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Research article

On the unique solution of a class of absolute value equations $Ax - B|Cx| = d$

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Abstract: In this paper, a class of absolute value equations (AVE) $Ax - B|Cx| = d$ with $A, B, C \in \mathbb{R}^{n\times n}$ is considered, which is a generalized form of the published works by Wu [1], Wu and Shen [2] and Mezzadri [3]. Some conditions to guarantee the unique solution of the above AVE are gained. The corresponding results of the above published works are generalized.

Keywords: sufficient condition; unique solution; absolute value equation
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1. Introduction

At present, the absolute value equation (AVE)

$$Ax - |x| = d \tag{1.1}$$

and its general form

$$Ax - B|x| = d \tag{1.2}$$

play an important role in the field of optimization (such as the complementarity problem, linear programming and convex quadratic programming), where $A, B \in \mathbb{R}^{n\times n}$. Hereby, the AVE (1.1) or (1.2) brought about widespread attention in the past several years.

Roughly speaking, the research of the AVE (1.1) or (1.2) is twofold: one, to design multifarious numerical methods for obtaining its numerical solution, see [4–13]; and, two, to present some conditions for the existence of solvability, bounds for the solutions, various equivalent reformulations, and so on, see [2, 3, 14–20].

In [1], for $A, B \in \mathbb{R}^{n\times n}$, Wu found a type of new generalized absolute value equations (NGAVE) below

$$Ax - |Bx| = d, \tag{1.3}$$

which is still from the linear complementarity problem. For the unique solution of the NGAVE (1.3), some necessary and sufficient conditions were presented in [1]. Obviously, the NGAVE (1.3) is quite other than the AVE (1.2). Whereas, if matrix $B = I$ in (1.2) and (1.3), where I stands for the identity matrix, then both reduce to the AVE (1.1).

Inspired by this flurry of activities, we further generalize the concept of absolute value equation and become interested in undertaking a further study of the following absolute value equation

$$Ax - B|Cx| = d, \tag{1.4}$$

where $A, B, C \in \mathbb{R}^{n\times n}$. Clearly, the AVE (1.4) is a generalized form of the above three types of absolute value equations.

Similarly to the above three types of absolute value equations, the analysis of the AVE (1.4) is interesting and challenging, as a consequence of the nonlinear and nondifferentiable term $B|Cx|$ in (1.4). To our knowledge, nobody has studied the AVE (1.4) as yet. This implies that so far it is *vacant* for some conditions to guarantee the unique solution of the AVE (1.4). Therefore, the aim of this paper is to fill in this gap in the literature. Namely, some conditions to guarantee the unique solution of the AVE (1.4) can be gained.

2. Main results

In this section, some conditions to guarantee the unique solution of the AVE (1.4) are presented. To achieve this goal, the following lemmas can be found in [1–3]. In the whole text, σ_1, σ_n and ρ, respectively, denote the largest singular value, the smallest singular value and the spectral radius of the matrix.

Lemma 2.1. [1] *The following statements are equal:*

(1) *the AVE* (1.3) *has a unique solution for any* $d \in \mathbb{R}^n$ *;*

(2) $det(F_1(A + B) + F_2(A - B)) \neq 0$, *where* $F_1, F_2 \in \mathbb{R}^{n\times n}$ *are two arbitrary nonnegative diagonal matrices with* $diag(F_1 + F_2) > 0$*;*

(3) $\{A + B, A - B\}$ *has the row* $\mathcal{W}$*-property;*

(4) $\det(A + B) \neq 0$ *and* $\{I, (A - B)(A + B)^{-1}\}$ *has the row* $\mathcal{W}$*-property;*

(5) $\det(A + B) \neq 0$ *and* $(A - B)(A + B)^{-1}$ *is a P-matrix;*

(6) $\det(A + (I - 2D)B) \neq 0$ *for any diagonal matrix* $D = diag(d_i)$ *with* $d_i \in [0, 1]$.

Lemma 2.2. [1] *Let* $\det(A) \neq 0$ *in* (1.3)*. When*

$$\rho((I - 2D)BA^{-1}) < 1 \tag{2.1}$$

for any diagonal matrix $D = diag(d_i)$ *with* $d_i \in [0, 1]$*, or*

$$\sigma_1(BA^{-1}) < 1, \tag{2.2}$$

or

$$\rho(|BA^{-1}|) < 1, \tag{2.3}$$

the AVE (1.3) *has a unique solution for any* $d \in \mathbb{R}^n$.

