\mu+1,\nu+1;c+1;x^\alpha), \quad (8.9)
$$

with $n = 2$, we have

$$
\begin{aligned}
{}_2F_1(\mu+2,\nu;c;x^\alpha) = & {}_2F_1(\mu,\nu;c;x^\alpha) + \frac{\nu x^\alpha}{c}\,{}_2F_1(\mu+1,\nu+1;c+1;x^\alpha) \\
& + \frac{\nu x^\alpha}{c}\,{}_2F_1(\mu+2,\nu+1;c+1;x^\alpha)
\end{aligned} \quad (8.10)
$$

Making use of (8.9) and (8.10) implies

$$
\begin{aligned}
{}_2F_1(\mu+2,\nu;c;x^\alpha) = & {}_2F_1(\mu,\nu;c;x^\alpha) + \frac{2\nu x^\alpha}{c}\,{}_2F_1(\mu+1,\nu+1;c+1;x^\alpha) \\
& + \frac{\nu(\nu+1)x^{2\alpha}}{c(c+1)}\,{}_2F_1(\mu+2,\nu+2;c+2;x^\alpha).
\end{aligned} \quad (8.11)
$$

Using (8.9) and (8.11) with $n = 3$, it follows that

$$\begin{aligned}{}_2F_1(\mu+3,\nu;c;x^\alpha) = {}_2F_1(\mu,\nu;c;x^\alpha) &+ \frac{3\nu x^\alpha}{c}\,{}_2F_1(\mu+1,\nu+1;c+1;x^\alpha)\\ &+ \frac{3\nu(\nu+1)x^{2\alpha}}{c(c+1)}\,{}_2F_1(\mu+2,\nu+2;c+2;x^\alpha)\\ &+ \frac{\nu(\nu+1)(\nu+2)x^{3\alpha}}{c(c+1)(c+2)}\,{}_2F_1(\mu+3,\nu+3;c+3;x^\alpha).\end{aligned} \tag{8.12}$$

Relation (8.12) can be written in the form

$${}_2F_1(\mu+3,\nu;c;x^\alpha) = \sum_{k=0}^{3}\binom{3}{k}\frac{(\nu)_k}{(c)_k}x^{\alpha k}\,{}_2F_1(\mu+k,\nu+k;c+k;x^\alpha).$$

In general, we may write that

$${}_2F_1(\mu+n,\nu;c;x^\alpha) = \sum_{k=0}^{n}\binom{n}{k}\frac{(\nu)_k}{(c)_k}x^{\alpha k}\,{}_2F_1(\mu+k,\nu+k;c+k;x^\alpha).$$

In order to prove (8.8), we note from (8.2) of theorem 8.1 (with $n = 1$) that

$${}_2F_1(\mu-1,\nu;c;x^\alpha) = {}_2F_1(\mu,\nu;c;x^\alpha) - \frac{\nu x^\alpha}{c}\,{}_2F_1(\mu,\nu+1;c+1;x^\alpha). \tag{8.13}$$

Similarly, (with $n = 2$) yields

$$\begin{aligned}{}_2F_1(\mu-2,\nu;c;x^\alpha) = {}_2F_1(\mu,\nu;c;x^\alpha) &- \frac{\nu x^\alpha}{c}\,{}_2F_1(\mu,\nu+1;c+1;x^\alpha)\\ &- \frac{\nu x^\alpha}{c}\,{}_2F_1(\mu-1,\nu+1;c+1;x^\alpha).\end{aligned} \tag{8.14}$$

Inserting (8.13) in (8.14), we get

$$\begin{aligned}{}_2F_1(\mu-2,\nu;c;x^\alpha) = {}_2F_1(\mu,\nu;c;x^\alpha) &- \frac{2\nu x^\alpha}{c}\,{}_2F_1(\mu,\nu+1;c+1;x^\alpha)\\ &+ \frac{\nu(\nu+1)x^{2\alpha}}{c(c+1)}\,{}_2F_1(\mu,\nu+2;c+2;x^\alpha).\end{aligned} \tag{8.15}$$

Using the Pochhammer symbol, we may write (8.15) as

$${}_2F_1(\mu-2,\nu;c;x^\alpha) = \sum_{k=0}^{2}(-1)^k\binom{2}{k}\frac{(\nu)_k}{(c)_k}x^{\alpha k}\,{}_2F_1(\mu,\nu+k;c+k;x^\alpha).$$

Thus, in general, we may write

$${}_2F_1(\mu-n,\nu;c;x^\alpha) = \sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{(\nu)_k}{(c)_k}x^{\alpha k}\,{}_2F_1(\mu,\nu+k;c+k;x^\alpha),$$

just as required in (8.8). □

Theorem 8.3. *For* $\alpha \in (0,1]$, *the following recursion formulas hold true for the CFGHF,* ${}_2F_1(\mu,\nu;c;x^\alpha)$

$$
\begin{gathered}
{}_2F_1(\mu,\nu;c+n;x^\alpha) = \frac{(c)_n}{(c-\nu)_n}\sum_{k=0}^{n}(-1)^k\begin{pmatrix} n \\ k \end{pmatrix}\frac{(\nu)_k}{(c)_k}\,{}_2F_1(\mu,\nu+k;c+k;x^\alpha), \\
\left(|x^\alpha|<1, c+n\notin\mathbb{Z}_0^-,\ n\in\mathbb{N}_0=\mathbb{N}\cup\{0\}\right)
\end{gathered}
\tag{8.16}
$$

