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Research article

A novel approach to find partitions of Z_m with equal sum subsets via complete graphs

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Abstract: In mathematics and computer sciences, the partitioning of a set into two or more disjoint subsets of equal sums is a well-known NP-complete problem, also referred to as partition problem. There are various approaches to overcome this problem for some particular choice of integers. Here, we use quadratic residue graph to determine the possible partitions of positive integers $m = 2^\beta, q^\beta, 2^\beta q, 2q^\beta, qp$, where p, q are odd primes and β is any positive integer. The quadratic residue graph is defined on the set $Z_m = \{\overline{0}, \overline{1}, \cdots, \overline{m-1}\}$, where Z_m is the ring of residue classes of m, i.e., there is an edge between $\overline{x}, \overline{y} \in Z_m$ if and only if $\overline{x}^2 \equiv \overline{y}^2 \pmod{m}$. We characterize these graphs in terms of complete graph for some particular classes of m.

Keywords: quadratic residues graph; complete graph; ring of integers
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1. Introduction

Graph plays a dynamic role in various sciences such as physics, biology, chemistry, and computer science [2–5, 8]. It is used in various frameworks related to social and information systems [4] and also solves many issues related to everyday life. In physics, there are various circuits constructed by considering different graphs [5]. The atomic number of many molecules is evaluated by using group symmetry graphs that are still unknown a few years ago [3]. In computer science, many problems have been discussed using graphs that were not easy to visualize earlier. For discrete mathematics and combinatorics, the applications of number theory and graph theory are of crucial importance. In this work, we employ number theory to investigate the special classes of graphs.

Rogers [18] discussed the action of a quadratic map on multiplicative groups under modulo a prime p by using the associated directed graph for which there is an edge from each element to its image. He established a formula to decompose a graph into cyclic components with their attached trees. The necessary and sufficient conditions for the existence of isolated fixed points have also been established. Somer and Krizek [19] studied the structures of graphs of quadratic congruences for composite modulus. Mahmood and Ahmad [10, 11] proposed many new results of graphs over residues modulo prime powers. Haris and Khalid [12–15] investigated the structure of power digraphs associated with the congruence $x^n \equiv y\ (mod\ m)$. Meemark and Wiroonsri [9] discussed the structure of $G(R, k)$ using a quadratic map, where R is the quotient ring of polynomials over finite fields and k is the modulus. Wei and Tang [20] introduced the concept of square mapping graphs of the Gaussian ring $Z_m[i]$. Ali et al. [21, 22] introduced new labeling algorithm on various classes of graphs with applications. Some basic and useful result discussed in [1, 6, 7, 16, 23, 24] as well.

Let p be a prime and a an integer coprime to p. Then a is called a quadratic residue $\pmod p$ if and only if the congruence $x^2 \equiv a \pmod p$ has a solution. Otherwise, a is called quadratic non-residue $\pmod p$. Two non-zero integers x and y are called zero divisors in the ring Z_m if and only if $xy \equiv 0 \pmod m$ [17]. Recall that a graph $\overbrace{G}(2, m)$, whose vertices are elements of ring Z_m and there will be an edge between $\overline{x}$ and $\overline{y}$ ($\overline{x} \neq \overline{y}$) if $\overline{x}^2 \equiv \overline{y}^2 \pmod m$ then, $\overbrace{G}(2, m)$ is termed as a quadratic graph. For $m = 30$ vertex set is

$\underline{\ \ }Z_n = \{\overline{0}, \overline{1}, \overline{2}, \overline{3}, \overline{4}, \overline{5}, \overline{6}, \overline{7}, \overline{8}, \overline{9}, \overline{10}, \overline{11}, \overline{12}, \overline{13}, \overline{14}, \overline{15}, \overline{16}, \overline{17}, \overline{18}, \overline{19}, \overline{20}, \overline{21}, \overline{22}, \overline{23}, \overline{24}, \overline{25}, \overline{26}, \overline{27},$
$\overline{28}, \overline{29}\}$,

by solving the congruences $\overline{x}^2 \equiv \overline{y}^2 \pmod{30}$ for each $\overline{x}, \overline{y} \in Z_n$ then, there are 2, 4, 6 copies of K_1, K_4 and K_2, respectively as shown in Figure 1. We note that each copy of K_2 and K_4 has equal sum 30 and 60, respectively. Here K_n is a complete graph obtained if each node connected with every other node except itself [4].

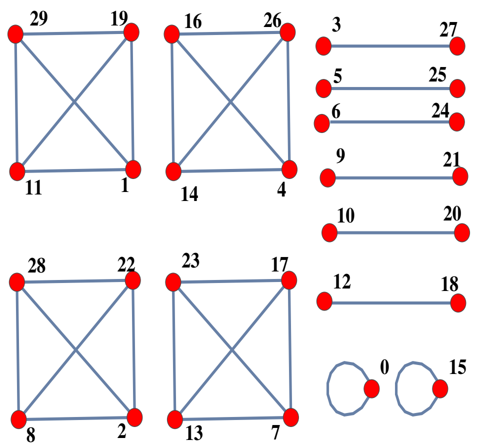

Figure 1. $\overbrace{G}(2, 30) = 2K_1 \bigoplus 6K_2 \bigoplus 4K_4$.

Theorem 1.1. *[17] Let p be an odd prime, k be a positive integer, and a an integer such that* $(a, p) = 1$. *Then*

1. *The congruence* $x^2 \equiv a\ (mod\ p^k)$ *has either no solution or exactly two incongruent solutions modulo* p^k.

2. *The congruence $x^2 \equiv a \ (mod\ p^k)$ has no solution if a is quadratic non-residue of p and exactly two incongruent solutions modulo p if a is quadratic residue of p.*

Theorem 1.2. *[17] Let a be an odd number. Then we have the following:*

1. *The congruence equation $x^2 \equiv a \ (mod\ 2)$ has the unique solution if and only if $x \equiv 1 \ (mod\ 2)$.*
2. *The congruence equation $x^2 \equiv a \ (mod\ 4)$ either has no solution if $a \equiv 3 \ (mod\ 4)$ or has two solutions $x \equiv 1, 3 \ (mod\ 4)$ if $a \equiv 1 \ (mod\ 4)$.*
3. *When $k \geq 3$, the equation $x^2 \equiv a \ (mod\ 2^k)$ either has no solution if $a \not\equiv 1 \ (mod\ 8)$; or has four solutions $x_1, -x_1, x_1 + 2^{k-1}, -(x_1 + 2^{k-1})$ if $a \equiv 1 \ (mod\ 8)$.*

2. Quadratic residues graphs over Z_{2^β} and Z_{q^β}

In this section, we characterize quadratic residue graphs for some well-known classes of integers 2^β and q^β, for each positive integer β and odd prime q.

Theorem 2.1. *Let $m = 2^\beta$ be an integer. Then*

$$\overbrace{G}(2, 2^\beta) = \begin{cases} 2K_\beta, & \text{if } \beta = 1, 2, \\ 2K_2 \bigoplus K_4, & \text{if } \beta = 3, \\ 4K_4, & \text{if } \beta = 4, \\ 6K_4 \bigoplus K_8, & \text{if } \beta = 5, \\ 4K_8 \bigoplus 8K_4, & \text{if } \beta = 6, \\ 6K_8 \bigoplus K_{16} \bigoplus 16K_4, & \text{if } \beta = 7, \\ 8K_8 \bigoplus 4K_{16} \bigoplus 32K_4, & \text{if } \beta = 8, \\ \bigoplus_{i=1}^{\frac{\beta-7}{2}} 2^{\beta-5-2(i-1)} K_{2^{3+(i-1)}} & \\ \bigoplus 6K_{2^{(\beta-1)/2}} \bigoplus K_{2^{(\beta+1)/2}} \bigoplus 2^{\beta-3} K_4, & \text{if } \beta \geq 9, \text{ and} \\ \qquad \beta \equiv 1 \ (mod\ 2), & \\ \bigoplus_{i=1}^{\frac{\beta-6}{2}} 2^{\beta-2i-3} K_{2^{2+i}} \bigoplus 2^2 K_{2^{\beta/2}} \bigoplus 2^{\beta-3} K_4, & \text{if } \beta \geq 10, \text{ and} \\ \qquad \beta \equiv 0 \ (mod\ 2), & \end{cases}$$

Proof. We discuss two cases to prove this theorem, the zero-divisors and unit elements of the ring Z_m. Firstly, we discuss zero-divisors, let $S = \{2m | m = 0, 1, 2, \cdots, 2^{\beta-1} - 1\}$ be the set of all zero-divisors of Z_{2^β} with including zero for each positive integer β. To find the number of solutions of $\eta^2 \equiv \beta^2 \ (\text{mod } 2^\beta)$, we start from $\eta^2 \equiv \beta^2 \ (\text{mod } 2)$, in this case just $S = \{0\}$. Therefore, $\eta^2 \equiv 0 \ (\text{mod } 2)$ has only one solution which is zero, but by the definition of quadratic zero-divisors graph there will be a no loop, so $\overbrace{G}(2, 2) = K_1$. For $\beta = 2$, $\eta^2 \equiv 0 \equiv 2^2 \ (\text{mod } 2^2)$ has two solution namely $\eta = 0, 2$, then $\overbrace{G}(2, 2^2) = K_2$. If $\beta = 3$, then there are two congruences

$$\eta^2 \equiv 0 \ (\text{mod } 2^3), \quad \text{and} \quad \eta^2 \equiv 4 \ (\text{mod } 2^3).$$

The roots of these congruences are $\eta = 0, 4$, and $\eta = 2, 6$, respectively. Thus, there exist two copies of K_2. For $\beta = 4$, we have

$$\eta^2 \equiv 0 \ (\text{mod } 2^4), \quad \text{and} \quad \eta^2 \equiv 4 \ (\text{mod } 2^4).$$

The corresponding roots of congruence are $\{0,4,8,12\}$ and $\{2,6,10,14\}$, respectively. Therefore, $\overbrace{G}(2,2^4) = 2K_4$. There are three congruences for $\beta = 5$

$$\eta^2 \equiv 0 \pmod{2^5},\ \eta^2 \equiv 4 \pmod{2^5},\ \text{ and }\ \eta^2 \equiv 16 \pmod{2^5}.$$

The solution sets of these congruences are $\{0,8,16,24\}$, $\{2,6,10,14,18,22,26,30\}$, and $\{4,12,20,28\}$, respectively. Therefore, there are two copies of K_4 and one copy of K_8. For $\beta = 6$, we have

$$\eta^2 \equiv 0 \pmod{2^6},\ \eta^2 \equiv 4 \pmod{2^6},\ \eta^2 \equiv 16 \pmod{2^6},\ \text{ and } \eta^2 \equiv 36 \pmod{2^6}.$$

The zeros of these congruences are $\{0,8,16,24,32,40,48,54\}$, $\{2,18,34,50\}\bigcup\{62,46,30,14\}$, $\{4,12,20,28,36,44,52,60\}$, and $\{6,22,38,54\}\bigcup\{58,42,26,10\}$, respectively. For $\beta = 7$, there are 7 different congruences. We have

$$\eta^2 \equiv (2t)^2 \pmod{2^7},\ t = 0,1,3,4,5,7, \tag{2.1}$$
$$\eta^2 \equiv 16 \pmod{2^7}. \tag{2.2}$$

Roots of these congruences are
$\{2t+16m|m = 0,1,2\cdots2^3-1\}$, $\{2t+32m|m = 0,1,2\cdots2^2-1\}\bigcup\{2^7-32m-2t|m = 0,1,2\cdots2^2-1\}$, $t = 1,3,4,5,7$, $\{4+8m|m = 0,1,2\cdots2^4-1\}$. Hence, there are six copies of K_8 and one copy of K_{16}. For $\beta = 8$, there are 12 congruences. We have

$$\eta^2 \equiv (2+4t)^2 \pmod{2^8},\ t = 0,1,\cdots,2^3-1, \tag{2.3}$$
$$\eta^2 \equiv (4t)^2 \pmod{2^8},\ t = 0,1,\cdots,2^2-1. \tag{2.4}$$

Zeroes of these congruences are
$\{2+4t+64m|m = 0,1,2\cdots2^2-1\}\bigcup\{2^8-64m-4t-2|m = 0,1,2\cdots2^2-1\}$, $t = 0,1,\cdots,2^3-1$, $\{4t+32m|m = 0,1,2\cdots2^4-1\}$, $\{4t+32m|m = 0,1,2\cdots2^3-1\}\bigcup\{2^8-32m-4t|m = 0,1,2\cdots2^3-1\}$, $t = 1,\cdots,2^2-1$. Thus, we have $\overbrace{G}(2,2^8) = 8K_4 \bigoplus 4K_{16}$. For $\beta = 9$, there are 23 different congruences. We have

$$\eta^2 \equiv (2+4t)^2 \pmod{2^9},\ t = 0,1,\cdots,2^4-1, \tag{2.5}$$
$$\eta^2 \equiv (4t)^2 \pmod{2^9},\ t = 0,1,3,4,5,7, \tag{2.6}$$
$$\eta^2 \equiv 64 \pmod{2^9}. \tag{2.7}$$

