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Research article

Generalizations of fractional Hermite-Hadamard-Mercer like inequalities for convex functions

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Abstract: In this work, we establish inequalities of Hermite-Hadamard-Mercer (HHM) type for convex functions by using generalized fractional integrals. The results of our paper are the extensions and refinements of Hermite-Hadamard (HH) and Hermite-Hadamard-Mercer (HHM) type inequalities. We discuss special cases of our main results and give new inequalities of HH and HHM type for different fractional integrals like, Riemann-Liouville (RL) fractional integrals, k-Riemann-Liouville (k-RL) fractional integrals, conformable fractional integrals and fractional integrals of exponential kernel.

Keywords: Hermite-Hadamard inequality; Hermite-Hadamard-Mercer inequality; Hölder inequality; Jensen-Mercer inequality; convex functions; fractional integrals
Mathematics Subject Classification: 26A33, 26A51, 26D07, 26D10, 26D15

1. Introduction and Preliminaries

For a convex function $f : \mathrm{I} \subseteq \mathbf{R} \to \mathbf{R}$ on I with $c, d \in \mathrm{I}$ and $c < d$, the Hermite–Hadamard inequality states that [1]:

$$f\left(\frac{c+d}{2}\right) \leq \frac{1}{d-c}\int_c^d f(t)dt \leq \frac{f(c)+f(d)}{2}. \tag{1.1}$$

The Hermite-Hadamard integral inequality (1.1) is one of the most famous and commonly used inequalities. The recently published papers [2–17] are focused on extending and generalizing the convexity, Hermite-Hadamard inequality, and other inequalities for convex functions.

The situation of the fractional calculus (integrals and derivatives) has won vast popularity and significance throughout the previous five decades or so, due generally to its demonstrated applications in numerous seemingly numerous and great fields of science and engineering [18–20].

Now, we recall the definitions of Riemann-Liouville (RL) and generalized Riemann–Liouville (GRL) fractional integrals given by Sarikaya and Ertuğral.

Definition 1.1 ([18–20])**.** Let $f \in L_1[c, d]$. The Riemann–Liouville (RL) fractional integrals ${}^{RL}I^{\nu}_{c+}f$ and ${}^{RL}I^{\nu}_{d-}f$ of order $\nu > 0$ with $c \geq 0$ are respectively defined by

$$ {}^{RL}I^{\nu}_{c+}f(x) = \frac{1}{\Gamma(\nu)} \int_c^x (x-t)^{\nu-1} f(t)dt, \quad c < x, \tag{1.2} $$

and

$$ {}^{RL}I^{\nu}_{d-}f(x) = \frac{1}{\Gamma(\nu)} \int_x^d (t-x)^{\nu-1} f(t)dt, \quad x < d, \tag{1.3} $$

with ${}^{RL}I^{0}_{c+}f(x) = {}^{RL}I^{0}_{d-}f(x) = f(x)$.

Definition 1.2 ([21])**.** Assume that the function $\hbar : [0, +\infty) \to [0, +\infty)$ satisfies the following condition:

$$ \int_0^1 \frac{\hbar(t)}{t} dt < +\infty. $$

Then, the left sided and right sided generalized Riemann–Liouville (GRL) fractional integrals, denoted by ${}^{GRL}_{\hbar}I_{c+}$ and ${}^{GRL}_{\hbar}I_{d-}$, are defined as follows:

$$ {}^{GRL}_{\hbar}I_{c+}f(x) = \int_c^x \frac{\hbar(x-t)}{x-t} f(t)dt, \quad c < x, \tag{1.4} $$

$$ {}^{GRL}_{\hbar}I_{d-}f(x) = \int_x^d \frac{\hbar(t-x)}{t-x} f(t)dt, \quad x < d. \tag{1.5} $$

Remark 1.1. *From the Definition 1.1 one can obtain some known definitions of fractional calculus as special cases. That is,*

- *If* $\hbar(t) = \frac{t^{\nu}}{\Gamma(\nu)}$, *then Definition 1.2 reduces to Definition 1.1.*
- *If* $\hbar(t) = \frac{t^{\frac{\nu}{k}}}{k\Gamma_k(\nu)}$, *then the GRL fractional integrals reduce to k–RL fractional integrals [22].*
- *If* $\hbar(t) = \frac{t}{\nu}\exp\left(-\frac{1-\nu}{\nu}t\right)$, *then the GRL fractional integrals reduce to the fractional integrals with exponential kernel [23].*
- *If* $\hbar(t) = t(y-t)^{\nu-1}$, *then the GRL fractional integrals reduce to the conformable fractional integrals [24].*

With a huge application of RL fractional integration and Hermite–Hadamard inequality, many researchers in the field of fractional calculus extended their research to the Hermite–Hadamard inequality, including RL fractional integration rather than ordinary integration; for example see [25–32].

On the one hand, it is well known that RL and GRL fractional integrals have the same importance in theory of integral inequalities, and the GRL fractional integrals are more convenient for calculation.

Therefore it is necessary to study the Hermite-Hadamard integral inequalities by using the GRL fractional integrals while by using the RL fractional integrals. Fortunately, studying the Hermite-Hadamard integral inequalities via the GRL fractional integrations can unify the research of ordinary and fractional integrations. So it is necessary and meaningful to study Hermite-Hadamard integral inequalities via the GRL fractional integrations (see for details [21, 33–36]).

In this paper, we consider the integral inequality of HHM type that depends on the Hermite-Hadamard and Jensen–Mercer inequalities. For this reason, we recall the Jensen–Mercer inequality: Let $0 < x_1 \leq x_2 \leq \cdots \leq x_n$ and $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ nonnegative weights such that $\sum_{i=1}^{n} \alpha_i = 1$. Then, the Jensen inequality [37, 38] is as follows, for a convex function f on the interval $[c, d]$, we have

$$f\left(\sum_{i=1}^{n} \alpha_i x_i\right) \leq \sum_{i=1}^{n} \alpha_i f(x_i), \tag{1.6}$$

for all $x_i \in [c, d]$ and $\alpha_i \in [0, 1]$, $i = 1, 2, ..., n$.

Theorem 1.1 ([11, 38]). *If f is convex function on $[c, d]$, then*

$$f\left(c + d - \sum_{i=1}^{n} \alpha_i x_i\right) \leq f(c) + f(d) - \sum_{i=1}^{n} \alpha_i f(x_i), \tag{1.7}$$

for each $x_i \in [c, d]$ and $\alpha_i \in [0, 1]$, $i = 1, 2, ..., n$ with $\sum_{i=1}^{n} \alpha_i = 1$.

For some results related to Jensen-Mercer inequality, see [39–41].

Based on the above observations and discussion, the primary purpose of this article is to establish several inequalities of HHM type for convex functions by using the GRL fractional integrals.

2. Main Results

Throughout this attempt, we consider the following notations:

$$\Lambda(t) := \int_0^t \frac{\hbar\left((y-x)u\right)}{u} du < +\infty \quad \text{and} \quad \Delta(t) := \int_0^t \frac{\hbar\left(\left(\frac{y-x}{2}\right)u\right)}{u} du < +\infty.$$

Theorem 2.1. *For a convex function $f : [c, d] \to \mathbb{R}$, we have the following inequalities for GRL:*

$$\begin{aligned} f\left(c + d - \frac{x+y}{2}\right) &\leq f(c) + f(d) - \frac{1}{2\Lambda(1)} \left[{}^{GRL}_{\hbar}I_{x+} f(y) + {}^{GRL}_{\hbar}I_{y-} f(x) \right] \\ &\leq f(c) + f(d) - f\left(\frac{x+y}{2}\right), \end{aligned} \tag{2.1}$$

and

$$\begin{aligned} f\left(c + d - \frac{x+y}{2}\right) &\leq \frac{1}{2\Lambda(1)} \left[{}^{GRL}_{\hbar}I_{(c+d-y)+} f(c+d-x) + {}^{GRL}_{\hbar}I_{(c+d-x)-} f(c+d-y) \right] \\ &\leq \frac{f(c+d-x) + f(c+d-y)}{2} \leq f(c) + f(d) - \frac{f(x) + f(y)}{2}. \end{aligned} \tag{2.2}$$

Proof. From Jensen-Mercer inequality, we have for $u, v \in [c, d]$:

$$f\left(c+d-\frac{u+v}{2}\right) \leq f(c)+f(d)-\frac{f(u)+f(v)}{2}. \quad (2.3)$$

Then, for $u = tx + (1-t)y$ and $v = ty + (1-t)x$, it follows that

$$f\left(c+d-\frac{x+y}{2}\right) \leq f(c)+f(d)-\frac{f(tx+(1-t)y)+f(ty+(1-t)x)}{2}, \quad (2.4)$$

for each $x, y \in [c, d]$ and $t \in [0, 1]$. By multiplying both sides of (2.4) by $\frac{\hbar((y-x)t)}{t}$ and integrating the result with respect to t over $[0, 1]$, we can obtain

$$\begin{aligned}
& f\left(c+d-\frac{x+y}{2}\right)\Lambda(1) \leq [f(c)+f(d)]\Lambda(1) \\
& -\frac{1}{2}\left[\int_0^1 \frac{\hbar((y-x)t)}{t}[f(tx+(1-t)y)+f(ty+(1-t)x)]\,dt\right] \\
& = [f(c)+f(d)]\int_0^1 \Lambda(1)-\frac{1}{2}\left[\int_0^1 \frac{\hbar((y-x)t)}{t} f(tx+(1-t)y)\,dt\right. \\
& + \left.\int_0^1 \frac{\hbar((y-x)t)}{t} f(ty+(1-t)x)\,dt\right] \\
& = [f(c)+f(d)]\Lambda(1)-\frac{1}{2}\left[\int_x^y \frac{\hbar(y-w)}{y-w} f(w)\,dw+\int_x^y \frac{\hbar(w-x)}{w-x} f(w)\,dw\right] \\
& = [f(c)+f(d)]\Lambda(1)-\frac{1}{2}\left[{}^{GRL}_{\hbar}I_{x+}f(w)+{}^{GRL}_{\hbar}I_{y-}f(w)\right].
\end{aligned}$$

This gives the first inequality in (2.1). To prove the second inequality in (2.1), first we have by the convexity of f:

$$f\left(\frac{u+v}{2}\right) \leq \frac{f(u)+f(v)}{2}. \quad (2.5)$$

By changing the variables $u = tx + (1-t)y$ and $v = ty + (1-t)x$ in (2.5), we have

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(tx+(1-t)y)+f(ty+(1-t)x)}{2}, \quad t \in [0, 1]. \quad (2.6)$$

Multiplying both sides of (2.6) by $\frac{\hbar((y-x)t)}{t}$ and integrating the result with respect to t over $[0, 1]$, we get

$$\begin{aligned}
f\left(\frac{x+y}{2}\right)\Lambda(1) & \leq \frac{1}{2}\left[\int_0^1 \frac{\hbar((y-x)t)}{t} f(tx+(1-t)y)\,dt+\int_0^1 \frac{\hbar((y-x)t)}{t} f(ty+(1-t)x)\,dt\right] \\
& = \frac{1}{2}\left[\int_x^y \frac{\hbar(y-w)}{y-w} f(w)+\int_x^y \frac{\hbar(w-x)}{w-x} f(w)\,dw\right] \\
& = \frac{1}{2}\left[{}^{GRL}_{\hbar}I_{x+}f(y)+{}^{GRL}_{\hbar}I_{y-}f(x)\right],
\end{aligned}$$

which implies that

$$-f\left(\frac{x+y}{2}\right) \geq -\frac{1}{2\Lambda(1)}\left[{}^{GRL}_{\hbar}I_{x+}f(y)+{}^{GRL}_{\hbar}I_{y-}f(x)\right]. \quad (2.7)$$

By adding $f(c)+f(d)$ on both sides of (2.7), we can obtain the second inequality in (2.1).