Proof. In view of (7.4), we have

$$
{}_2F_1(\mu,\nu;c+n;x^\alpha) = \frac{\Gamma(c+n)}{\Gamma(\nu)\Gamma(c+n-\nu)}\int_0^1 \tau^{\nu-1}(1-\tau)^{c-\nu-1}(1-x^\alpha\tau)^{-\mu}.(1-\tau)^n\,d\tau
$$

Using the binomial theorem, we obtain

$$
{}_2F_1(\mu,\nu;c+n;x^\alpha) = \frac{\Gamma(c+n)}{\Gamma(\nu)\Gamma(c+n-\nu)}\int_0^1\sum_{k=0}^{n}(-1)^k\begin{pmatrix} n \\ k \end{pmatrix}\tau^{\nu+k-1}(1-\tau)^{c-\nu-1}(1-x^\alpha\tau)^{-\mu}\,d\tau \tag{8.17}
$$

Using the definition of the Pochhammer symbol, we may write (8.17) as

$$
\begin{aligned}
{}_2F_1(\mu,\nu;c+n;x^\alpha) &= \frac{(c)_n}{(c-\nu)_n}\sum_{k=0}^{n}(-1)^k\begin{pmatrix} n \\ k \end{pmatrix}\frac{(\nu)_k}{(c)_k}.\frac{\Gamma(c+k)}{\Gamma(\nu+k)\Gamma(c-\nu)} \\
&\quad .\int_0^1\tau^{\nu+k-1}(1-\tau)^{c+k-\nu-k-1}(1-x^\alpha\tau)^{-\mu}\,d\tau
\end{aligned}
$$

Applying (7.4), we obtain

$$
{}_2F_1(\mu,\nu;c+n;x^\alpha) = \frac{(c)_n}{(c-\nu)_n}\sum_{k=0}^{n}(-1)^k\begin{pmatrix} n \\ k \end{pmatrix}\frac{(\nu)_k}{(c)_k}\,{}_2F_1(\mu,\nu+k;c+k;x^\alpha),
$$

just as required in theorem 8.3. □

9. Fractional Laplace transform of the CFGHF

In [34], Abdeljawad defined the fractional Laplace transform in the conformable sense as follows:

Definition 9.1. *[34] Let* $\alpha \in (0,1]$ *and* $f : [0,\infty) \to \mathbb{R}$ *be real valued function. Then the fractional Laplace transform of order* α *is defined by*

$$
L_\alpha[f(t)] = F_\alpha(s) = \int_0^\infty e^{-s\left(\frac{t^\alpha}{\alpha}\right)}f(t)\,d_\alpha t = \int_0^\infty e^{-s\left(\frac{t^\alpha}{\alpha}\right)}f(t)\,t^{\alpha-1}dt. \tag{9.1}
$$

Remark 9.1. *If* $\alpha = 1$, *then* (9.1) *is the classical definition of the Laplace transform of integer order.*

Also, the author in [34] gave the following interesting results.

Lemma 9.1. *[34] Let $\alpha \in (0,1]$ and $f:[0,\infty) \to \mathbb{R}$ be real valued function such that $L_\alpha[f(t)] = F_\alpha(s)$ exist. Then $F_\alpha(s) = L\left[f(\alpha t)^{\frac{1}{\alpha}}\right]$, where $L[f(t)] = \int\limits_0^\infty e^{-st} f(t)\,dt$.*

Lemma 9.2. *[34] The following the conformable fractional Laplace transform of certain functions:*

(1) $L_\alpha[1] = \frac{1}{s}, \quad s > 0,$

(2) $L_\alpha[t^p] = \alpha^{\frac{p}{\alpha}} \frac{\Gamma\left(1+\frac{p}{\alpha}\right)}{s^{1+\frac{p}{\alpha}}}, \quad s > 0,$

(3) $L_\alpha\left[e^{k\frac{t^\alpha}{\alpha}}\right] = \frac{1}{s-k}.$

Owing to the definition of CFGHF and applying the conformable fractional Laplace transform operator of an arbitrary order $\gamma \in (0,1]$, we have

$$L_\gamma\left[{}_2F_1(\mu,\nu;c;x^\alpha)\right] = L_\gamma\left[\sum_{n=0}^{\infty} \frac{(\mu)_n(\nu)_n}{(c)_n\, n!} x^{\alpha n}\right] = \sum_{n=0}^{\infty} \frac{(\mu)_n(\nu)_n}{(c)_n\, n!} L_\gamma\{x^{\alpha n}\} \tag{9.2}$$

Using (2) of lemma 9.2, we obtain

$$L_\gamma\left[{}_2F_1(\mu,\nu;c;x^\alpha)\right] = \sum_{n=0}^{\infty} \frac{(\mu)_n(\nu)_n}{(c)_n\, n!} \gamma^{\frac{n\alpha}{\gamma}} \frac{\Gamma\left(1+\frac{n\alpha}{\gamma}\right)}{s^{1+\frac{n\alpha}{\gamma}}} \tag{9.3}$$