Solutions of congruences (2.5)–(2.7) are $\{2+4t+128m|m = 0,1,2\cdots2^2-1\}\bigcup\{2^9-128m-4t-2|m = 0,1,2\cdots2^2-1\}$, $t = 0,1,\cdots,2^6-1$, $\{4t+32m|m = 0,1,2\cdots2^4-1\}$, $\{4t+64m|m = 0,1,2\cdots2^3-1\}\bigcup\{2^9-64m-4t|m = 0,1,2\cdots2^3-1\}$, $t = 1,3,4,5,7$, $\{8+16m|m = 0,1,2\cdots2^5-1\}$. That is, $\overbrace{G}(2,2^9) = 16K_8 \bigoplus 6K_{16} \bigoplus K_{32}$. For $\beta = 10$, there are 44 different congruences. We have

$$\eta^2 \equiv (2+4t)^2 \pmod{2^{10}},\ t = 0,1,\cdots,2^5-1, \tag{2.8}$$
$$\eta^2 \equiv (4+8t)^2 \pmod{2^{10}},\ t = 0,1,\cdots,2^3-1, \tag{2.9}$$
$$\eta^2 \equiv (8t)^2 \pmod{2^{10}},\quad t = 0,1,\cdots,2^2-1, \tag{2.10}$$

Sequences of roots of congruences (2.8)–(2.10) are $\{2+4t+256m|m = 0,1,2\cdots2^2-1\}\bigcup\{2^{10}-256m-4t-2|m = 0,1,2\cdots2^2-1\}$ $t = 0,1,\cdots,2^5-1$, $\{4+8t+128m|m = 0,1,2\cdots2^3-1\}\bigcup\{2^{10}-128n-$

$8t-4|m = 0,1,2\cdots 2^3-1\}$, $t = 0,1,\cdots,2^3-1$, $\{8t+64m|m = 0,1,2\cdots 2^5-1\}$, $\{8t+64m|m = 0,1,2\cdots 2^4-1\}\bigcup\{2^{10}-64m-8t|m = 0,1,2\cdots 2^4-1\}$, $t = 1,\cdots,2^2-1$. That is, $\overbrace{\ G\ }(2,2^{10}) = 32K_8 \bigoplus 8K_{16} \bigoplus 4K_{32}$. For $\beta = 11$,

$$\eta^2 \equiv (2+4t)^2 \pmod{2^{11}}, \quad t = 0,1,\cdots,2^6-1, \tag{2.11}$$
$$\eta^2 \equiv (4+8t)^2 \pmod{2^{11}}, \quad t = 0,1,\cdots,2^4-1, \tag{2.12}$$
$$\eta^2 \equiv (8t)^2 \pmod{2^{11}}, \quad t = 0,1,3,4,5,7, \tag{2.13}$$
$$\eta^2 \equiv 256 \pmod{2^{11}}. \tag{2.14}$$

Zeroes of congruences (2.11)–(2.14) are $\{2+4t+512m|m = 0,1,2\cdots 2^2-1\}\bigcup\{2^{11}-512m-4t-2|m = 0,1,2\cdots 2^2-1\}$, $t = 0,1,\cdots,2^6-1$, $\{4+8t+256m|m = 0,1,2\cdots 2^3-1\}\bigcup\{2^{11}-256m-8t-4|m = 0,1,2\cdots 2^3-1\}$, $t = 0,1,\cdots,2^4-1$, $\{8t+64m|m = 0,1,2\cdots 2^5-1\}$, $\{8t+128m|m = 0,1,2\cdots 2^4-1\}\bigcup\{2^{11}-128m-8t|m = 0,1,2\cdots 2^4-1\}$, $t = 1,3,4,5,7$, $\{16+32m|m = 0,1,2\cdots 2^6-1\}$.

Therefore, we have $\overbrace{\ G\ }(2,2^{11}) = 64K_8 \bigoplus 16K_{16} \bigoplus 6K_{32} \bigoplus K_{64}$. For $\beta = 12$, there are 172 congruences. We have

$$\eta^2 \equiv (2+4t)^2 \pmod{2^{12}}, \quad t = 0,1,\cdots,2^7-1, \tag{2.15}$$
$$\eta^2 \equiv (4+8t)^2 \pmod{2^{12}}, \quad t = 0,1,\cdots,2^5-1, \tag{2.16}$$
$$\eta^2 \equiv (8+16t)^2 \pmod{2^{12}}, \quad t = 0,1,\cdots,2^3-1, \tag{2.17}$$
$$\eta^2 \equiv (16t)^2 \pmod{2^{12}}, \quad t = 0,1,\cdots,2^2-1. \tag{2.18}$$

Sets of roots of congruences (2.15)–(2.18) are $\{2+4t+1024m|m = 0,1,2\cdots 2^2-1\}\bigcup\{2^{12}-1024m-4t-2|m = 0,1,2\cdots 2^2-1\}$, $t = 0,1,\cdots,2^7-1$, $\{4+8t+512m|m = 0,1,2\cdots 2^3-1\}\bigcup\{2^{12}-512m-8t-4|m = 0,1,2\cdots 2^3-1\}$, $t = 0,1,\cdots,2^5-1$, $\{8t+256m|m = 0,1,2\cdots 2^4-1\}\bigcup\{2^{12}-256m-8t|m = 0,1,2\cdots 2^4-1\}$, $t = 0,1,\cdots,2^3-1$, $\{16t+128m|m = 0,1,2\cdots 2^6-1\}$, $\{16t+128m|m = 0,1,2\cdots 2^5-1\}\bigcup\{2^{12}-128m-8t|m = 0,1,2\cdots 2^5-1\}$, $t = 1,\cdots,2^2-1$. That is, $\overbrace{\ G\ }(2,2^{12}) = 128K_8 \bigoplus 32K_{16} \bigoplus 8K_{32} \bigoplus 4K_{64}$.

The generalize sequence for $\beta \geq 9$, where β is an odd number, there are $\frac{2^4(2^{\beta-7}-1)+21}{3}$ number of congruences. we have

$$\eta^2 \equiv (2^i+2^{i+1}t)^2 \pmod{2^\beta}, \quad t = 0,1,\cdots,2^{\beta-2i-3}-1, \quad i = 1,2,3,\cdots,\frac{\beta-7}{2}, \tag{2.19}$$
$$\eta^2 \equiv (2^{\frac{\beta-5}{2}}t)^2 \pmod{2^\beta}, \quad t = 0,1,3,4,5,7, \tag{2.20}$$
$$\eta^2 \equiv 2^{\beta-3} \pmod{2^\beta}. \tag{2.21}$$

Sequences of roots of congruences (2.19)–(2.21) are $\{2^i+2^{i+1}t+2^{\beta-i-1}m|m = 0,1,2\cdots 2^{i+1}-1\}\bigcup\{2^\beta-2^{\beta-i-1}m-2^{i+1}t-2^i|m = 0,1,2\cdots 2^{i+1}-1\}$, $t = 0,1,\cdots,2^{\beta-2i-3}-1$, $i = 1,2,3,\cdots,\frac{\beta-7}{2}$, $\{2^{\frac{\beta-5}{2}}t+2^{\frac{\beta+1}{2}}m|m = 0,1,2,\cdots 2^{\frac{\beta-1}{2}}-1\}$, $\{2^{\frac{\beta-5}{2}}t+2^{\frac{\beta+3}{2}}m|m = 0,1,2,\cdots 2^{\frac{\beta-3}{2}}-1\}\bigcup\{2^\beta-2^{\frac{\beta+3}{2}}m-2^{\frac{\beta-5}{2}}k|m = 0,1,2,\cdots 2^{\frac{\beta-3}{2}}-1\}$, $t = 1,3,4,5,7$, $\{2^{\frac{\beta-3}{2}}+2^{\frac{\beta-1}{2}}m|m = 0,1,2\cdots 2^{\frac{\beta+1}{2}}-1\}$. That is, $\overbrace{\ G\ }(2,2^\beta) = \bigoplus_{i=1}^{\frac{\beta-7}{2}} 2^{\beta-5-2(i-1)}K_{2^{3+(i-1)}} \bigoplus 6K_{2^{(\beta-1)/2}} \bigoplus K_{2^{(\beta+1)/2}}$.

For second case when β is an even number and $\beta \geq 10$, we have $\frac{2^5(2^{\beta-8}-1)+36}{3}$ congruences as follows

$$\eta^2 \equiv (2^i + 2^{i+1}t)^2 \pmod{2^\beta}, \quad t = 0, 1, \cdots, 2^{\beta-2i-3} - 1, \quad i = 1, 2, 3, \cdots, \frac{\beta-6}{2}, \tag{2.22}$$

$$\eta^2 \equiv (2^{\frac{\beta-4}{2}}t)^2 \pmod{2^\beta}, \qquad t = 0, 1, \cdots, 2^2 - 1. \tag{2.23}$$

Sequences of zeroes of congruences (2.22)–(2.23) are $\{2^i+2^{i+1}t+2^{\beta-i-1}m|m = 0, 1, 2\cdots 2^{i+1}-1\}\bigcup\{2^\beta - 2^{\beta-i-1}m-2^{i+1}t-2^i|m = 0, 1, 2\cdots 2^{i+1}-1\}$, $t = 0, 1, \cdots, 2^{\beta-2i-3}-1$, $i = 1, 2, 3, \cdots, \frac{\beta-6}{2}$, $\{2^{\frac{\beta-4}{2}}t+2^{\frac{\beta+2}{2}}m|m = 0, 1, 2\cdots, 2^{\frac{\beta}{2}} - 1\}$, $\{2^{\frac{\beta-4}{2}}t + 2^{\frac{\beta+2}{2}}m|m = 0, 1, 2\cdots, 2^{\frac{\beta-2}{2}} - 1\}\bigcup\{2^\beta - 2^{\frac{\beta+2}{2}}m - 2^{\frac{\beta-4}{2}}t|m = 0, 1, 2^{\frac{\beta-2}{2}} - 1\}$, $t = 1, \cdots, 2^2 - 1$.

Thus, $\overbrace{G}(2, 2^\beta) = \bigoplus_{i=1}^{\frac{\beta-6}{2}} 2^{\beta-2i-3}K_{2^{2+i}} \bigoplus 2^2K_{2^{\beta/2}}$.

Now, we discuss the unit elements of Z_m. For $m = 2, 4$, the result is straightforward. For $\beta \geq 3$, the graph $\overbrace{G}(2, m)$ contains $\phi(2^\beta) = 2^{\beta-1}$ number of vertices. We calculate the least positive residues of the square of the integers, which are smaller than and relatively prime with m. Hence, there are $\phi(2^\beta) = 2^{\beta-1}$ obtained. By Theorem 1.2, the congruence $x^2 \equiv a \pmod{2^\beta}$ has either no solution or exactly four incongruent solutions. This implies that, there are always $\frac{\phi(m)}{4} = \frac{\phi(2^{\beta-1})}{4} = 2^{\beta-3}$ quadratic residues among all the vertices. Thus $\overbrace{G}(2, m) = 2^{\beta-3}K_4$. By combining both cases, we get the desired result. □

The quadratic residues graph for $n = 128$ is shown in Figure 2.

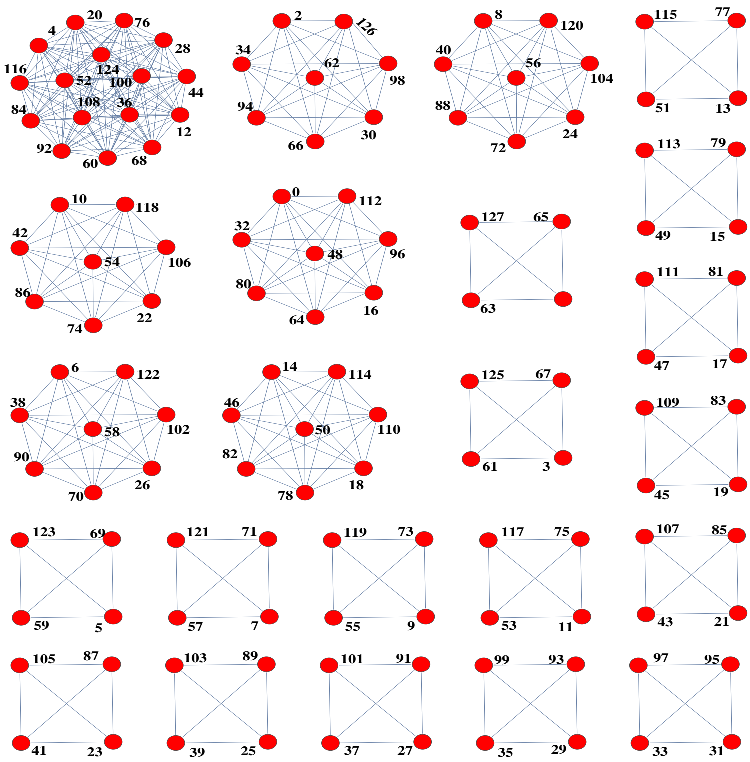

Figure 2. $\overbrace{G}(2, 2^7) = 6K_8 \bigoplus K_{16} \bigoplus 16K_4$.

