Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 6(9): 10105–10129.
DOI:10.3934/math.2021587
Received: 12 May 2021
Accepted: 02 July 2021
Published: 09 July 2021

Research article

Global existence and new decay results of a viscoelastic wave equation with variable exponent and logarithmic nonlinearities

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Abstract: In this paper, we consider the following viscoelastic problem with variable exponent and logarithmic nonlinearities:

$$u_{tt} - \Delta u + u + \int_0^t b(t-s)\Delta u(s)ds + |u_t|^{\gamma(\cdot)-2}u_t = u \ln |u|^{\alpha},$$

where $\gamma(.)$ is a function satisfying some conditions. We first prove a global existence result using the well-depth method and then establish explicit and general decay results under a wide class of relaxation functions and some specific conditions on the variable exponent function. Our results extend and generalize many earlier results in the literature.

Keywords: viscoelasticity; relaxation function; general decay; logarithmic nonlinearity; variable exponent
Mathematics Subject Classification: 35B37, 35L55, 74D05, 93D15, 93D20

1. Introduction

In this paper we are concerned with the following problem

$$\begin{cases} u_{tt} - \Delta u + u + \displaystyle\int_0^t b(t-s)\Delta u(s)ds + |u_t|^{\gamma(\cdot)-2}u_t = u \ln |u|^{\alpha} & \text{in } \Omega \times (0, +\infty), \\ u = \frac{\partial u}{\partial \nu} = 0, & \text{on } \partial\Omega \times (0, \infty), \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), & \text{in } \Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain of $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$, ν is the unit outer normal to $\partial\Omega$, u_0 and u_1 are the given data, b is a relaxation function and $\gamma(.)$ is a variable exponent.

Problem (1.1) **contains three class of problems:**

I. **Viscoelasticity with wide class of relaxation functions.**
The importance of the viscoelastic properties of materials has been realized because of the rapid developments in rubber and plastics industry. Many advances in the studies of constitutive relations, failure theories and life prediction of viscoelastic materials and structures were reported and reviewed in the last two decades [1]. There is an extensive literature on the stabilization of viscoelastic wave equations and many results have been established. There are a lot of contributions to generalize the decay rates by allowing an extended class of relaxation functions and give general decay rates. In fact, the journey of generalization of relaxation functions passed through several steps, we mention here the following stages:

1) As in [2], the relaxation function b satisfies , for two positive constants a_1 and a_2,

$$-a_1 b(t) \leq b'(t) \leq -a_2 b(t), \quad t \geq 0.$$

2) As in [3,4], the relaxation function b satisfies

$$b'(t) \leq -a(t)b(t), \ t \geq 0,$$

where $a : \mathbb{R}^+ \to \mathbb{R}^+$ is a nonincreasing differentiable function.

3) As in [5], the relaxation function b satisfies

$$b'(t) \leq -\chi(b(t)),$$

where χ is a positive function, $\chi(0) = \chi'(0) = 0$, and χ is strictly increasing and strictly convex near the origin.

4) As in [6], the relaxation function b satisfies

$$b'(t) \leq -a(t)b^p(t), \ \forall t \geq 0, \ 1 \leq p < \frac{3}{2}.$$

5) As in [7], the relaxation function b satisfies

$$b'(t) \leq -a(t)B(b(t)), \tag{1.2}$$

where $B \in C^1(\mathbb{R})$, with $B(0) = 0$ and B is linear or strictly increasing and strictly convex function C^2 near the origin.

II. **Variable-exponent nonlinearity.**
With the advancement of sciences and technology, many physical and engineering models required more sophisticated mathematical functional spaces to be studied and well understood. For example, in fluid dynamics, the elecrtorheological fluids (smart fluids) have the property that the viscosity changes (often drastically) when exposed to an electrical field. The Lebesgue and Sobolev spaces with variable exponents proved to be efficient tools to study such problems as well as other models like fluids with temperature-dependent viscosity, nonlinear viscoelasticity, filtration processes through a porous media and image processing. More details on these problems can be found in [8, 9]. For hyperbolic problems involving variable-exponent nonlinearities, we refer to [10–15]. For more results of other problems with the nonlinearity of power type, we refer the interested reader to see [16–18].

III. **Logarithmic source term.**
The logarithmic nonlinearity appears naturally in inflation cosmology and supersymmetric filed theories, quantum mechanics and nuclear physics [19, 20]. Problems with logarithmic nonlinearity have a lot of applications in many branches of physics such as nuclear physics, optics and geophysics [21–23].

In this paper, we consider problem (1.1) and prove the global existence of solutions, using the well-depth method. We then establish explicit and general decay results of the solution under suitable assumptions on the variable exponent $\gamma(.)$ and very general assumption on the relaxation function. To the best of our knowledge, such a problem has not been discussed before in the context of nonlinearity with variable exponents.

In addition to the introduction, this paper has four other sections. In Section 2, we present some preliminaries. The Existence is given in Section 3. In Section 4, we establish some technical lemmas needed for the proof of the main results. Our stability results and their proof are given in Section 5.

2. Preliminaries

In this section, we present some preliminaries about the logarithmic nonlinerity and the Lebesgue and Sobolev spaces with variable exponents (see [24–27]). Throughout this paper, c is used to denote a generic positive constant.

Definition 2.1. *Let $\beta : \Omega \to [1,\infty]$ be a measurable function, where Ω is a bounded domain of $\mathbb{R}^n$, then we have the following definitions:*

1) The Lebesgue space with a variable exponent $\beta(\cdot)$ is defined by

$$L^{\beta(\cdot)}(\Omega) := \Big\{v : \Omega \to \mathbb{R};\ \textit{measurable in } \Omega :\ \varrho_{\beta(\cdot)}(kv) < \infty,\ \textit{for some } k > 0\Big\},$$

where $\varrho_{\beta(\cdot)}(v) = \int_\Omega \frac{1}{\beta(x)}|v(x)|^{\beta(x)}dx$ is a modular.

2) The variable-exponent Sobolev space $W^{1,\beta(\cdot)}(\Omega)$ is:

$$W^{1,\beta(\cdot)}(\Omega) = \Big\{v \in L^{\beta(\cdot)}(\Omega)\ \ \textit{such that } \nabla v \textit{ exists and } |\nabla v| \in L^{\beta(\cdot)}(\Omega)\Big\}.$$

3) $W_0^{1,\beta(\cdot)}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{1,\beta(\cdot)}(\Omega)$.

Remark 2.2. *[9]*

1) $L^{\beta(\cdot)}(\Omega)$ is a Banach space equipped with the following Luxembourg-type norm

$$\|v\|_{\beta(\cdot)} := \inf\Big\{\lambda > 0 : \int_\Omega \Big|\frac{v(x)}{\lambda}\Big|^{\beta(x)} dx \le 1\Big\},$$

2) $W^{1,\beta(\cdot)}(\Omega)$ is a Banach space with respect to the norm

$$\|v\|_{W^{1,\beta(\cdot)}(\Omega)} = \|v\|_{\beta(\cdot)} + \|\nabla v\|_{\beta(\cdot)}.$$

Definition 2.3. *Let K be a convex function on $(0, r]$, then the convex conjugate of K, in the sense of Young (see [32]), is defined as follows:*

$$K^*(s) = s(K')^{-1}(s) - K\Big[(K')^{-1}(s)\Big], \quad \text{if } s \in (0, K'(r)] \tag{2.1}$$

and K^* satisfies the following generalized Young inequality

$$\alpha_1\alpha_2 \le K^*(\alpha_1) + K(\alpha_2), \quad \text{if } \alpha_1 \in (0, K'(r)],\ \alpha_2 \in (0, r]. \tag{2.2}$$

Let

$$\beta_1 := \text{essinf}_{x\in\Omega}\beta(x), \quad \beta_2 := \text{esssup}_{x\in\Omega}\beta(x).$$

Lemma 2.4. [9] *If $\beta : \Omega \to [1, \infty)$ is a measurable function with $\beta_2 < \infty$, then $C_0^\infty(\Omega)$ is dense in $L^{\beta(\cdot)}(\Omega)$.*

Remark 2.5 (**Log-Hölder continuity condition**)**.** *The exponent $p(\cdot) : \Omega \to [1, \infty]$ is said to be satisfying the log-Hölder continuity condition; if there exists a constant $c > 0$ such that, for all δ with $0 < \delta < 1$,*

$$|p(x) - p(y)| \le -\frac{c}{\log|x - y|}, \text{ for all } x, y \in \Omega, \text{ with } |x - y| < \delta. \tag{2.3}$$

Lemma 2.6. *[9][**Poincaré's Inequality**] Let Ω be a bounded domain of $\mathbb{R}^n$ and $p(\cdot)$ satisfies* (2.3)*, then*

$$\|v\|_{p(\cdot)} \le c_*\|\nabla v\|_{p(\cdot)}, \quad \text{for all } v \in W_0^{1,p(\cdot)}(\Omega).$$

In particular, the space $W_0^{1,p(\cdot)}(\Omega)$ has an equivalent norm given by $\|v\|_{W_0^{1,p(\cdot)}(\Omega)} = \|\nabla v\|_{p(\cdot)}$.

Lemma 2.7. *[9][**Embedding Property**] Let Ω be a bounded domain in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$. Assume that $p, k \in C(\overline{\Omega})$ such that*

$$1 < p_1 \le p(x) \le p_2 < +\infty, \qquad 1 < k_1 \le k(x) \le k_2 < +\infty, \quad \forall x \in \overline{\Omega},$$

and $k(x) < p^*(x)$ in $\overline{\Omega}$ with

$$p^*(x) = \begin{cases} \frac{np(x)}{n-p(x)}, & if p_2 < n; \\ +\infty, & if p_2 \ge n, \end{cases}$$

then we have continuous and compact embedding $W^{1,p(.)}(\Omega) \hookrightarrow L^{k(.)}(\Omega)$. So, there exists $c_e > 0$ such that

$$\|v\|_k \le c_e\|v\|_{W^{1,p(.)}}, \quad \forall v \in W^{1,p(.)}(\Omega).$$

Lemma 2.8. *[27] Let $\epsilon \in (0,1)$. Then there exists $\beta_\varepsilon > 0$ such that*

$$s|\ln s| \leq s^2 + \beta_\epsilon s^{1-\epsilon}, \ \forall s > 0. \tag{2.4}$$

We consider the following hypotheses:

($A1$) The relaxation function $b : \mathbb{R}^+ \to \mathbb{R}^+$ is a C^1 nonincreasing function satisfying

$$b(0) > 0, \ \ 1 - \int_0^\infty b(s)ds = \bar{b} > 0, \tag{2.5}$$

and there exists a C^1 function $B : (0,\infty) \to (0,\infty)$ which is strictly increasing and strictly convex C^2 function on $(0, r]$, $r \leq b(0)$, with $B(0) = B'(0) = 0$, such that

$$b'(t) \leq -a(t)B(b(t)), \quad \forall t \geq 0, \tag{2.6}$$

where a is a positive nonincreasing differentiable function.