Remark 9.2. *If $\gamma = \alpha$ in* (9.3) *we have*

$$L_\alpha\left[{}_2F_1(\mu,\nu;c;x^\alpha)\right] = \sum_{n=0}^{\infty} \frac{(\mu)_n(\nu)_n}{(c)_n\, n!} \frac{\alpha^n \Gamma(1+n)}{s^{1+n}} = \sum_{n=0}^{\infty} \frac{\alpha^n(\mu)_n(\nu)_n}{(c)_n\, s^{1+n}}. \tag{9.4}$$

Theorem 9.1. *Let $\alpha \in (0,1]$ and ${}_2F_1(\mu,\nu;c;x^\alpha)$ be a conformable fractional hypergeometric function, then*

$$L_\alpha\left[{}_2F_1\left(\mu,\nu;1;x^\alpha\left(1-e^{-\frac{t^\alpha}{\alpha}}\right)\right)\right] = \frac{1}{s}\,{}_2F_1(\mu,\nu;s+1;x^\alpha). \tag{9.5}$$

Proof. Using (3.3) and (9.1), one can see

$$\begin{aligned} L_\alpha\left[{}_2F_1\left(\mu,\nu;1;x^\alpha\left(1-e^{-\frac{t^\alpha}{\alpha}}\right)\right)\right] &= L_\alpha\left[\sum_{n=0}^{\infty} \frac{(\mu)_n(\nu)_n}{(1)_n\, n!} x^{\alpha n}\left(1-e^{-\frac{t^\alpha}{\alpha}}\right)^n\right] \\ &= \sum_{n=0}^{\infty} \frac{(\mu)_n(\nu)_n}{n!} x^{\alpha n} L_\alpha\left[\frac{1}{n!}\left(1-e^{-\frac{t^\alpha}{\alpha}}\right)^n\right] \end{aligned} \tag{9.6}$$

But

$$L_\alpha\left[\frac{1}{n!}\left(1-e^{-\frac{t^\alpha}{\alpha}}\right)^n\right] = L_\alpha\left[\frac{1}{n!}\sum_{k=0}^{n} \frac{(-n)_k}{k!} e^{-k\frac{t^\alpha}{\alpha}}\right] = \frac{1}{n!}\sum_{k=0}^{n} \frac{(-n)_k}{k!} L_\alpha\left\{e^{-k\frac{t^\alpha}{\alpha}}\right\}.$$

Using (3) of lemma 9.2, we have

$$L_\alpha\left[\frac{1}{n!}\left(1-e^{-\frac{t^\alpha}{\alpha}}\right)^n\right] = \frac{1}{n!}\sum_{k=0}^{n} \frac{(-n)_k}{k!} \frac{1}{s+k}$$

Since $(-n)_k = 0$ if $k > n$, then we can write

$$L_\alpha\left[\frac{1}{n!}\left(1-e^{-\frac{t^\alpha}{\alpha}}\right)^n\right] = \frac{1}{n!}\sum_{k=0}^{\infty}\frac{(-n)_k}{k!\,(s+k)} \tag{9.7}$$

Using $\frac{(s)_k}{s(s+1)_k} = \frac{1}{s+k}$, (9.7) becomes

$$\begin{aligned} L_\alpha\left[\frac{1}{n!}\left(1-e^{-\frac{t^\alpha}{\alpha}}\right)^n\right] &= \frac{1}{n!}\sum_{k=0}^{\infty}\frac{(-n)_k\,(s)_k}{s\,(s+1)_k\,k!} = \frac{1}{s.n!}\,{}_2F_1\,(-n, s; s+1; 1) \\ &= \frac{1}{s.n!}\frac{(1)_n}{(s+1)_n} = \frac{1}{s\,(s+1)_n} \end{aligned} \tag{9.8}$$

Substituting (9.8) into (9.6), we have

$$\begin{aligned} L_\alpha\left[{}_2F_1\left(\mu,\nu;1;x^\alpha\left(1-e^{-\frac{t^\alpha}{\alpha}}\right)\right)\right] &= \sum_{n=0}^{\infty}\frac{(\mu)_n\,(\nu)_n}{n!}\frac{1}{s\,(s+1)_n}x^{\alpha n} \\ &= \frac{1}{s}\,{}_2F_1\,(\mu,\nu;s+1;x^\alpha) \end{aligned}$$

as required. □

Theorem 9.2. *Let* $\alpha \in (0,1]$ *and* ${}_2F_1\,(\mu,\nu;c;x^\alpha)$ *be a conformable fractional hypergeometric function, then*

$$L_\alpha\,[\,t^{\alpha n}\sin(at^\alpha)] = \frac{a\alpha^{n+1}\Gamma(n+2)}{s^{n+2}}\,{}_2F_1\left(\frac{n+2}{2},\frac{n+3}{2};\frac{3}{2};-\left(\frac{\alpha a}{s}\right)^2\right). \tag{9.9}$$

Proof. First, we see that

$$\begin{aligned} L_\alpha\,[\,t^{\alpha n}\sin(at^\alpha)] &= L_\alpha\left[t^{\alpha n}\sum_{k=0}^{\infty}\frac{(-1)^k\,a^{2k+1}}{(2k+1)!}t^{\alpha(2k+1)}\right] \\ &= \sum_{k=0}^{\infty}\frac{(-1)^k\,a^{2k+1}}{(2k+1)!}L_\alpha\left\{t^{\alpha(n+2k+1)}\right\} \end{aligned}$$

