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*Research article*

# Volterra integral operator and essential norm on Dirichlet type spaces

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**Abstract:** In this paper, we study the boundedness and essential norm of Volterra integral operator $V_g$ and integral operator $S_g$ on Dirichlet type spaces $\mathcal{D}_{K,\alpha}$.

**Keywords:** Volterra type operator; Dirichlet type spaces; essential norm
**Mathematics Subject Classification:** 30D45, 30D50

## 1. Introduction

First, let us introduce some necessary notation. Let $\mathbb{D}$ be the unit disk in the complex plane $\mathbb{C}$, $H(\mathbb{D})$ be the class of functions analytic in $\mathbb{D}$ and $H^\infty$ be the class of bounded analytic functions on $\mathbb{D}$. The Bloch space $\mathcal{B}$ ( [34]) is the class of all $f \in H(\mathbb{D})$ for which

$$\|f\|_{\mathcal{B}} := |f(0)| + \sup_{z\in\mathbb{D}}(1-|z|^2)|f'(z)| < \infty.$$

The little Bloch space $\mathcal{B}_0$, consists of all $f \in \mathcal{H}(\mathbb{D})$ such that

$$\lim_{|z|\to 1^-}(1-|z|^2)|f'(z)| = 0.$$

The Hardy space $H^p(\mathbb{D})$ $(0 < p < \infty)$ ( [8, 10]) is the set of $f \in H(\mathbb{D})$ with

$$\|f\|^p_{H^p} = \sup_{0<r<1} \frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p d\theta < \infty.$$

Suppose that $0 < p < \infty$, $\alpha > -1$ and $dA_\alpha(z) = (1-|z|^2)^\alpha dA(z) = \frac{1}{\pi}(1-|z|^2)^\alpha dxdy$. The weighted Bergman space $A^p_\alpha(\mathbb{D})$ ( [34]) is the set of $f \in H(\mathbb{D})$ with

$$\|f\|^p_{A^p_\alpha} = \int_{\mathbb{D}} |f(z)|^p dA_\alpha(z) < \infty.$$

Let $\alpha \geq 0$. The Dirichlet type space $\mathcal{D}_\alpha$ is the set of $f \in H(\mathbb{D})$ with

$$\|f\|_{\mathcal{D}_\alpha}^2 = |f(0)|^2 + \|f'\|_{A_\alpha^2}^2 < \infty.$$

If $\alpha = 0$, it gives classic Dirichlet space $\mathcal{D}$. When $\alpha = 1$, it is Hardy space $H^2$. When $\alpha > 1$, it turns into weighted Bergman spaces $A_{\alpha-2}^2$. Thus, the interesting scope is $\alpha \in (0,1)$. For more information relating to $\mathcal{D}_\alpha$, we refer to [23, 25, 26].

In this paper, we use the weighted function in [9, 30]. We always suppose that $K : [0,\infty) \to [0,\infty)$ is a right-continuous and nondecreasing function. The weighted function $K$ also satisfies

$$\int_0^1 \frac{\varphi_K(s)}{s} ds < \infty \tag{A}$$

and

$$\int_1^\infty \frac{\varphi_K(s)}{s^2} ds < \infty, \tag{B}$$

where

$$\varphi_K(s) = \sup_{0\leq t\leq 1} K(st)/K(t), \quad 0 < s < \infty.$$

Let $\alpha \geq 0$ and Dirichlet type space $\mathcal{D}_{K,\alpha}$ denotes the spaces of function $f \in H(\mathbb{D})$ satisfying

$$\|f\|_{\mathcal{D}_{K,\alpha}}^2 = |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 \frac{(1-|z|^2)}{K(1-|z|^2)} dA_\alpha(z) < \infty.$$

When $\alpha > 0$, if the weighted function $K$ satisfies $(A)$ and $(B)$, we easily to see that $\mathcal{D}_\alpha \subseteq \mathcal{D}_{K,\alpha} \subseteq A_{\alpha-1}^2$. By [9], there exist a small $c > 0$, such that $C_1 t^{1-c} \leq K(t) \leq C_2 t^c$, where $0 < t < 1$, $C_1 > 0$ and $C_2 > 0$. Thus, when $\alpha \geq 1$, we easily to see that $A_{\alpha-2+c}^2 \subseteq \mathcal{D}_{K,\alpha} \subseteq A_{\alpha-1-c}^2$. Moreover, using high order characterization, it is not hard to check that $\mathcal{D}_{K,\alpha}$ turns into a Bergman type space, when $\alpha \geq 1$. Thus, the interesting scope is $\alpha \in [0,1)$. For more results of $\mathcal{D}_{K,\alpha}$ spaces, we refer to [3–5, 11, 15, 19, 20].