Theorem 2.2. *Let q be an odd prime. Then* $\widetilde{G}(2, q^{\beta})$

$$= \begin{cases} K_1 \bigoplus \frac{q-1}{2} K_2, & \text{if } \beta = 1, \\ K_q \bigoplus \frac{q(q-1)}{2} K_2, & \text{if } \beta = 2, \\ \left(\frac{q-1}{2}\right) K_{2p} \bigoplus K_q \bigoplus \frac{q^2(q-1)}{2} K_2, & \text{if } \beta = 3, \\ \left(\frac{q-1}{2}\right) q K_{2q} \bigoplus K_{q^2} \bigoplus \frac{q^3(q-1)}{2} K_2, & \text{if } \beta = 4, \\ \bigoplus_{i=1}^{\frac{\beta-1}{2}} \left(\frac{q^{\beta-2i} - q^{\beta-2i-1}}{2}\right) K_{2q^i} \bigoplus K_{q^{(\beta-1)/2}} \bigoplus \frac{q^{\beta-1}(q-1)}{2} K_2, & \text{if } \beta \geq 5, \\ \qquad \text{and } \beta \equiv 1 \ (mod\ 2), & \\ \bigoplus_{i=1}^{\frac{\beta-2}{2}} \left(\frac{q^{\beta-2i} - q^{\beta-2i-1}}{2}\right) K_{2q^i} \bigoplus K_{q^{\beta/2}} \bigoplus \frac{q^{\beta-1}(q-1)}{2} K_2, & \text{if } \beta \geq 6, \\ \qquad \text{and } \beta \equiv 0 \ (mod\ 2). & \end{cases}$$

Proof. To prove this theorem first we assume zero-divisors of the ring Z_m. Let q is an odd prime and $S = \{tq^{\beta} | t = 0, 1, 2, \cdots, q^{\beta-1} - 1\}$ be zero-divisors including zero of q^{β} for each positive integer β. To solve congruence $\eta^2 \equiv \beta^2 \pmod{q^{\beta}}$ for each $\beta \geq 1$, we start with $\eta^2 \equiv \beta^2 \pmod{q}$. In this case, 0 is only root of this congruence, but there is no edge between two vertices when they are same, so $\widetilde{G}(2, q) = K_1$. For $\beta = 2$, there is only one congruence namely $\eta^2 \equiv 0 \pmod{q^2}$, roots of this congruence are $\eta = 0, q, 2q, \cdots, (q-1)q$. There are q solutions and will be complete graph of order q. For $\beta = 3$, there are $\frac{q+1}{2}$ congruences. We have

$$\eta^2 \equiv 0 \pmod{q^3}, \tag{2.24}$$

$$\eta^2 \equiv (qt)^2 \pmod{q^3}, \quad t = 1, 2, \cdots, \frac{q-1}{2}. \tag{2.25}$$

Zeros of these congruences are $\{\eta = q^2(m-1) | m = 1, 2, \cdots, q\}$, and $\{\eta = q^2(m-1) + qt | \ t = 1, 2, \cdots, \frac{q-1}{2}, \quad m = 1, 2, \cdots, q\} \bigcup \{\eta = q^3 - q^2(m-1) - qt | \ t = 1, 2, \cdots, \frac{q-1}{2}, \quad m = 1, 2, \cdots, q\}$, respectively. For $\beta = 4$, the number of distinct congruences is $(q^2 - q + 1)/2$. These are

$$\eta^2 \equiv 0 \pmod{q^4}, \tag{2.26}$$

$$\eta^2 \equiv (qt)^2 \pmod{q^4}, \ t = 1, 2, \cdots, \frac{q^2-1}{2}, \quad \text{but } t \neq ql, \ \ l = 1, 2, \cdots, \frac{q-1}{2}. \tag{2.27}$$

Sequences of roots of these congruences are $\{\eta = q^2(m-1) | m = 1, 2, \cdots, q^2\}$, and $\{\eta = q^3(m-1) + qt | \ t = 1, 2, \cdots, \frac{q^2-1}{2}$, but $t \neq ql, \quad l = 1, 2, \cdots, \frac{q-1}{2}, \quad m = 1, 2, \cdots, q\} \bigcup \{\eta = q^4 - q^3(-1) - pt | \ t = 1, 2, \cdots, \frac{p^2-1}{2}$, but $t \neq pl, \ \ l = 1, 2, \cdots, \frac{q-1}{2}, \ \ m = 1, 2, \cdots, q\}$, respectively.

For $\beta = 5$, we have

$$\eta^2 \equiv 0 \pmod{q^5}, \tag{2.28}$$

$$\eta^2 \equiv (q^2 t)^2 \pmod{q^5}, \quad t = 1, 2, \cdots, \frac{q-1}{2} \tag{2.29}$$

$$\eta^2 \equiv (qt)^2 \pmod{q^5},\ t = 1,2,\cdots,\frac{q^3-1}{2}, \quad \text{but } t \neq pl,\ \ l = 1,2,\cdots,\frac{q^2-1}{2}. \tag{2.30}$$

Zeroes of these congruences are $\{\eta = q^3(m-1)|m = 1,2,\cdots,q^2\}$, $\{\eta = q^3(m-1) + q^2t|\ t = 1,2,\cdots,\frac{q-1}{2},\ \ m = 1,2,\cdots,q^2\} \bigcup \{\eta = q^5 - q^3(m-1) - q^2t|\ t = 1,2,\cdots,\frac{q-1}{2},\ \ m = 1,2,\cdots,q^2\}$, and $\{\eta = q^4(m-1) + qt|\ t = 1,2,\cdots,\frac{q^3-1}{2}$, but $t \neq ql,\ \ l = 1,2,\cdots,\frac{q^2-1}{2},\ \ m = 1,2,\cdots,q\} \bigcup \{\eta = q^5 - q^4(m-1) - qt|\ t = 1,2,\cdots,\frac{q^3-1}{2}$, but $t \neq ql,\ \ l = 1,2,\cdots,\frac{q^2-1}{2},\ \ m = 1,2,\cdots,q\}$, respectively.

If $\beta = 6$, then we have

$$\eta^2 \equiv 0 \pmod{q^6}, \tag{2.31}$$

$$\eta^2 \equiv (q^2t)^2 \pmod{q^6},\ t = 1,2,\cdots,\frac{q^2-1}{2}, \quad \text{but } t \neq ql,\ \ l = 1,2,\cdots,\frac{q-1}{2} \tag{2.32}$$

$$\eta^2 \equiv (qt)^2 \pmod{q^6},\ t = 1,2,\cdots,\frac{q^4-1}{2}, \quad \text{but } t \neq ql,\ \ l = 1,2,\cdots,\frac{q^3-1}{2}. \tag{2.33}$$

Sequences of roots of congruences (2.31)–(2.33) are $\{\eta = q^3(m-1)|m = 1,2,\cdots,q^3\}$, $\{\eta = q^4(m-1) + q^2t|\ t = 1,2,\cdots,\frac{q^2-1}{2}$, but $t \neq ql,\ \ l = 1,2,\cdots,\frac{q-1}{2},\ \ m = 1,2,\cdots,q^2\} \bigcup \{\eta = q^6 - q^3(m-1) - q^2t|\ t = 1,2,\cdots,\frac{q^2-1}{2}$, but $t \neq ql,\ \ l = 1,2,\cdots,\frac{q-1}{2},\ \ m = 1,2,\cdots,q^2\}$, and $\{\eta = q^5(m-1) + qt|\ t = 1,2,\cdots,\frac{q^4-1}{2}$, but $t \neq ql,\ \ l = 1,2,\cdots,\frac{q^3-1}{2},\ \ m = 1,2,\cdots,q\} \bigcup \{\eta = q^6 - q^5(m-1) - qt|\ t = 1,2,\cdots,\frac{q^4-1}{2}$, but $t \neq ql,\ \ l = 1,2,\cdots,\frac{q^3-1}{2},\ \ m = 1,2,\cdots,q\}$, respectively.

Now we are going to derive generalize sequence for both odd and even distinct congruences for $\beta \geq 5$ and $\beta \equiv 1 \pmod 2$. These are

$$\eta^2 \equiv (q^it)^2 \pmod{q^\beta},\ t = 1,2,\cdots,\frac{q^{\beta-2i}-1}{2}, \tag{2.34}$$

$$\text{but} \quad t \neq ql,\ \ l = 1,2,\cdots,\frac{q^{\beta-2i-1}-1}{2},\ \ i = 1,2,\cdots,\frac{\beta-1}{2},$$

$$\eta^2 \equiv 0 \pmod{q^\beta}. \tag{2.35}$$

Sequences of roots are $\bigcup_{i=1}^{\frac{\beta-1}{2}} \Big\{ \{\eta = q^{\beta-i}(m-1) + q^it|\ t = 1,2,\cdots,\frac{q^{\beta-2i}-1}{2}$, but $l \neq ql,\ \ l = 1,2,3,\cdots,\frac{q^{\beta-2i-1}-1}{2}$, $m = 1,2,\cdots,q^i\} \bigcup \{\eta = q^\beta - q^{\beta-i}(m-1) - q^it|\ t = 1,2,\cdots,\frac{q^{\beta-2i}-1}{2}$, but $t \neq ql,\ \ l = 1,2,\cdots,\frac{q^{\beta-2i-1}-1}{2},\ \ m = 1,2,\cdots,q^i\}\Big\}$, and $\{\eta = q^{\frac{\beta+1}{2}}(m-1)|m = 1,2,\cdots,q^{\frac{\beta-1}{2}}\}$, respectively. Therefore, for every positive integer $\beta \geq 5$ with $\beta \equiv 1 \pmod 2$, $\overbrace{G}(2,q^\beta) = \bigoplus_{i=1}^{\frac{\beta-1}{2}} \left(\frac{q^{\beta-2i}-q^{\beta-2i-1}}{2}\right) K_{2q^i} \bigoplus K_{q^{(\beta-1)/2}}$. In second case, when $\beta \geq 6$ and $\beta \equiv 1 \pmod 2$, the number of distinct congruences is $\frac{q^{\beta-1}+q+2}{2(q+1)}$. We have

$$\eta^2 \equiv (q^i t)^2 \pmod{q^\beta},\ t = 1, 2, \cdots, \frac{q^{\beta-2i}-1}{2}, \tag{2.36}$$
$$\text{but} \quad t \neq ql,\ l = 1, 2, \cdots, \frac{q^{\beta-2i-1}-1}{2},\ i = 1, 2, \cdots, \frac{\beta-2}{2},$$
$$\eta^2 \equiv 0 \pmod{q^\beta}. \tag{2.37}$$

Zeroes of congruences (2.36) and (2.37) are $\bigcup_{i=1}^{\frac{\beta-2}{2}} \Big\{\{\eta = q^{\beta-i}(m-1) + q^i t|\ t = 1, 2, \cdots, \frac{q^{\beta-2i}-1}{2}$, but $t \neq ql,\ l = 1, 2, 3, \cdots, \frac{q^{\beta-2i-1}-1}{2},\ m = 1, 2, \cdots, q^i\} \bigcup \{\eta = q^\beta - q^{\beta-i}(m-1) - q^i t|\ t = 1, 2, \cdots, \frac{q^{\beta-2i}-1}{2}$, but $t \neq ql,\ l = 1, 2, \cdots, \frac{q^{\beta-2i-1}-1}{2},\ m = 1, 2, \cdots, q^i\}\Big\}$, and $\{\eta = q^{\frac{\beta}{2}}(m-1)|m = 1, 2, \cdots, q^{\frac{\beta}{2}}\}$, respectively. Thus, for $\beta \equiv 6$ with $\beta \equiv 0 \pmod 2$, $\overbrace{G}(2, q^\beta) = \bigoplus_{i=1}^{\frac{\beta-2}{2}} \Big(\frac{q^{\beta-2i}-q^{\beta-2i-1}}{2}\Big) K_{2q^i} \bigoplus K_{q^{\beta/2}}$.

Now, we assume the set of unit elements of Z_m. The graph $\overbrace{G}(2, m)$ contains $\phi(q^\beta) = q^{\beta-1}(q-1)$, vertices, where $\beta \geq 1$. We determine the least positive residue of the square of the integers which are less than and relatively prime with m. Because, there are $\phi(q^\beta) = q^{\beta-1}(q-1)$ squares to be found. By Theorem 1.1, the congruence $x^2 \equiv a \pmod{q^\beta}$ has either no solution or exactly two incongruent solutions. This implies, there are always $\frac{\phi(q^\beta)}{2} = \frac{(q^{\beta-1}(q-1))}{2}$ quadratic residues among all the vertices. Thus, $\overbrace{G}(2, q^\beta) = \frac{(q^{\beta-1}(q-1))}{2} K_2$. By combining both cases, we get the desired result. □

Quadratic graph for $m = 162$ is shown in Figure 3.

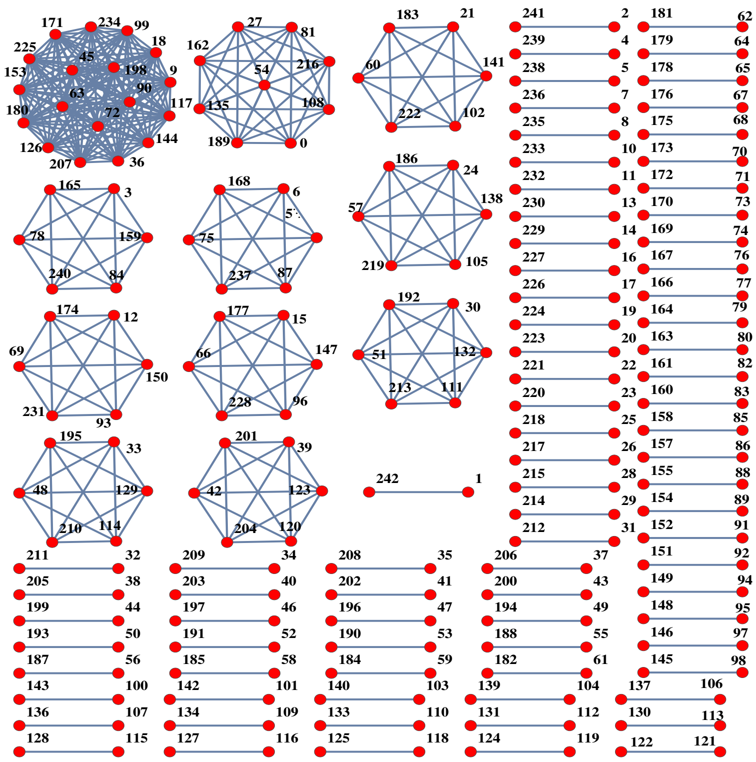

Figure 3. $\overbrace{G}(2, 243) = 9K_6 \bigoplus K_9 \bigoplus K_{18} \bigoplus 81K_2$.