($A2$) $\gamma : \overline{\Omega} \to [1,\infty)$ is a continuous function satisfies the log-Hölder continuity condition (Remark 2.5) such that

$$\gamma_1 := \operatorname{essinf}_{x\in\Omega}\gamma(x), \quad \gamma_2 := \operatorname{esssup}_{x\in\Omega}\gamma(x).$$

and $1 < \gamma_1 < \gamma(x) \leq \gamma_2$, where

$$\begin{cases} \gamma_2 < \infty, & n = 1,2; \\ \gamma_2 \leq \frac{2n}{n-2}, & n \geq 3. \end{cases}$$

($A3$) The constant α in (1.1) satisfies $0 < \alpha < \alpha_0$, where α_0 is the positive real number satisfying

$$\sqrt{\frac{2\pi\bar{b}}{\alpha_0}} = e^{-\frac{3}{2}-\frac{1}{\alpha_0}} \tag{2.7}$$

where $\|.\|_2 = \|.\|_{L^2(\Omega)}$.

Lemma 2.9. *[28, 29] (Logarithmic Sobolev inequality) Let u be any function in $H_0^1(\Omega)$ and d be any positive real number. Then*

$$\int_\Omega u^2 \ln|u|dx \leq \frac{1}{2}\|u\|_2^2 \ln\|u\|_2^2 + \frac{d^2}{2\pi}\|\nabla u\|_2^2 - (1 + \ln d)\|u\|_2^2. \tag{2.8}$$

Lemma 2.10. *There exists a unique $\alpha_0 > 0$ such that*

$$e^{-\frac{3}{2}-\frac{1}{s}} < \sqrt{\frac{2\pi\bar{b}}{s}}, \ \forall s \in (0,\alpha_0). \tag{2.9}$$

Proof. Let $g(s) = \sqrt{\frac{2\pi\bar{b}}{s}} - e^{-\frac{3}{2}-\frac{1}{s}}$, then g is a continuous and decreasing function on $(0,\infty)$, with

$$\lim_{s\to 0^+} g(s) = \infty \text{ and } \lim_{x\to\infty} g(x) = -e^{-\frac{3}{2}}.$$

Then, there exists a unique $\alpha_0 > 0$ such that $g(\alpha_0) = 0$ and (2.9) holds □

Remark 2.11. *Lemma 2.10 shows that the selection of α in* (A3) *is possible.*

Remark 2.12. *Using the facts that $B(0) = 0$ and B is strictly convex on $(0, r]$, then*

$$B(\theta s) \leq \theta B(s),\ 0 \leq \theta \leq 1 \text{ and } s \in (0, r]. \tag{2.10}$$

Remark 2.13. *[7] If B is a strictly increasing and strictly convex $\boldsymbol{C}^2$ function on $(0, r]$, with $B(0) = B'(0) = 0$, then there is a strictly convex and strictly increasing $\boldsymbol{C}^2$ function $\overline{B} : [0, +\infty) \longrightarrow [0, +\infty)$ which is an extension of B. For simplicity, in the rest of this paper, we use B instead of $\overline{B}$.*

3. Existence

In this section, we state the local existence theorem whose proof can be established by combining the arguments of [10, 30, 31]. Also, we state and prove a global existence result under smallness conditions on the initial data (u_0, u_1).

Theorem 3.1 (**Local Existence**). *Suppose conditions* (A1)–(A3) *hold and $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$. Then, there exists $T > 0$, such that problem* (1.1) *has a weak solution*

$$u \in L^\infty((0, T), H_0^1(\Omega)), \quad u_t \in L^\infty((0, T), L^2(\Omega)) \cap L^{\gamma(.)}(\Omega \times (0, T)).$$

Definition 3.2. *We define the following functionals which are needed for establishing the global existence*

$$\begin{aligned} E(t) = \frac{1}{2}\Big[\|u_t\|_2^2 + \Big(1 - \int_0^t b(s)ds\Big)\|\nabla u\|_2^2 + (b \circ \nabla u)(t) + \frac{\alpha + 2}{2}\|u\|_2^2\Big] \\ - \frac{1}{2}\int_\Omega u^2 \ln |u|^\alpha dx \end{aligned} \tag{3.1}$$

where for $v \in L^2_{loc}(\mathbb{R}^+; L^2(\Omega))$,

$$(b \circ v)(t) := \int_0^t b(t - s)\|v(t) - v(s)\|_2^2 ds.$$

$E(t)$ represents the modified energy functional associated to problem (1.1).

$$I(u) = I(u(t)) = \left(1 - \int_0^t b(s)ds\right)\|\nabla u\|_2^2 + \|u\|_2^2 + (bo\nabla u)(t) - \int_\Omega u^2 \ln |u|^\alpha dx \tag{3.2}$$

$$J(u) = J(u(t)) = \frac{1}{2}I(u(t)) + \frac{\alpha}{4}\|u\|_2^2, \tag{3.3}$$

then

$$E(t) = \frac{1}{2}\|u_t(t)\|_2^2 + J(u(t)). \tag{3.4}$$

Notation: *We define*

$$\rho_* = e^{\frac{2D_0 - \alpha}{\alpha}}, \qquad E_1 = \frac{1}{2}D_0\rho_*^2 - \frac{\alpha}{4}\rho_*^2 \ln \rho_*^2$$

and

$$D_0 = \frac{\alpha + 2}{2} + \alpha(1 + \ln d),$$

where $0 < d < \sqrt{\frac{2\pi\bar{b}}{\alpha}}$.

Lemma 3.3. *Assume that* $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$, *(A1) holds,*

$$\|u_0\|_2 < \rho_* \text{ and } 0 < E(0) < E_1. \tag{3.5}$$

Then, $I(u(t)) \geq 0$ *for all* $t \in [0, T)$.

Proof. First, we show that $\|u\|_2 < \rho_*$, $\forall t \in [0, T)$. By (2.5), (3.4) and (2.9), we obtain

$$\begin{aligned} E(t) &\geq J(u(t)) \\ &\geq \frac{\bar{b}}{2}\|\nabla u\|_2^2 + \frac{1}{2}\|u\|_2^2 + \frac{1}{2}(bo\nabla u)(t) - \frac{1}{2}\int_\Omega u^2 \ln |u|^\alpha dx + \frac{\alpha}{4}\|u\|_2^2 \\ &\geq \frac{1}{2}\left(\bar{b} - \frac{\alpha d^2}{2\pi}\right)\|\nabla u\|_2^2 + \frac{1}{2}\left(\frac{\alpha+2}{2} + \alpha(1 + \ln d) - \frac{\alpha}{2}\ln \|u\|_2^2\right)\|u\|_2^2 \end{aligned} \tag{3.6}$$

If we select $d < \sqrt{\frac{2\pi\bar{b}}{\alpha}}$, then (3.6) becomes

$$E(t) \geq Z(\rho) = \frac{1}{2}D_0\rho^2 - \frac{\alpha}{4}\rho^2 \ln \rho^2 \tag{3.7}$$

where $D_0 = \frac{\alpha+2}{2} + \alpha(1 + \ln d)$ and $\rho = \|u\|_2$. Using (3.7), we can deduce that that Z is increasing on $(0, \rho_*)$, decreasing on $(\rho_*, +\infty)$ and $Z(\rho) \to -\infty$ as $\rho \to +\infty$. Moreover,

$$\max_{0<\rho<+\infty} Z(\rho) = \frac{1}{2}D_0\rho_*^2 - \frac{\alpha}{4}\rho_*^2 \ln \rho_*^2 = Z(\rho_*) = E_1.$$

Suppose that $\|u(x,t)\|_2 < \rho_*$ is not true in $[0, T)$. Therefore, using the continuity of $u(t)$, it follows that there exists $0 < t_0 < T$ such that $\|u(x, t_0)\|_2 = \rho_*$. From Eq (3.7), we can see that

$$E(t_0) \geq Z(\|u(x, t_0)\|_2) = Z(\rho_*) = E_1,$$

which is a contradiction with $E(t) \leq E(0) < E_1$ for all $t \geq 0$. Recalling the definition of $I(u(t))$, and using (2.9) with $d < \sqrt{\frac{2\pi\bar{b}}{\alpha}}$, for all $t \in [0, T)$, lead to

$$\begin{aligned} I(u(t)) &\geq \bar{b}\|\nabla u\|_2^2 - \int_\Omega u^2 \ln |u|^\alpha dx \\ &\geq \left(\bar{b} - \frac{\alpha d^2}{2\pi}\right)\|\nabla u\|_2^2 + \left(1 + \alpha(1 + \ln d) - \frac{\alpha}{2}\ln \|u\|_2^2\right)\|u\|_2^2 \\ &\geq \left(\bar{b} - \frac{\alpha d^2}{2\pi}\right)\|\nabla u\|_2^2 + \|u\|_2^2 \geq 0. \end{aligned} \tag{3.8}$$

This completes the proof. □

Remark 3.4. *We can see that if* $\|u_0\|_2 < \rho_*$ *and* $E(0) < E_1$, *then* $J(u(t)) \geq 0$ *and consequently* $E(t) \geq 0$ *for all* $t \in [0, T)$. *Therefore, from* (3.8), *for* $t \in [0, T)$ *we have*

$$\begin{aligned} &\|u_t\|_2^2 \leq 2E(t) \leq 2E(0), \\ &\|\nabla u\|_2^2 \leq \frac{2\pi}{2\pi\bar{b} - \alpha d^2}I(t) \leq \frac{4\pi}{2\pi\bar{b} - \alpha d^2}E(t) \leq \frac{4\pi}{2\pi\bar{b} - \alpha d^2}E(0), \end{aligned} \tag{3.9}$$

which shows that the soultion is global and bounded in time.

4. Technical lemmas

In this section, we establish several lemmas needed for the proof of our main result.