Let $I$ be an arc of $\partial\mathbb{D}$ and $|I|$ be the normalized Lebesgue arc length of $I$. The Carleson square based on $I$, denoted by $S(I)$, is defined by

$$S(I) := \left\{z = re^{i\theta} \in \mathbb{D} : 1 - |I| \leq r < 1, e^{i\theta} \in I\right\}.$$

Let $\mu$ be a positive Borel measure on $\mathbb{D}$. For $0 < s < \infty$, $\mu$ is called an $s$-Carleson measure if

$$\sup_{I\subset\partial\mathbb{D}} \frac{\mu(S(I))}{|I|^s} < \infty.$$

We say that a function $f \in H^2(\mathbb{D})$ belongs to Morrey type space $\mathcal{H}_K^2$ if

$$\|f\|_{\mathcal{H}_K^2}^2 = |f(0)|^2 + \sup_{I\subseteq\partial\mathbb{D}} \frac{1}{K(|I|)} \int_I |f(\zeta) - f_I|^2 \frac{d\zeta}{2\pi} < \infty,$$

where

$$f_I = \frac{1}{|I|} \int_I f(\zeta) \frac{d\zeta}{2\pi}, \ I \subseteq \partial\mathbb{D}.$$

This space was introduced by H. Wulan and J. Zhou in [29]. When $K(t) = t$, it gives the $BMOA$ space, the space of those analytic functions $f$ in the Hardy space $H^p$ whose boundary functions having bounded mean oscillation on $\partial\mathbb{D}$. In the case $K(t) = t^\lambda$, $0 < \lambda < 1$, the space $\mathcal{H}_K^2$ gives classical Morrey space $\mathcal{L}^{2,\lambda}$. Morrey spaces $\mathcal{L}^{2,\lambda}$ were introduced by Morrey in [16]. From [29], we know that $f \in \mathcal{H}_K^2$ if and only if

$$\sup_{a\in\mathbb{D}} \frac{1-|a|^2}{K(1-|a|^2)} \int_{\mathbb{D}} |f'(z)|^2(1-|\varphi_a(z)|^2)dA(z) < \infty,$$

where $\varphi_a(z) = \frac{a-z}{1-\bar{a}z}$.

Let $\alpha \geq 0$ and we say that a function $f \in H(\mathbb{D})$ belongs to Morrey type space $\mathcal{H}_{K,\alpha}^2$ if

$$\|f\|_{\mathcal{H}_{K,\alpha}^2}^2 = |f(0)|^2 + \sup_{a\in\mathbb{D}} \frac{1-|a|^2}{K(1-|a|^2)} \int_{\mathbb{D}} |f'(z)|^2(1-|\varphi_a(z)|^2)dA_\alpha(z) < \infty.$$

It is easy to verify $\mathcal{H}_{K,\alpha}^2$ is a Banach space under the above norm.

Let $f, g \in H(\mathbb{D})$. The Volterra integral operator $V_g$ and the integral operator $S_g$ are defined by

$$V_gf(z) := \int_0^z g'(w)f(w)dw, \quad S_gf(z) := \int_0^z g(w)f'(w)dw, \quad z \in \mathbb{D},$$

respectively. For $g \in H(\mathbb{D})$, the multiplication operator $M_g$ is defined by $M_gf(z) = f(z)g(z)$. It is easy to see that $M_g$ is related with $S_g$ and $V_g$ by

$$M_gf(z) = f(0)g(0) + S_gf(z) + V_gf(z).$$

It it well known that $V_g$ is bounded on the Hardy space $H^p$ (Bergman space $A^p$) if and only if $g \in BMOA$ ($g \in \mathcal{B}$). $V_g$ is bounded on $BMOA$ if and only if $g \in BMOA_{\log}$ (see [24]). For more information relating to Volterra integral operator $V_g$, we refer to [1, 2, 7, 12–14, 17, 21, 22, 28, 31, 33].

In this note, we study Volterra integral operator $V_g$ acting from $\mathcal{D}_{K,\alpha}$ to $\mathcal{H}_{K,\alpha}^2$, that is, we prove that $V_g : \mathcal{D}_{K,\alpha} \to \mathcal{H}_{K,\alpha}^2$ is bounded if and only if $g \in \mathcal{B}$, when $0 < \alpha < 1$. Meanwhile, the boundedness of $S_g$ and the essential norm of $V_g$ and $S_g$ from $\mathcal{D}_{K,\alpha}$ to $\mathcal{H}_{K,\alpha}^2$ are also studied.

In this paper, the symbol $f \approx g$ means that $f \lesssim g \lesssim f$. We say that $f \lesssim g$ if there exists a constant $C$ such that $f \leq Cg$.

## 2. Auxiliary results

In this section, we are going to give some auxiliary results.

**Lemma 1.** *Let (A) and (B) hold for $K$. Suppose that $\alpha > 0$ and $f \in \mathcal{D}_{K,\alpha}$, then*

$$|f(z)| \lesssim \|f\|_{\mathcal{D}_{K,\alpha}} \sqrt{\frac{K(1-|z|^2)}{(1-|z|^2)^{1+\alpha}}}, \quad z \in \mathbb{D}.$$

*Proof.* The proof is similar to [33], thus we omit it here. The proof is completed. □

**Lemma 2.** *Let* $(B)$ *hold for* $K$. *Suppose that* $\alpha > 0$. *Then*

$$f_a(z) = \frac{(1-|a|^2)\sqrt{K(1-|a|^2)}}{(1-\overline{a}z)^{\frac{3+\alpha}{2}}} \in \mathcal{D}_{K,\alpha}$$

*and*

$$F_a(z) = \frac{(1-|a|^2)\sqrt{K(1-|a|^2)}}{\overline{a}(1-\overline{a}z)^{\frac{3+\alpha}{2}}} \in \mathcal{D}_{K,\alpha},$$

*where* $z,\ a \in \mathbb{D}$.