Using (2) of lemma 9.2, it follows that

$$\begin{aligned} L_\alpha\,[\,t^{\alpha n}\sin(at^\alpha)] &= \sum_{k=0}^{\infty}\frac{(-1)^k\,a^{2k+1}}{(2k+1)!}\alpha^{n+2k+1}\frac{\Gamma(n+2k+2)}{s^{n+2k+2}} \\ &= \frac{a\alpha^{n+1}\Gamma(n+2)}{s^{n+2}}\sum_{k=0}^{\infty}\frac{\Gamma(n+2k+2)}{\Gamma(n+2)\,(2k+1)!}\left(\frac{-\alpha^2a^2}{s^2}\right)^k \\ &= \frac{a\alpha^{n+1}\Gamma(n+2)}{s^{n+2}}\sum_{k=0}^{\infty}\frac{(n+2)_{2k}}{(2)_{2k}}\left(\frac{-\alpha^2a^2}{s^2}\right)^k \end{aligned}$$

But $(n+2)_{2k} = \left(\frac{n+2}{2}\right)_k.\left(\frac{n+3}{2}\right)_k$ and $(2)_{2k} = (1)_k.\left(\frac{3}{2}\right)_k = \left(\frac{3}{2}\right)_k k!$. Therefore,

$$\begin{aligned} L_\alpha\,[\,t^{\alpha n}\sin(at^\alpha)] &= \frac{a\alpha^{n+1}\Gamma(n+2)}{s^{n+2}}\sum_{k=0}^{\infty}\frac{\left(\frac{n+2}{2}\right)_k\cdot\left(\frac{n+3}{2}\right)_k}{\left(\frac{3}{2}\right)_k k!}\left(\frac{-\alpha^2a^2}{s^2}\right)^k \\ &= \frac{a\alpha^{n+1}\Gamma(n+2)}{s^{n+2}}\,{}_2F_1\left(\frac{n+2}{2},\frac{n+3}{2};\frac{3}{2};-\left(\frac{\alpha a}{s}\right)^2\right). \end{aligned}$$

□

10. Applications

The general solution of a wide class of conformable fractional differential equations of mathematical physics can be written in terms of the CFGHF after using a suitable change of independent variable. This technique will be illustrated through the following interesting discussion.

Abul-Ez et al. [33] gave the hypergeometric representation of the conformable fractional Legendre polynomials $P_{\alpha n}(x)$, as

$$P_{\alpha n}(x) = {}_2F_1\left(-n, n+1; 1; \frac{1-x^{\alpha}}{2}\right).$$

This formula can be easily obtained through the CFGHE as follows.

Note that, the conformable fractional Legendre polynomials $P_{\alpha n}(x)$ satisfy the conformable fractional differential equation

$$\left(1-x^{2\alpha}\right) D_x^{\alpha} D_x^{\alpha} P_{\alpha n}(x) - 2\alpha x^{\alpha} D_x^{\alpha} P_{\alpha n}(x) + \alpha^2 n(n+1) P_{\alpha n}(x) = 0. \tag{10.1}$$

With the help of $t^{\alpha} = \frac{1-x^{\alpha}}{2}$, we get

$$D_x^{\alpha} P_{\alpha n} = \left(\frac{-1}{2}\right) D_t^{\alpha} P_{\alpha n}, \text{ and } D_x^{\alpha} D_x^{\alpha} P_{\alpha n} = \frac{1}{4} D_t^{\alpha} D_t^{\alpha} P_{\alpha n}.$$

Using (10.1), we obtain

$$t^{\alpha}(1-t^{\alpha}) D_t^{\alpha} D_t^{\alpha} P_{\alpha n} + \alpha \{1-2t^{\alpha}\} D_t^{\alpha} P_{\alpha n} + \alpha^2 n(n+1) P_{\alpha n} = 0. \tag{10.2}$$

Comparing the last Eq (10.2) with the CFGHE (3.1), we obtain the parameters μ, ν and c, such that

$$\mu = -n, \quad \nu = n+1 \text{ and } c = 1,$$

Hence, we may write the conformable fractional Legendre polynomials as

$$\begin{aligned} P_{\alpha n}(x) &= {}_2F_1(-n, n+1; 1; t^{\alpha}) \\ &= {}_2F_1\left(-n, n+1; 1; \frac{1-x^{\alpha}}{2}\right). \end{aligned}$$

Example 10.1. *Consider the following conformable fractional differential equation*

$$\left(1-e^{x^{\alpha}}\right) D_x^{\alpha} D_x^{\alpha} y + \frac{\alpha}{2} D_x^{\alpha} y + \alpha^2 e^{x^{\alpha}} y = 0 \tag{10.3}$$

Then the general solution of (10.3) *can be easily deduced as follows.*
Let $t^{\alpha} = \left(1-e^{x^{\alpha}}\right)$, *then we have*

$$D_x^{\alpha} y = -e^{x^{\alpha}} D_t^{\alpha} y = -(1-t^{\alpha}) D_t^{\alpha} y$$

and

$$D_x^{\alpha} D_x^{\alpha} y = -\alpha e^{x^{\alpha}} D_t^{\alpha} y + e^{2x^{\alpha}} D_t^{\alpha} D_t^{\alpha} y = (1-t^{\alpha})^2 D_t^{\alpha} D_t^{\alpha} y - \alpha(1-t^{\alpha}) D_t^{\alpha} y$$