Now we give the proof of inequalities (2.2). Since f is convex function, then for all $u,v\in[c,d]$, we have

$$f\left(c+d-\frac{u+v}{2}\right)=f\left(\frac{c+d-u+c+d-v}{2}\right)\le\frac{1}{2}\left[f(c+d-u)+f(c+d-v)\right]. \tag{2.8}$$

Then, for $c+d-u=t(c+d-x)+(1-t)(c+d-y)$ and $c+d-v=t(c+d-y)+(1-t)(c+d-x)$, it follows that

$$f\left(c+d-\frac{u+v}{2}\right)\le\frac{1}{2}\Big[f(t(c+d-x)+(1-t)(c+d-y))\\ +f(t(c+d-y)+(1-t)(c+d-x))\Big]. \tag{2.9}$$

for each $x,y\in[c,d]$ and $t\in[0,1]$. Now, by multiplying both sides of (2.9) by $\frac{\hbar((y-x)t)}{t}$ and integrating the obtaining inequality with respect to t over $[0,1]$, we obtain

$$\begin{aligned}
&f\left(c+d-\frac{u+v}{2}\right)\Lambda(1)\le\frac{1}{2}\left[\int_0^1\frac{\hbar((y-x)t)}{t}f(t(c+d-x)+(1-t)(c+d-y))\,dt\right.\\
&\left.+\int_0^1\frac{\hbar((y-x)t)}{t}f(t(c+d-y)+(1-t)(c+d-x))\,dt\right]\\
&=\frac{1}{2}\left[\int_{c+d-y}^{c+d-x}\frac{\hbar(w-(c+d-y))}{w-(c+d-y)}f(w)\,dw+\int_{c+d-y}^{c+d-x}\frac{\hbar((c+d-x)-w)}{(c+d-x)-w}f(w)\,dw\right]\\
&=\frac{1}{2}\left[{}^{GRL}_{\hbar}I_{(c+d-y)+}f(c+d-x)+{}^{GRL}_{\hbar}I_{(c+d-x)-}f(c+d-y)\right],
\end{aligned}$$

and this completes the proof of the first inequality in (2.2). To prove the second inequality in (2.2), first we use the convexity of f to get

$$f(t(c+d-x)+(1-t)(c+d-y))\le tf(c+d-x)+(1-t)f(c+d-y), \tag{2.10}$$

$$f(t(c+d-y)+(1-t)(c+d-x))\le(1-t)f(c+d-x)+tf(c+d-y). \tag{2.11}$$

Adding (2.10) and (2.11), we get

$$f(t(c+d-x)+(1-t)(c+d-y))+f(t(c+d-y)+(1-t)(c+d-x))\\ \le f(c+d-x)+f(c+d-y)\le 2[f(c)+f(d)]-[f(x)+f(y)]. \tag{2.12}$$

Multiplying both sides of (2.12) by $\frac{\hbar((y-x)t)}{t}$ and integrating the result with respect to t over $[0,1]$, we obtain

$$\begin{aligned}
&\int_0^1\frac{\hbar((y-x)t)}{t}f(t(c+d-x)+(1-t)(c+d-y))\,dt\\
&\qquad+\int_0^1\frac{\hbar((y-x)t)}{t}f(t(c+d-y)+(1-t)(c+d-x))\,dt\\
&\qquad\le\Lambda(1)f(c+d-x)+\Lambda(1)f(c+d-y)
\end{aligned}$$

$$\leq 2\Lambda(1)\left[f(c)+f(d)\right]-\Lambda(1)\left[f(x)+f(y)\right].$$

By using the change of variables of integration and then by multiplying the result by $\frac{1}{2\Lambda(1)}$, we can obtain the second and third inequalities in (2.2). This completes the proof of Theorem 2.1. □

Remark 2.1. *Let the assumptions of Theorem 2.1 be satisfied. Then,*

- *If $\hbar(t)=t$, then Theorem 2.1 reduces to [42, Theorem 2.1].*
- *If $\hbar(t)=\frac{t^{\nu}}{\Gamma(\nu)}$, then Theorem 2.1 reduces to [43, Theorem 2].*
- *If we set $\hbar(t)=t$, $x=c$ and $y=d$ in (2.2), then (2.2) becomes (1.1).*
- *If $\hbar(t)=\frac{t^{\frac{\nu}{k}}}{k\Gamma_k(a)}$ in Theorem 2.1 (Eq. (2.2)), we get*

$$f\left(c+d-\frac{x+y}{2}\right)\leq\frac{\Gamma_k(\nu+k)}{2(y-x)^{\frac{\nu}{k}}}\left[{}^{RL}_{\hbar}I_{(c+d-y)+,k}f(c+d-x)+{}^{RL}_{\hbar}I_{(c+d-x)-,k}f(c+d-y)\right]$$
$$\leq\frac{f(c+d-x)+f(c+d-y)}{2}\leq f(c)+f(d)-\frac{f(x)+f(y)}{2}.$$

- *If we set $\hbar(t)=\frac{t^{\nu}}{\Gamma(\nu)}$, $x=c$ and $y=d$ in (2.2), then we have*

$$f\left(\frac{c+d}{2}\right)\leq\frac{\Gamma(\nu+1)}{2(b-a)^{\nu}}\left[{}^{RL}I^{\nu}_{c+}f(d)+{}^{RL}I^{\nu}_{d-}f(c)\right]\leq\frac{f(c)+f(d)}{2},$$

  which is derived in [25].
- *If we set $\hbar(t)=\frac{t^{\frac{\nu}{k}}}{k\Gamma_k(\nu)}$, $x=c$ and $y=d$ in (2.2), then we have*

$$f\left(\frac{c+d}{2}\right)\leq\frac{\Gamma_k(\nu+k)}{2(d-c)^{\frac{\nu}{k}}}\left[{}^{RL}I^{\nu}_{c+,k}f(d)+{}^{RL}I^{\nu}_{d-,k}f(c)\right]\leq\frac{f(c)+f(d)}{2},$$

  which is derived in [44].
- *If $x=c$ and $y=d$, then inequalities (2.1) reduces to the following inequalities:*

$$f\left(\frac{c+d}{2}\right)\leq f(c)+f(d)-\frac{1}{2\Lambda(1)}\left[{}^{GRL}_{\hbar}I_{c+}f(y)+{}^{GRL}_{\hbar}I_{d-}f(x)\right]\leq f(c)+f(d)-f\left(\frac{c+d}{2}\right).$$

- *If $x=c$ and $y=d$, then inequalities (2.2) reduces to [21, Theorem 5].*

Corollary 2.1. *For a convex function $f:[c,d]\to\mathbb{R}$, we have the following inequalities of HHM type for conformable fractional integrals:*

$$f\left(c+d-\frac{x+y}{2}\right)\leq f(c)+f(d)-\frac{\nu}{2(y^{\nu}-x^{\nu})}\int_x^y f(t)d_{\nu}t\leq f(c)+f(d)-f\left(\frac{x+y}{2}\right),\tag{2.13}$$

and

$$f\left(c+d-\frac{x+y}{2}\right)\leq\frac{\nu}{(y^{\nu}-x^{\nu})}\int_{c+d-y}^{c+d-x}f(t)d_{\nu}t\leq\frac{f(c+d-x)+f(c+d-y)}{2}$$
$$\leq f(c)+f(d)-\frac{f(x)+f(y)}{2}.\tag{2.14}$$

Proof. By setting $\hbar(t) = t(y-t)^{\nu-1}$ in Theorem 2.1, we can directly obtain the proof. □

Remark 2.2. *If we set $x = c$ and $y = d$ in (2.14), then we have the well-known conformable fractional HH integral inequality:*

$$f\left(\frac{c+d}{2}\right) \leq \frac{\nu}{d^{\nu}-c^{\nu}} \int_c^d f(t)\, d_{\nu}t \leq \frac{f(c)+f(d)}{2},$$

which is derived by Adil Khan et al. in [45].

Corollary 2.2. *For a convex function $f : [c,d] \to \mathbb{R}$, we have the following inequalities of HHM type for fractional integrals with exponential kernel:*

$$f\left(c+d-\frac{x+y}{2}\right) \leq f(c)+f(d) - \frac{(\nu-1)}{2\left[\exp\left(-\frac{1-\nu}{\nu}(y-x)\right)-1\right]}\left[{}^{exp}I^{\nu}_{x+}f(y) + {}^{exp}I^{\nu}_{y-}f(x)\right]$$
$$\leq f(c)+f(d) - f\left(\frac{x+y}{2}\right), \quad (2.15)$$

and

$$f\left(c+d-\frac{x+y}{2}\right) \leq \frac{(\nu-1)}{2\left[\exp\left(-\frac{1-\nu}{\nu}(y-x)\right)-1\right]}\left[{}^{exp}I^{\nu}_{(c+d-y)+}f(c+d-x) + {}^{exp}I^{\nu}_{(c+d-x)-}f(c+d-y)\right]$$
$$\leq \frac{f(c+d-x)+f(c+d-y)}{2} \leq f(c)+f(d) - \frac{f(x)+f(y)}{2}. \quad (2.16)$$

Proof. By setting $\hbar(t) = \frac{t}{\nu}\exp\left(-\frac{1-\nu}{\nu}t\right)$ in Theorem 2.1, we can easily obtain the proof of Corollary 2.2. □

Remark 2.3. *If we set $x = c$ and $y = d$ in (2.16), then we have the HH inequalities for fractional integrals with exponential kernel:*

$$f\left(\frac{c+d}{2}\right) \leq \frac{(\nu-1)}{2\left[\exp\left(-\frac{1-\nu}{\nu}(d-c)\right)-1\right]}\left[{}^{exp}I^{\nu}_{c+}f(d) + {}^{exp}I^{\nu}_{d-}f(c)\right] \leq \frac{f(c)+f(d)}{2},$$

which is derived by Ahmad et al. in [46].