*Proof.* Since $(B)$ holds, then from [9], there is some $c \in (0,1)$, such that

$$\varphi_K(t) \lesssim t^{1-c},\ t \geq 1. \tag{1}$$

Combining with $K$ which is nondecreasing and Lemma 3.10 of [34], we obtain

$$\begin{aligned}
&\int_{\mathbb{D}} |f_a'(z)|^2 \frac{1-|z|^2}{K(1-|z|^2)} dA_\alpha(z) = (1-|a|^2)^2 \int_{\mathbb{D}} \left( \frac{(1-|z|^2)K(1-|a|^2)}{|1-\overline{a}z|^{5+\alpha} K(1-|z|^2)} \right) dA_\alpha(z) \\
\lesssim&(1-|a|^2)^2 \int_{\mathbb{D}} \left( \frac{(1-|z|^2)K(1-|a|)}{|1-\overline{a}z|^{5+\alpha} K(1-|z|)} \right) dA_\alpha(z) \\
\lesssim&(1-|a|^2)^2 \int_{\mathbb{D}} \left( \frac{(1-|z|^2)K(|1-\overline{a}z|)}{|1-\overline{a}z|^{5+\alpha} K(1-|z|)} \right) dA_\alpha(z) \\
\lesssim&(1-|a|^2)^2 \int_{\mathbb{D}} \left( \frac{(1-|z|^2)}{|1-\overline{a}z|^{5+\alpha}} \right) \varphi_K \left( \frac{|1-\overline{a}z|}{1-|z|} \right) dA_\alpha(z) \\
\lesssim&(1-|a|^2)^2 \int_{\mathbb{D}} \left( \frac{(1-|z|^2)^{1+\alpha}(|1-\overline{a}z|)^{1-c}}{|1-\overline{a}z|^{5+\alpha}(1-|z|)^{1-c}} \right) dA(z) \lesssim 1.
\end{aligned}$$

where the third inequality is deduced by $1-|a| \leq |1-\overline{a}z|$ and $K$ is nondecreasing, the last second inequality is deduced by $1-|z| \leq |1-\overline{a}z|$ and (1). Thus, $f_a \in \mathcal{D}_{K,\alpha}$. Similar proof can be applied to $F_a$, thus we omit here. The proof is completed. □

**Lemma 3.** ( [34]) *Suppose that* $\alpha > -1$ *and* $\mu$ *is a non-negative measure on* $\mathbb{D}$. *Then* $\mu$ *is a* $(2+\alpha)$-*Carleson measure if and only if the following inequality*

$$\int_{\mathbb{D}} |f(z)|^2 d\mu \lesssim \|f\|^2_{A^2_\alpha}$$

*holds for all* $f \in A^2_\alpha$.

**Lemma 4.** ( [32]) *Let* $p > 1$ *and* $f \in H(\mathbb{D})$. *Then* $f \in \mathcal{B}$ *if and only if the measure* $d\mu_f = |f'(z)|^2(1-|z|^2)^p dA(z)$ *is a* $p$-*Carleson measure.*

**Lemma 5.** ( [6]) *Suppose that* $1 < p < \infty$, $\alpha > -1$, $\beta \geq 0$ *with* $\beta < 2+\alpha$. *Let* $f \in H(\mathbb{D})$ *and* $z, w \in \mathbb{D}$. *Then*

$$\int_{\mathbb{D}} |f(z)-f(0)|^p \frac{(1-|z|^2)^\alpha}{|1-\overline{w}z|^\beta} dA(z) \lesssim \int_{\mathbb{D}} |f'(z)|^p \frac{(1-|z|^2)^{p+\alpha}}{|1-\overline{w}z|^\beta} dA(z).$$

**Lemma 6.** *Let (A) and (B) hold for K. Suppose that* $0 < \alpha < 1$. *Then* $f \in \mathcal{H}^2_{K,\alpha}$ *if and only if*

$$\sup_{I \subseteq \partial\mathbb{D}} \frac{1}{K(|I|)} \int_{S(I)} |f'(z)|^2 (1-|z|^2)^{1+\alpha} dA(z) < \infty. \tag{2}$$

*Proof.* The proof is similar to Lemma 2.1 of [18]. Thus we omit here. The proof is complete. □

## 3. Boundedness of $V_g$ and $S_g$ operators

**Theorem 1.** *Let (A) and (B) hold for K. Suppose that* $g \in H(\mathbb{D})$ *and* $0 < \alpha < 1$. *Then* $V_g$ *is bounded from* $\mathcal{D}_{K,\alpha}$ *to* $\mathcal{H}^2_{K,\alpha}$ *if and only if* $g \in \mathcal{B}$. *Moreover, the operator norm satisfies* $\|V_g\| \approx \|g\|_{\mathcal{B}}$.