Now, in view of (10.3)*, it can be easily seen that,*

$$t^{\alpha}\left[(1-t^{\alpha})^{2}\,D_{t}^{\alpha}D_{t}^{\alpha}y-\alpha\,(1-t^{\alpha})\,D_{t}^{\alpha}y\right]-\frac{\alpha}{2}\,(1-t^{\alpha})\,D_{t}^{\alpha}y+\alpha^{2}\,(1-t^{\alpha})\,y=0 \tag{10.4}$$

Simplifying (10.4)*, we get*

$$t^{\alpha}\,(1-t^{\alpha})\,D_{t}^{\alpha}D_{t}^{\alpha}y+\alpha\left\{\frac{-1}{2}-t^{\alpha}\right\}D_{t}^{\alpha}y+\alpha^{2}y=0 \tag{10.5}$$

Comparing (10.5) *with the CFGHE* (3.1)*, we obtain* $\mu+\nu=0,\ \mu\nu=-1$ *and* $c=\frac{-1}{2}$*. Thus,* $\mu=1$ *and* $\gamma=-1$*. Therefore, the general solution of the CFDE* (10.3) *can be given in the form*

$$\begin{aligned} y &= A\,{}_2F_1\,(\mu,\nu;c;t^{\alpha})+B\,t^{\alpha(1-c)}\,{}_2F_1\,(\mu-c+1,\nu-c+1;2-c;t^{\alpha}) \\ &= A\,{}_2F_1\left(1,-1;\frac{-1}{2};1-e^{x^{\alpha}}\right)+B\,\left[1-e^{x^{\alpha}}\right]^{\frac{3}{2}}\,{}_2F_1\left(\frac{5}{2},\frac{1}{2};\frac{5}{2};1-e^{x^{\alpha}}\right) \end{aligned}$$

where A and B are arbitrary constants.

Example 10.2. *Consider the class of conformable fractional differential equation which contains two arbitrary regular* α*-singular points* $x=\lambda_1$ *and* $x=\lambda_2$*:*

$$(x^{\alpha}-\lambda_1)(x^{\alpha}-\lambda_2)D_{x}^{\alpha}D_{x}^{\alpha}y+(\lambda_3+\lambda_4x^{\alpha})D_{x}^{\alpha}y+\lambda_5y=0,\quad \alpha\in(0,1], \tag{10.6}$$

where $\lambda_i(i=1,2,\ldots,5)\in\mathbb{R}$ *and* $\lambda_1\neq\lambda_2$*.*

Taking $t^{\alpha}=(x^{\alpha}-\lambda_1)/(\lambda_2-\lambda_1)$*. Then*

$$x^{\alpha}=(\lambda_2-\lambda_1)t^{\alpha}+\lambda_1,\quad D_{x}^{\alpha}y=\frac{1}{\lambda_2-\lambda_1}D_{t}^{\alpha}y$$

and

$$D_{x}^{\alpha}D_{x}^{\alpha}y=\frac{1}{(\lambda_2-\lambda_1)^{2}}D_{t}^{\alpha}D_{t}^{\alpha}y.$$

Substituting in (10.6) *the obtained equation becomes*

$$t^{\alpha}(1-t^{\alpha})D_{t}^{\alpha}D_{t}^{\alpha}y+\left(\frac{\lambda_3+\lambda_1\lambda_4}{\lambda_1-\lambda_2}-\lambda_4t^{\alpha}\right)D_{t}^{\alpha}y-\lambda_5y=0. \tag{10.7}$$

Now, we can write (10.7) *as the CFGHE such that*

$$c=\frac{\lambda_3+\lambda_1\lambda_4}{\alpha(\lambda_1-\lambda_2)},\quad \mu+\nu+1=\frac{\lambda_4}{\alpha},\quad \text{and}\ \ \mu\nu=\frac{\lambda_5}{\alpha^{2}}. \tag{10.8}$$

Hence, the general solution of (10.6) *can be obtained about the regular singular points* $t=0$ *and* $t=1$*, which means that we can find the general solutions about* $x=\lambda_1$ *and* $x=\lambda_2$*.*

As a special case, putting $\lambda_1=1$*,* $\lambda_2=-1$*,* $\lambda_3=0$*,* $\lambda_4=\alpha$*, and* $\lambda_5=-\alpha^2n^2$ *in Eq.* (10.6)*, we obtain the conformable fractional Chebyshev differential equation*

$$(1-x^{2\alpha})D_{x}^{\alpha}D_{x}^{\alpha}y-\alpha x^{\alpha}D_{x}^{\alpha}y+\alpha^{2}n^{2}y=0,\quad \alpha\in(0,1]. \tag{10.9}$$

From (10.7)*, it follows that* $c = 1/2$*,* $\mu + \nu = 0$*, and* $\mu\nu = -n^2$*. Hence,* $\mu = -n$*,* $\nu = n$*, and* $c = 1/2$*. The general solution of* (10.9) *about* $x = 1$ *is*

$$y(x) = A\,{}_2F_1\left(-n, n; \frac{1}{2}; \frac{1-x^\alpha}{2}\right) + B\left(\frac{1-x^\alpha}{2}\right)^{\frac{1}{2}}\,{}_2F_1\left(-n+\frac{1}{2}, n+\frac{1}{2}; \frac{3}{2}; \frac{1-x^\alpha}{2}\right),$$

and the general solution of (10.9) *about* $x = -1$ *is*

$$y(x) = A\,{}_2F_1\left(-n, n; \frac{1}{2}; \frac{1+x^\alpha}{2}\right) + B\left(\frac{1+x^\alpha}{2}\right)^{\frac{1}{2}}\,{}_2F_1\left(-n+\frac{1}{2}, n+\frac{1}{2}; \frac{3}{2}; \frac{1+x^\alpha}{2}\right).$$