Lemma 4.1. *The energy functional associated to problem* (1.1) *satisfies, for any* $t \geq 0$,

$$E'(t) = \frac{1}{2}(b' \circ \nabla u)(t) - \frac{1}{2}b(t)\|\nabla u\|_2^2 - \int_\Omega |u_t|^{\gamma(x)}dx \leq 0. \tag{4.1}$$

Proof. Multiplying (1.1) by u_t, integrating over Ω and using the boundary conditions, imply (4.1). □

Lemma 4.2. *[31] Assume that b satisfies* $(A1)$*. Then, for* $u \in H_0^1(\Omega)$,

$$\int_\Omega \left(\int_0^t b(t-s)(u(t)-u(s))ds\right)^2 dx \leq c(bo\nabla u)(t),$$

and

$$\int_\Omega \left(\int_0^t b'(t-s)(u(t)-u(s))ds\right)^2 dx \leq -c(b'o\nabla u)(t).$$

Lemma 4.3. *[7] Assume* $(A1)$ *holds. Then, for any* $t \geq t_0$,, *we have*

$$a(t)\int_0^{t_0} b(s)\|\nabla u(t) - \nabla u(t-s)\|_2^2 ds \leq -cE'(t).$$

Lemma 4.4. *Assume that* $(A1)$*–*$(A3)$ *and* (3.5) *hold, then the functional*

$$I_1(t) := \int_\Omega u u_t dx$$

satisfies, along with the solution of (1.1), the estimates:

$$\begin{aligned} I_1'(t) \leq \|u_t\|_2^2 - \|u\|_2^2 - \frac{\bar{b}}{4}\|\nabla u(t)\|_2^2 + c(bo\nabla u)(t) \\ + c\int_\Omega |u_t|^{\gamma(x)}dx + \int_\Omega u^2 \ln|u|^\alpha dx, \quad \textit{for } \gamma_1 \geq 2 \end{aligned} \tag{4.2}$$

and

$$\begin{aligned} I_1'(t) \leq \|u_t\|_2^2 - \|u\|_2^2 - \frac{\bar{b}}{4}\|\nabla u(t)\|_2^2 + c(bo\nabla u)(t) \\ + c\int_\Omega |u_t|^{\gamma(x)}dx + \left(\int_\Omega |u_t|^{\gamma(x)}\right)^{\gamma_1 - 1} + \int_\Omega u^2 \ln|u|^\alpha dx, \textit{ for } 1 < \gamma_1 < 2. \end{aligned} \tag{4.3}$$

Proof. Differentiate I_1 and use the differential equation in (1.1), to get

$$\begin{aligned} I_1'(t) = \|u_t\|_2^2 - \|u\|_2^2 - \left(1 - \int_0^t b(s)ds\right)\|\nabla u\|_2^2 + \int_\Omega \nabla u(t)\int_0^t b(t-s)\left(\nabla u(s) - \nabla u(t)\right)ds dx \\ - \int_\Omega u|u_t|^{\gamma(x)-2}u_t dx + \int_\Omega u^2 \ln|u|^\alpha dx. \end{aligned} \tag{4.4}$$

Young's inequality and (4.2) give

$$\int_\Omega \nabla u. \int_0^t b(t-s)(\nabla u(s) - \nabla u(t))dsdx \\ \leq \delta_0 \int_\Omega |\nabla u|^2 dx + \frac{c}{4\delta_0}(bo\nabla u)(t). \tag{4.5}$$

Estimation of the term $-\int_\Omega u|u_t|^{\gamma(x)-2}u_t dx$:
We use Young's inequality with $p(x) = \frac{\gamma(x)}{\gamma(x)-1}$ and $p'(x) = \gamma(x)$ so, for all $x \in \Omega$, we have

$$|u_t|^{\gamma(x)-2}u_t u \leq \delta |u|^{\gamma(x)} + c_\delta(x)|u_t|^{\gamma(x)},$$

where

$$c_\delta(x) = \delta^{1-\gamma(x)}(\gamma(x))^{-\gamma(x)}(\gamma(x)-1)^{\gamma(x)-1}.$$

Hence,

$$-\int_\Omega u|u_t|^{\gamma(x)-2}u_t dx \leq \delta \int_\Omega |u|^{\gamma(x)}dx + \int_\Omega c_\delta(x)|u_t|^{\gamma(x)}dx. \tag{4.6}$$

Now, using (3.1), (4.1), (3.9) and Lemma 2.7, we obtain

$$\begin{aligned}\int_\Omega |u|^{\gamma(x)}dx &\leq \int_{\Omega_+} |u|^{\gamma(x)}dx + \int_{\Omega_-} |u|^{\gamma(x)}dx \\ &\leq \int_{\Omega_+} |u|^{\gamma_2}dx + \int_{\Omega_-} |u|^{\gamma_1}dx \\ &\leq \int_\Omega |u|^{\gamma_2}dx + \int_\Omega |u|^{\gamma_1}dx \\ &\leq \left(c_e^{\gamma_1}\|\nabla u\|_2^{\gamma_1} + c_e^{\gamma_2}\|\nabla u\|_2^{\gamma_2}\right) \\ &\leq \left(c_e^{\gamma_1}\|\nabla u\|_2^{\gamma_1-2} + c_e^{\gamma_2}\|\nabla u\|_2^{\gamma_2-2}\right)\|\nabla u\|_2^2 \\ &\leq \left(c_e^{\gamma_1}\left(\frac{4\pi}{2\pi\bar{b}-\alpha d^2}E(0)\right)^{\gamma_1-2} + c_e^{\gamma_2}\left(\frac{4\pi}{2\pi\bar{b}-\alpha d^2}E(0)\right)^{\gamma_2-2}\right)\|\nabla u\|_2^2 \\ &\leq c\|\nabla u\|_2^2,\end{aligned} \tag{4.7}$$

where

$$\Omega_+ = \{x \in \Omega : |u(x,t)| \geq 1\} \text{ and } \Omega_- = \{x \in \Omega : |u(x,t)| < 1\},$$

and

$$c = \left(c_e^{\gamma_1}\left(\frac{4\pi}{2\pi\bar{b}-\alpha d^2}E(0)\right)^{\gamma_1-2} + c_e^{\gamma_2}\left(\frac{4\pi}{2\pi\bar{b}-\alpha d^2}E(0)\right)^{\gamma_2-2}\right).$$

Then, (4.6) and (4.7) yield

$$-\int_\Omega u|u_t|^{\gamma(x)}u_t dx \leq \delta c\|\nabla u\|_2^2 + \int_\Omega c_\delta(x)|u_t|^{\gamma(x)}dx. \tag{4.8}$$

Combining the above results with fixing $\delta_0 = \frac{\bar{b}}{2}$ and $\delta = \frac{\bar{b}}{4c}$ completes the proof of (4.2).
For the proof of (4.3), we re-estimate the fifth term in (4.4) as follows:
First, we define

$$\Omega_1 = \{x \in \Omega : \gamma(x) < 2\} \text{ and } \Omega_2 = \{x \in \Omega : \gamma(x) \geq 2\}.$$

Then, we get

$$-\int_\Omega u|u_t|^{\gamma(x)-2}u_t dx = -\int_{\Omega_1} u|u_t|^{\gamma(x)-2}u_t dx - \int_{\Omega_2} u|u_t|^{\gamma(x)-2}u_t dx. \tag{4.9}$$

Using the definition of Ω_1, we have

$$2\gamma(x) - 2 < \gamma(x), \text{ and } 2\gamma(x) - 2 \geq 2\gamma_1 - 2. \tag{4.10}$$

Therefore, using Young's and Poincaré's inequalities and (4.10), we obtain

$$\begin{aligned} -\int_{\Omega_1} u|u_t|^{\gamma(x)-2}u_t dx &\leq \theta \int_{\Omega_1} |u|^2 dx + \frac{1}{4\theta}\int_{\Omega_1} |u_t|^{2\gamma(x)-2}dx \\ &\leq \theta c_*^2 \|\nabla u\|_2^2 + c\Big[\int_{\Omega_1^+} |u_t|^{2\gamma(x)-2}dx + \int_{\Omega_1^-} |u_t|^{2\gamma(x)-2}dx\Big] \\ &\leq \theta c_*^2 \|\nabla u\|_2^2 + c\Big[\int_{\Omega_1^+} |u_t|^{\gamma(x)}dx + \int_{\Omega_1^-} |u_t|^{2\gamma_1-2}dx\Big] \\ &\leq \theta c_*^2 \|\nabla u\|_2^2 + c\Big[\int_{\Omega} |u_t|^{\gamma(x)}dx + \Big(\int_{\Omega_1^-} |u_t|^{2}dx\Big)^{\gamma_1-1}\Big] \\ &\leq \theta c_*^2 \|\nabla u\|_2^2 + c\Big[\int_{\Omega} |u_t|^{\gamma(x)}dx + \Big(\int_{\Omega_1^-} |u_t|^{\gamma(x)}dx\Big)^{\gamma_1-1}\Big] \\ &\leq \theta c_*^2 \|\nabla u\|_2^2 + c\Big[\int_{\Omega} |u_t|^{\gamma(x)}dx + \Big(\int_{\Omega} |u_t|^{\gamma(x)}dx\Big)^{\gamma_1-1}\Big], \end{aligned} \tag{4.11}$$

where

$$\Omega_1^+ = \{x \in \Omega_1 : |u_t(x,t)| \geq 1\} \text{ and } \Omega_1^- = \{x \in \Omega_1 : |u_t(x,t)| < 1\}. \tag{4.12}$$

After setting $\theta = \frac{\bar{b}}{8c_*^2}$, (4.11) becomes

$$-\int_{\Omega_1} u|u_t|^{\gamma(x)-2}u_t dx \leq \frac{\bar{b}}{8}\|\nabla u\|_2^2 + c\Big[\int_\Omega |u_t|^{\gamma(x)}dx + \Big(\int_\Omega |u_t|^{\gamma(x)}dx\Big)^{\gamma_1-1}\Big]. \tag{4.13}$$

Next, for any δ we have, by the case $\gamma(x) \geq 2$,

$$-\int_{\Omega_2} u|u_t|^{\gamma(x)-2}u_t dx \leq \delta c\|\nabla u\|_2^2 + \int_\Omega c_\delta(x)|u_t|^{\gamma(x)}dx. \tag{4.14}$$

Therefore, by combining (4.9)–(4.14), we arrive at

$$\begin{aligned} I_1'(t) &\leq \|u_t\|_2^2 - \left(\frac{3\bar{b}}{8} - c\delta\right)\|\nabla u(t)\|_2^2 + c(bo\nabla u)(t) \\ &\quad + c\left[\int_\Omega (1 + c_\delta(x))\,|u_t|^{\gamma(x)}dx + \left(\int_\Omega |u_t|^{\gamma(x)}\right)^{\gamma_1-1}\right] + \int_\Omega u^2 \ln|u|^\alpha dx. \end{aligned}$$

By fixing $\delta = \frac{\bar{b}}{8c}$, $c_\delta(x)$ remains bounded and, consequently, we obtain (4.3). □