*Proof.* For any $I \in \partial\mathbb{D}$, let $a = (1-|I|)\zeta \in \mathbb{D}$, where $\zeta$ is the center of $I$. Then

$$(1-|a|^2) \approx |1-\bar{a}z|, \quad |K(1-|a|^2) \approx K(|I|), \quad z \in S(I). \tag{3}$$

Let $f_a$ be defined as in Lemma 2. Then

$$|f_a(z)|^2 \approx \frac{K(|I|)}{|I|^{1+\alpha}}, \ z \in S(I).$$

Suppose that $V_g$ is bounded from $\mathcal{D}_{K,\alpha}$ to $\mathcal{H}^2_{K,\alpha}$ . By Lemmas 4 and 6, we have

$$\begin{aligned}
&\frac{1}{|I|^{\alpha+1}} \int_{S(I)} |g'(z)|^2 (1-|z|^2)^{\alpha+1} dA(z) \\
\lesssim &\frac{1}{K(|I|)} \int_{S(I)} |f_a(z)|^2 |g'(z)|^2 (1-|z|^2)^{\alpha+1} dA(z) \\
\lesssim &\frac{1}{K(|I|)} \int_{S(I)} |(V_g f_a)'(z)|^2 (1-|z|^2)^{\alpha+1} dA(z) \\
\lesssim &\|V_g f_a\|^2_{\mathcal{H}^2_{K,\alpha}} < \infty.
\end{aligned}$$

Thus, $g \in \mathcal{B}$.

On the other hand, suppose that $g \in \mathcal{B}$, by Lemma 4, we have $d\mu_g = |g'(z)|^2(1-|z|^2)^{\alpha+1} dA(z)$ is a $(\alpha+1)$-Carleson measure. Let $f \in \mathcal{D}_{K,\alpha}$. From Lemma 6, we only need to prove that

$$L =: \frac{1}{K(|I|)} \int_{S(I)} |(V_g f)'(z)|^2 (1-|z|^2)^{1+\alpha} dA(z) < \infty.$$

Since

$$\begin{aligned}
L &= \frac{1}{K(|I|)} \int_{S(I)} |f(z)|^2 |g'(z)|^2 (1-|z|^2)^{1+\alpha} dA(z) \\
&\lesssim \frac{1}{K(|I|)} \int_{S(I)} |f(a)|^2 |g'(z)|^2 (1-|z|^2)^{1+\alpha} dA(z)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{K(|I|)}\int_{S(I)}|f(z)-f(a)|^2|g'(z)|^2(1-|z|^2)^{1+\alpha}dA(z)\\
&= M+N.
\end{aligned}
$$

Using Lemma 1 and (3), we see that

$$
M \lesssim \|g\|_{\mathcal{B}}^2\|f\|_{\mathcal{D}_{K,\alpha}}^2.
$$

By Lemma 3, we have $A^2_{\alpha-1}\subseteq L^2(d\mu_g)$. Note that

$$
\|f\|^2_{A^2_{\alpha-1}} \approx \int_{\mathbb{D}}|f'(z)|^2(1-|z|^2)^{\alpha+1}dA(z)\le \|f\|^2_{\mathcal{D}_{K,\alpha}}.
$$

Thus, $\mathcal{D}_{K,\alpha}\subseteq A^2_{\alpha-1}$. Bearing in mind these facts, we are going to estimate $N$. Let $z=\varphi_a(w)$. Since $|\varphi_a'(w)|(1-|w|^2)=1-|\varphi_a(w)|^2$, using Lemmas 3, 4, 5, we obtain