The strategy used in the preceding examples can be easily applied to solve some famous differential equations such as, Fibonacci, and Lucas differential equations in the framework of fractional calculus. Handled by Fibonacci, and Lucas differential equations have advantages due to their own importance in applications. The Fibonacci polynomial is a polynomial sequence, which can be considered as a generalization circular for the Fibonacci numbers. It is used in many applications, e.g., biology, statistics, physics, and computer science [47]. The Fibonacci and Lucas sequences of both polynomials and numbers are of great importance in a variety of topics, such as number theory, combinatorics, and numerical analysis. For these studies, we refer to [47–50]. Table 1 provides briefly the general solutions of such famous differential equations.

Table 1. General solutions of some famous CDEs.

Conformable fractional Fibonacci differential equation	
CF Fibonacci DE	$\left(x^{2\alpha}+4\right)D_x^\alpha D_x^\alpha y + 3\alpha x^\alpha D_x^\alpha y - \alpha^2\left(n^2-1\right)y = 0$
Suitable transformation	$t^\alpha = \left(1+\frac{x^{2\alpha}}{4}\right)$
Transformed equation	$t^\alpha\left(1-t^\alpha\right)D_t^\alpha D_t^\alpha y + \alpha\left\{\frac{3}{2}-2t^\alpha\right\}D_t^\alpha y - \alpha^2\frac{\left(1-n^2\right)}{4}y = 0.$
Parameters (μ, ν and c)	$\mu = \frac{1-n}{2}$, $\nu = \frac{1+n}{2}$ and $c = \frac{3}{2}$
General solution	$y = A\,{}_2F_1\left(\frac{1-n}{2}, \frac{1+n}{2}; \frac{3}{2}; 1+\frac{x^{2\alpha}}{4}\right) + B\left[1+\frac{x^{2\alpha}}{4}\right]^{\frac{-1}{2}}\,{}_2F_1\left(\frac{-n}{2}, \frac{n}{2}; \frac{1}{2}; 1+\frac{x^{2\alpha}}{4}\right)$
Conformable fractional Lucas differential equation	
CF Lucas DE	$\left(x^{2\alpha}+4\right)D_x^\alpha D_x^\alpha y + \alpha x^\alpha D_x^\alpha y - \alpha^2 n^2 y = 0,$
Suitable transformation	$t^\alpha = \left(1+\frac{x^{2\alpha}}{4}\right)$
Transformed equation	$t^\alpha\left(1-t^\alpha\right)D_t^\alpha D_t^\alpha y + \alpha\left\{\frac{1}{2}-t^\alpha\right\}D_t^\alpha y + \alpha^2\frac{n^2}{4}y = 0.$
Parameters (μ, ν and c)	$\mu = \frac{n}{2}$, $\nu = \frac{-n}{2}$ and $c = \frac{1}{2}$
General solution	$y = A\,{}_2F_1\left(\frac{n}{2}, \frac{-n}{2}; \frac{1}{2}; 1+\frac{x^{2\alpha}}{4}\right) + B\left[1+\frac{x^{2\alpha}}{4}\right]^{\frac{1}{2}}\,{}_2F_1\left(\frac{1+n}{2}, \frac{1-n}{2}; \frac{3}{2}; 1+\frac{x^{2\alpha}}{4}\right)$

11. Conclusions

The Gaussian hypergeometric function ${}_2F_1$ has been studied extensively from its mathematical point of view [51]. This occurs, naturally, due to its many applications on a large variety of physical and mathematical problems. For example, in quantum mechanics, the investigation of the Schrödinger equation for some systems involving Pöschl-Teller, Wood-Saxon, and Hulthén potentials leads to solutions expressed in terms of the hypergeometric functions [52]. Another significant case is related to the angular momentum theory since the eigenfunctions of the angular momentum operators are written in terms of ${}_2F_1$ functions [53]. One essential tool related to such problems is then provided by the derivatives of the ${}_2F_1$ function with respect to the parameters μ, ν, and c since they allow one, for example, to write a Taylor expansion around given values μ_0, ν_0, or c_0. As a result, the importance of the Gaussian hypergeometric differential equation motivates one to provide a detailed study on the CFGHF. The solutions of the CFGHE are given to improve and generalize those given in [31]. Besides, many interesting properties and useful formulas of CFGHF are presented. Finally, supported examples show that a class of conformable fractional differential equations of mathematical physics can be solved through the CFGHF.

Interestingly, the obtained results of the current work have treated various famous aspects such as generating functions, differential forms, contiguous relations, and recursion formulas. Moreover, they have been generalized and developed in the context of the fractional setting. These aspects play essential roles in themselves and their diverse applications. In fact, most of the special functions of mathematical physics and engineering, for instance, the Jacobi and Laguerre polynomials, can be expressed in terms of the Gauss hypergeometric function and other related hypergeometric functions. Therefore, the numerous generating functions involving extensions and generalizations of the Gauss hypergeometric function can play essential roles in the theory of special functions of applied mathematics and mathematical physics, see [54].