Theorem 2.2. *For a convex function $f : [c,d] \to \mathbb{R}$, we have the following inequalities for GRL:*

$$f\left(c+d-\frac{x+y}{2}\right) \leq \frac{1}{2\Delta(1)}\left[{}^{GRL}_{\hbar}I_{\left(c+d-\frac{x+y}{2}\right)-}f(c+d-y) + {}^{GRL}_{\hbar}I_{\left(c+d-\frac{x+y}{2}\right)+}f(c+d-x)\right]$$
$$\leq f(c)+f(d) - \frac{f(x)+f(y)}{2}. \quad (2.17)$$

Proof. From the convexity of f, we have

$$f\left(c+d-\frac{u+v}{2}\right) = f\left(\frac{c+d-u+c+d-v}{2}\right) \leq \frac{1}{2}f(c+d-u)+f(c+d-v). \quad (2.18)$$

By setting $u = \frac{t}{2}x + \frac{2-t}{2}y$, $v = \frac{2-t}{2}x + \frac{t}{2}y$, it follows that

$$f\left(c+d-\frac{x+y}{2}\right) \leq \frac{1}{2}\left[f\left(c+d-\left(\frac{t}{2}x+\frac{2-t}{2}y\right)\right)+f\left(c+d-\left(\frac{2-t}{2}x+\frac{t}{2}y\right)\right)\right], \tag{2.19}$$

for all $x, y \in [c, d]$ and $t \in [0, 1]$. Multiplying both sides of (2.19) by $\frac{\hbar\left(\left(\frac{y-x}{2}\right)t\right)}{t}$ and integrating its result with respect to t over $[0, 1]$, we get

$$\begin{aligned}
& f\left(c+d-\frac{x+y}{2}\right)\Delta(1) \leq \frac{1}{2}\left[\int_0^1 \frac{\hbar\left(\left(\frac{y-x}{2}\right)t\right)}{t} f\left(c+d-\left(\frac{t}{2}x+\frac{2-t}{2}y\right)\right)dt\right. \\
& \left.+\int_0^1 \frac{\hbar\left(\left(\frac{y-x}{2}\right)t\right)}{t} f\left(c+d-\left(\frac{2-t}{2}x+\frac{t}{2}y\right)\right)dt\right] \\
& = \frac{1}{2}\left[\int_{c+d-y}^{c+d-\frac{x+y}{2}} \frac{\hbar(w-(c+d-y))}{w-(c+d-y)} f(w)\,dw + \int_{c+d-\frac{x+y}{2}}^{c+d-x} \frac{\hbar((c+d-x)-w)}{(c+d-x)-w} f(w)\,dw\right] \\
& = \frac{1}{2}\left[{}^{GRL}_{\hbar}I_{\left(c+d-\frac{x+y}{2}\right)-} f(c+d-y) + {}^{GRL}_{\hbar}I_{\left(c+d-\frac{x+y}{2}\right)+} f(c+d-x)\right].
\end{aligned}$$

Thus, the first inequality in (2.17) is proved. To prove the second inequality in (2.17), by using the Jensen–Mercer inequality, we can deduce:

$$f\left(c+d-\left(\frac{t}{2}x+\frac{2-t}{2}y\right)\right) \leq f(c)+f(d)-\left[\frac{t}{2}f(x)+\frac{2-t}{2}f(y)\right] \tag{2.20}$$

$$f\left(c+d-\left(\frac{2-t}{2}x+\frac{t}{2}y\right)\right) \leq f(c)+f(d)-\left[\frac{2-t}{2}f(x)+\frac{t}{2}f(y)\right]. \tag{2.21}$$

By adding (2.20) and (2.21), we obtain

$$f\left(c+d-\left(\frac{t}{2}x+\frac{2-t}{2}y\right)\right)+f\left(c+d-\left(\frac{2-t}{2}x+\frac{t}{2}y\right)\right) \leq 2\left[f(c)+f(d)\right]-f(x)+f(y). \tag{2.22}$$

Multiplying both sides of inequality (2.22) by $\frac{\hbar\left(\left(\frac{y-x}{2}\right)t\right)}{t}$ and integrating the result with respect to t over $[0, 1]$, we get

$$\begin{aligned}
& \int_0^1 \frac{\hbar\left(\left(\frac{y-x}{2}\right)t\right)}{t} f\left(c+d-\left(\frac{t}{2}x+\frac{2-t}{2}y\right)\right)dt + \int_0^1 \frac{\hbar\left(\left(\frac{y-x}{2}\right)t\right)}{t} f\left(c+d-\left(\frac{2-t}{2}x+\frac{t}{2}y\right)\right)dt \\
& \qquad \leq 2\Delta(1)\left[f(c)+f(d)\right]-\Delta(1)\left[f(x)+f(y)\right].
\end{aligned}$$

By using change of variables of integration and multiplying the result by $\frac{1}{2\Delta(1)}$, we can easily obtain second inequality in (2.17). □

Remark 2.4. *Assume that the assumptions of Theorem 2.2 are satisfied.*

- *If $\hbar(t) = t$, then inequalities (2.17) becomes inequalities [42, Theorem 2.1].*
- *If we put $\hbar(t) = t$, $x = c$ and $y = d$ in Theorem 2.2, then inequalities (2.17) becomes inequalities (1.1).*

- *If* $\hbar(t) = \frac{t^{\nu}}{\Gamma(\nu)}$, *then Theorem 2.2 reduces to [43, Theorem 3].*
- *If we put* $\hbar(t) = \frac{t^{\nu}}{\Gamma(\nu)}$, $x = c$ *and* $y = d$ *in Theorem 2.2, then Theorem 2.2 reduces to [26, Theorem 4].*
- *If* $\hbar(t) = \frac{t^{\frac{\nu}{k}}}{k\Gamma_k(\nu)}$ *in Theorem 2.2, we get*

$$\begin{aligned} f\left(c+d-\frac{x+y}{2}\right) &\\ \leq \frac{2^{\frac{\nu}{k}-1}\Gamma_k(\nu+k)}{(y-x)^{\frac{\nu}{k}}} &\left[{}^{RL}_{\hbar}I_{\left(c+d-\frac{x+y}{2}\right)-,k}f(c+d-y) + {}^{RL}_{\hbar}I_{\left(c+d-\frac{x+y}{2}\right)+,k}f(c+d-x)\right] \\ &\leq f(c)+f(d)-\frac{f(x)+f(y)}{2}. \end{aligned}$$

- *If we put* $\hbar(t) = \frac{t^{\frac{\nu}{k}}}{k\Gamma_k(\nu)}$, $x = c$ *and* $y = d$ *in Theorem 2.2, then Theorem 2.2 reduces to [44, Theorem 1.1].*
- *If* $x = c$ *and* $y = d$, *then Theorem 2.2 becomes*

$$f\left(\frac{c+d}{2}\right) \leq \frac{1}{2\Delta(1)}\left[{}^{GRL}_{\hbar}I_{\left(\frac{c+d}{2}\right)-}f(c) + {}^{GRL}_{\hbar}I_{\left(\frac{c+d}{2}\right)+}f(d)\right] \leq \frac{f(c)+f(d)}{2}.$$

Corollary 2.3. *For a convex function* $f : [c,d] \to \mathbb{R}$, *we have the following inequalities of HHM type for conformable fractional integrals:*

$$f\left(c+d-\frac{x+y}{2}\right) \leq \frac{\nu}{\left[y^{\nu}-\left(\frac{x+y}{2}\right)^{\nu}\right]}\int_{c+d-y}^{c+d-x} f(t)d_{\nu}t \leq f(c)+f(d)-\frac{f(x)+f(y)}{2}. \tag{2.23}$$

Proof. By setting $\hbar(t) = t(y-t)^{\nu-1}$ in Theorem 2.2, then we have proof of Corollary 2.3. □

Remark 2.5. *If we set* $x = c$ *and* $y = d$ *in (2.23), then we get*

$$f\left(\frac{c+d}{2}\right) \leq \frac{\nu}{\left[d^{\nu}-\left(\frac{c+d}{2}\right)^{\nu}\right]}\int_{c}^{d} f(t)d_{\nu}t \leq \frac{f(c)+f(d)}{2}.$$

Corollary 2.4. *For a convex function* $f : [c,d] \to \mathbb{R}$, *we have the following inequalities of HHM type for fractional integrals with exponential kernel:*

$$\begin{aligned} f\left(c+d-\frac{x+y}{2}\right) \leq \frac{(\nu-1)}{2\left[\exp\left(-\frac{1-\nu}{\nu}\frac{(y-x)}{2}\right)-1\right]}&\left[{}^{exp}I^{\nu}_{\left(c+d-\frac{x+y}{2}\right)-}f(c+d-y)\right. \\ &\left.+ {}^{exp}I^{\nu}_{\left(c+d-\frac{x+y}{2}\right)+}f(c+d-x)\right] \leq f(c)+f(d)-\frac{f(x)+f(y)}{2}. \end{aligned} \tag{2.24}$$

Proof. By setting $\hbar(t) = \frac{t}{\nu}\exp\left(-\frac{1-\nu}{\nu}t\right)$ in Theorem 2.2, we get proof of Corollary 2.4. □

Remark 2.6. *If we set* $x = c$ *and* $y = d$ *in (2.24), then we get*

$$f\left(\frac{c+d}{2}\right) \leq \frac{(\nu-1)}{2\left[\exp\left(-\frac{1-\nu}{\nu}\frac{(d-c)}{2}\right)-1\right]}\left[{}^{exp}I^{\nu}_{\left(\frac{c+d}{2}\right)-}f(c) + {}^{exp}I^{\nu}_{\left(\frac{c+d}{2}\right)+}f(d)\right] \leq \frac{f(c)+f(d)}{2}.$$

3. Related equalities and inequalities

In view of the inequalities (2.1) and (2.17), we can generate some related results in this section.