$$
\begin{aligned}
N &\approx \frac{(1-|a|^2)^4}{K(1-|a|^2)}\int_{S(I)}\left|\frac{f(z)-f(a)}{(1-\overline{a}z)^2}\right|^2 d\mu_g(z)\\
&\le \frac{(1-|a|^2)^4}{K(1-|a|^2)}\int_{\mathbb{D}}\left|\frac{f(z)-f(a)}{(1-\overline{a}z)^2}\right|^2 d\mu_g(z)\\
&\lesssim \frac{(1-|a|^2)^{2+2}}{K(1-|a|^2)}\int_{\mathbb{D}}\left|\frac{f(z)-f(a)}{(1-\overline{a}z)^2}\right|^2 (1-|z|^2)^{\alpha-1}dA(z)\\
&\le \frac{(1-|a|^2)^2}{K(1-|a|^2)}\int_{\mathbb{D}}\frac{|f(z)-f(a)|^2(1-|a|^2)^2}{|1-\overline{a}z|^4}(1-|z|^2)^{\alpha-1}dA(z)\\
&\lesssim \frac{(1-|a|^2)^2}{K(1-|a|^2)}\int_{\mathbb{D}}|(f\circ\varphi_a)(w)-(f\circ\varphi_a)(0)|^2(1-|\varphi_a(w)|^2)^{\alpha-1}dA(w)\\
&\le \frac{(1-|a|^2)^{1+\alpha}}{K(1-|a|^2)}\int_{\mathbb{D}}|(f\circ\varphi_a)(w)-(f\circ\varphi_a)(0)|^2(1-|w|^2)^{\alpha-1}dA(w)\\
&\le \frac{(1-|a|^2)^{1+\alpha}}{K(1-|a|^2)}\int_{\mathbb{D}}|(f\circ\varphi_a)'(w)|^2(1-|w|^2)^{\alpha+1}dA(w)\\
&\le \frac{(1-|a|^2)^{1+\alpha}}{K(1-|a|^2)}\int_{\mathbb{D}}|f'(\varphi_a(w)|^2(1-|\varphi_a(w)|^2)^2(1-|w|^2)^{\alpha-1}dA(w)\\
&\le \frac{(1-|a|^2)^{1+\alpha}}{K(1-|a|^2)}\int_{\mathbb{D}}|f'(z)|^2(1-|z|^2)^2(1-|\varphi_a(z)|^2)^{\alpha-1}\frac{(1-|a|^2)^2}{|1-\overline{a}z|^4}dA(w)\\
&= \int_{\mathbb{D}}|f'(z)|^2\frac{(1-|z|^2)^{\alpha+1}}{K(1-|z|^2)}\frac{K(1-|z|^2)}{K(1-|a|^2)}\frac{(1-|a|^2)^{2(1+\alpha)}}{|1-\overline{a}z|^{2+2\alpha}}dA(w)\\
&\lesssim \int_{\mathbb{D}}|f'(z)|^2\frac{(1-|z|^2)^{\alpha+1}}{K(1-|z|^2)}\left(\frac{K(|1-\overline{a}z|)}{K(1-|a|)}\right)\frac{(1-|a|^2)^{2(1+\alpha)}}{|1-\overline{a}z|^{2+2\alpha}}dA(z)\\
&\lesssim \int_{\mathbb{D}}|f'(z)|^2\frac{(1-|z|^2)^{\alpha+1}}{K(1-|z|^2)}\varphi_K\left(\frac{|1-\overline{a}z|}{1-|a|}\right)\frac{(1-|a|^2)^{2(1+\alpha)}}{|1-\overline{a}z|^{2+2\alpha}}dA(z)\\
&\lesssim \int_{\mathbb{D}}|f'(z)|^2\frac{(1-|z|^2)^{\alpha+1}}{K(1-|z|^2)}\frac{(|1-\overline{a}z|)^{1-c}}{(1-|a|^2)^{1-c}}\frac{(1-|a|^2)^{2(1+\alpha)}}{|1-\overline{a}z|^{2+2\alpha}}dA(z)\lesssim \|f\|^2_{\mathcal{D}_{K,\alpha}},
\end{aligned}
$$

where the last second inequality is deduced by (1). Combining the estimates $M$ and $N$, we conclude that $V_g : \mathcal{D}_{K,\alpha} \to \mathcal{H}^2_{K,\alpha}$ is bounded. □

**Theorem 2.** *Let (A) and (B) hold for K. Suppose that* $g \in H(\mathbb{D})$ *and* $0 < \alpha < 1$. *Then* $S_g$ *is bounded from* $\mathcal{D}_{K,\alpha}$ *to* $\mathcal{H}^2_{K,\alpha}$ *if and only if* $g \in H^\infty$. *Moreover, the operator norm satisfies* $\|S_g\| \approx \sup_{z\in\mathbb{D}} |g(z)|$.

*Proof.* Suppose that $S_g$ is bounded from $\mathcal{D}_{K,\alpha}$ to $\mathcal{H}^2_{K,\alpha}$. Let $a \in \mathbb{D}$ and

$$F_a(z) = \frac{(1-|a|^2)\sqrt{K(1-|a|^2)}}{\overline{a}(1-\overline{a}z)^{\frac{3+\alpha}{2}}}.$$

By Lemma 2, we have $F_a \in \mathcal{D}_{K,\alpha}$ and $\|F_a\|_{\mathcal{D}_{K,\alpha}} \lesssim 1$. For $a \in \mathbb{D}$ and $r > 0$, let $D(a, r)$ denote the Bergman metric disk centered at $a$ with radius $r$. From [34] we see that

$$\frac{(1-|a|^2)^2}{|1-\overline{a}z|^4} \approx \frac{1}{(1-|z|^2)^2} \approx \frac{1}{(1-|a|^2)^2}$$

when $z \in D(a, r)$. Using subharmonic property of $|g|^2$, we have

$$\begin{aligned}
\infty >&\|S_g F_a\|^2_{\mathcal{H}^2_{K,\alpha}}\\
\gtrsim& \sup_{b\in\mathbb{D}} \frac{1-|b|^2}{K(1-|b|^2)} \int_{\mathbb{D}} |F'_a(z)|^2 |g(z)|^2 \left(1-|\varphi_b(z)|^2\right) dA_\alpha(z)\\
\gtrsim& \frac{1-|a|^2}{K(1-|a|^2)} \int_{\mathbb{D}} |F'_a(z)|^2 |g(z)|^2 \left(1-|\varphi_a(z)|^2\right) dA_\alpha(z)\\
\gtrsim& \frac{1-|a|^2}{K(1-|a|^2)} \int_{D(a,r)} |F'_a(z)|^2 |g(z)|^2 \left(1-|\varphi_a(z)|^2\right) dA_\alpha(z)\\
=& \frac{1}{(1-|a|^2)^2} \int_{D(a,r)} |g(z)|^2 dA(z) \gtrsim |g(a)|^2.
\end{aligned}$$