Lemma 4.5. *Assume that* $(A1)$–$(A3)$ *and* (3.5) *hold, then for any* $\delta > 0$*, the functional*

$$I_2(t) := -\int_\Omega u_t \int_0^t b(t-s)(u(t)-u(s))dsdx$$

satisfies, along the solution of (1.1), the estimates:

$$\begin{aligned} I_2'(t) \le\ & \delta\|\nabla u\|_2^2 - \Big(\int_0^t b(s)ds - \delta\Big)\|u_t\|_2^2 + \int_\Omega c_\delta(x)|u_t|^{\gamma(x)}dx \\ & + \frac{c}{\delta}(-b' o \nabla u)(t) + \frac{c}{\delta}(b \circ \nabla u)(t) + c_{\epsilon,\delta}(bo\nabla u)^{\frac{1}{1+\epsilon}}(t), \ \textit{for}\ \gamma_1 \ge 2, \end{aligned} \tag{4.15}$$

and for $1 < \gamma_1 < 2$*, we have the following estimate*

$$\begin{aligned} I_2'(t) \le\ & \delta\|\nabla u\|_2^2 - \Big(\int_0^t b(s)ds - \delta\Big)\|u_t\|_2^2 + c(b \circ \nabla u)(t) + c_{\epsilon,\delta}(bo\nabla u)^{\frac{1}{1+\epsilon}}(t) \\ & + \frac{c}{\delta}(-b' o \nabla u)(t) + \frac{c}{\delta}\Big[\int_\Omega |u_t|^{\gamma(x)}dx + \Big(\int_\Omega |u_t|^{\gamma(x)}dx\Big)^{\gamma_1 - 1}\Big] \end{aligned} \tag{4.16}$$

Proof. Direct differentiation of I_2, using (1.1), yields

$$\begin{aligned} I_2'(t) =\ & \int_\Omega \nabla u \int_0^t b(t-s)(\nabla u(t) - \nabla u(s))dsdx \\ & - \int_\Omega u \int_0^t b(t-s)(u(t)-u(s))dsdx \\ & - \int_\Omega \Big(\int_0^t b(t-s)\nabla u(s)ds\Big)\Big(\int_0^t b(t-s)(\nabla u(t) - \nabla u(s))ds\Big)dx \\ & - \int_\Omega u_t \int_0^t b'(t-s)(u(t)-u(s))dsdx - \left(\int_0^t b(s)ds\right)\|u_t\|_2^2 \\ & + \int_\Omega |u_t|^{\gamma(x)-2}u_t \int_0^t b(t-s)(u(t)-u(s))dsdx \\ & - \alpha \int_\Omega u \ln|u| \int_0^t b(t-s)(u(t)-u(s))dsdx \\ =\ & \left(1 - \int_0^t b(s)ds\right)\int_\Omega \nabla u \int_0^t b(t-s)(\nabla u(t) - \nabla u(s))dsdx \\ & - \int_\Omega u \int_0^t b(t-s)(u(t)-u(s))dsdx \\ & + \int_\Omega \Big(\int_0^t b(t-s)(\nabla u(t) - \nabla u(s))ds\Big)^2 dx \\ & - \int_\Omega u_t \int_0^t b'(t-s)(u(t)-u(s))dsdx - \left(\int_0^t b(s)ds\right)\|u_t\|_2^2 \\ & + \int_\Omega |u_t|^{\gamma(x)-2}u_t \int_0^t b(t-s)(u(t)-u(s))dsdx \\ & - \alpha \int_\Omega u \ln|u| \int_0^t b(t-s)(u(t)-u(s))dsdx. \end{aligned} \tag{4.17}$$

Using Young's inequality and Lemma 4.2, we obtain

$$\begin{aligned}&\left(1-\int_0^t b(s)ds\right)\int_\Omega \nabla u.\int_0^t b(t-s)(\nabla u(t)-\nabla u(s))dsdx\\&\le c\delta\|\nabla u\|_2^2+\frac{c}{\delta}(bo\nabla u)(t).\end{aligned}\tag{4.18}$$

The use of Lemma 4.2, Young's and Poincaré's inequalities leads to

$$\int_\Omega u\int_0^t b(t-s)(u(t)-u(s))dsdx\le c\delta\|\nabla u\|_2^2+\frac{c}{\delta}(bo\nabla u)(t)\tag{4.19}$$

Exploiting Lemma (4.2) and Young's inequality, we obtain

$$-\int_\Omega u_t\int_0^t b'(t-s)(u(t)-u(s))dsdx\le\delta\|u_t\|_2^2+\frac{c}{\delta}(-b'o\nabla u)(t).\tag{4.20}$$

Next, for almost every $x\in\Omega$ fixed, we have

$$\begin{aligned}\int_0^t b(t-s)|u(t)-u(s)|ds&\le\left(\int_0^t b(s)ds\right)^{\frac{\gamma(x)-1}{\gamma(x)}}\left(\int_0^t b(t-s)|u(t)-u(s)|^{\gamma(x)}ds\right)^{\frac{1}{\gamma(x)}}\\&\le(1-\bar{b})^{\frac{\gamma(x)-1}{\gamma(x)}}\left(\int_0^t b(t-s)|u(t)-u(s)|^{\gamma(x)}ds\right)^{\frac{1}{\gamma(x)}}.\end{aligned}\tag{4.21}$$

Therefore, for almost every $x\in\Omega$, we have

$$\left|\int_0^t b(t-s)|u(t)-u(s)|ds\right|^{\gamma(x)}\le(1-\bar{b})^{\gamma_1-1}\int_0^t b(t-s)|u(t)-u(s)|^{\gamma(x)}ds.\tag{4.22}$$

By using Young's, Hölder's and Poincaré's inequalities and Lemma 4.2, we get

$$\begin{aligned}&\int_\Omega|u_t|^{\gamma(x)-2}u_t\int_0^t b(t-s)(u(t)-u(s))dsdx\\&\le\delta\int_\Omega\left|\int_0^t b(t-s)(u(t)-u(s))ds\right|^{\gamma(x)}dx+\int_\Omega c_\delta(x)|u_t|^{\gamma(x)}dx\\&\le\delta\big(1-\bar{b}\big)^{\gamma_1-1}\int_\Omega\int_0^t b(t-s)|(u(t)-u(s)|^{\gamma(x)}dsdx+\int_\Omega c_\delta(x)|u_t|^{\gamma(x)}dx,\end{aligned}\tag{4.23}$$

where

$$c_\delta(x)=\delta^{1-\gamma(x)}(\gamma(x))^{-\gamma(x)}(\gamma(x)-1)^{\gamma(x)-1}.$$

Similarly, we have

$$\begin{aligned}&\int_\Omega\int_0^t b(t-s)|(u(t)-u(s)|^{\gamma(x)}dsdx\\&\le\int_{\Omega_+}\int_0^t b(t-s)|(u(t)-u(s)|^{\gamma_2}dsdx+\int_{\Omega_-}\int_0^t b(t-s)|(u(t)-u(s)|^{\gamma_1}dsdx\\&\le\int_0^t b(t-s)\|(u(t)-u(s)\|_{\gamma_2}^{\gamma_2}ds+\int_0^t b(t-s)\|(u(t)-u(s)\|_{\gamma_1}^{\gamma_1}ds\\&\le\left[c_e^{\gamma_2}\left(\frac{4\pi}{2\pi\bar{b}-\alpha d^2}E(0)\right)^{\frac{\gamma_2-2}{2}}+c_e^{\gamma_1}\left(\frac{4\pi}{2\pi\bar{b}-\alpha d^2}E(0)\right)^{\frac{\gamma_1-2}{2}}\right]\int_0^t b(t-s)\|(u(t)-u(s)\|_2^2ds.\end{aligned}\tag{4.24}$$

Therefore,

$$\int_\Omega |u_t|^{\gamma(x)-2}u_t \int_0^t b(t-s)(u(t)-u(s))dsdx \leq c\delta\Big(1-\bar{b}\Big)^{\gamma_1-1}(b\circ\nabla u)(t) + \int_\Omega c_\delta(x)|u_t|^{\gamma(x)}dx, \tag{4.25}$$

where $c = \left[c_e^{\gamma_2}\left(\frac{4\pi}{2\pi\bar{b}-\alpha d^2}E(0)\right)^{\frac{\gamma_2-2}{2}} + c_e^{\gamma_1}\left(\frac{4\pi}{2\pi\bar{b}-\alpha d^2}E(0)\right)^{\frac{\gamma_1-2}{2}}\right]$.

For the last term in (4.17), the use of (2.4), Young's, Cauchy-Schwarz' and Poincaré's inequalities, the embedding theorem and Lemma 4.2 leads to, for any $\delta > 0$,

$$\begin{aligned}
&\int_\Omega u\ln|u|^\alpha \int_0^t b(t-s)(u(t)-u(s))dsdx \leq \\
&\alpha\int_\Omega \Big(u^2+\beta_\epsilon|u|^{1-\epsilon}\Big)\left|\int_0^t b(t-s)(u(t)-u(s))dsdx\right| \\
&\leq c\int_\Omega |u|^2\left|\int_0^t b(t-s)(u(t)-u(s))ds\right|dx \\
&\quad +\delta\int_\Omega u^2dx + c_{\epsilon,\delta}\int_\Omega\left|\int_0^t b(t-s)(u(t)-u(s))ds\right|^{\frac{2}{1+\epsilon}}dx \\
&\leq c\delta\|\nabla u\|_2^2 + \frac{c}{\delta}\int_\Omega\left|\int_0^t b(t-s)(u(t)-u(s))ds\right|^2dx \\
&\quad + c_{\epsilon,\delta}\int_\Omega\left|\int_0^t b(t-s)(u(t)-u(s))ds\right|^{\frac{2}{1+\epsilon}}dx \\
&\leq c\delta\|\nabla u\|_2^2 + \frac{c}{\delta}(bo\nabla u)(t) + c_{\epsilon,\delta}(bo\nabla u)^{\frac{1}{1+\epsilon}}(t).
\end{aligned}$$

Combining the above estimates with (4.17), we obtain (4.15).