Lemma 3.1. *Let $f : [c, d] \to \mathbb{R}$ be a differentiable function on (c, d) such that $f' \in L[c, d]$. Then, the following equality holds for GRL:*

$$\begin{aligned}
&\frac{f(c+d-y)+f(c+d-x)}{2} - \frac{1}{2\Lambda(1)}\left[{}^{GRL}_{\hbar}I_{(c+d-y)+}f(c+d-x) + {}^{GRL}_{\hbar}I_{(c+d-x)-}f(c+d-y)\right] \\
&= \frac{(y-x)}{2\Lambda(1)}\int_0^1 [\Lambda(t)-\Lambda(1-t)]\, f'(c+d-(tx+(1-t)y))\,dt \\
&= \frac{(y-x)}{2\Lambda(1)}\int_0^1 \Lambda(t)\left[f'(c+d-(tx+(1-t)y)) - f'(c+d-(ty+(1-t)x))\right]dt. \quad (3.1)
\end{aligned}$$

Proof. By the help of the right hand side of (3.1), we have

$$\begin{aligned}
&\frac{(y-x)}{2\Lambda(1)}\int_0^1 \Lambda(t)\left[f'(c+d-(tx+(1-t)y)) - f'(c+d-(ty+(1-t)x))\right]dt \\
&= \frac{(y-x)}{2\Lambda(1)}\left[\int_0^1 \Lambda(t) f'(c+d-(tx+(1-t)y))\,dt - \int_0^1 \Lambda(t) f'(c+d-(ty+(1-t)x))\,dt\right] \\
&= \frac{(y-x)}{2\Lambda(1)}[S_1 - S_2]. \quad (3.2)
\end{aligned}$$

By applying integration by parts, one can obtain

$$\begin{aligned}
S_2 &= \int_0^1 \Lambda(t) f'(c+d-(ty+(1-t)x))\,dt \\
&= -\Lambda(1)\frac{f(c+d-y)}{y-x} + \frac{1}{y-x}\int_0^1 \frac{\hbar((y-x)t)}{t} f(c+d-(ty+(1-t)x)) \\
&= -\Lambda(1)\frac{f(c+d-y)}{y-x} + \frac{1}{y-x} \\
&= -\Lambda(1)\frac{f(c+d-y)}{y-x} + \frac{1}{y-x}\left[{}^{GRL}_{\hbar}I_{(c+d-y)+}f(c+d-x)\right].
\end{aligned}$$

Similarly, one can obtain

$$\begin{aligned}
S_1 &= \int_0^1 \Lambda(t) f(c+d-(tx+(1-t)y))\,dt \\
&= \Lambda(1)\frac{f(c+d-x)}{y-x} - \frac{1}{y-x}\left[{}^{GRL}_{\hbar}I_{(c+d-x)-}f(c+d-y)\right].
\end{aligned}$$

By making use of S_1 and S_2 in (3.2), we get the identity (3.1). □

Remark 3.1. *Let the assumptions of Lemma 3.1 be satisfied.*

- *If $\hbar(t) = t$, then Lemma 3.1 reduces to [43, Corollary 1].*

- *If* $\hbar(t)=\frac{t^{\nu}}{\Gamma(\nu)}$, *then Lemma 3.1 reduces to [43, Lemma 1].*
- *If* $\hbar(t)=\frac{t^{\frac{\nu}{k}}}{k\Gamma_k(\nu)}$ *in Lemma 3.1 , we get*

$$\begin{aligned}&\frac{f(c+d-x)+f(c+d-y)}{2}\\&\quad-\frac{\Gamma_k(\nu+k)}{2(y-x)^{\frac{\nu}{k}}}\left[{}^{RL}_{\hbar}I_{(c+d-y)+,k}f(c+d-x)+{}^{RL}_{\hbar}I_{(c+d-x)-,k}f(c+d-y)\right]\\&\qquad=\frac{y-x}{2}\int_0^1\left[t^{\frac{\nu}{k}}-(1-t)^{\frac{\nu}{k}}\right]f'(c+d-(tx+(1-t)y))\,dt. \qquad (3.3)\end{aligned}$$

- *If* $x=c$ *and* $y=d$*, then Lemma 3.1 reduces to [47, Lemma 2.1].*

Corollary 3.1. *Let the assumptions of Lemma 3.1 be satisfied, then the following equality holds for the conformable fractional integrals:*

$$\begin{aligned}&\frac{f(c+d-y)+f(c+d-x)}{2}-\frac{\nu}{y^{\nu}-x^{\nu}}\int_{c+d-y}^{c+d-x}f(t)d_{\nu}t\\&=\frac{(y-x)}{2\mathbf{\Lambda}_1(1)}\int_0^1\left[\mathbf{\Lambda}_1(t)-\mathbf{\Lambda}_1(1-t)\right]f'(c+d-(tx+(1-t)y))\,dt\\&=\frac{(y-x)}{2\mathbf{\Lambda}_1(1)}\int_0^1\mathbf{\Lambda}_1(t)\left[f'(c+d-(tx+(1-t)y))-f'(c+d-(ty+(1-t)x))\right]dt, \qquad (3.4)\end{aligned}$$

where

$$\mathbf{\Lambda}_1(t)=\frac{y^{\nu}-(tx+(1-t)y)^{\nu}}{\nu}.$$

Proof. By setting $\hbar(t)=t(y-t)^{\nu-1}$ in Lemma 3.1, then we have proof of Corollary 3.1. □

Remark 3.2. *By setting* $x=c$ *and* $y=d$ *in (3.4), we get*

$$\begin{aligned}\frac{f(c)+f(d)}{2}-\frac{\nu}{d^{\nu}-c^{\nu}}\int_c^d f(t)d_{\nu}t&=\frac{(d-c)}{2\mathbf{\Lambda}_2(1)}\int_0^1\left[\mathbf{\Lambda}_2(t)-\mathbf{\Lambda}_2(1-t)\right]f'(td+(1-t)c)\,dt\\&=\frac{(d-c)}{2\mathbf{\Lambda}_2(1)}\int_0^1\mathbf{\Lambda}_2(t)\left[f'(td+(1-t)c)-f'(tc+(1-t)d)\right]dt,\end{aligned}$$

where

$$\mathbf{\Lambda}_2(t)=\frac{y^{\nu}-(tc+(1-t)d)^{\nu}}{\nu}.$$

Corollary 3.2. *Let the assumptions of Lemma 3.1 be satisfied, then the following equality holds for the fractional integrals with exponential kernel:*

$$\begin{aligned}&\frac{f(c+d-y)+f(c+d-x)}{2}-\frac{(\nu-1)}{2\left[\exp\left(-\frac{1-\nu}{\nu}(y-x)\right)-1\right]}\\&\times\left[{}^{exp}I^{\nu}_{(c+d-y)+}f(c+d-x)+{}^{exp}I^{\nu}_{(c+d-x)-}f(c+d-y)\right]\\&=\frac{(y-x)}{2\mathbf{\Lambda}_3(1)}\int_0^1\left[\mathbf{\Lambda}_3(t)-\mathbf{\Lambda}_3(1-t)\right]f'(c+d-(tx+(1-t)y))\,dt\end{aligned}$$

$$= \frac{(y-x)}{2\mathbf{\Lambda}_3(1)} \int_0^1 \mathbf{\Lambda}_3(t) \left[f'(c+d-(tx+(1-t)y)) - f'(c+d-(ty+(1-t)x)) \right] dt, \tag{3.5}$$

where

$$\mathbf{\Lambda}_3(t) = \frac{\exp\left(-\frac{1-\nu}{\nu}(y-x)t\right) - 1}{\nu - 1}.$$

Proof. By setting $\hbar(t) = \frac{t}{\nu}\exp\left(-\frac{1-\nu}{\nu}t\right)$ in Lemma 3.1, we get proof of Corollary 3.2. □

Remark 3.3. *If we set $x = c$ and $y = d$ in (3.5), we get*

$$\frac{f(c)+f(d)}{2} - \frac{(\nu-1)}{2\left[\exp\left(-\frac{1-\nu}{\nu}(d-c)\right)-1\right]} \left[{}^{exp}I^{\nu}_{c+} f(d) + {}^{exp}I^{\nu}_{d-} f(c) \right]$$
$$= \frac{(d-c)}{2\mathbf{\Lambda}_4(1)} \int_0^1 \left[\mathbf{\Lambda}_4(t) - \mathbf{\Lambda}_4(1-t)\right] f'(td+(1-t)c)\, dt$$
$$= \frac{(d-c)}{2\mathbf{\Lambda}_4(1)} \int_0^1 \mathbf{\Lambda}_4(t) \left[f'(td+(1-t)c) - f'(tc+(1-t)d) \right] dt,$$

where

$$\mathbf{\Lambda}_4(t) = \frac{\exp\left(-\frac{1-\nu}{\nu}(d-c)t\right) - 1}{\nu - 1}.$$

Lemma 3.2. *Let $f : [c,d] \to \mathbb{R}$ be a differentiable function on (c,d) such that $f' \in L[c,d]$. Then, the following equality holds for GRL:*

$$\frac{1}{2\Delta(1)} \left[{}^{GRL}_{\hbar}I_{\left(c+d-\frac{x+y}{2}\right)+} f(c+d-x) + {}^{GRL}_{\hbar}I_{\left(c+d-\frac{x+y}{2}\right)-} f(c+d-y) \right] - f\left(c+d-\frac{x+y}{2}\right)$$
$$= \frac{(y-x)}{4\Delta(1)} \int_0^1 \Delta(t) \left[f'\left(c+d-\left(\frac{2-t}{2}x+\frac{t}{2}y\right)\right) - f'\left(c+d-\left(\frac{t}{2}x+\frac{2-t}{2}y\right)\right) \right] dt. \tag{3.6}$$

Proof. The proof of Lemma 3.2 is similar to Lemma 3.1, so we omit it. □

Remark 3.4. *Let the assumptions of Lemma 3.2 be satisfied.*

- *If $\hbar(t) = \frac{t^{\nu}}{\Gamma(\nu)}$, then Lemma 3.2 reduces to [43, Lemma 2].*
- *If $\hbar(t) = \frac{t^{\frac{\nu}{k}}}{k\Gamma_k(\nu)}$ in Lemma 3.2, we get*

$$\frac{2^{\frac{\nu}{k}-1}\Gamma_k(\nu+k)}{(y-x)^{\frac{\nu}{k}}} \left[{}^{RL}_{\hbar}I_{\left(c+d-\frac{x+y}{2}\right)-,k} f(c+d-y) + {}^{RL}_{\hbar}I_{\left(c+d-\frac{x+y}{2}\right)+,k} f(c+d-x) \right]$$
$$- f\left(c+d-\frac{x+y}{2}\right)$$
$$= \frac{y-x}{4} \int_0^1 t^{\frac{\nu}{k}} \left[f'\left(c+d-\left(\frac{2-t}{2}x+\frac{t}{2}y\right)\right) - f'\left(c+d-\left(\frac{t}{2}x+\frac{2-t}{2}y\right)\right) \right] dt. \tag{3.7}$$