That is,

$$\|S_g\|^2 \gtrsim \|S_g\|^2 \|F_a\|^2_{\mathcal{D}_{K,\alpha}} \gtrsim \|S_g F_a\|^2_{\mathcal{H}^2_{K,\alpha}} \gtrsim |g(a)|^2.$$

Since $a \in \mathbb{D}$ is arbitrary, we have

$$\|g\|^2_{H^\infty} \lesssim \|S_g\|^2 < \infty.$$

On the other hand. Let $g \in H^\infty$. Using (1), we can deduce that for $f \in \mathcal{D}_{K,\alpha}$,

$$\begin{aligned}
&\frac{1-|a|^2}{K(1-|a|^2)} \int_{\mathbb{D}} |f'(z)|^2 |g(z)|^2 \left(1-|\varphi_a(z)|^2\right) dA_\alpha(z)\\
\lesssim& \|g\|^2_{H^\infty} \int_{\mathbb{D}} |f'(z)|^2 \frac{(1-|a|^2)^2 K(1-|z|^2)}{|1-\overline{a}z|^2 K(1-|a|^2)} \frac{(1-|z|^2)}{K(1-|z|^2)} dA_\alpha(z)\\
\lesssim& \|g\|^2_{H^\infty} \int_{\mathbb{D}} |f'(z)|^2 \frac{(1-|a|^2)^2 K(|1-\overline{a}z|)}{|1-\overline{a}z|^2 K(1-|a|)} \frac{(1-|z|^2)}{K(1-|z|^2)} dA_\alpha(z)\\
\lesssim& \|g\|^2_{H^\infty} \int_{\mathbb{D}} |f'(z)|^2 \frac{(1-|a|^2)^2 \varphi_K\left(\frac{|1-\overline{a}z|}{1-|a|}\right)}{|1-\overline{a}z|^2} \frac{(1-|z|^2)}{K(1-|z|^2)} dA_\alpha(z)\\
\lesssim& \|g\|^2_{H^\infty} \int_{\mathbb{D}} |f'(z)|^2 \frac{(1-|a|^2)^2 (|1-\overline{a}z|)^{1-c}}{|1-\overline{a}z|^2 (1-|a|^2)^{1-c}} \frac{(1-|z|^2)}{K(1-|z|^2)} dA_\alpha(z)\\
\lesssim& \|g\|^2_{H^\infty} \|f\|^2_{\mathcal{D}_{K,\alpha}}.
\end{aligned}$$

The proof is completed. □

**Remark.** *Note that*

$$M_g f(z) = f(0)g(0) + S_g f(z) + V_g f(z).$$

*Hence, if (A) and (B) hold for K, then $M_g$ is bounded from $\mathcal{D}_{K,\alpha}$ to $\mathcal{H}^2_{K,\alpha}$ if and only if $g \in H^\infty$.*

## 4. Essential norm

Let us recall the definition of essential norm. Suppose that $X$ be a Banach space and $T$ is a bounded linear operator on $X$. The essential norm of $T$ is the distance of $T$ to the closed ideals of compact operators, that is

$$\|T\|_e = inf\{\|T - S\| : S \ is\ a\ compact\ operator\ on\ X\}.$$

Note that $T$ is compact if and only if $\|T\|_e = 0$.

**Lemma 7.** *Suppose that $0 < \alpha < 1$ and $K$ satisfies the conditions (A) and (B). Let $g \in \mathcal{B}$. Then $V_{g_r} : \mathcal{D}_{K,\alpha} \to \mathcal{H}^2_{K,\alpha}$ is compact. Here $g_r(z) = g(rz), 0 < r < 1, z \in \mathbb{D}$.*

*Proof.* Let $\{f_n\}$ be any function sequence such that $\|f_n\|_{\mathcal{D}_{K,\alpha}} \lesssim 1$ and $f_n \to 0$ uniformly on compact subsets of $\mathbb{D}$ as $n \to \infty$. We need only to show that