Lemma 2.3. [2,3] *The following statements are equal:*

(1) *the AVE* (1.2) *has a unique solution for any* $d \in \mathbb{R}^n$*;*

(2) *det*$((A - B)F_1 + (A + B)F_2) \neq 0$*, where* $F_1, F_2 \in \mathbb{R}^{n\times n}$ *are two arbitrary nonnegative diagonal matrices with diag*$(F_1 + F_2) > 0$*;*

(3) $\{A - B, A + B\}$ *has the column* $\mathcal{W}$*-property;*

(4) $\det(A - B) \neq 0$ *and* $\{I, (A - B)^{-1}(A + B)\}$ *has the column* $\mathcal{W}$*-property;*

(5) $\det(A - B) \neq 0$ *and* $(A - B)^{-1}(A + B)$ *is a P-matrix;*

(6) $\det(A + B(I - 2D)) \neq 0$ *for any diagonal matrix* $D = diag(d_i)$ *with* $d_i \in [0, 1]$.

Lemma 2.4. [2,3] *Let* $\det(A) \neq 0$ *in* (1.2)*. When*

$$\rho(A^{-1}B(I - 2D)) < 1 \tag{2.4}$$

for any diagonal matrix $D = diag(d_i)$ *with* $d_i \in [0, 1]$*, or*

$$\sigma_1(A^{-1}B) < 1, \tag{2.5}$$

or

$$\rho(|A^{-1}B|) < 1, \tag{2.6}$$

the AVE (1.2) *has a unique solution for any* $d \in \mathbb{R}^n$*.*

Based on the above lemmas, naturally, we need to consider two cases for the AVE (1.4): (1) matrix B is nonsingular; (2) matrix C is nonsingular.

Case I. When $\det(B) \neq 0$, the AVE (1.4) can be equivalently expressed as the following AVE

$$B^{-1}Ax - |Cx| = B^{-1}d. \tag{2.7}$$

Based on Lemmas 2.1 and 2.2, for the AVE (1.4), we can obtain the following results, see Theorems 2.1 and 2.2.

Theorem 2.1. *If* $\det(B) \neq 0$*, then the following statements are equal:*

(1) *the AVE* (1.4) *has a unique solution for any* $d \in \mathbb{R}^n$*;*

(2) *det*$(F_1(B^{-1}A+C)+F_2(B^{-1}A-C)) \neq 0$*, where* $F_1, F_2 \in \mathbb{R}^{n\times n}$ *are two arbitrary nonnegative diagonal matrices with diag*$(F_1 + F_2) > 0$*;*

(3) $\{B^{-1}A + C, B^{-1}A - C\}$ *has the row* $\mathcal{W}$*-property;*

(4) $\det(B^{-1}A + C) \neq 0$ *and* $\{I, (B^{-1}A - C)(B^{-1}A + C)^{-1}\}$ *has the row* $\mathcal{W}$*-property;*

(5) $\det(B^{-1}A + C) \neq 0$ *and* $(B^{-1}A - C)(B^{-1}A + C)^{-1}$ *is a P-matrix;*

(6) $\det(A + B(I - 2D)C) \neq 0$ *for any diagonal matrix* $D = diag(d_i)$ *with* $d_i \in [0, 1]$.

Theorem 2.2. *Let* $\det(A) \neq 0$ *and* $\det(B) \neq 0$ *in* (1.4)*. When*

$$\rho((I-2D)CA^{-1}B) < 1 \tag{2.8}$$

for any diagonal matrix $D = diag(d_i)$ *with* $d_i \in [0,1]$*, or*

$$\sigma_1(CA^{-1}B) < 1, \tag{2.9}$$

or

$$\rho(|CA^{-1}B|) < 1, \tag{2.10}$$

the AVE (1.4) *has a unique solution for any* $d \in \mathbb{R}^n$*.*

Clearly, Theorems 2.1 and 2.2 are generalization forms of the results in Lemmas 2.1 and 2.2, respectively.

Case II. When $\det(C) \neq 0$, the AVE (1.4) can be equivalently expressed as the following AVE

$$AC^{-1}y - B|y| = d, \text{with } y = Cx. \tag{2.11}$$

Based on Lemmas 2.3 and 2.4, for the AVE (1.4), we can obtain the following results, see Theorems 2.3 and 2.4.