For the proof of (4.16), we re-estimate the fifth term in (4.17) as follows:

$$\begin{aligned}
&\int_\Omega |u_t|^{\gamma(x)-2}u_t\int_0^t b(t-s)(u(t)-u(s))dsdx \\
&\leq \delta\int_\Omega\left|\int_0^t b(t-s)(u(t)-u(s))ds\right|^2dx + \frac{c}{\delta}\int_\Omega |u_t|^{2\gamma(x)-2}dx \\
&\leq \delta(1-\bar{b})(b\circ u)(t) + \frac{c}{\delta}\int_\Omega |u_t|^{2\gamma(x)-2}dx \\
&\leq c\delta(b\circ\nabla u)(t) + \frac{c}{\delta}\int_{\Omega_1}|u_t|^{2\gamma(x)-2}dx + \frac{c}{\delta}\int_{\Omega_2}|u_t|^{2\gamma(x)-2}dx \\
&\leq c\delta(b\circ\nabla u)(t) + \frac{c}{\delta}\Big(\int_\Omega |u_t|^{\gamma(x)}dx + \Big(\int_\Omega |u_t|^{\gamma(x)}dx\Big)^{\gamma_1-1}\Big).
\end{aligned} \tag{4.26}$$

Then (4.16) is established. □

Lemma 4.6. *Given $t_0 > 0$. Assume that $(A1)$–$(A3)$ and (3.5) hold. Then,*

$$L(t) := N_1E(t) + N_2I_1(t) + I_2(t)$$

satisfies, for a suitable choice of N_1, $N_2 > 0$ *and for some positive constants* λ_0 *and c, the estimates, for any* $t \geq t_0$,

$$L'(t) \leq -\lambda_0 E(t) + c(bo\nabla u)(t) + c_\epsilon (bo\nabla u)^{\frac{1}{1+\epsilon}}(t), \ for\ \gamma_1 \geq 2, \tag{4.27}$$

and

$$L'(t) \leq -cE(t) + c(b \circ \nabla u)(t) + c_\epsilon (bo\nabla u)^{\frac{1}{1+\epsilon}}(t) + c\Big(- E'(t)\Big)^{\gamma_1 - 1}, \ for\ 1 < \gamma_1 < 2. \tag{4.28}$$

Proof. Since b is positive and $b(0) > 0$ then, for any $t_0 > 0$, we have

$$\int_0^t b(s)ds \geq \int_0^{t_0} b(s)ds = b_0 > 0, \ \forall t \geq t_0.$$

By using (4.1), (4.2) and (4.15), then, for $t \geq t_0$ and any $\lambda_0 > 0$, we have

$$\begin{aligned} L'(t) \leq & -\lambda_0 E(t) - \left(N_2\delta - \frac{\bar{b}}{2} + \frac{\lambda_0(1-b_0)}{2}\right) \|\nabla u\|_2^2 - \left(N_2(b_0 - \delta) - 1 - \frac{\lambda_0}{2}\right) \|u_t\|_2^2 \\ & + c(bo\nabla u)(t) + \left(\frac{1}{2}N_1 - \frac{4c}{\ell}N_2^2\right)(b'o\nabla u)(t) \\ & + \left(1 - \frac{\lambda_0}{2}\right)\int_\Omega u^2 \ln|u|^\alpha dx + \left(1 - \frac{\lambda_0(\alpha+2)}{4}\right)\|u\|_2^2. \end{aligned}$$

Using the Logarithmic Sobolev inequality, for $0 < \lambda_0 < \frac{1}{2}$, we get

$$\begin{aligned} L'(t) \leq & -\lambda_0 E(t) - \left(N_2\delta - \frac{\bar{b}}{2} + \frac{\lambda_0(1-b_0)}{2} - \left(1 - \frac{\lambda_0}{2}\right)\frac{\alpha d^2}{2\pi}\right)\|\nabla u\|_2^2 \\ & - \left(N_2(b_0 - \delta) - 1 - \frac{\lambda_0}{2}\right)\|u_t\|_2^2 \\ & + c(bo\nabla u)(t) + \left(\frac{1}{2}N_1 - \frac{4c}{\bar{b}}N_2^2\right)(b'o\nabla u)(t) \\ & - \left(1 - \frac{\alpha}{2}\left(1 - \frac{\lambda_0}{2}\right)\ln\|u\|_2^2 + \alpha(1 + \ln d)\left(1 - \frac{\lambda_0}{2}\right) - \frac{\lambda_0(\alpha+2)}{4}\right)\|u\|_2^2. \end{aligned}$$

At this point, we select λ_0 and α so small that

$$1 - \frac{\alpha}{2}\left(1 - \frac{\lambda_0}{2}\right)\ln\|u\|_2^2 + \alpha(1 + \ln d)\left(1 - \frac{\lambda_0}{2}\right) - \frac{\lambda_0(\alpha+2)}{4} > 0.$$

Then, we choose N_2 large enough so that:

$$N_2\delta - \frac{\bar{b}}{2} + \frac{\lambda_0(1-b_0)}{2} - \left(1 - \frac{\lambda_0}{2}\right)\frac{\alpha d^2}{2\pi} > 0$$

and

$$N_2(b_0 - \delta) - 1 - \frac{\lambda_0}{2} > 0,$$

and then N_1 large enough that

$$N_1 - \frac{4c}{\bar{b}}N_2^2 > 0.$$

Therefore, we arrive at the desired result (4.27). On the other hand, we can choose N_1 even larger (if needed) so that

$$L \sim E. \tag{4.29}$$

□

5. Decay results

In this section, we establish our main decay results. For this purpose, we need the following remarks and lemma.

Remark 5.1. *Using* (3.6) *and* (4.1) *, we get*

$$\begin{aligned}(bo\nabla u)(t) &= (bo\nabla u)^{\frac{\epsilon}{1+\epsilon}}(t)(bo\nabla u)^{\frac{1}{1+\epsilon}}(t)\\ &\leq c(bo\nabla u)^{\frac{1}{1+\epsilon}}(t).\end{aligned} \tag{5.1}$$

Remark 5.2. *In the case of B* ***is linear*** *and since a is nonincreasing, we have*

$$\begin{aligned}a(t)(b\circ\nabla u)^{\frac{1}{1+\epsilon}}(t) &= (a^{\epsilon}(t)a(t)(b\circ\nabla u)(t))^{\frac{1}{1+\epsilon}}\\ &\leq (a^{\epsilon}(0)a(t)(b\circ\nabla u)(t))^{\frac{1}{1+\epsilon}}\\ &\leq c\,(a(t)(b\circ\nabla u)(t))^{\frac{1}{1+\epsilon}}\\ &\leq c(-E'(t))^{\frac{1}{1+\epsilon}}.\end{aligned} \tag{5.2}$$

Lemma 5.3. *If* $(A1)$–$(A2)$ *are satisfied, then we have the following estimate*

$$(bo\nabla u)(t) \leq \frac{t}{\varepsilon_0}B^{-1}\left(\frac{\varepsilon_0\psi(t)}{ta(t)}\right), \quad \forall t>0, \tag{5.3}$$

where ε_0 *is small enough and the functional* ψ *is defined by*

$$\psi(t) := (-b'o\nabla u)(t) \leq -cE'(t), \tag{5.4}$$

Proof. To establish (5.3), let us define the following functional

$$\Lambda(t) := \frac{\varepsilon_0}{t}\int_0^t \|\nabla u(t)-\nabla u(t-s)\|_2^2 ds, \quad \forall t>0. \tag{5.5}$$

Then, using (3.1), (4.1) and the dentition of $\Lambda(t)$, we have

$$\begin{aligned}\Lambda(t) &\leq \frac{2\varepsilon_0}{t}\Big(\int_0^t \|\nabla u(t)\|_2^2 + \int_0^t \|\nabla u(t-s)\|_2^2 ds\Big).\\ &\leq \frac{4\varepsilon_0}{\bar{b}t}\Big(\int_0^t (E(t)+E(t-s))ds\Big)\\ &\leq \frac{8\varepsilon_0}{\bar{b}t}\int_0^t E(s)ds\\ &\leq \frac{8\varepsilon_0}{\bar{b}t}\int_0^t E(0)ds = \frac{8\varepsilon_0}{\bar{b}}E(0) < +\infty.\end{aligned} \tag{5.6}$$

Thus, ε_0 can be chosen so small so that, for all $t > 0$,

$$\Lambda(t) < 1. \tag{5.7}$$

Without loss of the generality, for all $t > 0$, we assume that $\Lambda(t) > 0$, otherwise we get an exponential decay from (4.27). The use of Jensen's inequality and using (5.4), (2.10) and (5.7) gives

$$\begin{aligned}
\psi(t) &= \frac{1}{\varepsilon_0\Lambda(t)}\int_0^t \Lambda(t)(-b'(s))\int_\Omega \varepsilon_0|\nabla u(t)-\nabla u(t-s)|^2dxds\\
&\geq \frac{1}{\varepsilon_0\Lambda(t)}\int_0^t \Lambda(t)a(s)B(b(s))\int_\Omega \varepsilon_0|\nabla u(t)-\nabla u(t-s)|^2dxds\\
&\geq \frac{a(t)}{\varepsilon_0\Lambda(t)}\int_0^t B(\Lambda(t)b(s))\int_\Omega \varepsilon_0|\nabla u(t)-\nabla u(t-s)|^2dxds\\
&\geq \frac{ta(t)}{\varepsilon_0}B\Big(\frac{\varepsilon_0}{t}\int_0^t b(s)\int_\Omega |\nabla u(t)-\nabla u(t-s)|^2dxds\Big),
\end{aligned} \tag{5.8}$$

hence (5.3) is established. □

Theorem 5.4 (**The case: $\gamma_1 \geq 2$).** *Assume that* (*A*1)–(*A*3) *and* (3.5) *hold. Let* $(u_0, u_1) \in H_0^1(\Omega)\times L^2(\Omega)$. *Then, there exist positive constants c, t_0 and t_1 such that the solution of* (1.1) *satisfies,*

$$E(t) \leq c\left(1+\int_{t_0}^t a^{1+\epsilon}(s)ds\right)^{\frac{-1}{\epsilon}}, \ \forall t\geq t_0, \quad \textit{if } B \textit{ is linear} \tag{5.9}$$

and

$$E(t) \leq ct^{\frac{1}{1+\epsilon}}\mathcal{B}_2{}^{-1}\left(\frac{c}{t^{\frac{1}{1+\epsilon}}\int_{t_1}^t a(s)ds}\right), \ \forall t\geq t_1, \quad \textit{if } B \textit{ is nonlinear,} \tag{5.10}$$

where $\mathcal{B}_2(s) = s\mathcal{B}'(\varepsilon_1 s)$ *and* $\mathcal{B}(t) = \left(\left[B^{-1}\right]^{\frac{1}{1+\epsilon}}\right)^{-1}(t)$.