- *If $x = c$ and $y = d$, then Lemma 3.2 becomes*

$$\frac{1}{2\Delta(1)}\left[{}^{GRL}_{\hbar}I_{\left(\frac{c+d}{2}\right)-}f(c)+{}^{GRL}_{\hbar}I_{\left(\frac{c+d}{2}\right)+}f(d)\right]-f\left(\frac{c+d}{2}\right)$$
$$=\frac{d-c}{4\Delta(1)}\int_0^1\Delta(t)\left[f'\left(\frac{t}{2}c+\frac{2-t}{2}d\right)-f'\left(\frac{2-t}{2}c+\frac{t}{2}d\right)\right]dt.$$

Corollary 3.3. *Let the assumptions of Lemma 3.2 be satisfied, then the following equality holds for the conformable fractional integrals:*

$$\frac{\nu}{\left[y^{\nu}-\left(\frac{x+y}{2}\right)^{\nu}\right]}\int_{c+d-y}^{c+d-x}f(t)d_{\nu}t=\frac{(y-x)}{4\Delta_1(1)}\int_0^1\Delta_1(t)\left[f'\left(c+d-\left(\frac{2-t}{2}x+\frac{t}{2}y\right)\right)\right.$$
$$\left.-f'\left(c+d-\left(\frac{t}{2}x+\frac{2-t}{2}y\right)\right)\right]dt, \quad (3.8)$$

where

$$\Delta_1(t)=\frac{y^{\nu}-\left(y-\left(\frac{y-x}{2}\right)t\right)^{\nu}}{\nu}.$$

Proof. By setting $\hbar(t)=t(y-t)^{\nu-1}$ in Lemma 3.2, we have proof of Corollary 3.3. □

Remark 3.5. *If we set $x=c$ and $y=d$ in (3.8), we get*

$$\frac{\nu}{\left[d^{\nu}-\left(\frac{d+c}{2}\right)^{\nu}\right]}\int_c^d f(t)d_{\nu}t=\frac{(d-c)}{4\Delta_2(1)}\int_0^1\Delta_2(t)\left[f'\left(\frac{2-t}{2}d+\frac{t}{2}c\right)-f'\left(\frac{t}{2}d+\frac{2-t}{2}c\right)\right]dt,$$

where

$$\Delta_2(t)=\frac{d^{\nu}-\left(d-\left(\frac{d-c}{2}\right)t\right)^{\nu}}{\nu}.$$

Corollary 3.4. *Let the assumptions of Lemma 3.2 be satisfied, then the following equality holds for the fractional integrals with exponential kernel:*

$$\frac{(\nu-1)}{2\left[\exp\left(-\frac{1-\nu}{\nu}\frac{(y-x)}{2}\right)-1\right]}\left[{}^{exp}I^{\nu}_{\left(c+d-\frac{x+y}{2}\right)+}f(c+d-x)\right.$$
$$\left.+{}^{exp}I^{\nu}_{\left(c+d-\frac{x+y}{2}\right)-}f(c+d-y)\right]-f\left(c+d-\frac{x+y}{2}\right)$$
$$=\frac{(y-x)}{4\Delta_3(1)}\int_0^1\Delta_3(t)\left[f'\left(c+d-\left(\frac{2-t}{2}x+\frac{t}{2}y\right)\right)-f'\left(c+d-\left(\frac{t}{2}x+\frac{2-t}{2}y\right)\right)\right]dt, \quad (3.9)$$

where

$$\Delta_3(t)=\frac{\exp\left(-\frac{1-\nu}{\nu}\frac{(y-x)t}{2}\right)-1}{\nu-1}.$$

Proof. By setting $\hbar(t)=\frac{t}{\nu}\exp\left(-\frac{1-\nu}{\nu}t\right)$ in Lemma 3.2, we get proof of Corollary 3.4. □

Remark 3.6. *If we put $x = c$ and $y = d$ in (3.9), we get*

$$\frac{(\nu-1)}{2\left[\exp\left(-\frac{1-\nu}{\nu}\frac{(d-c)}{2}\right)-1\right]}\left[{}^{exp}I^{\nu}_{\left(\frac{c+d}{2}\right)+}f(d)+{}^{exp}I^{\nu}_{\left(\frac{c+d}{2}\right)-}f(c)\right]-f\left(\frac{c+d}{2}\right)$$
$$=\frac{(d-c)}{4\Delta_4(1)}\int_0^1\Delta_4(t)\left[f'\left(\frac{2-t}{2}d+\frac{t}{2}c\right)-f'\left(\frac{t}{2}d+\frac{2-t}{2}c\right)\right]dt,$$

where

$$\Delta_4(t)=\frac{\exp\left(-\frac{1-\nu}{\nu}\frac{(d-c)t}{2}\right)-1}{\nu-1}.$$

Theorem 3.1. *Let $f:[c,d]\to\mathbb{R}$ be a differentiable function on (c,d) such that $|f'|$ is convex on $[c,d]$. Then, the following inequality holds for GRL:*

$$\left|{}^{GRL}_{\hbar}\mathcal{F}(c,d;x,y)\right|\le\frac{(y-x)}{2\Lambda(1)}\left[\left[|f'(c)|+|f'(d)|\right]\int_0^1|\Lambda(t)-\Lambda(1-t)|\,dt\right.$$
$$\left.-\left[|f'(x)|+|f'(y)|\right]\int_0^1 t\,|\Lambda(t)-\Lambda(1-t)|\,dt\right],\quad(3.10)$$

where

$$\left|{}^{GRL}_{\hbar}\mathcal{F}(c,d;x,y)\right|:=\left|\frac{f(c+d-y)+f(c+d-x)}{2}-\frac{1}{2\Lambda(1)}\left[{}^{GRL}_{\hbar}I_{(c+d-y)+}f(c+d-x)\right.\right.$$
$$\left.\left.+{}^{GRL}_{\hbar}I_{(c+d-x)-}f(c+d-y)\right]\right|.$$

Proof. In view of Lemma 3.1, we have

$$\left|{}^{GRL}_{\hbar}\mathcal{F}(c,d;x,y)\right|\le\frac{(y-x)}{2\Lambda(1)}\int_0^1|\Lambda(t)-\Lambda(1-t)|\,|f'(c+d-(tx+(1-t)y))|\,dt.$$

Then, by using the Jensen–Mercer inequality, we obtain

$$\left|{}^{GRL}_{\hbar}\mathcal{F}(c,d;x,y)\right|\le\frac{(y-x)}{2\Lambda(1)}\int_0^1|\Lambda(t)-\Lambda(1-t)|\left[|f'(c)|+|f'(d)|-t\,|f'(x)|-(1-t)\,|f'(y)|\right]dt$$
$$=\frac{(y-x)}{2\Lambda(1)}\left[\int_0^1|\Lambda(t)-\Lambda(1-t)|\left[|f'(c)|+|f'(d)|\right]dt\right.$$
$$\left.-|f'(x)|\int_0^1 t\,|\Lambda(t)-\Lambda(1-t)|\,dt-|f'(y)|\int_0^1(1-t)\,|\Lambda(t)-\Lambda(1-t)|\,dt\right]$$
$$=\frac{(y-x)}{2\Lambda(1)}\left[\left[|f'(c)|+|f'(d)|\right]\int_0^1|\Lambda(t)-\Lambda(1-t)|\,dt\right.$$
$$\left.-\left[|f'(x)|+|f'(y)|\right]\int_0^1 t\,|\Lambda(t)-\Lambda(1-t)|\,dt\right],$$

which completes the proof of Theorem 3.1. □

Remark 3.7. *Let the assumptions of Theorem 3.1 be satisfied. Then,*

- *If* $\hbar(t) = \frac{t^{\nu}}{\Gamma(\nu)}$, *then Theorem 3.1 reduces to [43, Theorem 4].*
- *If* $\hbar(t) = \frac{t^{\frac{\nu}{k}}}{k\Gamma_k(\nu)}$ *in Theorem 3.1, we get*

$$\left|\frac{f(c+d-x)+f(c+d-y)}{2} - \frac{\Gamma_k(\nu+k)}{2(y-x)^{\frac{\nu}{k}}}\left[{}^{RL}_{\hbar}I_{(c+d-y)+,k}f(c+d-x)\right.\right.$$
$$\left.\left.+{}^{RL}_{\hbar}I_{(c+d-x)-,k}f(c+d-y)\right]\right| \le \frac{y-x}{\nu+k}\left(k-\frac{k}{2^{\frac{\nu}{k}}}\right)\left[|f'(c)|+|f'(d)|-\frac{|f'(x)|+|f'(y)|}{2}\right]. \quad (3.11)$$

- *If* $x = c$ *and* $y = d$, *then Theorem 3.1 reduces to [21, Theorem 6].*

Corollary 3.5. *Let the assumptions of Theorem 3.1 be satisfied. Then, we have*

$$\left|\frac{f(c+d-y)+f(c+d-x)}{2} - \frac{1}{y-x}\int_{c+d-y}^{c+d-x} f(x)\,dx\right| \le \frac{1}{4}\left[|f'(c)|+|f'(d)|-\frac{|f'(x)|+|f'(y)|}{2}\right]. \quad (3.12)$$

Proof. If we set $\hbar(t) = t$ in Theorem 3.1, then we have proof of Corollary 3.5. □

Remark 3.8. *If we use* $x = c$ *and* $y = d$ *in Corollary 3.5, then Corollary 3.5 reduces to [47, Theorem 2.2].*

Corollary 3.6. *Let the assumptions of Theorem 3.1 be satisfied. Then, we have the following inequality holds for conformable fractional integrals:*

$$\left|\frac{f(c+d-y)+f(c+d-x)}{2} - \frac{\nu}{y^{\nu}-x^{\nu}}\int_{c+d-y}^{c+d-x} f(t)d_{\nu}t\right| \le \frac{\nu(y-x)}{2(y^{\nu}-x^{\nu})}$$
$$\times\left[\left[|f'(c)|+|f'(d)|\right]\int_0^1 |\Lambda_1(t)-\Lambda_1(1-t)|\,dt - \left[|f'(x)|+|f'(y)|\right]\int_0^1 t\,|\Lambda_1(t)-\Lambda_1(1-t)|\,dt\right]. \quad (3.13)$$

Proof. By setting $\hbar(t) = t(y-t)^{\nu-1}$ in Theorem 3.1, we have proof of Corollary 3.6. □