3. Quadratic residues graphs over $Z_{2^\beta q}$, Z_{qp} and Z_{2q^β}

In this section, we characterize quadratic residues graphs for $n = 2^\beta q, 2q^\beta, qp$.

Theorem 3.1. *Let q be an odd prime. Then* $\overbrace{G}(2, 2^\beta q)$

$$
= \begin{cases}
2K_1 \bigoplus (q-1)K_2, & \text{if } \beta = 1, \\
2K_2 \bigoplus (q-1)K_4, & \text{if } \beta = 2, \\
2K_2 \bigoplus qK_4 \bigoplus \frac{q-1}{2}K_8, & \text{if } \beta = 3, \\
4K_4 \bigoplus 2(q-1)K_8, & \text{if } \beta = 4, \\
6K_4 \bigoplus (3q-2)K_8 \bigoplus \frac{q-1}{2}K_{16}, & \text{if } \beta = 5, \\
8K_4 \bigoplus 4qK_8 \bigoplus 2(q-1)K_{16}, & \text{if } \beta = 6, \\
16K_4 \bigoplus 2(4q-1)K_8 \bigoplus (3q-2)K_{16} \bigoplus \frac{q-1}{2}K_{32}, & \text{if } \beta = 7, \\
32K_4 \bigoplus 8(2q-1)K_8 \bigoplus 4qK_{16} \bigoplus 2(q-1)K_{32}, & \text{if } \beta = 8, \\
64K_4 \bigoplus 16(2q-1)K_8 \bigoplus (8q-2)K_{16} \bigoplus (3q-2)K_{32} \bigoplus \frac{q-1}{2}K_{64}, & \text{if } \beta = 9, \\
\bigoplus_{i=1}^{2} 2^{\beta-2i-1}K_{2^{i+1}} \bigoplus_{i=1}^{\frac{\beta-8}{2}} \left(2^{\beta-4-2i}q - 2^{\beta-5-2i}\right)K_{2^{3+i}} \bigoplus & \\
4qK_{2^{\frac{\beta}{2}}} \bigoplus 2(q-1)K_{2^{\frac{\beta+2}{2}}} \bigoplus \frac{2^{\beta-1}(q-1)}{8}K_8, & \text{if } \beta \geq 10, \\
\qquad \text{and } \beta \equiv 0 \ (mod\ 2), & \\
\bigoplus_{i=1}^{2} 2^{\beta-2i-1}K_{2^{i+1}} \bigoplus_{i=1}^{\frac{\beta-9}{2}} \left(2^{\beta-4-2i}q - 2^{\beta-5-2i}\right)K_{2^{3+i}} \bigoplus & \\
(8q-2)K_{2^{\frac{\beta-1}{2}}} \bigoplus (3q-2)K_{2^{\frac{\beta+1}{2}}} \bigoplus \frac{q-1}{2}K_{2^{\frac{\beta+3}{2}}} \bigoplus \frac{2^{\beta-1}(q-1)}{8}K_8, & \text{if } \beta \geq 11, \\
\qquad \text{and } \beta \equiv 1 \ (mod\ 2). &
\end{cases}
$$

Proof. Let $n = 2^\beta q$ be an integer, where q is an odd prime. For $\beta = 1$, $S = \{2m |\ m = 0, 1, 2, \cdots q - 1\} \bigcup \{q\}$ is set of zero-divisors of Z_{2q} including 0. There are $\frac{q+1}{2}$ distinct congruences. We have

$$\eta^2 \equiv 0 \ (\text{mod } 2q), \ \eta^2 \equiv q \ (\text{mod } 2q), \ \text{and } \eta^2 \equiv (2t)^2 \ (\text{mod } 2q), \quad t = 1, 2, \cdots, \frac{q-1}{2},$$

$\gamma = 0$, $\gamma = q$, and $\gamma = 2t$, $2q - 2t$, $t = 1, 2, 3, \cdots, \frac{q-1}{2}$ zeroes of congruences, respectively. When $\beta = 2$, there exit following congruences given as

$$\eta^2 \equiv 0 \ (\text{mod } 2^2 q), \ \eta^2 \equiv q \ (\text{mod } 2^2 q), \ \text{and } \eta^2 \equiv (2t)^2 \ (\text{mod } 2^2 q), \quad t = 1, 2, \cdots, \frac{q-1}{2}.$$

Sets of roots of these congruence are $\{0, 2q\}$, $\{q, 3q\}$, $\{2t, 2q - 2t\} \bigcup \{2^2 q - 2t, 2^2 q - 2q + 2t\}$, $t = 1, 2, \cdots, \frac{q-1}{2}$, respectively. There are $q + 2$ congruences for $\beta = 3$. They are $\eta^2 \equiv 0 \ (\text{mod } 2^3 q)$, $\eta^2 \equiv (2q)^2 \ (\text{mod } 2^3 q)$, $\eta^2 \equiv q^2 \ (\text{mod } 2^3 q)$, and $\eta^2 \equiv (2t)^2 \ (\text{mod } 2^3 q)$, $t = 1, 2, \cdots, q-1$. The zeroes of these congruences are $\{0, 2^2 q\}$, $\{2q, 2^3 q - 2q\}$, $\{q, 3q, 5q, 7q\}$ and $\{2t, 2^2 q - 2t\} \bigcup \{2^3 q - 2t, 2^3 q - 2^2 q + 2t\}$, $t = 1, 2, \cdots, q-1$, respectively. For $\beta = 4$, there are $q + 3$ congruences. We have

$$\eta^2 \equiv 0 \ (\text{mod } 2^4 q), \ \eta^2 \equiv q^2 \ (\text{mod } 2^4 q), \ \eta^2 \equiv (2q)^2 \ (\text{mod } 2^4 q), \ \eta^2 \equiv (3q)^2 \ (\text{mod } 2^4 q),$$

$$\textit{and} \quad \eta^2 \equiv (2t)^2 \ (\text{mod } 2^4 q), \quad t = 1, 2, \cdots, q-1,$$

zeroes are $\{0, 2^2q, 2^3q, 2^2q + 2^3q\}$, $\{q, 7p, 9p, 15q\}$, $\{2q, 6q, 10q, 14q\}$, $\{3q, 5q, 11q, 13q\}$ and $\{2t, 2^2q - 2t, 2^2q + 2t, 2^3q - 2t\} \bigcup \{2^4q - 2t, 2^4q - 2^2q + 2t, 2^4q - 2^2q - 2t, 2^4q - 2^3q + 2t\}$, $\quad t = 1, 2, \cdots, q - 1$, respectively.

For $\beta = 5$, there are $(3q + 11)/2$ congruences

$$\begin{aligned}
\eta^2 &\equiv 0 \pmod{2^5q}, && (3.1)\\
\eta^2 &\equiv (qt)^2 \pmod{2^5q}, t = 1, 3, 5, 7, && (3.2)\\
\eta^2 &\equiv (4q)^2 \pmod{2^5q}, && (3.3)\\
\eta^2 &\equiv (2q)^2 \pmod{2^5q}, && (3.4)\\
\eta^2 &\equiv (2t)^2 \pmod{2^5q}, \quad t = 2i, \ i = 1, 2, 3, \cdots, q - 1, && (3.5)\\
\eta^2 &\equiv (2t)^2 \pmod{2^5q}, \quad t = 2i - 1, \ i = 1, 2, 3, \cdots, \frac{q-1}{2}. && (3.6)
\end{aligned}$$

Zeroes of congruences (3.1)−(3.6) are $\{0, 2^3q, 2^4q, 2^3q+2^4q\}$, $\bigcup_{t\in\{1,3,5,7\}} \{qt, 2^4q+qt\} \bigcup \{2^5q-qt, 2^5q-2^4q-qt\}$, $\{4q, 12q, 20q, 28q\}$, $\{2q, 6q, 10q, 14q, 18q, 22q, 26q, 30q\}$, $\bigcup_{\substack{t=2l,\\ 1\le l\le q-1}} \Big\{\{2^3qj+2t, \ \ j = 0, 1, 2, \cdots, 2^2-1\} \cup \{2^5q-2^3qj-2t, \ \ j = 0, 1, 2, \cdots, 2^2-1\}\Big\}$, $\bigcup_{l=i}^{\frac{q-1}{2}} \Big\{\{2^2qi+4l-2, \ \ i = 0, 1, 2, \cdots, 2^3-1\} \cup \{2^5q-2^2qi-4l+2, \ \ i = 0, 1, 2, \cdots, 2^3-1\}\Big\}$.

For $\beta = 6$, we have

$$\begin{aligned}
\eta^2 &\equiv (qt)^2 \pmod{2^6q}, \quad t = 2i - 1, \ i = 1, 2, 3, \cdots, 2^3, && (3.7)\\
\eta^2 &\equiv (2t)^2 \pmod{2^6q}, \quad t = qi, \ i = 0, 1, 2, 3, && (3.8)\\
\eta^2 &\equiv (2t)^2 \pmod{2^6q}, \quad t = 4l, \ \ l = 1, 2 \cdots \frac{q-1}{2}, && (3.9)\\
\eta^2 &\equiv (2)^2 \pmod{2^6q}, \quad t = 3q - 4i - qj, \ i = 1, 2, 3, \cdots, \lfloor\frac{3q}{4}\rfloor, \\
& \qquad j = 0, 1, 2, \cdots, \lfloor\frac{3q-4i}{q}\rfloor. && (3.10)
\end{aligned}$$

Solution sets of congruences (3.7)−(3.10) are

$$\begin{aligned}
&\bigcup_{t=2i-1, i=1}^{2^3} \{qt, 2^5q + qt\} \bigcup \{2^6q - qt, 2^6q - 2^5q - qt\},\\
&\{0, 2^3q, 2^4q, 2^5q, 2^3q + 2^4q, 2^3q + 2^5q, 2^4q + 2^5q, 2^3q + 2^4q + 2^5q\},
\end{aligned}$$

$$\begin{aligned}
&\bigcup_{t=qi, i=1}^{3} \{2t, 2^4t - 2t, 2^4t + 2t, 2^5t - 2t\} \bigcup \{2^6q - 2t, 2^6q - 2^4t\\
&+ \ 2t, 2^6q - 2^4t - 2t, 2^6q - 2^5t + 2t\},
\end{aligned}$$

$$\bigcup_{t=4l,l=1}^{\frac{q-1}{2}} \{2^3qj+2t,\ j=0,1,2,\cdots,2^3-1\}\bigcup\{2^6q-2^3qj-2t,\ j=0,1,2,\cdots,2^3-1\},$$

$$\bigcup_{1\le i\le\lfloor\frac{3q}{4}\rfloor,j=0}^{\lfloor\frac{3q-4i}{q}\rfloor} \{2^4ql+6p-8i-2qj,\ l=0,1,\cdots,2^2-1\}\bigcup\{2^6q-2^4ql-6q+8i+2qj,\ l=0,1,\cdots,2^2-1\}$$
$$\bigcup\{2^4ql+2^4+6q-8i-2qj,\ l=0,1,\cdots,2^2-1\}\bigcup\{2^6q-2^4ql-2^4-6q+8i+2qj,\ l=0,1,\cdots,2^2-1\}.$$

For $\beta=7$, we obtain

$$\eta^2 \equiv (qt)^2 \pmod{2^7q},\quad t=2i-1,\ i=1,2,3,\cdots,2^4, \tag{3.11}$$
$$\eta^2 \equiv (2t)^2 \pmod{2^7q},\quad t=qi,\ i=0,1,3,4,5,7, \tag{3.12}$$
$$\eta^2 \equiv (2t)^2 \pmod{2^7q},\quad t=2q, \tag{3.13}$$
$$\eta^2 \equiv (2t)^2 \pmod{2^7q},\quad t=4l,\ \ l=1,2,\cdots,q-1, \tag{3.14}$$
$$\eta^2 \equiv (2t)^2 \pmod{2^7q},\quad t=7q-8i-2qj,\ i=1,2,3,\cdots,\lfloor\frac{7q}{8}\rfloor,\quad j=0,1,2,\cdots,\lfloor\frac{7q-8i}{2q}\rfloor, \tag{3.15}$$
$$\eta^2 \equiv (2t)^2 \pmod{2^7q},\quad t=4l-2,\ \ l=1,2\cdots\frac{q-1}{2}. \tag{3.16}$$