Proof. **Case 1**: B is linear

We multiply (4.27) by $a(t)$ and use (5.1) and (5.2) to get

$$a(t)L'(t) \leq -\lambda_0 a(t)E(t) + c\left(-E'(t)\right)^{\frac{1}{1+\epsilon}}, \quad \forall t\geq t_0. \tag{5.11}$$

Multiply (5.11) by $a^\epsilon(t)E^\epsilon(t)$, and recall that $a' \leq 0$, to obtain

$$a^{\epsilon+1}(t)E^\epsilon(t)L'(t) \leq -\lambda_0 a^{\epsilon+1}(t)E^{\epsilon+1}(t) + c\,(aE)^\epsilon(t)\left(-E'(t)\right)^{\frac{1}{\epsilon+1}}, \quad \forall t\geq t_0.$$

Use of Young's inequality, with $q = \epsilon+1$ and $q^* = \frac{\epsilon+1}{\epsilon}$, gives, for any $\varepsilon' > 0$,

$$\begin{aligned}
a^{\epsilon+1}(t)E^\epsilon(t)L'(t) &\leq -\lambda_0 a^{\epsilon+1}(t)E^{\epsilon+1}(t) + c\left(\varepsilon' a^{\epsilon+1}(t)E^{\epsilon+1} - c_{\varepsilon'}E'(t)\right)\\
&= -(\lambda_0-\varepsilon' c)a^{\epsilon+1}(t)E^{\epsilon+1} - cE'(t), \quad \forall t\geq t_0.
\end{aligned}$$

We then choose $0 < \varepsilon' < \frac{\lambda_0}{c}$ and use that $a' \leq 0$ and $E' \leq 0$, to get, for $c_1 = \lambda_0 - \varepsilon' c$,

$$\left(a^{\epsilon+1}E^{\epsilon}L\right)'(t) \leq a^{\epsilon+1}(t)E^{\epsilon}(t)L_1'(t) \leq -c_1 a^{\epsilon+1}(t)E^{\epsilon+1}(t) - cE'(t), \quad \forall t \geq t_0,$$

which implies

$$\left(a^{\epsilon+1}E^{\epsilon}L + cE\right)'(t) \leq -c_1 a^{\epsilon+1}(t)E^{\epsilon+1}(t), \quad \forall t \geq t_0,$$

where $L_1 = a^{\epsilon+1}E^{\epsilon}L + cE$. Then $L_1 \sim E$ (thanks to (4.29)) and

$$L_1'(t) \leq -ca^{\epsilon+1}(t)L_1^{\epsilon+1}(t), \ \forall t \geq t_0.$$

Integrating over (t_0, t) and using the fact that $L_1 \sim E$, we obtain (5.9).

Case 2: B is non-linear.
Using (4.27), (5.1) and (5.3), we obtain, $\forall t \geq t_0$,

$$L'(t) \leq -\lambda_0 E(t) + ct^{\frac{1}{1+\epsilon}}\left[B^{-1}\left(\frac{\varepsilon_0 \psi(t)}{ta(t)}\right)\right]^{\frac{1}{1+\epsilon}}. \tag{5.12}$$

Combining the strictly increasing property of $\overline{B}$ and the fact that $\frac{1}{t} < 1$ whenever $t > 1$, we obtain

$$B^{-1}\left(\frac{\varepsilon_0 \psi(t)}{ta(t)}\right) \leq B^{-1}\left(\frac{\varepsilon_0 \psi(t)}{t^{\frac{1}{1+\epsilon}}a(t)}\right) \tag{5.13}$$

then, (5.12) becomes, for $\forall t \geq t_1 = \max\{t_0, 1\}$,

$$L'(t) \leq -\lambda_0 E(t) + ct^{\frac{1}{1+\epsilon}}\left[B^{-1}\left(\frac{\varepsilon_0 \psi(t)}{t^{\frac{1}{1+\epsilon}}a(t)}\right)\right]^{\frac{1}{1+\epsilon}}. \tag{5.14}$$

Set

$$\mathcal{B}(t) = \left(\left[B^{-1}\right]^{\frac{1}{1+\epsilon}}\right)^{-1}(t), \quad \chi(t) = \frac{\varepsilon_0 \psi(t)}{t^{\frac{1}{1+\epsilon}}a(t)} \tag{5.15}$$

Using the facts that $\mathcal{B}' > 0$ and $\mathcal{B}'' > 0$ on $(o, r]$, (5.14) reduces to

$$L'(t) \leq -\lambda_0 E(t) + ct^{\frac{1}{1+\epsilon}}\mathcal{B}^{-1}(\chi(t)), \quad \forall t \geq t_1 \tag{5.16}$$

Now, for $\varepsilon_1 < r$ and using (5.36) and the fact that $E' \leq 0$, $\mathcal{B}' > 0, \mathcal{B}'' > 0$ on $(0, r]$, we find that the functional L_2, defined by

$$L_2(t) := \mathcal{B}'\left(\frac{\varepsilon_1}{t^{\frac{1}{1+\epsilon}}} \cdot \frac{E(t)}{E(0)}\right)L(t),$$

satisfies, for some $c_1, c_2 > 0$.

$$c_1 L_2(t) \leq E(t) \leq c_2 L_2(t) \tag{5.17}$$

and, for all $t \geq t_1$,

$$L_2'(t) \leq -\lambda_0 E(t)\mathcal{B}'\left(\frac{\varepsilon_1}{t^{\frac{1}{1+\epsilon}}} \cdot \frac{E(t)}{E(0)}\right) + ct^{\frac{1}{1+\epsilon}}\mathcal{B}'\left(\frac{\varepsilon_1}{t^{\frac{1}{1+\epsilon}}} \cdot \frac{E(t)}{E(0)}\right)\mathcal{B}^{-1}(\chi(t)). \tag{5.18}$$

So, using (2.1) and (2.2) with $\alpha_1 = \mathcal{B}'\left(\frac{\varepsilon_1}{t^{\frac{1}{1+\epsilon}}} \cdot \frac{E(t)}{E(0)}\right)$ and $\alpha_2 = \mathcal{B}^{-1}(\chi(t))$, we arrive at

$$\begin{aligned} L_2'(t) &\leq -\lambda_0 E(t)\mathcal{B}'\left(\frac{\varepsilon_1}{t^{\frac{1}{1+\epsilon}}} \cdot \frac{E(t)}{E(0)}\right) + ct^{\frac{1}{1+\epsilon_0}}\mathcal{B}^*\left(G'\left(\frac{\varepsilon_1}{t^{\frac{1}{1+\epsilon}}} \cdot \frac{E(t)}{E(0)}\right)\right) \\ &\quad + ct^{\frac{1}{1+\epsilon}}\chi(t) \\ &\leq -\lambda_0 E(t)\mathcal{B}'\left(\frac{\varepsilon_1}{t^{\frac{1}{1+\epsilon_0}}} \cdot \frac{E(t)}{E(0)}\right) + c\varepsilon_1 \frac{E(t)}{E(0)}\mathcal{B}'\left(\frac{\varepsilon_1}{t^{\frac{1}{1+\epsilon_0}}} \cdot \frac{E(t)}{E(0)}\right) \\ &\quad + ct^{\frac{1}{1+\epsilon_0}}\chi(t). \end{aligned} \tag{5.19}$$

Then, multiplying (5.19) by $a(t)$ and using (5.4), (5.15), we get

$$\begin{aligned} a(t)L_2'(t) \leq& -\lambda_0 a(t)E(t)\mathcal{B}'\left(\frac{\varepsilon_1}{t^{\frac{1}{1+\epsilon_0}}} \cdot \frac{E(t)}{E(0)}\right) + c\varepsilon_1 a(t)\frac{E(t)}{E(0)}\mathcal{B}'\left(\frac{\varepsilon_1}{t^{\frac{1}{1+\epsilon_0}}} \cdot \frac{E(t)}{E(0)}\right) \\ &- cE'(t), \forall t \geq t_1. \end{aligned}$$

Using the non-increasing property of a, we obtain, for all $t \geq t_1$,

$$\begin{aligned} (aL_2 + cE)'(t) \leq& -\lambda_0 a(t)E(t)\mathcal{B}'\left(\frac{\varepsilon_1}{t^{\frac{1}{1+\epsilon}}} \cdot \frac{E(t)}{E(0)}\right) \\ &+ c\varepsilon_1 a(t)\frac{E(t)}{E(0)}\mathcal{B}'\left(\frac{\varepsilon_1}{t^{\frac{1}{1+\epsilon}}} \cdot \frac{E(t)}{E(0)}\right) \end{aligned}$$

Therefore, by setting $L_3 := aL_2 + cE \sim E$, we conclude that

$$L_3'(t) \leq -\lambda_0 a(t)E(t)\mathcal{B}'\left(\frac{\varepsilon_1}{t^{\frac{1}{1+\epsilon}}} \cdot \frac{E(t)}{E(0)}\right) + c\varepsilon_1 a(t) \cdot \frac{E(t)}{E(0)}\mathcal{B}'\left(\frac{\varepsilon_1}{t^{\frac{1}{1+\epsilon}}} \cdot \frac{E(t)}{E(0)}\right).$$

This gives, for a suitable choice of ε_1,

$$L_3'(t) \leq -ca(t)\left(\frac{E(t)}{E(0)}\right)\mathcal{B}'\left(\frac{\varepsilon_1}{t^{\frac{1}{1+\epsilon}}} \cdot \frac{E(t)}{E(0)}\right), \qquad \forall t \geq t_1$$

or

$$c\left(\frac{E(t)}{E(0)}\right)\mathcal{B}'\left(\frac{\varepsilon_1}{t^{\frac{1}{1+\epsilon}}} \cdot \frac{E(t)}{E(0)}\right)a(t) \leq -L_3'(t), \qquad \forall t \geq t_1 \tag{5.20}$$

An integration of (5.20) yields

$$\int_{t_1}^{t} c\left(\frac{E(s)}{E(0)}\right)\mathcal{B}'\left(\frac{\varepsilon_1}{s^{\frac{1}{1+\epsilon}}} \cdot \frac{E(s)}{E(0)}\right)a(s)ds \leq -\int_{t_1}^{t} L_3'(s)ds \leq L_3(t_1). \tag{5.21}$$

Using the facts that $\mathcal{B}', \mathcal{B}'' > 0$ and the non-increasing property of E, we deduce that the map $t \mapsto E(t)\mathcal{B}'\left(\frac{\varepsilon_1}{t^{\frac{1}{1+\epsilon}}} \cdot \frac{E(t)}{E(0)}\right)$ is non-increasing and consequently, we have

$$\begin{aligned} &c\left(\frac{E(t)}{E(0)}\right)\mathcal{B}'\left(\frac{\varepsilon_1}{t^{\frac{1}{1+\epsilon_0}}} \cdot \frac{E(t)}{E(0)}\right)\int_{t_1}^{t} a(s)ds \\ &\quad \leq \int_{t_1}^{t} c\left(\frac{E(s)}{E(0)}\right)\mathcal{B}'\left(\frac{\varepsilon_1}{s^{\frac{1}{1+\epsilon}}} \cdot \frac{E(s)}{E(0)}\right)a(s)ds \leq L_3(t_1), \qquad \forall t \geq t_1 \end{aligned} \tag{5.22}$$