$$\lim_{n\to\infty} \|T_{g_r} f_n\|_{\mathcal{H}^2_{K,\alpha}} = 0.$$

Since

$$|g_r'(z)| \lesssim \frac{\|g\|_{\mathcal{B}}}{1 - r^2},\ z \in \mathbb{D}.$$

Combining with (1), we have

$$\begin{aligned}
&\sup_{a\in\mathbb{D}} \frac{1 - |a|^2}{K(1 - |a|^2)} \int_{\mathbb{D}} |f_n(z)|^2 |g_r'(z)|^2 \left(1 - |\varphi_a(z)|^2\right) dA_\alpha(z) \\
\lesssim& \frac{\|g\|^2_{\mathcal{B}}}{(1 - r^2)^2} \sup_{a\in\mathbb{D}} \frac{1 - |a|^2}{K(1 - |a|^2)} \int_{\mathbb{D}} |f_n(z)|^2 \left(1 - |\varphi_a(z)|^2\right) dA_\alpha(z) \\
\lesssim& \frac{\|g\|^2_{\mathcal{B}}}{(1 - r^2)^2} \int_{\mathbb{D}} |f_n(z)|^2 \frac{1 - |z|^2}{K(1 - |z|^2)} \left(\frac{(1 - |a|^2)^2 K(1 - |z|)}{K(1 - |a|)|1 - \bar{a}z|^2}\right) dA_\alpha(z) \\
\lesssim& \frac{\|g\|^2_{\mathcal{B}}}{(1 - r^2)^2} \int_{\mathbb{D}} |f_n(z)|^2 \frac{1 - |z|^2}{K(1 - |z|^2)} \left(\frac{(1 - |a|)^2 \frac{K(|1-\bar{a}z|)}{K(1-|a|)}}{|1 - \bar{a}z|^2}\right) dA_\alpha(z) \\
\lesssim& \frac{\|g\|^2_{\mathcal{B}}}{(1 - r^2)^2} \int_{\mathbb{D}} |f_n(z)|^2 \frac{1 - |z|^2}{K(1 - |z|^2)} \left(\frac{(1 - |a|)^2 \varphi_K\left(\frac{|1-\bar{a}z|}{1-|a|}\right)}{|1 - \bar{a}z|^2}\right) dA_\alpha(z) \\
\lesssim& \frac{\|g\|^2_{\mathcal{B}}}{(1 - r^2)^2} \int_{\mathbb{D}} |f_n(z)|^2 \frac{1 - |z|^2}{K(1 - |z|^2)} \left(\frac{(1 - |a|)^2 \frac{|1-\bar{a}z|^{1-c}}{(1-|a|)^{1-c}}}{|1 - \bar{a}z|^2}\right) dA_\alpha(z) \\
\lesssim& \frac{\|g\|^2_{\mathcal{B}}}{(1 - r^2)^2} \int_{\mathbb{D}} |f_n(z)|^2 \frac{1 - |z|^2}{K(1 - |z|^2)} \left(\frac{(1 - |a|)^{1+c}}{|1 - \bar{a}z|^{1+c}}\right) dA_\alpha(z) \\
\lesssim& \frac{\|g\|^2_{\mathcal{B}}}{(1 - r^2)^2} \int_{\mathbb{D}} |f_n(z)|^2 \frac{1 - |z|^2}{K(1 - |z|^2)} dA_\alpha(z).
\end{aligned}$$

Note that $\|f_n\|_{\mathcal{D}_{K,\alpha}} \lesssim 1$ and Lemma 1, the argument is then finished by the Dominated Convergence Theorem. □

Let $X$ and $Y$ be two Banach spaces with $X \subset Y$. If $f \in Y$, then the distance from $f$ to $X$ is defined as

$$dist_Y(f, X) = \inf_{g\in X} \|f - g\|_Y.$$

We also need the following lemma.

**Lemma 8.** ( [27]) *If $f \in \mathcal{B}$, then*

$$\limsup_{|z|\to 1^-}(1 - |z|^2)|f'(z)| \approx \mathrm{dist}_{\mathcal{B}}(f, \mathcal{B}_0) \approx \limsup_{r\to 1^-} \|f - f_r\|_{\mathcal{B}}.$$

**Theorem 3.** *Suppose $0 < \alpha < 1$, $g \in \mathcal{B}$ and $K$ satisfy the conditions (A) and (B). Then $V_g : \mathcal{D}_{K,\alpha} \to \mathcal{H}^2_{K,\alpha}$ satisfies*

$$\|V_g\|_e \approx dist(g, \mathcal{B}_0) \approx \limsup_{|z|\to 1^-}(1 - |z|^2)|g'(z)|.$$

*Proof.* Let $\{I_n\}$ be the subarc sequence of $\partial\mathbb{D}$, such that $|I_n| \to 0$ as $n \to \infty$, $w_n = (1 - |I_n|)\zeta_n \in \mathbb{D}$, where $\zeta_n$ is the center of $I_n$. $n = 1, 2, ....$ Then

$$1 - |w_n| \approx |1 - \overline{w_n} z| \approx |I_n|, \quad z \in S(I_n).$$

Thus, by [9], we know that

$$K(1 - |w_n|) \approx K(|I_n|), \quad z \in S(I_n).$$

Take

$$f_n(z) = \frac{(1 - |a_n|^2)\sqrt{K(1 - |a_n|^2)}}{(1 - \overline{a_n} z)^{\frac{3+\alpha}{2}}}.$$

Then $f_n \to 0$ uniformly on the compact subsets of $\mathbb{D}$ as $n \to \infty$ and $\|f_n\|_{\mathcal{D}_{K,\alpha}} \lesssim 1$. Thus, for any compact operator $S$ from $\mathcal{D}_{K,\alpha}$ to $\mathcal{H}^2_{K,\alpha}$, we have