Zeroes of congruences (3.11)−(3.16) are

$$\bigcup_{t=2i-1,i=1}^{2^4} \{qt,2^6q+qt\}\bigcup\{2^7q-qt,2^7q-2^6q-qt\},$$
$$\{0,2^4q,2^5q,2^6q,2^4q+2^5q,2^4q+2^6q,2^5q+2^6q,2^4q+2^5q+2^6q\},$$

$$\bigcup_{\substack{t=qi,\\ i\in\{1,3,4,5,7\}}} \Big\{\{2t,2^5t-2t,2^5t+2t,2^6t-2t\}\cup\{2^7q-2t,2^7q-2^5t+2t,2^7q-2^5t-2t,2^7q-2^6t+2t\}\Big\},\{2^3qj+2t,t=2q,\ j=0,1,2,\cdots,2^4-1\},$$

$$\bigcup_{\substack{t=4l,\\ 1\le l\le q-1}} \Big\{\{2^4qj+2t,\ \ j=0,1,2,\cdots,2^3-1\}\cup\{2^7q-2^4qj-2t,\ \ j=0,1,2,\cdots,2^3-1\}\Big\},$$

$$\bigcup_{1\le i\le\lfloor\frac{7q}{8}\rfloor,\ j=0}^{\lfloor\frac{7q-8i}{2q}\rfloor} \Big\{\{2^5ql+14q-16i-4qj,\ \ l=0,1,2,\cdots,2^2-1\}\cup\{2^7q-2^5ql-14q+16i+4qj, l=0,1,2,\cdots,2^2-1\}\bigcup\{2^5ql+2^3q+2^2q+14q-16i-4qj,\ \ l=0,1,2,\cdots,2^2-1\}\cup\{2^7q-2^5ql-2^3q-2^2q-14q+16i+4qj,\ \ l=0,1,2,\cdots,2^2-1\}\Big\},$$

$$\bigcup_{l=i}^{\frac{q-1}{2}} \Big\{\{2^3qi+8l-4,\ \ i=0,1,2,\cdots,2^4-1\}\cup\{2^7q-2^3qi-8l+4,\ i=0,1,2,\cdots,2^4-1\}\Big\}.$$

For $\beta=8$, the following congruence equations turning out to be

$$\eta^2 \equiv (qt)^2 \pmod{2^8q},\ \ t=2i-1,\ i=1,2,3,\cdots,2^5, \tag{3.17}$$

$$\eta^2 \equiv (2t)^2 \pmod{2^8q},\ \ t=pl,\ \ l=2i-1\ i=1,2,3,\cdots,2^3, \tag{3.18}$$

$$\eta^2 \equiv (2t)^2 \pmod{2^8q},\ \ t=2qi,\ \ i=0,1,2,,\cdots,2^2-1, \tag{3.19}$$

$$\eta^2 \equiv (2t)^2 \pmod{2^8q},\ \ t=15q-16i-2qj,\ i=1,2,3,\cdots,\lfloor\frac{15q}{16}\rfloor,\ j=0,1,2,\cdots,\lfloor\frac{15q-16i}{2q}\rfloor, \tag{3.20}$$

$$\eta^2 \equiv (2t)^2 \pmod{2^8q},\ \ t=8l,\ \ l=1,2\cdots\frac{q-1}{2}, \tag{3.21}$$

$$\eta^2 \equiv (2t)^2 \pmod{2^8q},\ \ t=6q-8i-2qj,\ i=1,2,3,\cdots,\lfloor\frac{6q}{8}\rfloor,\ j=0,1,2,\cdots,\lfloor\frac{6q-8i}{2q}\rfloor. \tag{3.22}$$

Zeroes of congruences (3.17)−(3.22) are

$$\bigcup_{t=2i-1,i=1}^{2^5} \{qt,2^7q+qt\}\bigcup\{2^8q-qt,2^8q-2^7q-qt\},$$

$$\bigcup_{t=q(2i-1),i=1}^{2^3} \{2^6qj+2t,\ \ j=0,1,2,\cdots,2^2-1\}\bigcup\{2^8q-2^6qj-2t, j=0,1,2,\cdots,2^2-1\},$$

$$\bigcup_{k=2qi,\ i=1}^{3} \{2^5qj+2t,\ \ j=0,1,2,\cdots,2^3-1\}\bigcup\{2^8q-2^5qj-2t, j=0,1,2,\cdots,2^3-1\},\{2^4qj,\ \ j=0,1,2,\cdots,2^4-1\},$$

$$\bigcup_{1\le i\le\lfloor 15q/16\rfloor,\ j=0}^{\lfloor(15q-16i)/2q\rfloor} \Big\{\{2^6ql+30q-32i-4qj,\ l=0,1,\cdots,2^2-1\}\cup\{2^8q-2^6ql -30q+32i+4qj, l=0,1,\cdots,2^2-1\}\bigcup\{2^6ql+2^3q+2^2q+30q-32i-4qj,\ l=0,1,\cdots,2^2-1\}\bigcup\{2^8q-2^6ql-2^3q-2^2q-30q+32i+4qj,\ l=0,1,\cdots,2^2-1\}\Big\},$$

$$\bigcup_{1\le i\le\lfloor 15q/16\rfloor,\ j=0}^{\lfloor(15q-16i)/2q\rfloor} \Big\{\{2^6ql+30q-32i-4qj,\ l=0,1,\cdots,2^2-1\}\cup\{2^8q-2^6ql-30q+32i+4qj, l=0,1,\cdots,2^2-1\}\bigcup\{2^6ql+2^3q+2^2q+30q-32i-4qj,\ l=0,1,\cdots,2^2-1\}\cup\{2^8q-2^6ql-2^3q-2^2q-30q+32i+4qj,\ l=0,1,\cdots,2^2-1\}\Big\},$$

$$\bigcup_{t=8l,\ l=1}^{(q-1)/2} \{2^4qj+2t,\ j=0,1,2,\cdots,2^4-1\}\bigcup\{2^8q-2^4qj-2t, j=0,1,2,\cdots,2^4-1\},$$

$$\bigcup_{1\le i\le\lfloor 6q/8\rfloor,\ j=0}^{\lfloor(6q-8i)/2q\rfloor} \Big\{\{2^4ql+12q-16i-4qj, l=0,1,2,\cdots,2^4-1\}\cup\{2^8q-2^4ql-12q+16i+4qj, l=0,1,2,\cdots,2^4-1\}\Big\}.$$

For $\beta=9$, we have

$$\eta^2 \equiv (qt)^2 \ (\mathrm{mod}\ 2^9q),\ \ t=2i-1,\ i=1,2,3,\cdots,2^6, \tag{3.23}$$

$$\eta^2 \equiv (2t)^2 \ (\mathrm{mod}\ 2^9q),\ \ t=ql,\ \ l=2i-1\ i=1,2,3,\cdots,2^4, \tag{3.24}$$

$$\eta^2 \equiv (2t)^2 \ (\mathrm{mod}\ 2^9q),\ \ t=2qi,\ i=0,1,3,4,5,7, \tag{3.25}$$

$$\eta^2 \equiv (2t)^2 \ (\mathrm{mod}\ 2^9q),\ \ t=31q-32i-2qj,\ i=1,2,3,\cdots,\lfloor\frac{31p}{32}\rfloor, \quad j=0,1,3,\cdots,\lfloor\frac{31q-32i}{2q}\rfloor, \tag{3.26}$$

$$\eta^2 \equiv (2t)^2 \ (\mathrm{mod}\ 2^9q),\ \ t=4q, \tag{3.27}$$

$$\eta^2 \equiv (2t)^2 \ (\mathrm{mod}\ 2^9q),\ \ t=8l,\ \ l=1,2,\cdots,q-1, \tag{3.28}$$

$$\eta^2 \equiv (2t)^2 \ (\mathrm{mod}\ 2^9 q), \quad t = 14q - 16i - 4qj, \ i = 1,2,3,\cdots,\lfloor\frac{14q}{16}\rfloor,$$
$$j = 0,1,3,\cdots,\lfloor\frac{14q-16i}{4q}\rfloor, \tag{3.29}$$
$$\eta^2 \equiv (2t)^2 \ (\mathrm{mod}\ 2^9 q), \quad t = 8l - 4, \quad l = 1,2\cdots\frac{q-1}{2}. \tag{3.30}$$

Sets of solution of congruences (3.23)−(3.30) are

$$\bigcup_{t=2i-1,i=1}^{2^6} \{qt, 2^8 q + qt\} \bigcup \{2^9 q - qt, 2^9 q - 2^8 q - qt\},$$

$$\bigcup_{t=q(2i-1),\ i=1}^{2^4} \{2^7 qj + 2t, \quad j = 0,1,2,\cdots,2^2-1\} \bigcup \{2^9 q - 2^7 qj - 2t, j = 0,$$
$$1,2,\cdots,2^2-1\},$$

$$\bigcup_{\substack{t=2qi,\\ i\in\{1,3,4,5,7\}}} \{2^6 qj + 2t, \quad j = 0,1,2,\cdots,2^3-1\} \bigcup \{2^9 q - 2^6 qj - 2t, j = 0,$$
$$1,2,\cdots,2^3-1\}, \{2^5 qj, \quad j = 0,1,2,\cdots,2^4-1\},$$

$$\bigcup_{1\le i\le\lfloor 31p/32\rfloor,\ j=0}^{\lfloor(31q-32i)/2q\rfloor} \Big\{\{2^7 ql + 62q - 64i - 4qj, \quad l = 0,1,2,\cdots,2^2-1\} \cup \{2^9 q - \tag{3.31}$$
$$2^7 ql - 62q + 64i + 4qj, l = 0,1,2,\cdots,2^2-1\} \bigcup \{2^7 ql + 2^6 q + \tag{3.32}$$
$$2^3 q + 2^2 q + 62q - 64i - 4qj, \quad l = 0,1,2,\cdots,2^2-1\} \cup \{2^9 q - \tag{3.33}$$
$$2^7 ql - 2^6 q - 2^3 q - 2^2 q - 62q + 64i + 4qj, \quad l = 0,1,2,\cdots,2^2-1\}\Big\}, \tag{3.34}$$

$$\bigcup_{\substack{t=8l,\\ 1\le l\le q-1}} \Big\{\{2^5 qj + 2t, \quad j = 0,1,2,\cdots,2^4-1\} \cup \{2^9 q - 2^5 qj - 2t, j = 0, \tag{3.35}$$
$$1,2,\cdots,2^4-1\}\Big\}, \{2^4 qj + 2t, t = 4q, \ j = 0,1,2,\cdots,2^5-1\}, \tag{3.36}$$

$$\bigcup_{1\le i\le\lfloor 14q/16\rfloor,\ j=0}^{\lfloor(14q-16i)/4q\rfloor} \Big\{\{2^6 ql + 28q - 32i - 8qj, \ l = 0,1,2,\cdots,2^3-1\} \cup \{2^9 q - 2^6 ql$$
$$-28q + 32i + 8qj, l = 0,1,2,\cdots,2^3-1\} \bigcup \{2^6 ql + 2^4 q + 2^3 q +$$
$$28q - 32i - 8qj, \ l = 0,1,2,\cdots,2^3-1\} \cup \{2^9 q - 2^6 ql - 2^4 q -$$
$$2^3 q - 28q + 32i + 8qj, l = 0,1,2,\cdots,2^3-1\}\Big\},$$

$$\bigcup_{l=1}^{(q-1)/2} \Big\{\{2^4qi+16l-8,\ \ i=0,1,2,\cdots,2^5-1\}\cup\{2^9q-2^4qi-16l+8, i=0,1,2,\cdots,2^5-1\}\Big\}.$$

For $\beta=10$, we get

$$\eta^2 \equiv (qt)^2 \pmod{2^{10}q},\ \ t=2i-1,\ i=1,2,3,\cdots,2^7, \tag{3.37}$$
$$\eta^2 \equiv (2t)^2 \pmod{2^{10}q},\ \ t=ql,\ \ l=2i-1\ i=1,2,3,\cdots,2^5, \tag{3.38}$$
$$\eta^2 \equiv (2t)^2 \pmod{2^{10}q},\ \ t=2qi,\ i=2j-1,\ \ j=1,3,\cdots,2^3, \tag{3.39}$$
$$\eta^2 \equiv (2t)^2 \pmod{2^{10}q},\ \ t=63q-64i-2qj,\ i=1,2,3,\cdots,\lfloor\frac{63q}{64}\rfloor,\ j=0,1,3,\cdots,\lfloor\frac{63q-64i}{2q}\rfloor, \tag{3.40}$$
$$\eta^2 \equiv (2t)^2 \pmod{2^{10}q},\ \ t=4qi,\ \ i=0,1,2,,\cdots,2^2-1, \tag{3.41}$$
$$\eta^2 \equiv (2t)^2 \pmod{2^{10}q},\ \ t=30q-32i-4qj,\ i=1,2,3,\cdots,\lfloor\frac{30q}{32}\rfloor,\ j=0,1,3,\cdots,\lfloor\frac{30q-32i}{4q}\rfloor, \tag{3.42}$$
$$\eta^2 \equiv (2t)^2 \pmod{2^{10}q},\ \ t=16l,\ \ l=1,2\cdots\frac{q-1}{2}, \tag{3.43}$$
$$\eta^2 \equiv (2t)^2 \pmod{2^{10}q},\ \ t=12q-16i-4qj,\ i=1,2,3,\cdots,\lfloor\frac{12q}{16}\rfloor,\ j=0,1,3,\cdots,\lfloor\frac{12q-16i}{4q}\rfloor. \tag{3.44}$$