The derivatives of any order of the GHF ${}_2F_1(\mu, \nu; c; x)$ with respect to the parameters μ, ν, and c, which can be expressed in terms of generalizations of multivariable Kampe de Fériet functions, have many applications (see the work of [55]). We may recall that the contiguous function relation applications range from the evaluation of hypergeometric series to the derivation of the summation and transformation formulas for such series; these can be used to evaluate the contiguous functions to a hypergeometric function, see [43]. Furthermore, using some contiguous function relations for the classical Gaussian hypergeometric series ${}_2F_1$, several new recursion formulas for the Appell functions F_2 with essential applications have been the subject of some research work, see for example [46] and reference therein. In conclusion, it is rather interesting to consider a broad generalization of the Gaussian hypergeometric function in future work either in the framework of fractional calculus or in a higher-dimensional setting. Our concluded results can be used for a wide variety of cases.

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Conflict of interest

The authors declare no conflict of interest.

References

1. M. Abul-Ez, Bessel polynomial expansions in spaces of holomorphic functions, *J. Math. Anal. Appl.*, **221** (1998), 177–190.
2. M. Abdalla, M. Abul-Ez, J. Morais, On the construction of generalized monogenic Bessel polynomials, *Math. Method. Appl. Sci.*, **41** (2018), 9335–9348.
3. L. Aloui, M. Abul-Ez, G. Hassan, Bernoulli special monogenic polynomials with the difference and sum polynomial bases, *Complex Var. Elliptic*, **59** (2014), 631–650.
4. I. Podlubny, *Fractional differential equations: an introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications*, Elsevier, 1998.
5. I. Podlubny, *Fractional differential equations*, San Diego: Academic Press, 1999.
6. R. Hilfer, *Applications of fractional calculus in physics*, Singapore: World scientific, 2000.
7. R. L. Magin, Fractional calculus in bioengineering, *Crit Rev Biomed Eng.*, **32** (2004), 1–104.
8. O. P. Agrawal, A general formulation and solution scheme for fractional optimal control problems, *Nonlinear Dynam.*, **38** (2004), 323–337.
9. A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and applications of fractional differential equations*, Elsevier, 2006.
10. K. S. Nisar, O. A. Ilhan, J. Manafian, M. Shahriari, D. Soybaş, Analytical behavior of the fractional Bogoyavlenskii equations with conformable derivative using two distinct reliable methods, *Results Phys.*, **22** (2021), 103975.
11. H. Tajadodi, Z. A. Khan, J. F. Gómez-Aguilar, A. Khan, H. Khan, Exact solutions of conformable fractional differential equations, *Results Phys.*, **22** (2021), 103916.
12. R. W. Ibrahim, D. Altulea, R. M. Elobaid. Dynamical system of the growth of COVID-19 with controller, *Adv. Differ. Equ.*, **1** (2021), 1–12.
13. J. Yang, M. Fečkan, J. Wang, Consensus problems of linear multi-agent systems involving conformable derivative, *Appl. Math. Comput.*, **394** (2020), 125809.
14. M. J. Lazo, D. F. Torres, Variational calculus with conformable fractional derivatives, *IEEE/CAA J. Automatic Sinica*, **4** (2016), 340–352.
15. T. Chiranjeevi, R. K. Biswas, Closed-form solution of optimal control problem of a fractional order system, *J. King Saud Univ. Sci.*, **31** (2019), 1042–1047.
16. A. Atangana, M. A. Khan, Validity of fractal derivative to capturing chaotic attractors, *Chaos Soliton. Fract.*, **126** (2019), 50–59.
17. G. Consolini, M. Materassi, A stretched logistic equation for pandemic spreading, *Chaos Soliton. Fract.*, **140** (2020), 110–113.
18. R. Agarwal, S. D. Purohit, Mathematical model for anomalous subdiffusion using conformable operator, *Chaos Soliton. Fract.*, **140** (2020), 110–199.
19. M. Zayed, M. Abul-Ez, M. Abdalla, N. Saad, On the fractional order Rodrigues formula for the shifted Legendre-type matrix polynomials, *Mathematics*, **8** (2020), 136.