Remark 3.9. *If we set* $x = c$ *and* $y = d$, *then we have*

$$\left|\frac{f(c)+f(d)}{2} - \frac{\nu}{d^{\nu}-c^{\nu}}\int_c^d f(t)d_{\nu}t\right| \le \frac{\nu(d-c)}{2(d^{\nu}-c^{\nu})}\left[\left[|f'(c)|+|f'(d)|\right]\int_0^1 t\,|\Lambda_2(t)-\Lambda_2(1-t)|\,dt\right].$$

Corollary 3.7. *Let the assumptions of Theorem 3.1 be satisfied. Then, we have the following inequality for fractional integrals with exponential kernel:*

$$\left|\frac{f(c+d-y)+f(c+d-x)}{2} - \frac{(\nu-1)}{2\left[\exp\left(-\frac{1-\nu}{\nu}(y-x)\right)-1\right]}\right.$$
$$\left.\times\left[{}^{exp}I^{\nu}_{(c+d-y)+}f(c+d-x) + {}^{exp}I^{\nu}_{(c+d-x)-}f(c+d-y)\right]\right|$$

$$\leq \frac{(\nu-1)(y-x)}{2\Big[\exp\Big(-\frac{1-\nu}{\nu}(y-x)\Big)-1\Big]}\Bigg[\left[|f'(c)|+|f'(d)|\right]\int_0^1 |\Lambda_3(t)-\Lambda_3(1-t)|\,dt$$
$$-\left[|f'(x)|+|f'(y)|\right]\int_0^1 t\,|\Lambda_3(t)-\Lambda_3(1-t)|\,dt\Bigg]. \quad (3.14)$$

Proof. By setting $\hbar(t)=\frac{t}{\nu}\exp\Big(-\frac{1-\nu}{\nu}t\Big)$ in Theorem 3.1, we get proof of Corollary 3.7. □

Remark 3.10. *If we set $x=c$ and $y=d$ in (3.14), then we have*

$$\left|\frac{f(c)+f(d)}{2}-\frac{(\nu-1)}{2\Big[\exp\Big(-\frac{1-\nu}{\nu}(d-c)\Big)-1\Big]}\left[{}^{exp}I^{\nu}_{c+}f(d)+{}^{exp}I^{\nu}_{d-}f(c)\right]\right|$$
$$\leq \frac{(\nu-1)(d-c)}{2\Big[\exp\Big(-\frac{1-\nu}{\nu}(d-c)\Big)-1\Big]}\left[\left[|f'(c)|+|f'(d)|\right]\int_0^1 t\,|\Lambda_4(t)-\Lambda_4(1-t)|\,dt\right].$$

Theorem 3.2. *Let $f:[c,d]\to\mathbb{R}$ be a differentiable function on (c,d) such that $|f'|^q$ is convex on $[c,d]$ for some $q>1$. Then, the following inequality holds for GRL:*

$$\left|{}^{GRL}_{\hbar}\mathcal{F}(c,d;x,y)\right| \leq \frac{(y-x)}{2\Lambda(1)}\left(\int_0^1 |\Lambda(t)-\Lambda(1-t)|^p\,dt\right)^{\frac{1}{p}}$$
$$\times\left(|f'(c)|^q+|f'(d)|^q-\frac{|f'(x)|^q+|f'(y)|^q}{2}\right)^{\frac{1}{q}}, \quad (3.15)$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. In view of Lemma 3.1 and the well–known Hölder's inequality, one can obtain

$$\left|{}^{GRL}_{\hbar}\mathcal{F}(c,d;x,y)\right| \leq \frac{(y-x)}{2\Lambda(1)}\left(\int_0^1 |\Lambda(t)-\Lambda(1-t)|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1 |f'(c+d-(tx+(1-t)y))|^q\,dt\right)^{\frac{1}{q}}.$$

We can apply the Jensen–Mercer inequality due to the convexity of $|f'|^q$, to get

$$\left|{}^{GRL}_{\hbar}\mathcal{F}(c,d;x,y)\right| \leq \frac{(y-x)}{2\Lambda(1)}\left(\int_0^1 |\Lambda(t)-\Lambda(1-t)|^p\,dt\right)^{\frac{1}{p}}$$
$$\times\left(\int_0^1 \left[|f'(c)|^q+|f'(d)|^q-\left(t\,|f'(x)|^q+(1-t)\,|f'(y)|^q\right)\right]dt\right)^{\frac{1}{q}}$$
$$=\frac{(y-x)}{2\Lambda(1)}\left(\int_0^1 |\Lambda(t)-\Lambda(1-t)|^p\,dt\right)^{\frac{1}{p}}\left(|f'(c)|^q+|f'(d)|^q-\frac{|f'(x)|^q+|f'(y)|^q}{2}\right)^{\frac{1}{q}},$$

which completes the proof of Theorem 3.2. □

Corollary 3.8. *Let the assumptions of Theorem 3.2 be satisfied, then we have*

$$\left|\frac{f(c+d-y)+f(c+d-x)}{2}-\frac{1}{y-x}\int_{c+d-y}^{c+d-x}f(x)\,dx\right|\le\frac{(y-x)}{2(1+p)^{\frac{1}{p}}}$$
$$\times\left(|f'(c)|^q+|f'(d)|^q-\frac{|f'(x)|^q+|f'(y)|^q}{2}\right)^{\frac{1}{q}}. \quad (3.16)$$

Remark 3.11. *If we use $x = c$ and $y = d$ in Corollary 3.8, then Corollary 3.8 reduces to [47, Theorem 2.3].*

Proof. By using $\hbar(t) = t$ in inequality (3.15), we can obtain inequality (3.16). □

Corollary 3.9. *Let the assumptions of Theorem 3.2 be satisfied, then we have the following inequality holds for RL:*

$$\left|\frac{f(c+d-y)+f(c+d-x)}{2}-\frac{\Gamma(\nu+1)}{2(y-x)^{\nu}}\left[{}^{RL}I^{\nu}_{(c+d-y)+}f(c+d-x)+{}^{RL}I^{\nu}_{(c+d-x)-}f(c+d-y)\right]\right|$$
$$\le\frac{(y-x)}{2(\nu p+1)^{\frac{1}{p}}}\left(|f'(c)|^q+|f'(d)|^q-\frac{|f'(x)|^q+|f'(y)|^q}{2}\right)^{\frac{1}{q}}. \quad (3.17)$$

Proof. By setting $\hbar(t) = \frac{t^{\nu}}{\Gamma(\nu)}$ in inequality (3.15), we obtain inequality (3.17). □

Remark 3.12. *If we set $x = c$ and $y = d$ in Corollary 3.9, then we have*

$$\left|\frac{f(c)+f(d)}{2}-\frac{\Gamma(\nu+1)}{2(d-c)^{\nu}}\left[{}^{RL}I^{\nu}_{c+}f(d)+{}^{RL}I^{\nu}_{d-}f(c)\right]\right|\le\frac{(d-c)}{2(\nu p+1)^{\frac{1}{p}}}\left(\frac{|f'(c)|^q+|f'(d)|^q}{2}\right)^{\frac{1}{q}}.$$

Corollary 3.10. *Let the assumptions of Theorem 3.2 be satisfied, then we have for k–RL:*

$$\left|\frac{f(c+d-y)+f(c+d-x)}{2}-\frac{\Gamma_k(\nu+k)}{2(y-x)^{\frac{\nu}{k}}}\left[{}^{RL}I^{\nu}_{(c+d-y)+,k}f(c+d-x)+{}^{RL}I^{\nu}_{(c+d-x)-,k}f(c+d-y)\right]\right|$$
$$\le\frac{(y-x)}{2\left(\frac{\nu}{k}p+1\right)^{\frac{1}{p}}}\left(|f'(c)|^q+|f'(d)|^q-\frac{|f'(x)|^q+|f'(y)|^q}{2}\right)^{\frac{1}{q}}. \quad (3.18)$$

Proof. By setting $\hbar(t) = \frac{t^{\frac{\nu}{k}}}{k\Gamma_k(\nu)}$ in inequality (3.15), we can obtain inequality (3.18). □

Remark 3.13. *If we set $x = c$ and $y = d$ in Corollary 3.10, then we obtain*

$$\left|\frac{f(c)+f(d)}{2}-\frac{\Gamma_k(\nu+k)}{2(d-c)^{\frac{\nu}{k}}}\left[{}^{RL}I^{\nu}_{c+,k}f(d)+{}^{RL}I^{\nu}_{d-,k}f(c)\right]\right|\le\frac{(d-c)}{2\left(\frac{\nu}{k}p+1\right)^{\frac{1}{p}}}\left(\frac{|f'(c)|^q+|f'(d)|^q}{2}\right)^{\frac{1}{q}}.$$

Corollary 3.11. *Let the assumptions of Theorem 3.2 be satisfied, then we have the following inequality for the conformable fractional integrals:*

$$\left|\frac{f(c+d-y)+f(c+d-x)}{2}-\frac{\nu}{y^{\nu}-x^{\nu}}\int_{c+d-y}^{c+d-x}f(t)d_{\nu}t\right|\le\frac{\nu(y-x)}{2(y^{\nu}-x^{\nu})}$$
$$\times\left(\int_0^1|\Lambda_1(t)-\Lambda_1(1-t)|^p\,dt\right)^{\frac{1}{p}}\left(|f'(c)|^q+|f'(d)|^q-\frac{|f'(x)|^q+|f'(y)|^q}{2}\right)^{\frac{1}{q}}. \quad (3.19)$$

Proof. By setting $\hbar(t) = t(y-t)^{\nu-1}$ in Theorem 3.2, we get proof of Corollary 3.11. □

Remark 3.14. *If we set $x = c$ and $y = d$ in (3.19), then we have*

$$\left|\frac{f(c)+f(d)}{2} - \frac{\nu}{d^\nu - c^\nu}\int_c^d f(t)d_\nu t\right| \le \frac{\nu(d-c)}{2(d^\nu - c^\nu)}\left(\int_0^1 |\Lambda_2(t) - \Lambda_2(1-t)|^p dt\right)^{\frac{1}{p}} \times \left(\frac{|f'(c)|^q + |f'(d)|^q}{2}\right)^{\frac{1}{q}}.$$

Corollary 3.12. *Let the assumptions of Theorem 3.2 be satisfied, then we have the following inequality for the fractional integrals with exponential kernel:*

$$\begin{aligned}&\left|\frac{f(c+d-y)+f(c+d-x)}{2} - \frac{(\nu-1)}{2\left[\exp\left(-\frac{1-\nu}{\nu}(y-x)\right)-1\right]}\right.\\ &\qquad\left.\times\left[{}^{exp}I^\nu_{(c+d-y)+}f(c+d-x) + {}^{exp}I^\nu_{(c+d-x)-}f(c+d-y)\right]\right|\\ &\le \frac{(\nu-1)(y-x)}{2\left[\exp\left(-\frac{1-\nu}{\nu}(y-x)\right)-1\right]}\left(\int_0^1 |\Lambda_3(t)-\Lambda_3(1-t)|^p dt\right)^{\frac{1}{p}}\\ &\qquad\times\left(|f'(c)|^q + |f'(d)|^q - \frac{|f'(x)|^q + |f'(y)|^q}{2}\right)^{\frac{1}{q}}. \qquad (3.20)\end{aligned}$$