$$\lim_{n\to\infty} \|S f_n\|_{\mathcal{H}^2_{K,\alpha}} \to 0.$$

Therefore

$$\begin{aligned}
\|V_g - S\| &\gtrsim \limsup_{n\to\infty} \left(\|V_g f_n\|_{\mathcal{H}^2_{K,\alpha}} - \|S f_n\|_{\mathcal{H}^2_{K,\alpha}}\right) \\
&= \limsup_{n\to\infty} \|V_g f_n\|_{\mathcal{H}^2_{K,\alpha}} \\
&\approx \limsup_{n\to\infty} \left(\frac{1}{K(|I_n|)} \int_{S(I_n)} |(V_g f_n)'(z)|^2 (1 - |z|^2)^{1+\alpha} dA(z)\right)^{\frac{1}{2}} \\
&= \limsup_{n\to\infty} \left(\frac{1}{K(|I_n|)} \int_{S(I_n)} |f_n(z)|^2 |g'(z)|^2 (1 - |z|^2)^{1+\alpha} dA(z)\right)^{\frac{1}{2}} \\
&\approx \limsup_{n\to\infty} \left(\frac{1}{|I_n|^{1+\alpha}} \int_{S(I_n)} |g'(z)|^2 (1 - |z|^2)^{1+\alpha} dA(z)\right)^{\frac{1}{2}} \\
&\gtrsim \limsup_{n\to\infty}(1 - |w_n|^2)|g'(w_n)|.
\end{aligned}$$

On the other hand, by Lemma 7, $V_{g_r} : \mathcal{D}_{K,\alpha} \to \mathcal{H}^2_{K,\alpha}$ is compact operator. Combining this with Theorem 1 and the linearity of $V_g$ respect to $g$ implies

$$\|V_g\|_e \le \|V_g - V_{g_r}\| = \|V_{g-g_r}\| \approx \|g - g_r\|_{\mathcal{B}}.$$

Together with Lemma 8, we have

$$\|V_g\|_e \lesssim \limsup_{|z|\to 1^-}(1-|z|^2)|g'(z)| \approx dist(g, \mathcal{B}_0).$$

The proof is completed. □

**Corollary 1.** *Suppose* $0 < \alpha < 1$ *and* $K$ *satisfies the conditions (A) and (B). If* $g \in H(\mathbb{D})$*, then* $V_g : \mathcal{D}_{K,\alpha} \to \mathcal{H}^2_{K,\alpha}$ *is compact if and only if* $g \in \mathcal{B}_0$.

**Theorem 4.** *Suppose* $0 < \alpha < 1$ *and* $K$ *satisfies the conditions (A) and (B). If* $g \in H(\mathbb{D})$ *and* $S_g$ *is bounded from* $\mathcal{D}_{K,\alpha}$ *to* $\mathcal{H}^2_{K,\alpha}$*, then*

$$\|S_g\|_e \approx \sup_{z\in\mathbb{D}} |g(z)|.$$

*Proof.* For compact operators $S$, it follows that

$$\|S_g\|_e = \inf_S \|S_g - S\| \le \|S_g\| \lesssim \sup_{z\in\mathbb{D}} |g(z)|.$$

On the other hand, we choose the sequence $\{a_n\} \subset \mathbb{D}$ such that $|a_n| \to 1$. We define

$$f_n(z) := \frac{(1-|a_n|^2)\sqrt{K(1-|a_n|^2)}}{(1-\overline{a_n}z)^{\frac{3+\alpha}{2}}}, \quad z \in \mathbb{D}.$$

It follows from the proof of Lemma 2 that $\|f_n\|_{\mathcal{D}_{K,\alpha}} \lesssim 1$. It is easy to check that $f_n$ converges to zero uniformly on any compact subsets of $\mathbb{D}$. Then $\|Sf_n\|_{\mathcal{H}^2_{K,\alpha}} \to 0$ as $n \to \infty$ for any compact operator $S$ from $\mathcal{D}_{K,\alpha}$ to $\mathcal{H}^2_{K,\alpha}$. So

$$\begin{aligned}
\|S_g - S\| &\gtrsim \limsup_{n\to\infty} \|(S_g - S)f_n\|_{\mathcal{H}^2_{K,\alpha}} \\
&\ge \limsup_{n\to\infty} (\|S_g f_n\|_{\mathcal{H}^2_{K,\alpha}} - \|Sf_n\|_{\mathcal{H}^2_{K,\alpha}}) \\
&= \limsup_{n\to\infty} \|S_g f_n\|_{\mathcal{H}^2_{K,\alpha}}.
\end{aligned}$$

From the proof of Theorem 2, we have

$$\|S_g f_n\|_{\mathcal{H}^2_{K,\alpha}} \gtrsim |g(a_n)|.$$

Since $\{a_n\} \subseteq \mathbb{D}$ is arbitrary, we have

$$\|S_g\|_e \gtrsim \sup_{z\in\mathbb{D}} |g(z)|.$$

The proof is completed. □

**Corollary 2.** *Suppose* $0 < \alpha < 1$ *and* $K$ *satisfy the conditions (A) and (B). If* $g \in H(\mathbb{D})$*, then* $S_g : \mathcal{D}_{K,\alpha} \to \mathcal{H}^2_{K,\alpha}$ *is compact if and only if* $g = 0$.

## 5. Conclusions

In this paper, we give some equivalent characterizations of Volterra integral operator and essential norm from Dirichlet type spaces $\mathcal{D}_{K,\alpha}$ to Morrey type spaces $\mathcal{H}^2_{K,\alpha}$.


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## Conflict of interest

We declare that we have no conflict of interest.

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