20. E. Ünal, A. Gökdoğan, Uyumlu kesir mertebeden chebyshev diferensiyel denklemleri ve kesirsel chebyshev polinomları, *Afyon Kocatepe Üniversitesi Fen ve Mühendislik Bilimleri Dergisi*, **16** (2016), 576–584.
21. H. Ç. Yaslan, Numerical solution of the conformable space-time fractional wave equation, *Chinese J. Phys.*, **56** (2018), 2916–2925.
22. E. F. D. Goufo, Application of the Caputo-Fabrizio fractional derivative without singular kernel to Korteweg-de Vries-Burgers equation, *Math. Model. Anal.*, **21** (2016), 188–198.
23. E. F. D. Goufo, Strange attractor existence for non-local operators applied to four-dimensional chaotic systems with two equilibrium points, *Chaos*, **29** (2019), 023117.
24. E. F. D. Goufo, M. Mbehou, M. M. K. Pene, A peculiar application of Atangana–Baleanu fractional derivative in neuroscience: Chaotic burst dynamics, *Chaos Soliton. Fract.*, **115** (2018), 170–176.
25. P. Agarwal, M. Chand, S. D. Purohit, A note on generating functions involving the generalized Gauss hypergeometric functions, *Natl. Acad. Sci. Lett.*, **37** (2014), 457–459.
26. M. A. Chaudhry, A. Qadir, M. Rafique, S. M. Zubair, Extension of Euler's beta function, *J. Comput. Appl. Math.*, **78** (1997), 19–32.
27. M. A. Chaudhry, A. Qadir, H. M. Srivastava, R. B. Paris, Extended hypergeometric and confluent hypergeometric functions, *Appl. Math. Comput.*, **159** (2004), 589–602.
28. E. Özergin, *Some properties of hypergeometric functions*, Doctoral dissertation, Eastern Mediterranean University (EMU), 2011.
29. E. Özergin, M. A. Özarslan, A. Altın, Extension of gamma, beta and hypergeometric functions, *J. Comput. Appl. Math.*, **235** (2011), 4601–4610.
30. R. Khalil, M. Al Horani, A. Yousef, M. Sababheh, A new definition of fractional derivative, *J. Comput. Appl. Math.*, **264** (2014), 65–70.
31. M. A. Hammad, H. Alzaareer, H. Al-Zoubi, H. Dutta, Fractional Gauss hypergeometric differential equation, *J. Interdiscip. Math.*, **22** (2019), 1113–1121.
32. A. Ali, M. Islam, A. Noreen, Solution of fractional k-Hypergeometric differential equation, *J. Math. Anal.*, **14** (2020), 125–132.
33. M. Abul-Ez, M. Zayed, A. Youssef, M. De la Sen, On conformable fractional Legendre polynomials and their convergence properties with applications, *Alex. Eng. J.*, **59** (2020), 5231–5245.
34. T. Abdeljawad, On conformable fractional calculus, *J. Comput. Appl. Math.*, **279** (2015), 57–66.
35. D. Zhao, M. Luo, General conformable fractional derivative and its physical interpretation, *Calcolo*, **54** (2017), 903–917.
36. H. Kiskinov, M. Petkova, A. Zahariev, About the Cauchy problem for nonlinear system with conformable derivatives and variable delays, *AIP Conference Proceedings*, **2172** (2019), 050006.
37. D. R. Anderson, D. Ulness, Newly defined conformable derivatives, *Adv. Dyn. Syst. Appl.*, **10** (2015), 109–137.
38. A. A. Martynyuk, I. M. Stamova, Fractional-like derivative of Lyapunov-type functions and applications to the stability analysis of motion, *Electron. J. Differ. Eq.,* **2018** (2018), 1–12.

39. A. Martynyuk, G. Stamov, I. Stamova, Integral estimates of the solutions of fractional-like equations of perturbed motion, *Nonlinear Anal-Model*, **24** (2019), 138–149.

40. A. Martynyuk, G. Stamov, I. Stamova, Practical stability analysis with respect to manifolds and boundedness of differential equations with fractional-like derivatives, *Rocky Mt. J. Math.*, **49** (2019), 211–233.

41. M. A. Hammad, R. Khalil, Systems of linear fractional differential equations, *Asian Journal of Mathematics and Computer Research*, **12** (2016), 120–126.

42. A. El-Ajou, A modification to the conformable fractional calculus with some applications, *Alex. Eng. J.*, **59** (2020), 2239–2249.

43. S. Mubeen, G. Rahman, A. Rehman, M. Naz, Contiguous function relations for-hypergeometric functions, *International Scholarly Research Notices*, **2014** (2014), 410801.

44. E. D. Rainville, *Special functions*, Chelsea: The Macmillan Co., 1960.

45. K. S. Rao, V. Lakshminarayanan, *Generalized hypergeometric functions*, IOP Publishing, 2018.

46. S. B. Opps, N. Saad, H. M. Srivastava, Recursion formulas for Appell's hypergeometric function F_2 with some applications to radiation field problems, *Appl. Math. Comput.*, **207** (2009), 545–558.

47. T. Koshy, *Fibonacci and Lucas numbers with applications*, John Wiley & Sons, 2019.

48. W. Wang, H. Wang, Some results on convolved (p, q)-Fibonacci polynomials, *Integr. Transf. Spec. F.*, **26** (2015), 340–356.

49. N. Taskara, K. Uslu, H. H. Gulec, On the properties of Lucas numbers with binomial coefficients, *Appl. Math. Lett.*, **23** (2010), 68–72.

50. H. H. Gulec, N. Taskara, K. Uslu, A new approach to generalized Fibonacci and Lucas numbers with binomial coefficients, *Appl. Math. Comput.*, **220** (2013), 482–486.

51. A. Erdlyi, W. Magnus, F. Oberhettinger, F. G. Tricomi, *Higher transcendental functions, Vols. I-III*, New York: McGraw-Hill, 1953.

52. S. Flügge, *Practical quantum mechanics I*, Berlin: Springer-Verlag, 1971.

53. M. Abramowitz, I. A. Stegun, *Handbook of mathematical functions*, New York: Dover publications inc., 1972.

54. H. M. Srivastava, P. Agarwal, S. Jain, Generating functions for the generalized Gauss hypergeometric functions, *Appl. Math. Comput.*, **247** (2014), 348–352.

55. L. U. Ancarani, G. Gasaneo, Derivatives of any order of the Gaussian hypergeometric function ${}_2F_1(a, b, c; z)$ with respect to the parameters a, b and c, *J. Phys. A Math. Theor.*, **42** (2009), 395208.

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