Proof. By setting $\hbar(t) = \frac{t}{\nu}\exp\left(-\frac{1-\nu}{\nu}t\right)$ in Theorem 3.2, we have proof of Corollary 3.12. □

Remark 3.15. *If we set $x = c$ and $y = d$ in (3.20), then we have*

$$\begin{aligned}&\left|\frac{f(c)+f(d)}{2} - \frac{(\nu-1)}{2\left[\exp\left(-\frac{1-\nu}{\nu}(d-c)\right)-1\right]}\left[{}^{exp}I^\nu_{c+}f(d) + {}^{exp}I^\nu_{d-}f(c)\right]\right|\\ &\le \frac{(\nu-1)(d-c)}{2\left[\exp\left(-\frac{1-\nu}{\nu}(d-c)\right)-1\right]}\left(\int_0^1 |\Lambda_4(t)-\Lambda_4(1-t)|^p dt\right)^{\frac{1}{p}}\left(\frac{|f'(c)|^q + |f'(d)|^q}{2}\right)^{\frac{1}{q}}.\end{aligned}$$

Theorem 3.3. *Let $f : [c,d] \to \mathbb{R}$ be a differentiable function on (c,d) such that $|f|$ is convex on $[c,d]$. Then, the following inequality holds for GRL:*

$$\left|{}^{GRL}_{\hbar}\mathcal{G}(c,d;x,y)\right| \le \frac{(y-x)}{2\Delta(1)}\left[|f'(c)| + |f'(d)| - \frac{|f'(x)| + |f'(y)|}{2}\right]\int_0^1 |\Delta(t)|\,dt, \qquad (3.21)$$

where

$$\begin{aligned}\left|{}^{GRL}_{\hbar}\mathcal{G}(c,d;x,y)\right| := \Bigg|\frac{1}{2\Delta(1)}\Big[&{}^{GRL}_{\hbar}I_{\left(c+d-\frac{x+y}{2}\right)+}f(c+d-x)\\ &+ {}^{GRL}_{\hbar}I_{\left(c+d-\frac{x+y}{2}\right)-}f(c+d-y)\Big] - f\left(c+d-\frac{x+y}{2}\right)\Bigg|.\end{aligned}$$

Proof. From Lemma 3.2 , we have

$$\left|{}^{GRL}_{\hbar}\mathcal{G}(c,d;x,y)\right| \leq \frac{(y-x)}{4\Delta(1)}\left[\int_0^1 |\Delta(t)|\left|f'\left(c+d-\left(\frac{2-t}{2}x+\frac{t}{2}y\right)\right)\right|dt\right.$$
$$\left.+\int_0^1 |\Delta(t)|\left|f'\left(c+d-\left(\frac{t}{2}x+\frac{2-t}{2}y\right)\right)\right|dt\right]$$

Then, by using the Jensen–Mercer inequality, we get

$$\begin{aligned}\left|{}^{GRL}_{\hbar}\mathcal{G}(c,d;x,y)\right| &\leq \frac{(y-x)}{4\Delta(1)}\left[\int_0^1 |\Delta(t)|\left(|f'c|+|f'(d)|-\left(\frac{2-t}{2}|f'(x)|+\frac{t}{2}|f'(y)|\right)\right)dt\right.\\&\left.+\int_0^1 |\Delta(t)|\left(|f'(c)|+|f'(d)|-\left(\frac{t}{2}|f'(x)|+\frac{2-t}{2}|f'(y)|\right)\right)dt\right]\\&=\frac{(y-x)}{4\Delta(1)}\left[\int_0^1 |\Delta(t)|\left[2|f'(c)|+2|f'(d)|-(|f'(x)|+|f'(y)|)\right]dt\right]\\&=\frac{(y-x)}{2\Delta(1)}\left[|f'(c)|+|f'(d)|-\frac{|f'(x)|+|f'(y)|}{2}\right]\int_0^1|\Delta(t)|\,dt,\end{aligned}$$

which completes the proof of Theorem 3.3. □

Remark 3.16. *Let the assumptions of Theorem 3.3 be satisfied. Then, the following special cases can be considered.*

- *If $\hbar(t)=t$, then Theorem 3.3 reduces to [43, Corollary 2].*
- *If $\hbar(t)=t$, $x=c$ and $y=d$, then Theorem 3.3 reduces to [48, Theorem 2.2].*
- *If $\hbar(t)=\frac{t^\nu}{\Gamma(\nu)}$, then Theorem 3.3 reduces to [43, Theorem 5].*
- *If $\hbar(t)=\frac{t^\nu}{\Gamma(\nu)}$, $x=c$ and $y=d$, then Theorem 3.3 reduces to [26, Theorem 5] with $q=1$.*
- *If $\hbar(t)=\frac{t^{\frac{\nu}{k}}}{k\Gamma_k(\nu)}$ in Theorem 3.3, we get*

$$\left|\frac{2^{\frac{\nu}{k}-1}\Gamma_k(\nu+k)}{(y-x)^{\frac{\nu}{k}}}\left[{}^{RL}_{\hbar}I_{\left(c+d-\frac{x+y}{2}\right)-,k}f(c+d-y)+{}^{RL}_{\hbar}I_{\left(c+d-\frac{x+y}{2}\right)+,k}f(c+d-x)\right]\right.$$
$$\left.-f\left(c+d-\frac{x+y}{2}\right)\right|\leq\frac{k(y-x)}{2(\nu+k)}\left[|f'(c)|+|f'(d)|-\frac{|f'(x)|+|f'(y)|}{2}\right]. \quad (3.22)$$

- *If we set $\hbar(t)=\frac{t^{\frac{\nu}{k}}}{k\Gamma_k(\nu)}$, $x=c$ and $y=d$ in Theorem 3.3, then we have*

$$\left|\frac{2^{\frac{\nu-k}{k}}\Gamma_k(\nu+k)}{(d-c)^{\frac{\nu}{k}}}\left[{}^{RL}I^{\nu}_{\left(\frac{c+d}{2}\right)+,k}f(d)+{}^{RL}I^{\nu}_{\left(\frac{c+d}{2}\right)-,k}f(c)\right]-f\left(\frac{c+d}{2}\right)\right|\leq\frac{k(d-c)}{2(\nu+k)}\left[\frac{|f'(c)|+|f'(d)|}{2}\right].$$

Corollary 3.13. *Let the assumptions of Theorem 3.3 be satisfied. Then, the following inequality holds for the conformable fractional integrals:*

$$\left|\frac{\nu}{\left[y^\nu-\left(\frac{x+y}{2}\right)^\nu\right]}\int_{c+d-y}^{c+d-x}f(t)d_\nu t\right|\leq\frac{\nu(y-x)}{2\left[y^\nu-\left(\frac{x+y}{2}\right)^\nu\right]}$$
$$\times\left[|f'(c)|+|f'(d)|-\frac{|f'(x)|+|f'(y)|}{2}\right]\int_0^1|\Delta_1(t)|\,dt. \quad (3.23)$$

Proof. By setting $\hbar(t) = t(y-t)^{\nu-1}$ in Theorem 3.3, we can get proof of Corollary 3.13. □

Remark 3.17. *If we set $x = c$ and $y = d$ in (3.23), then we have*

$$\left|\frac{\nu}{\left[d^{\nu}-\left(\frac{c+d}{2}\right)^{\nu}\right]}\int_{c}^{d} f(t)d_{\nu}t\right| \leq \frac{\nu(d-c)}{2\left[d^{\nu}-\left(\frac{c+d}{2}\right)^{\nu}\right]}\left[\frac{|f'(c)|+|f'(d)|}{2}\right]\int_{0}^{1}|\Delta_{2}(t)|\,dt.$$

Corollary 3.14. *Let the assumptions of Theorem 3.3 be satisfied. Then, the following inequality holds for the fractional integrals with exponential kernel:*

$$\begin{aligned}&\left|\frac{(\nu-1)}{2\left[\exp\left(-\frac{1-\nu}{\nu}\frac{(y-x)}{2}\right)-1\right]}\left[\left[{}^{exp}I^{\nu}_{\left(c+d-\frac{x+y}{2}\right)+}f(c+d-x)\right.\right.\right.\\&\qquad\left.\left.\left.+{}^{exp}I^{\nu}_{\left(c+d-\frac{x+y}{2}\right)-}f(c+d-y)\right]\right]-f\left(c+d-\frac{x+y}{2}\right)\right|\\&\qquad\leq\frac{(\nu-1)(y-x)}{2\left[\exp\left(-\frac{1-\nu}{\nu}\frac{(y-x)}{2}\right)-1\right]}\left[|f'(c)|+|f'(d)|-\frac{|f'(x)|+|f'(y)|}{2}\right]\int_{0}^{1}|\Delta_{3}(t)|\,dt. \quad (3.24)\end{aligned}$$

Proof. By setting $\hbar(t) = \frac{t}{\nu}\exp\left(-\frac{1-\nu}{\nu}t\right)$ in Theorem 3.3, we can obtain proof of Corollary 3.14. □

Remark 3.18. *If we set $x = c$ and $y = d$ in (3.24), then we have*

$$\begin{aligned}&\left|\frac{(\nu-1)}{2\left[\exp\left(-\frac{1-\nu}{\nu}\frac{(d-c)}{2}\right)-1\right]}\left[\left[{}^{exp}I^{\nu}_{\left(\frac{c+d}{2}\right)+}f(d)+{}^{exp}I^{\nu}_{\left(\frac{c+d}{2}\right)-}f(c)\right]\right]-f\left(\frac{c+d}{2}\right)\right|\\&\qquad\leq\frac{(\nu-1)(d-c)}{2\left[\exp\left(-\frac{1-\nu}{\nu}\frac{(d-c)}{2}\right)-1\right]}\left[\frac{|f'(c)|+|f'(d)|}{2}\right]\int_{0}^{1}|\Delta_{4}(t)|\,dt.\end{aligned}$$

Theorem 3.4. *Let $f : [c,d] \to \mathbb{R}$ be a differentiable function on (c,d) such that $|f'|^q$ is convex on $[c,d]$ for some $q > 1$. Then, the following inequality holds for GRL:*

$$\begin{aligned}\left|{}^{GRL}_{\hbar}\mathcal{G}(c,d;x,y)\right| &\leq \frac{(y-x)}{4\Delta(1)}\left(\int_{0}^{1}|\Delta(t)|^{p}\,dt\right)^{\frac{1}{p}}\left[\left(|f'(c)|^{q}+|f'(d)|^{q}-\frac{3|f'(x)|^{q}+|f'(y)|^{q}}{4}\right)^{\frac{1}{q}}\right.\\&\qquad\left.+\left(|f'(c)|^{q}+|f'(d)|^{q}-\left(\frac{|f'(x)|^{q}+3|f'(y)|^{q}}{4}\right)\right)^{\frac{1}{q}}\right], \quad (3.25)\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q} = 1$.