Zeroes of congruences (3.37)−(3.44) are

$$\bigcup_{t=2i-1,\ i=1}^{2^7} \{qt, 2^9+qt\}\bigcup\{2^{10}q-qt, 2^{10}q-2^9q-qt\},$$

$$\bigcup_{t=q(2i-1),i=1}^{2^5} \{2^8qj+2t,\ \ j=0,1,2,\cdots,2^2-1\}\bigcup\{2^{10}q-2^8qj-2t, j=0,1,2,\cdots,2^2-1\},$$

$$\bigcup_{t=2q(2i-1),i=1}^{2^3} \{2^7qj+2t,\ \ j=0,1,2,\cdots,2^3-1\}\bigcup\{2^{10}q-2^7qj-2t, j=0,1,2,\cdots,2^3-1\},$$

$$\bigcup_{t=4qi,\ i=1}^{3} \{2^6qj+2t,\ \ j=0,1,2,\cdots,2^4-1\}\bigcup\{2^{10}q-2^6qj-2t, j=0,$$
$$1,2,\cdots,2^4-1\},\{2^5qj,\ \ j=0,1,2,\cdots,2^5-1\},$$

$$\bigcup_{1\le i\le\lfloor 30q/32\rfloor,\ j=0}^{\lfloor(30q-32i)/4q\rfloor} \Big\{\{2^7qt+60q-64i-8qj,\ \ l=0,1,2,\cdots,2^3-1\}\cup\{2^{10}q-$$
$$2^7ql-60q+64i+8qj, l=0,1,2,\cdots,2^3-1\}\bigcup\{2^7ql+2^4q+$$
$$2^3q+60q-64i-8qj,\ \ l=0,1,2,\cdots,2^3-1\}\cup\{2^{10}q-2^7ql-$$
$$2^4q-2^3q-60q+64i+8qj, l=0,1,2,\cdots,2^3-1\}\Big\},$$

$$\bigcup_{t=16l, l=1}^{(q-1)/2} \{2^5qj+2t,\ j=0,1,2,\cdots,2^5-1\}\bigcup\{2^{10}q-2^5qj-2t, j=0,$$
$$1,2,\cdots,2^5-1\},$$

$$\bigcup_{1\le i\le\lfloor 12q/16\rfloor,\ j=0}^{\lfloor(12q-16i)/4q\rfloor} \Big\{\{2^5ql+24q-32i-8qj, l=0,1,2,\cdots,2^5-1\}\cup\{2^{10}q-2^5ql$$
$$-24q+32i+8qj, l=0,1,2,\cdots,2^5-1\}\Big\}.$$

For $\beta=11$, we obtain

$$\eta^2 \equiv (qt)^2 \pmod{2^{11}q},\ \ t=2i-1,\ i=1,2,3,\cdots,2^8, \tag{3.45}$$
$$\eta^2 \equiv (2t)^2 \pmod{2^{11}q},\ \ t=ql,\ \ l=2i-1\ i=1,2,3,\cdots,2^6, \tag{3.46}$$
$$\eta^2 \equiv (2t)^2 \pmod{2^{11}q},\ \ t=2qi,\ i=2j-1,\ \ j=1,3,\cdots,2^4, \tag{3.47}$$
$$\eta^2 \equiv (2t)^2 \pmod{2^{11}q},\ \ t=127q-128i-2qj,\ i=1,2,3,\cdots,\lfloor\frac{127q}{128}\rfloor,$$
$$j=0,1,3,\cdots,\lfloor\frac{127q-128i}{2q}\rfloor, \tag{3.48}$$
$$\eta^2 \equiv (2t)^2 \pmod{2^{11}q},\ \ t=4qi,\ \ i=0,1,3,4,5,7, \tag{3.49}$$

$$\eta^2 \equiv (2t)^2 \pmod{2^{11}q},\ \ t=62q-64i-4qj,\ i=1,2,3,\cdots,\lfloor\frac{62q}{64}\rfloor,$$
$$j=0,1,3,\cdots,\lfloor\frac{62q-64i}{4q}\rfloor, \tag{3.50}$$
$$\eta^2 \equiv (2t)^2 \pmod{2^{11}q},\ \ t=8q, \tag{3.51}$$
$$\eta^2 \equiv (2t)^2 \pmod{2^{11}q},\ \ t=16l,\ \ l=1,2,\cdots,q-1, \tag{3.52}$$

$$\eta^2 \equiv (2t)^2 \pmod{2^{11}q}, \quad t = 28q - 32i - 8qj, \ i = 1, 2, 3, \cdots, \lfloor \frac{28q}{32} \rfloor,$$
$$j = 0, 1, 3, \cdots, \lfloor \frac{28q - 32i}{8q} \rfloor, \tag{3.53}$$
$$\eta^2 \equiv (2t)^2 \pmod{2^{11}q}, \quad t = 16l - 8, \quad l = 1, 2 \cdots \frac{q-1}{2}. \tag{3.54}$$

Sets of solution of congruences (3.45)−(3.54) are

$$\bigcup_{t=2q(2i-1),i=1}^{2^4} \{2^8qj + 2t, \quad j = 0, 1, 2, \cdots, 2^3 - 1\} \bigcup \{2^{11}q - 2^8qj - 2t, j = 0, 1, 2, \cdots, 2^3 - 1\},$$

$$\bigcup_{1\le i\le\lfloor 127q/128\rfloor,\ j=0}^{\lfloor(127q-128i)/2q\rfloor} \Big\{\{2^9qt + 254q - 256i - 4qj, \quad t = 0, 1, \cdots, 2^2 - 1\} \cup \{2^{11}q - 2^9ql - 254q + 256i + 4qj, l = 0, 1, \cdots, 2^2 - 1\} \bigcup \{2^9ql + 2^7q + 2^6q + 2^3q + 2^2q + 254q - 256i - 4qj, \quad l = 0, 1, \cdots, 2^2 - 1\} \cup \{2^{11}q - 2^9ql - 2^7q - 2^6q - 2^3q - 2^2q - 254q + 256i + 4qj, l = 0, 1, \cdots, 2^2 - 1\}\Big\},$$

$$\bigcup_{\substack{t=2qi,\\ i\in\{1,3,4,5,7\}}} \{2^7qj + 2t, \quad j = 0, 1, 2, \cdots, 2^4 - 1\} \bigcup \{2^{11}q - 2^7qj - 2t, j = 0, 1, 2, \cdots, 2^4 - 1\}, \{2^6qj, \quad j = 0, 1, 2, \cdots, 2^5 - 1\},$$

$$\bigcup_{1\le i\le\lfloor 62q/64\rfloor,\ j=0}^{\lfloor(62q-64i)/4q\rfloor} \Big\{\{2^8ql + 124q - 128i - 8qj, \quad l = 0, 1, 2, \cdots, 2^3 - 1\} \cup \{2^{11}q - 2^8ql - 124q + 128i + 8qj, l = 0, 1, 2, \cdots, 2^3 - 1\} \bigcup \{2^8ql + 2^7q + 2^4q + 2^3q + 124q - 128i - 8qj, \quad l = 0, 1, 2, \cdots, 2^3 - 1\} \cup \{2^{11}q - 2^8ql - 2^7q - 2^4q - 2^3q - 124q + 128i + 8qj, \quad l = 0, 1, 2, \cdots, 2^3 - 1\}\Big\},$$

$$\bigcup_{\substack{t=16l,\\ 1\le l\le q-1}} \Big\{\{2^6qj + 2t, \quad j = 0, 1, 2, \cdots, 2^5 - 1\} \cup \{2^{11}q - 2^6qj - 2t, j = 0, 1, 2, \cdots, 2^5 - 1\}\Big\}, \{2^5qj + 2t, t = 8q, \ j = 0, 1, 2, \cdots, 2^6 - 1\},$$

$$\bigcup_{1\le i\le\lfloor 28q/32\rfloor,\ j=0}^{\lfloor(14q-16i)/8q\rfloor} \Big\{\{2^7ql+56q-64i-16qj,\ l=0,1,2,\cdots,2^4-1\}\cup\{2^{11}q-2^7ql -56q+64i+16qj, l=0,1,2,\cdots,2^4-1\}\bigcup\{2^7ql+2^5q+2^4q+ 56q-64i-16qj,\ l=0,1,2,\cdots,2^4-1\}\cup\{2^{11}q-2^7ql-2^5q- 2^4q-56q+64i+16qj, l=0,1,2,\cdots,2^4-1\}\Big\},$$

$$\bigcup_{l=1}^{(q-1)/2} \Big\{\{2^5qi+32l-16,\ \ i=0,1,2,\cdots,2^6-1\}\cup\{2^{11}q-2^5qi-32l+ 16, i=0,1,2,\cdots,2^6-1\}\Big\}.$$

For $\beta\ge 10$ with $\beta\equiv 0 \pmod 2$, there are $2^{\beta-5}5+\frac{(2^{\beta-8}-1)2^{\beta-5}}{3\cdot 2^{\beta-8}}(2q-1)+6q-2$ congruences. We have

$$\eta^2 \equiv (pt)^2 \pmod{2^\beta q},\ \ t=2i-1,\ i=1,2,3,\cdots,2^{\beta-3}, \tag{3.55}$$

$$\eta^2 \equiv (2t)^2 \pmod{2^\beta q},\ \ t=ql,\ \ l=2i-1\ i=1,2,3,\cdots,2^{\beta-5}. \tag{3.56}$$

For $\nu=1,2,3,\cdots,\frac{\beta-8}{2}$,

$$\eta^2 \equiv (2t)^2 \pmod{2^\beta q},\ \ t=2^\nu qi,\ i=2j-1,\ \ j=1,3,\cdots,2^{\beta-7}, \tag{3.57}$$

$$\eta^2 \equiv (2t)^2 \pmod{2^\beta q},\ \ t=\left(2^{\beta-5+\nu}-2^{\nu-1}\right)q-(2^{\beta-5+\nu})i-2^\nu qj,$$
$$i=1,2,\cdots,\lfloor\frac{2^{\beta-5+\nu}-2^{\nu-1}q}{2^{\beta-5+\nu}}\rfloor,$$
$$j=0,1,\cdots,\lfloor\frac{(2^{\beta-5+\nu}-2^{\nu-1})q-(2^{\beta-5+\nu})i}{2^\nu q}\rfloor, \tag{3.58}$$

$$\eta^2 \equiv (2t)^2 \pmod{2^\beta q},\ \ t=2^{\frac{\beta-6}{2}}qi,\ \ i=0,1,2,,\cdots,2^2-1, \tag{3.59}$$

$$\eta^2 \equiv (2t)^2 \pmod{2^\beta q},\ \ t=(2^{\frac{\beta}{2}}-2^{\frac{\beta-8}{2}})q-2^{\frac{\beta}{2}}i-2^{\frac{\beta-6}{2}}qj,$$
$$i=1,2,3,\cdots,\lfloor\frac{(2^{\frac{\beta}{2}}-2^{\frac{\beta-8}{2}})q}{2^{\frac{\beta}{2}}}\rfloor,$$
$$j=0,1,2,\cdots,\lfloor\frac{(2^{\frac{\beta}{2}}-2^{\frac{\beta-8}{2}})q-2^{\frac{\beta}{2}}i}{2^{\frac{\beta-6}{2}}q}\rfloor, \tag{3.60}$$

$$\eta^2 \equiv (2t)^2 \pmod{2^\beta q},\ \ t=2^{\frac{\beta-2}{2}}l,\ \ l=1,2\cdots\frac{q-1}{2}, \tag{3.61}$$

$$\eta^2 \equiv (2t)^2 \pmod{2^\beta q},\ \ t=(2^{\frac{\beta-2}{2}}-2^{\frac{\beta-6}{2}})q-2^{\frac{\beta-2}{2}}i-2^{\frac{\beta-6}{2}}qj,$$
$$i=1,2,3,\cdots,\lfloor\frac{(2^{\frac{\beta-2}{2}}-2^{\frac{\beta-6}{2}})q}{2^{\frac{\beta-2}{2}}}\rfloor,$$
$$j=0,1,2,\cdots,\lfloor\frac{(2^{\frac{\beta-2}{2}}-2^{\frac{\beta-6}{2}})q-2^{\frac{\beta-2}{2}}i}{2^{\frac{\beta-6}{2}}q}\rfloor. \tag{3.62}$$

Zeroes of congruences (3.55)−(3.62) are

$$\bigcup_{t=2i-1,\ i=1}^{2^{\beta-3}} \{qt,2^{\beta-1}q+qt\}\bigcup\{2^\beta q-qt,2^{\beta-1}q-2^{\beta-1}q-qt\},$$

$$\bigcup_{t=q(2i-1),i=1}^{2^{\beta-5}} \{2^{\beta-2}qj+2t,\quad j=0,1,2,\cdots,2^2-1\}\bigcup\{2^{\beta}q-2^{\beta-2}qj-2t, j=0, 1,2,\cdots,2^2-1\},$$

For $v=1,2,3,\cdots,\frac{\beta-8}{2}$,

$$\bigcup_{t=2^{v}q(2i-1),i=1}^{2^{\beta-7}}\ \bigcup_{v=1}^{(\beta-8)/2} \{2^{\beta-2-v}qj+2t,\quad j=0,1,2,\cdots,2^3-1\}\bigcup\{2^{\beta}q-2^{\beta-2-v}qj -2t, j=0,1,2,\cdots,2^3-1\},$$

$$\bigcup_{1\le i\le\lfloor A\rfloor,\ j=0}^{\lfloor B\rfloor}\ \bigcup_{v=1}^{(\beta-8)/2} \Big\{\{2^{\beta-1-v}ql+2t,\quad l=0,1,\cdots,2^2-1\}\cup\{2^{\beta}q-2^{\beta-1-v}ql -2t, l=0,1,\cdots,2^2-1\}\bigcup\{2^{\beta-1-v}qk+2^{v-1}C+2t, l=0, 1,\cdots,2^2-1\}\cup\{2^{\beta}q-2^{\beta-1-v}ql-2^{v-1}C-2t, l=0,1,\cdots, 2^2-1\}\Big\},$$

where $A=\lfloor\frac{2^{\beta-5+v}-2^{v-1}q}{2^{\beta-5+v}}\rfloor$, $B=\lfloor\frac{(2^{\beta-5+v}-2^{v-1})q-(2^{\beta-5+v})i}{2^{v}q}\rfloor$, $C=2^7q+2^6q+2^3q+2^2q$ and $t=\Big(2^{\beta-5+v}-2^{v-1}\Big)q-(2^{\beta-5+v})i-2^{v}qj$.