Multiplying each side of (5.22) by $\frac{1}{t^{\frac{1}{1+\epsilon}}}$, we have

$$\left(\frac{1}{t^{\frac{1}{1+\epsilon}}}\cdot\frac{E(t)}{E(0)}\right)\mathcal{B}'\left(\frac{\varepsilon_1}{t^{\frac{1}{1+\epsilon}}}\cdot\frac{E(t)}{E(0)}\right)\int_{t_1}^{t}a(s)ds\leq\frac{c}{t^{\frac{1}{1+\epsilon}}},\qquad\forall t\geq t_1\tag{5.23}$$

Next, we set $\mathcal{B}_2(s)=s\mathcal{B}'(\varepsilon_1 s)$ which is strictly increasing, and consequently we obtain,

$$\mathcal{B}_2\left(\frac{1}{t^{\frac{1}{1+\epsilon}}}\cdot\frac{E(t)}{E(0)}\right)\int_{t_1}^{t}a(s)ds\leq\frac{c}{t^{\frac{1}{1+\epsilon}}},\qquad\forall t\geq t_1\tag{5.24}$$

Finally, we infer

$$E(t)\leq ct^{\frac{1}{1+\epsilon}}\mathcal{B}_2^{-1}\left(\frac{c}{t^{\frac{1}{1+\epsilon}}\int_{t_1}^{t}a(s)ds}\right).\tag{5.25}$$

This finishes the proof. □

The following examples illustrate the results of Theorem 5.4:

Example 1. *Let $b(t)=c_1e^{-c_2(1+t)}$, where $c_2>0$ and $c_1>0$ is small enough so that $(A1)$ holds. Then $b'(t)=-a(t)B(b(t))$ where $B(t)=t$ and $a(t)=c$. Therefore, we can use* (5.9) *to deduce*

$$E(t)\leq\frac{c}{(1+t)^{\frac{1}{\epsilon}}}.\tag{5.26}$$

Example 2. *Let $b(t)=\frac{c_1}{(1+t)^q}$, where $q>1+\epsilon$ and c_1 is chosen so that hypothesis $(A1)$ is satisfied. Then*

$$b'(t)=-aB(b(t)),\quad\text{with}\quad B(s)=s^{\frac{q+1}{q}},$$

where a is a fixed constant. Then, (5.10) *gives,*

$$E(t)\leq\frac{c}{t^{\frac{q-1-\epsilon}{(1+\epsilon)^2(q+1)}}}.\tag{5.27}$$

To establish the stability result in the case $\mathbf{1}<\gamma_{\mathbf{1}}<\mathbf{2}$, we need the following lemma:

Lemma 5.5. *The energy functional $E(t)$ satisfies the following estimate:*

$$\Big[-E'(t)\Big]^{\frac{1}{1+\varepsilon}}+\Big[-E'(t)\Big]^{\gamma_1-1}\leq c\Big[-E'(t)\Big]^{\gamma_\varepsilon},\tag{5.28}$$

where $\gamma_\varepsilon=\min\{\gamma_1-1,\frac{1}{1+\varepsilon}\}$.

Proof. Using (2.5), (3.1), (3.3), (3.6) and Lemma 3.3, we have

$$E(t)=J(t)+\frac{1}{2}\|u_t(t)\|_2^2\geq J(t)\geq\frac{\bar{b}}{2}\|\nabla u(t)\|_2^2,$$

then, using (4.1),

$$\|\nabla u(t)\|_2^2\leq\frac{2}{\bar{b}}E(t)\leq\frac{2}{\bar{b}}E(0).\tag{5.29}$$

So, from (4.1), (4.7) and using Young's inequality, we get

$$\begin{aligned} |E'(t)| &= \frac{1}{2}b(t)\|\nabla u(t)\|_2^2 - \frac{1}{2}(b' o\nabla u)(t) - \int_\Omega |u_t|^{\gamma(x)}dx \\ &\leq \frac{1}{2}b(t)\|\nabla u(t)\|_2^2 - \int_0^t b'(t-s)\left(\|\nabla u(t)\|_2^2 + \|\nabla u(s)\|_2^2\right)ds + c\|\nabla u\|_2^2 \\ &\leq \frac{2}{l}\left(\frac{1}{2}b(t) + 2b(0) - 2b(t) + c\right)E(0) \\ &\leq cE(0). \end{aligned} \tag{5.30}$$

Setting $\gamma_\varepsilon = \min\{\gamma_1 - 1, \frac{1}{1+\varepsilon}\}$ and using (5.30), we obtain

$$\begin{aligned} \Big[-E'(t)\Big]^{\frac{1}{1+\varepsilon}} + \Big[-E'(t)\Big]^{\gamma_1-1} &\leq \Big[-E'(t)\Big]^{\gamma_\varepsilon}\Big[-E'(t)\Big]^{\frac{1}{1+\varepsilon}-\gamma_\varepsilon} \\ &\quad + \Big[-E'(t)\Big]^{\gamma_\varepsilon}\Big[-E'(t)\Big]^{\gamma_1-1-\gamma_\varepsilon} \\ &\leq \left((cE(0))^{\frac{1}{1+\varepsilon}-\gamma_\varepsilon} + (cE(0))^{\gamma_1-1-\gamma_\varepsilon}\right)\Big[-E'(t)\Big]^{\gamma_\varepsilon}, \end{aligned} \tag{5.31}$$

which completes the proof of Lemma 5.5.

□

Theorem 5.6 (**The case: $1 < \gamma_1 < 2$**). *Assume that* (*A*1)–(*A*3) *and* (3.5) *hold. Let* $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$. *Then, there exist positive constants* C, k_2, k_3 *such that the energy functional associated to problem (1.1) satisfies*

$$E(t) \leq C\left(\int_{t_0}^t a^{\frac{1}{\gamma_\varepsilon}}(s)ds\right)^{\frac{\gamma_\varepsilon-1}{\gamma_\varepsilon}}, \quad \forall t \geq t_0, \text{ if } B \text{ is linear}, \tag{5.32}$$

and, if B is nonlinear, we have

$$E(t) \leq k_3 t^{\frac{1}{1+\epsilon}}\mathcal{B}_3{}^{-1}\left(\frac{k_2}{t^{\frac{1}{1+\epsilon}}\int_{t_1}^t a(s)ds}\right), \qquad \forall\, t > t_1, \tag{5.33}$$

where $\gamma_\varepsilon = \min\{\gamma_1 - 1, \frac{1}{1+\varepsilon}\}$, $\mathcal{B}_3(s) = s\mathcal{B}'(\varepsilon_3 s)$*and* $\mathcal{B}(s) = \left(\left[B^{-1}\right]^{\frac{1}{1+\epsilon}}\right)^{-1}(s)$.

Proof. **Case B is linear.**
Multiplying (4.28) by $a(t)$ and combining (2.6), (3.1), (5.2) and (5.28), we obtain, for some $m_1 > 0$,

$$\begin{aligned} a(t)L'(t) &\leq -m_1 a(t)E(t) + c\Big[-E'(t)\Big]^{\frac{1}{1+\varepsilon}} + ca(t)\Big[-E'(t)\Big]^{\gamma_1-1} \\ &\leq -m_1 a(t)E(t)c + c\Big[-E'(t)\Big]^{\gamma_\varepsilon}, \quad \forall t > t_0. \end{aligned} \tag{5.34}$$

Let $\mathcal{L} := aL + cE \sim E$, multiply both sides of the above estimate by $a^q E^q$, with $q = \frac{1}{\gamma_\varepsilon} - 1$ and apply Young's inequality, to get,

$$a^q E^q(t)\mathcal{L}'(t) \le -(m_1 - \epsilon_2)a^{q+1}(t)E^{q+1}(t) - cE'(t), \quad \forall t \ge t_0.$$

Set $\mathcal{L}_1 := a^q E^q \mathcal{L} + cE \sim E$, take ϵ_2 small enough and use the non-increasing property of E we obtain, for some $m_2, m_3 > 0$,

$$\mathcal{L}_1'(t) \le -m_2 a^{q+1}(t)E^{q+1}(t) \le -m_3 a^{q+1}(t)\mathcal{L}_2^{q+1}(t), \qquad \forall t \ge t_0.$$

A simple integration over (t_0, t) and using the equivalence $L \sim E$, we obtain,

$$E(t) \le C\left(\int_{t_0}^t a^{\frac{1}{\gamma\varepsilon}}(s)ds\right)^{\frac{\gamma\varepsilon-1}{\gamma\varepsilon}}, \qquad \forall t \ge t_0.$$

Case B is nonlinear.
Using (4.27), (5.1) and (5.3), we obtain, $\forall t \ge t_0$,

$$L'(t) \le -\lambda_0 E(t) + ct^{\frac{1}{1+\epsilon}}\left[B^{-1}\left(\frac{\varepsilon_0 I(t)}{ta(t)}\right)\right]^{\frac{1}{1+\epsilon}} + c\Big[-E'(t)\Big]^{\gamma_1-1}. \tag{5.35}$$

Using (5.13)–(5.15), (5.35) reduces to

$$L'(t) \le -\lambda_0 E(t) + ct^{\frac{1}{1+\epsilon}}\mathcal{B}^{-1}(\chi(t)) + c\Big[-E'(t)\Big]^{\gamma_1-1}, \quad \forall t \ge t_1 \tag{5.36}$$

Now, for $\varepsilon_3 < r$ and using (5.36) and the fact that $E' \le 0$, $H' > 0, H'' > 0$ on $(0, r]$, we find that the functional $\mathcal{F}$, defined by

$$\mathcal{F}(t) := \mathcal{B}'\left(\frac{\varepsilon_3}{t^{\frac{1}{1+\epsilon}}} \cdot \frac{E(t)}{E(0)}\right)L(t),$$

satisfies

$$\mathcal{F} \sim E \tag{5.37}$$

and, for all $t \ge t_1$,

$$\begin{aligned}\mathcal{F}'(t) \le & -\lambda_0 E(t)\mathcal{B}'\left(\frac{\varepsilon_3}{t^{\frac{1}{1+\epsilon}}} \cdot \frac{E(t)}{E(0)}\right) + ct^{\frac{1}{1+\epsilon}}\mathcal{B}'\left(\frac{\varepsilon_3}{t^{\frac{1}{1+\epsilon}}} \cdot \frac{E(t)}{E(0)}\right)\mathcal{B}^{-1}(\chi(t)) \\ & + c\mathcal{B}'\left(\frac{\varepsilon_3}{t^{\frac{1}{1+\epsilon}}} \cdot \frac{E(t)}{E(0)}\right)\Big[-E'(t)\Big]^{\gamma_1-1}.\end{aligned} \tag{5.38}$$