Proof. From Lemma 3.2 and well-known Hölder's inequality, we obtain

$$\begin{aligned}\left|{}^{GRL}_{\hbar}\mathcal{G}(c,d;x,y)\right| &\leq \frac{(y-x)}{4\Delta(1)}\left(\int_{0}^{1}|\Delta(t)|^{p}\,dt\right)^{\frac{1}{p}}\left[\left(\int_{0}^{1}\left|f'\left(c+d-\left(\frac{2-t}{2}x+\frac{t}{2}y\right)\right)\right|^{q}dt\right)^{\frac{1}{q}}\right.\\&\qquad\left.+\left(\int_{0}^{1}\left|f'\left(c+d-\left(\frac{t}{2}x+\frac{2-t}{2}y\right)\right)\right|^{q}dt\right)^{\frac{1}{q}}\right].\end{aligned}$$

By applying the Jensen–Mercer inequality due to convexity of $|f'|^q$, we can obtain

$$\begin{aligned}\left|{}^{GRL}_{\hbar}\mathcal{G}(c,d;x,y)\right| &\leq \frac{(y-x)}{4\Delta(1)}\left(\int_0^1 |\Delta(t)|^p\,dt\right)^{\frac{1}{p}}\left[\left(\int_0^1\left[|f'(c)|^q+|f'(d)|^q-\left(\frac{2-t}{2}|f'(x)|^q+\frac{t}{2}|f'(y)|^q\right)\right]dt\right)^{\frac{1}{q}}\right.\\&\left.+\left(\int_0^1\left[|f'(c)|^q+|f'(d)|^q-\left(\frac{t}{2}|f'(x)|^q+\frac{2-t}{2}|f'(y)|^q\right)\right]dt\right)^{\frac{1}{q}}\right]\\&=\frac{(y-x)}{4\Delta(1)}\left(\int_0^1 |\Delta(t)|^p\,dt\right)^{\frac{1}{p}}\left[\left(|f'(c)|^q+|f'(d)|^q-\frac{3|f'(x)|^q+|f'(y)|^q}{4}\right)^{\frac{1}{q}}\right.\\&\left.+\left(|f'(c)|^q+|f'(d)|^q-\left(\frac{|f'(x)|^q+3|f'(y)|^q}{4}\right)\right)^{\frac{1}{q}}\right],\end{aligned}$$

and this completes proof of the Theorem 3.4. □

Remark 3.19. *Let the assumptions of Theorem 3.4 be satisfied. Then, the following special cases can be considered.*

- *If $\hbar(t)=t$, then Theorem 3.4 reduces to [43, Corollary 3].*
- *If $\hbar(t)=t$, $x=c$ and $y=d$, then Theorem 3.4 reduces to [48, Theorem 2.3].*
- *If $\hbar(t)=\frac{t^\nu}{\Gamma(\nu)}$, then Theorem 3.4 reduces to [43, Theorem 6].*
- *If $\hbar(t)=\frac{t^\nu}{\Gamma(\nu)}$, $x=c$ and $y=d$, then Theorem 3.4 reduces to [26, Theorem 6].*
- *If $\hbar(t)=\frac{t^{\frac{\nu}{k}}}{k\Gamma_k(\nu)}$ in Theorem 3.4, we get*

$$\begin{aligned}&\left|\frac{2^{\frac{\nu}{k}-1}\Gamma_k(\nu+k)}{(y-x)^{\frac{\nu}{k}}}\left[{}^{RL}_{\hbar}I_{\left(c+d-\frac{x+y}{2}\right)-,k}f(c+d-y)+{}^{RL}_{\hbar}I_{\left(c+d-\frac{x+y}{2}\right)+,k}f(c+d-x)\right]\right.\\&\left.-f\left(c+d-\frac{x+y}{2}\right)\right|\leq\frac{y-x}{4}\left(\frac{k}{\nu p+k}\right)^{\frac{1}{p}}\left[\left(|f'(c)|^q+|f'(d)|^q-\frac{3|f'(x)|^q+|f'(y)|^q}{4}\right)^{\frac{1}{q}}\right.\\&\left.+\left(|f'(c)|^q+|f'(d)|^q-\frac{|f'(x)|^q+3|f'(y)|^q}{4}\right)^{\frac{1}{q}}\right].\end{aligned}\tag{3.26}$$

- *If $\hbar(t)=\frac{t^{\frac{\nu}{k}}}{k\Gamma_k(\nu)}$, $x=c$ and $y=d$, then we have*

$$\begin{aligned}&\left|\frac{2^{\frac{\nu-k}{k}}\Gamma_k(\nu+k)}{(d-c)^{\frac{\nu}{k}}}\left[{}^{RL}I^{\nu}_{\left(\frac{c+d}{2}\right)+,k}f(d)+{}^{RL}I^{\nu}_{\left(\frac{c+d}{2}\right)-,k}f(c)\right]-f\left(\frac{c+d}{2}\right)\right|\\&\leq\frac{(d-c)}{4}\left(\frac{k}{\nu p+k}\right)^{\frac{1}{p}}\left[\left(\frac{|f'(c)|^q+3|f'(d)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{3|f'(c)|^q+|f'(d)|^q}{4}\right)^{\frac{1}{q}}\right].\end{aligned}$$

Corollary 3.15. *Let the assumptions of Theorem 3.4 be satisfied. Then, the following inequality holds for the conformable fractional integrals:*

$$\left|\frac{\nu}{\left[y^\nu-\left(\frac{x+y}{2}\right)^\nu\right]}\int_{c+d-y}^{c+d-x}f(t)d_\nu t\right|\leq\frac{\nu(y-x)}{4\left(y^\nu-\left(\frac{x+y}{2}\right)^\nu\right)}\left(\int_0^1|\Delta_1(t)|^p\,dt\right)^{\frac{1}{p}}$$

$$\times\left[\left(|f'(c)|^q+|f'(d)|^q-\frac{3|f'(x)|^q+|f'(y)|^q}{4}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(|f'(c)|^q+|f'(d)|^q-\left(\frac{|f'(x)|^q+3|f'(y)|^q}{4}\right)\right)^{\frac{1}{q}}\right]. \quad (3.27)$$

Proof. By setting $\hbar(t)=t(y-t)^{\nu-1}$ in Theorem 3.4, we can obtain proof of Corollary 3.15. □

Remark 3.20. *If we set $x=c$ and $y=d$ in (3.27), then we have*

$$\left|\frac{\nu}{\left[d^\nu-\left(\frac{c+d}{2}\right)^\nu\right]}\int_c^d f(t)d_\nu t\right|\leq\frac{\nu(d-c)}{4\left(d^\nu-\left(\frac{c+d}{2}\right)^\nu\right)}\left(\int_0^1|\Delta_2(t)|^p\,dt\right)^{\frac{1}{p}}\left[\left(\frac{|f'(c)|^q+3|f'(d)|^q}{4}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{3|f'(c)|^q+|f'(d)|^q}{4}\right)^{\frac{1}{q}}\right].$$

Corollary 3.16. *Let the assumptions of Theorem 3.4 be satisfied. Then, the following inequality holds for the fractional integrals with exponential kernel:*

$$\left|\frac{(\nu-1)}{2\left[\exp\left(-\frac{1-\nu}{\nu}\frac{(y-x)}{2}\right)-1\right]}\left[\left[{}^{exp}I^\nu_{\left(c+d-\frac{x+y}{2}\right)+}f(c+d-x)\right.\right.\right.$$
$$\left.\left.\left.+\,{}^{exp}I^\nu_{\left(c+d-\frac{x+y}{2}\right)-}f(c+d-y)\right]\right]-f\left(c+d-\frac{x+y}{2}\right)\right|$$
$$\leq\frac{(\nu-1)(y-x)}{4\left[\exp\left(-\frac{1-\nu}{\nu}\frac{(y-x)}{2}\right)-1\right]}\left(\int_0^1|\Delta_3(t)|^p\,dt\right)^{\frac{1}{p}}\left[\left(|f'(c)|^q+|f'(d)|^q-\frac{3|f'(x)|^q+|f'(y)|^q}{4}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(|f'(c)|^q+|f'(d)|^q-\left(\frac{|f'(x)|^q+3|f'(y)|^q}{4}\right)\right)^{\frac{1}{q}}\right]. \quad (3.28)$$

Proof. By setting $\hbar(t)=\frac{t}{\nu}\exp\left(-\frac{1-\nu}{\nu}t\right)$ in Theorem 3.4, we can obtain proof of Corollary 3.16. □

Remark 3.21. *If we set $x=c$ and $y=d$ in (3.28), then we have*

$$\left|\frac{(\nu-1)}{2\left[\exp\left(-\frac{1-\nu}{\nu}\frac{(d-c)}{2}\right)-1\right]}\left[\left[{}^{exp}I^\nu_{\left(\frac{c+d}{2}\right)+}f(d)+{}^{exp}I^\nu_{\left(\frac{c+d}{2}\right)-}f(c)\right]\right]-f\left(\frac{c+d}{2}\right)\right|$$
$$\leq\frac{(\nu-1)(d-c)}{4\left[\exp\left(-\frac{1-\nu}{\nu}\frac{(d-c)}{2}\right)-1\right]}\left(\int_0^1|\Delta_4(t)|^p\,dt\right)^{\frac{1}{p}}\left[\left(\frac{|f'(c)|^q+3|f'(d)|^q}{4}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{3|f'(c)|^q+|f'(d)|^q}{4}\right)^{\frac{1}{q}}\right].$$

4. Conclusions

In this work inequalities of Hermite-Hadamard-Mercer type via generalized fractional integrals are obtained. It is also proved that the results in this paper are generalization of the several existing comparable results in literature. As future direction, one may finds some new interesting inequalities through different types of convexities. Our results may stimulate further research in different areas of pure and applied sciences.

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Conflict of interest

The authors declare that they have no conflict of interest.

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