$$\bigcup_{t=2^{\frac{\beta-6}{2}}qi,\ i=1}^{3} \{2^{\frac{\beta+2}{2}}qj+2t,\quad j=0,1,2,\cdots,2^{\frac{\beta-2}{2}}-1\}\bigcup\{2^{\beta}q-2^{\frac{\beta+2}{2}}qj-2t, j= 0,1,2,\cdots,2^{\frac{\beta-2}{2}}-1\},\{2^{\frac{\beta}{2}}qj,\quad j=0,1,2,\cdots,2^{\frac{\beta}{2}}-1\},$$

$$\bigcup_{1\le i\le\lfloor D\rfloor,\ j=0}^{\lfloor E\rfloor} \Big\{\{2^{\frac{\beta+4}{2}}ql+2t,\quad l=0,1,2,\cdots,2^{\frac{\beta-4}{2}}-1\}\cup\{2^{\beta}q-2^{\frac{\beta+4}{2}}ql-2t,\quad l= 0,1,2,\cdots,2^{\frac{\beta-4}{2}}-1\}\bigcup\{2^{\frac{\beta+4}{2}}ql+2^{(\beta-10)/2}F+2t,\ l=0,1,2,\cdots, 2^{\frac{\beta-4}{2}}-1\}\cup\{2^{\beta}q--2^{(\beta-10)/2}F-2t, l=0,1,2,\cdots,2^{\frac{\beta-4}{2}}-1\}\Big\},$$

where $D=\frac{(2^{\frac{\beta}{2}}-2^{\frac{\beta-8}{2}})q}{2^{\frac{\beta}{2}}}$, $E=\frac{(2^{\frac{\beta}{2}}-2^{\frac{\beta-8}{2}})q-2^{\frac{\beta}{2}}i}{2^{\frac{\beta-6}{2}}q}$, $F=2^4q+2^3q$ and $t=(2^{\frac{\beta}{2}}-2^{\frac{\beta-8}{2}})q-2^{\frac{\beta}{2}}i-2^{\frac{\beta-6}{2}}qj$.

$$\bigcup_{t=2^{(\beta-2)/2}l,l=1}^{(q-1)/2} \{2^{\beta/2}qj+2t,\ j=0,1,2,\cdots,2^{\beta/2}-1\}\bigcup\{2^{\beta}q-2^{\beta/2}qj-2t, j= 0,1,2,\cdots,2^{\beta/2}-1\},$$

$$\bigcup_{1\le i\le\lfloor G\rfloor,\ j=0}^{\lfloor H\rfloor} \Big\{\{2^{\beta/2}ql+2t,\quad l=0,1,2,\cdots,2^{\beta/2}-1\}\cup\{2^{\beta}q-2^{\beta}ql-2t,\quad l= 0,1,2,\cdots,2^{\beta/2}-1\}\Big\},$$

where $G = \frac{(2^{\frac{\beta-2}{2}}-2^{\frac{\beta-6}{2}})q}{2^{\frac{\beta-2}{2}}}$, $H = \frac{(2^{\frac{\beta-2}{2}}-2^{\frac{\beta-6}{2}})q-2^{\frac{\beta-2}{2}}i}{2^{\frac{\beta-6}{2}}q}$ and $t = (2^{\frac{\beta-2}{2}} - 2^{\frac{\beta-6}{2}})q - 2^{\frac{\beta-2}{2}}i - 2^{\frac{\beta-6}{2}}qj$.

For $\beta \geq 11$ with $\beta \equiv 1 \pmod 2$, there are $2^{\beta-5}5 + \frac{(2^{\beta-8}-1)2^{\beta-5}}{3\cdot 2^{\beta-8}}(2q-1) + \frac{23q-9}{2}$ congruences. We have

$$\eta^2 \equiv (qt)^2 \pmod{2^\beta q}, \quad t = 2i-1, \ i = 1,2,3,\cdots,2^{\beta-3}, \tag{3.63}$$
$$\eta^2 \equiv (2t)^2 \pmod{2^\beta q}, \quad t = ql, \quad l = 2i-1 \ i = 1,2,3,\cdots,2^{\beta-5}. \tag{3.64}$$

For $\nu = 1,2,3,\cdots,\frac{\beta-9}{2}$,

$$\eta^2 \equiv (2t)^2 \pmod{2^\beta q}, \quad t = 2^\nu qi, \ i = 2j-1, \quad j = 1,3,\cdots,2^{\beta-7}, \tag{3.65}$$
$$\eta^2 \equiv (2t)^2 \pmod{2^\beta q}, \quad t = \left(2^{\beta-5+\nu} - 2^{\nu-1}\right)q - (2^{\beta-5+\nu})i - 2^\nu qj,$$
$$i = 1,2,\cdots,\lfloor\frac{2^{\beta-5+\nu} - 2^{\nu-1}q}{2^{\beta-5+\nu}}\rfloor,$$
$$j = 0,1,\cdots,\lfloor\frac{(2^{\beta-5+\nu} - 2^{\nu-1})q - (2^{\beta-5+\nu})i}{2^\nu q}\rfloor, \tag{3.66}$$
$$\eta^2 \equiv (2t)^2 \pmod{2^\beta q}, \quad t = 2^{\frac{\beta+1}{2}}qi, \quad i = 0,1,3,4,5,7, \tag{3.67}$$
$$\eta^2 \equiv (2t)^2 \pmod{2^\beta q}, \quad t = (2^{\frac{\beta+1}{2}} - 2^{\frac{\beta-9}{2}})q - 2^{\frac{\beta+1}{2}}i - 2^{\frac{\beta-7}{2}}qj,$$
$$i = 1,2,3,\cdots,\lfloor\frac{2^{\frac{\beta+1}{2}} - 2^{\frac{\beta-9}{2}})q}{2^{\frac{\beta+1}{2}}}\rfloor,$$
$$j = 0,1,3,\cdots,\lfloor\frac{(2^{\frac{\beta+1}{2}} - 2^{\frac{\beta-9}{2}})q - 2^{\frac{\beta+1}{2}}i}{2^{\frac{\beta-7}{2}}q}\rfloor, \tag{3.68}$$
$$\eta^2 \equiv (2t)^2 \pmod{2^\beta q}, \quad t = 2^{\frac{\beta-5}{2}}q, \tag{3.69}$$
$$\eta^2 \equiv (2t)^2 \pmod{2^\beta q}, \quad t = 2^{\frac{\beta-3}{2}}l, \quad l = 1,2,\cdots,q-1, \tag{3.70}$$
$$\eta^2 \equiv (2t)^2 \pmod{2^\beta q}, \quad t = (2^{\frac{\beta-1}{2}} - 2^{\frac{\beta-7}{2}})q - 2^{\frac{\beta-1}{2}}i - 2^{\frac{\beta-5}{2}}qj,$$
$$i = 1,2,3,\cdots,\lfloor\frac{(2^{\frac{\beta-1}{2}} - 2^{\frac{\beta-7}{2}})q}{2^{\frac{\beta-1}{2}}}\rfloor,$$
$$j = 0,1,3,\cdots,\lfloor\frac{(2^{\frac{\beta-1}{2}} - 2^{\frac{\beta-7}{2}})q - 2^{\frac{\beta-1}{2}}i}{2^{\frac{\beta-5}{2}}q}\rfloor, \tag{3.71}$$
$$\eta^2 \equiv (2t)^2 \pmod{2^\beta q}, \quad t = 2^{\frac{\beta-3}{2}}l - 2^{\frac{\beta-5}{2}}, \quad l = 1,2\cdots\frac{q-1}{2}. \tag{3.72}$$

Zeroes of congruences (3.63)−(3.72) are

$$\bigcup_{t=2i-1,\ i=1}^{2^{\beta-3}} \{qt, 2^{\beta-1}q + qt\} \bigcup \{2^\beta q - qt, 2^{\beta-1}q - 2^{\beta-1}q - qt\},$$
$$\bigcup_{t=q(2i-1), i=1}^{2^{\beta-5}} \{2^{\beta-2}qj + 2t, \quad j = 0,1,2,\cdots,2^2-1\} \bigcup \{2^\beta q - 2^{\beta-2}qj$$
$$-2t, j = 0,1,2,\cdots,2^2-1\},$$

For $\nu = 1, 2, 3, \cdots, \frac{\beta-9}{2}$,

$$\bigcup_{t=2^{\nu}q(2i-1),i=1}^{2^{\beta-7}} \bigcup_{\nu=1}^{(\beta-9)/2} \{2^{\beta-2-\nu}qj + 2t, \quad j = 0, 1, 2, \cdots, 2^3 - 1\} \bigcup \{2^{\beta}q - 2^{\beta-2-\nu}qj - 2t, j = 0, 1, 2, \cdots, 2^3 - 1\},$$

$$\bigcup_{1\leq i\leq\lfloor A\rfloor,\ j=0}^{\lfloor B\rfloor} \bigcup_{\nu=1}^{(\beta-9)/2} \Big\{\{2^{\beta-1-\nu}ql + 2t, \quad l = 0, 1, \cdots, 2^2 - 1\} \cup \{2^{\beta}q - 2^{\beta-1-\nu}ql - 2t, \quad l = 0, 1, \cdots, 2^2 - 1\} \bigcup \{2^{\beta-1-\nu}qt + 2^{\nu-1}C + 2t, l = 0, 1, \cdots, 2^2 - 1\} \cup \{2^{\beta}q - 2^{\beta-1-\nu}ql - 2^{\nu-1}C - 2t, l = 0, 1, \cdots, 2^2 - 1\}\Big\},$$

where $A = \lfloor\frac{2^{\beta-5+\nu}-2^{\nu-1}q}{2^{\beta-5+\nu}}\rfloor$, $B = \lfloor\frac{(2^{\beta-5+\nu}-2^{\nu-1})q-(2^{\beta-5+\nu})i}{2^{\nu}q}\rfloor$, $C = 2^7q + 2^6q + 2^3q + 2^2q$ and $t = \Big(2^{\beta-5+\nu} - 2^{\nu-1}\Big)q - (2^{\beta-5+\nu})i - 2^{\nu}qj$.

$$\bigcup_{\substack{t=2^{(\beta+1)/2}qi,\\ i\in\{1,3,4,5,7\}}} \{2^{(\beta+3)/2}qj + 2t, \quad j = 0, 1, 2, \cdots, 2^{(\beta-3)/2} - 1\} \bigcup \{2^{\beta}q - 2^{(\beta+3)/2}qj - 2t, j = 0, 1, 2, \cdots, 2^{(\beta-3)/2} - 1\}, \{2^{(\beta+1)/2}qj, \quad j = 0, 1, 2, \cdots, 2^{(\beta-1)/2} - 1\},$$

$$\bigcup_{1\leq i\leq\lfloor L\rfloor,\ j=0}^{\lfloor M\rfloor} \Big\{\{2^{(\beta+5)/2}ql + 2t, \quad l = 0, 1, 2, \cdots, 2^{(\beta-5)/2} - 1\} \cup \{2^{\beta}q - 2^{(\beta+5)/2}ql - 2t, \quad l = 0, 1, 2, \cdots, 2^{(\beta-5)/2} - 1\} \bigcup \{2^{(\beta+5)/2}ql + 2^{(\beta-11)/2}N + 2t, l = 0, 1, 2, \cdots, 2^{(\beta-5)/2} - 1\} \cup \{2^{\beta}q - 2^{(\beta+5)/2}ql - 2^{(\beta-11)/2}N - 2t, l = 0, 1, 2, \cdots, 2^{(\beta-5)/2} - 1\}\Big\},$$

where $L = \frac{2^{\frac{\beta+1}{2}}-2^{\frac{\beta-9}{2}})q}{2^{\frac{\beta+1}{2}}}$, $M = \frac{(2^{\frac{\beta+1}{2}}-2^{\frac{\beta-9}{2}})q-2^{\frac{\beta+1}{2}}i}{2^{\frac{\beta-7}{2}}q}$, $N = 2^7q + 2^4q + 2^3q$ and $t = (2^{\frac{\beta+1}{2}} - 2^{\frac{\beta-9}{2}})q - 2^{\frac{\beta+1}{2}}i - 2^{\frac{\beta-7}{2}}qj$.

$$\bigcup_{\substack{t=2^{(\beta-3)/2}l,\\ 1\leq l\leq q-1}} \Big\{\{2^{(\beta+1)/2}qj + 2t, \quad j = 0, 1, \cdots, 2^{(\beta-1)/2} - 1\} \cup \{2^{\beta}q - 2^{(\beta+1)/2}qj - 2t, j = 0, 1, \cdots, 2^{(\beta-1)/2} - 1\}\Big\}, \{2^{(\beta-1)/2}qj + 2t, t = (\beta-5)/2q, j = 0, 1, 2, \cdots, 2^{(\beta+1)/2} - 1\},$$

$$\bigcup_{1\leq i\leq\lfloor Q\rfloor,\ j=0}^{\lfloor R\rfloor} \Big\{\{2^{(\beta+3)/2}ql + 2t, \ l = 0, 1, 2, \cdots, 2^{(\beta-3)/2} - 1\} \cup \{2^{\beta}q - 2^{(\beta+3)/2}ql - 2t, l = 0, 1, 2, \cdots, 2^{(\beta-3)/2} - 1\} \bigcup \{2^{(\beta+3)/2}ql + 2^{(\beta-11)/2}S + 2t, l = 0, 1, 2, \cdots, 2^{(\beta-3)/2} - 1\} \cup \{2^{\beta}q - 2^{(\beta+3)/2}ql - 2^{(\beta-11)/2}S - 2t, l = 0, 1, 2, \cdots, 2^{(\beta-3)/2} - 1\}\Big\},$$

where $Q = \frac{(2^{\frac{\beta-1}{2}}-2^{\frac{\beta-7}{2}})q}{2^{\frac{\beta-1}{2}}}$, $R = \frac{(2^{\frac{\beta-1}{2}}-2^{\frac{\beta-7}{2}})q-2^{\frac{\beta-1}{2}}i}{2^{\frac{\beta-5}{2}}q}$, $S = 2^5q + 2^4q$ and $t = (2^{\frac{\beta-1}{2}} - 2^{\frac{\beta-7}{2}})q - 2^{\frac{\beta-1}{2}}i - 2^{\frac{\beta-5}{2}}qj$.