After applying with the generalized Young inequality we arrive at

$$\begin{aligned}\mathcal{F}'(t) \le & -\lambda_0 E(t)\mathcal{B}'\left(\frac{\varepsilon_3}{t^{\frac{1}{1+\epsilon}}} \cdot \frac{E(t)}{E(0)}\right) + ct^{\frac{1}{1+\epsilon_0}}\mathcal{B}^*\left(\mathcal{B}'\left(\frac{\varepsilon_3}{t^{\frac{1}{1+\epsilon}}} \cdot \frac{E(t)}{E(0)}\right)\right) \\ & + c\mathcal{B}'\left(\frac{\varepsilon_3}{t^{\frac{1}{1+\epsilon}}} \cdot \frac{E(t)}{E(0)}\right)\Big[-E'(t)\Big]^{\gamma_1-1} + ct^{\frac{1}{1+\epsilon}}\chi(t) \\ \le & -\lambda_0 E(t)\mathcal{B}'\left(\frac{\varepsilon_3}{t^{\frac{1}{1+\epsilon}}} \cdot \frac{E(t)}{E(0)}\right) + c\varepsilon_1\frac{E(t)}{E(0)}\mathcal{B}'\left(\frac{\varepsilon_3}{t^{\frac{1}{1+\epsilon}}} \cdot \frac{E(t)}{E(0)}\right) \\ & + ct^{\frac{1}{1+\epsilon}}\chi(t) - c_\varepsilon E' + \varepsilon\,[\mathcal{B}']^{\frac{1}{2-\gamma_1}}\left(\frac{\varepsilon_3}{t^{\frac{1}{1+\epsilon}}} \cdot \frac{E(t)}{E(0)}\right).\end{aligned} \tag{5.39}$$

Using the facts that $\frac{1}{2-\gamma_1} > 1$ and $\mathcal{B}'\left(\frac{\varepsilon_3}{t^{\frac{1}{1+\epsilon}}} \cdot \frac{E(t)}{E(0)}\right)$ is bounded, we have

$$[\mathcal{B}']^{\frac{1}{2-\gamma_1}}\left(\frac{\varepsilon_3}{t^{\frac{1}{1+\epsilon}}} \cdot \frac{E(t)}{E(0)}\right) \leq c\mathcal{B}'\left(\frac{\varepsilon_3}{t^{\frac{1}{1+\epsilon}}} \cdot \frac{E(t)}{E(0)}\right). \tag{5.40}$$

Then, multiplying (5.39) by $a(t)$, using (5.15), (5.40) and the fact that $E(t) > 0$, we get

$$\begin{aligned} a(t)\mathcal{F}_1'(t) \leq & -\lambda_0 a(t)E(t)\mathcal{B}'\left(\frac{\varepsilon_3}{t^{\frac{1}{1+\epsilon}}} \cdot \frac{E(t)}{E(0)}\right) + c\varepsilon_5 a(t) \cdot \frac{E(t)}{E(0)}\mathcal{B}'\left(\frac{\varepsilon_3}{t^{\frac{1}{1+\epsilon}}} \cdot \frac{E(t)}{E(0)}\right) \\ & + c\varepsilon a(t)E(t)\mathcal{B}'\left(\frac{\varepsilon_3}{t^{\frac{1}{1+\epsilon}}} \cdot \frac{E(t)}{E(0)}\right) - cE'(t), \forall t \geq t_1. \end{aligned}$$

where $\mathcal{F}_1 = \mathcal{F} + c_\varepsilon E'$. Using the non-increasing property of a, we obtain, for all $t \geq t_1$,

$$\begin{aligned} (a\mathcal{F}_1 + cE)'(t) \leq & -\lambda_0 a(t)E(t)H'\left(\frac{\varepsilon_3}{t^{\frac{1}{1+\epsilon}}} \cdot \frac{E(t)}{E(0)}\right) + c\varepsilon_5 a(t)\frac{E(t)}{E(0)}\mathcal{B}'\left(\frac{\varepsilon_3}{t^{\frac{1}{1+\epsilon}}} \cdot \frac{E(t)}{E(0)}\right) \\ & + c\varepsilon a(t)E(t)\mathcal{B}'\left(\frac{\varepsilon_3}{t^{\frac{1}{1+\epsilon}}} \cdot \frac{E(t)}{E(0)}\right). \end{aligned}$$

Therefore, by setting $\mathcal{F}_2 := a\mathcal{F}_1 + cE \sim E$, we conclude that

$$\begin{aligned} \mathcal{F}_2'(t) \leq & -\lambda_0 a(t)E(t)\mathcal{B}'\left(\frac{\varepsilon_3}{t^{\frac{1}{1+\epsilon}}} \cdot \frac{E(t)}{E(0)}\right) + c\varepsilon_3 a(t) \cdot \frac{E(t)}{E(0)}\mathcal{B}'\left(\frac{\varepsilon_3}{t^{\frac{1}{1+\epsilon}}} \cdot \frac{E(t)}{E(0)}\right) \\ & + c\varepsilon a(t)E(t)\mathcal{B}'\left(\frac{\varepsilon_3}{t^{\frac{1}{1+\epsilon}}} \cdot \frac{E(t)}{E(0)}\right). \end{aligned}$$

This gives, for a suitable choice of ε_3 and ε,

$$\mathcal{F}_2'(t) \leq -ka(t)\left(\frac{E(t)}{E(0)}\right)\mathcal{B}'\left(\frac{\varepsilon_3}{t^{\frac{1}{1+\epsilon}}} \cdot \frac{E(t)}{E(0)}\right), \qquad \forall t \geq t_1$$

or

$$k\left(\frac{E(t)}{E(0)}\right)\mathcal{B}'\left(\frac{\varepsilon_3}{t^{\frac{1}{1+\epsilon_0}}} \cdot \frac{E(t)}{E(0)}\right)a(t) \leq -\mathcal{F}_2'(t), \qquad \forall t \geq t_1 \tag{5.41}$$

An integration of (5.41) yields

$$\int_{t_1}^{t} k\left(\frac{E(s)}{E(0)}\right)\mathcal{B}'\left(\frac{\varepsilon_3}{s^{\frac{1}{1+\epsilon}}} \cdot \frac{E(s)}{E(0)}\right)a(s)ds \leq -\int_{t_1}^{t} \mathcal{F}_2'(s)ds \leq \mathcal{F}_2(t_1). \tag{5.42}$$

Using the facts that $\mathcal{B}', \mathcal{B}'' > 0$ and the non-increasing property of E, we deduce that the map $t \mapsto E(t)\mathcal{B}'\left(\frac{\varepsilon_3}{t^{\frac{1}{1+\epsilon}}} \cdot \frac{E(t)}{E(0)}\right)$ is non-increasing and consequently, we have

$$\begin{aligned} & k\left(\frac{E(t)}{E(0)}\right)\mathcal{B}'\left(\frac{\varepsilon_3}{t^{\frac{1}{1+\epsilon}}} \cdot \frac{E(t)}{E(0)}\right)\int_{t_1}^{t} a(s)ds \\ & \quad \leq \int_{t_1}^{t} k\left(\frac{E(s)}{E(0)}\right)\mathcal{B}'\left(\frac{\varepsilon_3}{s^{\frac{1}{1+\epsilon}}} \cdot \frac{E(s)}{E(0)}\right)a(s)ds \leq \mathcal{F}_2(t_1), \qquad \forall t \geq t_1 \end{aligned} \tag{5.43}$$

Multiplying each side of (5.43) by $\frac{1}{t^{\frac{1}{1+\epsilon}}}$, we have

$$\left(\frac{k}{t^{\frac{1}{1+\epsilon}}}\cdot\frac{E(t)}{E(0)}\right)\mathcal{B}'\left(\frac{\varepsilon_3}{t^{\frac{1}{1+\epsilon_0}}}\cdot\frac{E(t)}{E(0)}\right)\int_{t_1}^{t} a(s)ds \leq \frac{k_2}{t^{\frac{1}{1+\epsilon}}}, \qquad \forall t \geq t_1 \tag{5.44}$$

Using the fact that $\mathcal{B}_3(s) = s\mathcal{B}'(\varepsilon_3 s)$ is strictly increasing, we obtain

$$k\mathcal{B}_3\left(\frac{1}{t^{\frac{1}{1+\epsilon}}}\cdot\frac{E(t)}{E(0)}\right)\int_{t_1}^{t} a(s)ds \leq \frac{k_2}{t^{\frac{1}{1+\epsilon}}}, \qquad \forall t \geq t_1 \tag{5.45}$$

Finally, we infer

$$E(t) \leq k_3 t^{\frac{1}{1+\epsilon}}\mathcal{B}_3^{-1}\left(\frac{k_2}{t^{\frac{1}{1+\epsilon}}\int_{t_1}^{t} a(s)ds}\right). \tag{5.46}$$

This finishes the proof. □

The following examples illustrate the results of Theorem 5.6:

Example 3. *Let $b(t) = c_1 e^{-c_2(1+t)}$, where $c_2 > 0$ and $c_1 > 0$ is small enough so that $(A1)$ holds. Then $b'(t) = -a(t)B(b(t))$ where $B(t) = t$ and $a(t) = c$. Therefore, (5.32) gives for $t > t_0$ and $\epsilon \in (0, 1)$,*

$$E(t) \leq c(t - t_0)^{\frac{\gamma\epsilon - 1}{\gamma\epsilon}}. \tag{5.47}$$

Example 4. *Let $b(t) = \frac{c_1}{(1+t)^q}$, where $q > 1 + \epsilon$ and c_1 is chosen so that hypothesis $(A1)$ is satisfied. Then*

$$b'(t) = -aB(b(t)), \quad \text{with} \quad B(s) = s^{\frac{q+1}{q}},$$

where a is a fixed constant. Then, (5.33) gives, for $t > t_1$ and $\epsilon \in (0, 1)$,

$$E(t) \leq \frac{c}{t^{\frac{q-1-\epsilon}{(1+\epsilon)^2(q+1)}}}. \tag{5.48}$$

Remark 5.7. *The classical power-type nonlinearity term in [33] provides a canonical description for the dynamics analysis of a quasi-wave propagation in a nonlinear process, therefore, the fast cumulative of such nonlinear interactions results in a significant effect to the solution under large spatial and temporal scales. However, the logarithmic nonlinearity in (1.1) only expresses slowly cumulative of nonlinear, thus giving another kind of description for dynamic process. Let us note here that though the logarithmic nonlinearity is somehow weaker than the polynomial nonlinearity, both the existence and stability result are not obtained by straightforward application of the method used for polynomial nonlinearity.*

Acknowledgment

The authors would like to express their profound gratitude to King Fahd University of Petroleum and Minerals (KFUPM)- Interdisciplinary Research Center (IRC) for Construction and Building Materials for their continuous supports. The authors also thank the referee for her/his very careful reading and valuable comments. This work is funded by KFUPM under Project #SB191037.

Conflict of interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

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