$$\bigcup_{l=1}^{(q-1)/2} \Big\{\{2^{(\beta-1)/2}qi + 2^{(\beta-1)/2}l - 2^{(\beta-3)/2},\ \ i = 0, 1, 2, \cdots, 2^{(\beta+1)/2} - 1\} \cup \{2^{\beta}q - 2^{(\beta-1)/2}qi - 2^{(\beta-1)/2}l + 2^{(\beta-3)/2}, i = 0, 1, 2, \cdots, 2^{(\beta+1)/2} - 1\}\Big\}.$$

Now, we consider the set of unit elements of Z_m. The $x^2 \equiv a \pmod 2$ has one solution and $x^2 \equiv a \pmod{2^2}$ has two solution. By Theorem 1.2, if $\beta \geq 3$, the congruence $x^2 \equiv a \pmod{2^\beta}$ has either no solution or exactly 4 incongruent solutions. Furthermore, again by Theorem 1.1, for an odd prime the congruence $x^2 \equiv a \pmod q$ has either no solution or exactly 2 incongruent solutions. By using Chinese remainder theorem, if $\beta = 0$ or 1 then, $x^2 \equiv a \pmod{2^\beta}$ has either no solution or exactly 2 incongruent solutions. If $\beta = 2$ then, $x^2 \equiv a \pmod{2^\beta}$ has either no solution or exactly 4 incongruent solutions. Lastly, if $\beta \geq 3$, then $x^2 \equiv a \pmod{2^\beta}$ has either no solution or exactly 8 incongruent solutions. Hence, $\overbrace{G}(2, 2^\beta q) = \frac{\phi(n)}{8}K_8, \beta \geq 3$. We achieve the desired outcome by combining both cases. □

Proposition 3.2. *If q and p are odd primes, then* $\overbrace{G}(2, qp) = K_1 \bigoplus \frac{q+p-2}{2}K_2 \bigoplus \frac{(q-1)(p-1)}{4}K_4$.

Proof. To prove this first we discuss the zero-divisors elements of ring Z_m. Let $\{0, nq, yp | n = 1, 2, 3, \cdots, p-1, y = 1, 2, 3, \cdots, q-1\}$ be a set of zero-divisors of Z_{qp} with zero. There are $\frac{q+p}{2}$ distinct congruences. We have

$$\eta^2 \equiv 0 \pmod{qp}, \tag{3.73}$$
$$\eta^2 \equiv (qt)^2 \pmod{qp},\ \ t = 1, 2, 3, \cdots, \frac{p-1}{2}, \tag{3.74}$$
$$\eta^2 \equiv (pt)^2 \pmod{qp},\ \ t = 1, 2, 3, \cdots, \frac{q-1}{2}. \tag{3.75}$$

Zeroes of these congruences are $\eta = 0$, $\eta = qt, qp - qt$, $t = 1, 2, 3, \cdots, \frac{p-1}{2}$, and $\eta = pt, qp - pt$, $t = 1, 2, 3, \cdots, \frac{q-1}{2}$. Thus, $\overbrace{G}(2, qp) = K_1 \bigoplus \frac{q+p-2}{2}K_2$. Now we discuss the unit elements of Z_m. By Theorem 1.1, for distinct odd primes q and p, the congruence $x^2 \equiv a \pmod q$ has either no solution or exactly 2 incongruent solutions. Similarly, the congruence $x^2 \equiv a \pmod p$ has either no solution or exactly 2 incongruent solutions. By using Chines remainder theorem we have, $x^2 \equiv a \pmod{q^\beta}$ has either no solution or exactly 4 solutions. Thus, $\overbrace{G}(2, qp) = \frac{\phi(qp)}{4}K_4 = \frac{\phi(q-1)(p-1))}{4}K_4$. We get the desired outcome by combining both cases. □

Theorem 3.3. *Let q be an odd prime. Then,* $\widetilde{G}(2, 2q^{\beta})$

$$= \begin{cases} 2K_1 \bigoplus (q-1)K_2, & \text{if } \beta = 1, \\ q(q-1)K_2 \bigoplus 2K_q, & \text{if } \beta = 2, \\ (q-1)q^2K_2 \bigoplus 2K_p \bigoplus (q-1)K_{2q}, & \text{if } \beta = 3, \\ (q-1)q^{\beta-1}K_2 \bigoplus_{i=1}^{\frac{\beta-2}{2}} q^{\beta-1-2i}(q-1)K_{2q^i} \bigoplus 2K_{q^{\beta/2}}, & \text{if } \beta \geq 4, \text{ and} \\ \qquad\qquad \beta \equiv 0 \ (mod\ 2), & \\ (q-1)q^{\beta-1}K_2 \bigoplus_{i=1}^{\frac{\beta-1}{2}} q^{\beta-1-2i}(q-1)K_{2q^i} \bigoplus 2K_{q^{(\beta-1)/2}}, & \text{if } \beta \geq 5 \text{ and} \\ \qquad\qquad \beta \equiv 1 \ (mod\ 2). & \end{cases}$$

Proof. The proof is on similar lines as illustrated in the proof of Theorem 3.1. □

Figures 4 and 5 reflect Theorems 3.1 and 3.3, respectively.

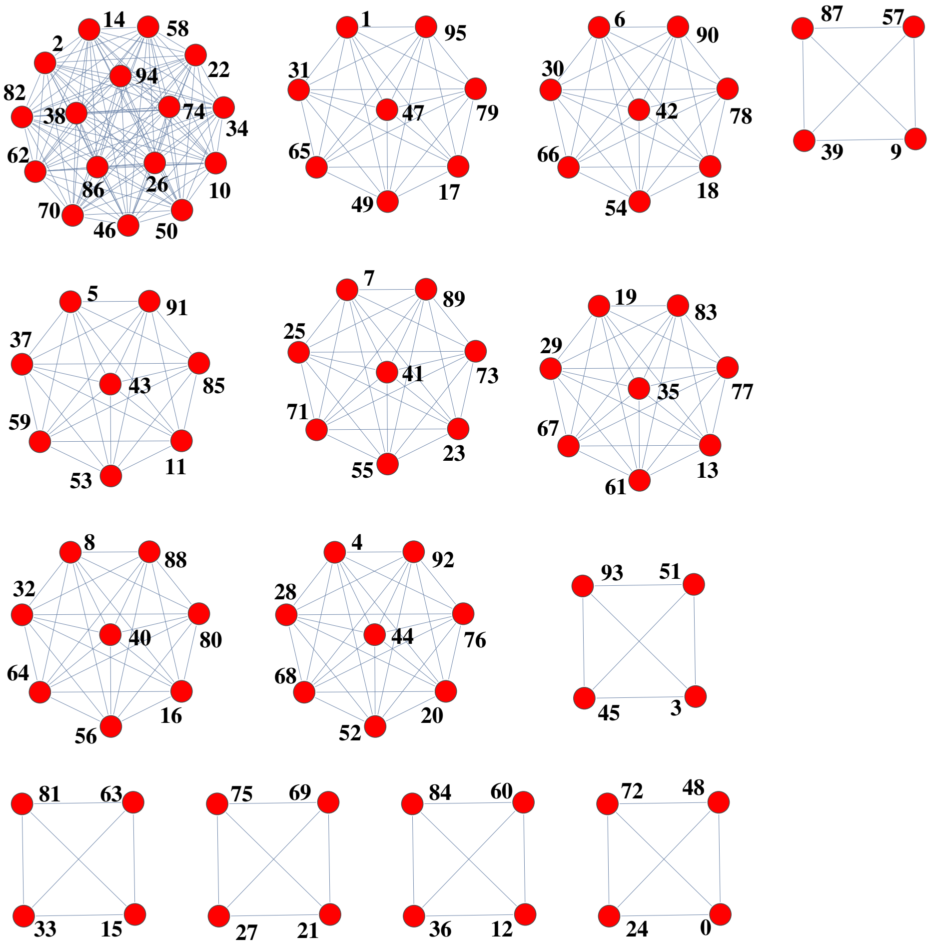

Figure 4. $\widetilde{G}(2, 96) = 6K_4 \bigoplus 7K_8 \bigoplus K_{16}$.

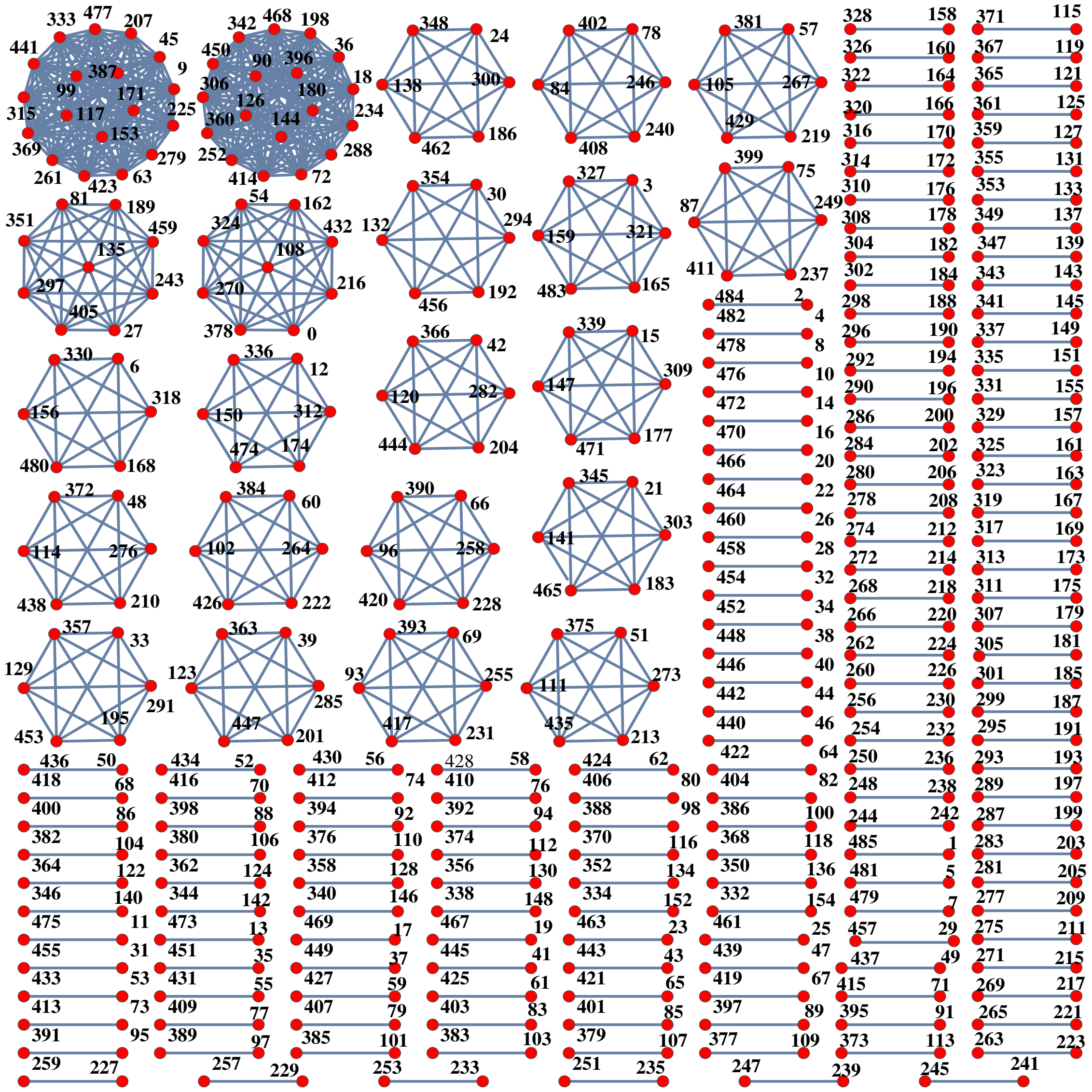

Figure 5. $\widetilde{G}(2, 486) = 2K_{18} \bigoplus 2K_9 \bigoplus 18K_6 \bigoplus 162K_2$.

4. Conclusions

In this article, we investigated the mapping $x^\alpha \equiv y^\alpha, \pmod{m}$ for $\alpha = 2$ over the ring of integers. A problem of partitions of a given set into the form of subsets with equal sums is NP problem. A paradigmatic approach was introduced to find equal sum partitions of quadratic maps via complete graphs. Moreover, we characterized quadratic graphs associated with the mapping $x^\alpha \equiv y^\alpha, \pmod{m}$, $\alpha = 2$ for well-known classes $m = 2^\beta, q^\beta, 2^\beta q, 2q^\beta, qp$, in terms of complete graphs, where q, p is an odd prime. Later on, we intend to extend our research to higher values of α over various rings. We hope that this work will open new inquiry opportunities in various fields for other researchers and knowledge seekers.

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Conflict of interest

The authors declare that they have no conflict of interest regarding the publication of the research article.

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