Theorem 2.3. *If* $\det(C) \neq 0$*, then the following statements are equal:*

(1) *the AVE* (1.4) *has a unique solution for any* $d \in \mathbb{R}^n$*;*

(2) $det((AC^{-1}+B)F_1+(AC^{-1}-B)F_2) \neq 0$*, where* $F_1, F_2 \in \mathbb{R}^{n\times n}$ *are two arbitrary nonnegative diagonal matrices with* $diag(F_1 + F_2) > 0$*;*

(3) $\{AC^{-1} - B, AC^{-1} + B\}$ *has the column* $\mathcal{W}$*-property;*

(4) $\det(AC^{-1} - B) \neq 0$ *and* $\{I, (AC^{-1} - B)^{-1}(AC^{-1} + B)\}$ *has the column* $\mathcal{W}$*-property;*

(5) $\det(AC^{-1} - B) \neq 0$ *and* $(AC^{-1} - B)^{-1}(AC^{-1} + B)$ *is a P-matrix;*

(6) $\det(A + B(I - 2D)C) \neq 0$ *for any diagonal matrix* $D = diag(d_i)$ *with* $d_i \in [0,1]$.

Theorem 2.4. *Let* $\det(A) \neq 0$ *and* $\det(C) \neq 0$ *in* (1.4)*. When*

$$\rho((I-2D)CA^{-1}B) < 1 \tag{2.12}$$

for any diagonal matrix $D = diag(d_i)$ *with* $d_i \in [0,1]$*, or*

$$\sigma_1(CA^{-1}B) < 1, \tag{2.13}$$

or

$$\rho(|CA^{-1}B|) < 1, \tag{2.14}$$

the AVE (1.4) *has a unique solution for any* $d \in \mathbb{R}^n$*.*

Of course, Theorems 2.3 and 2.4 are also generalization forms of the results in Lemmas 2.3 and 2.4, respectively.

In addition, based on Theorem 2.2 and Theorem 2.4, clearly, Corollary 2.1 can be otained.

Corollary 2.1. *Let* $\det(A) \neq 0$ *in* (1.4)*, and at least one of matrices B and C in* (1.4) *be nonsingular. When*

$$\rho((I - 2D)CA^{-1}B) < 1 \tag{2.15}$$

for any diagonal matrix $D = diag(d_i)$ with $d_i \in [0, 1]$, or

$$\sigma_1(CA^{-1}B) < 1, \tag{2.16}$$

or

$$\rho(|CA^{-1}B|) < 1, \tag{2.17}$$

the AVE (1.4) *has a unique solution for any $d \in \mathbb{R}^n$.*

By investigating the condition of Corollary 2.1, to guarantee a unique solution for the AVE (1.4), Corollary 2.1 not only requires the nonsingular matrix A, but also at least one of matrices B and C in (1.4) is nonsingular. On this occasion, we can use the conditions (2.15), (2.16) or (2.17) to judge the unique solution of the AVE (1.4).

Next, we will further relax the condition of Corollary 2.1. To this end, here, $sign(x)$ denotes the vector, which consists of $1, 0, -1$ dependent on whether $x > 0$, $x = 0$ and $x < 0$. Here, we set $y = Cx$. By making use of $|y| = Ey$ with $E = diag(sign(y))$, the AVE (1.4) is equal to

$$\begin{cases} Ax - BEy = d, \\ Cx - y = 0, \end{cases}$$

which is a two-by-two block linear equation below

$$\begin{bmatrix} A & -BE \\ C & -I \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} d \\ 0 \end{bmatrix}. \tag{2.18}$$

That is to say, we just need to establish some conditions to guarantee the unique solution of the two-by-two block linear Eq (2.18).

To establish the condition of the two-by-two block linear Eq (2.18), we consider the following general form of (2.18), i.e.,

$$\begin{bmatrix} A & -B\bar{E} \\ C & -I \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} d \\ 0 \end{bmatrix}, \tag{2.19}$$

where $\bar{E} = diag(\bar{e}_i)$ with any vector of components $\bar{e}_i \in [-1, 1]$. Clearly, the two-by-two block linear Eq (2.18) is a special case of the two-by-two block linear Eq (2.19). Once we give the sufficient condition for the unique solution of the two-by-two block linear Eq (2.19), naturally, the sufficient condition of the unique solution of the two-by-two block linear Eq (2.18) is obtained as well.

By the simple computation, we have

$$\begin{bmatrix} A & -B\bar{E} \\ C & -I \end{bmatrix} = \begin{bmatrix} A & 0 \\ C & -I \end{bmatrix} \begin{bmatrix} I & -A^{-1}B\bar{E} \\ 0 & I - CA^{-1}B\bar{E} \end{bmatrix}. \tag{2.20}$$

Based on Eq (2.20), clearly, if matrix $I - CA^{-1}B\bar{E}$ is nonsingular, then matrix

$$\begin{bmatrix} A & -B\bar{E} \\ C & -I \end{bmatrix}$$

is also nonsingular. So, we know that the two-by-two block linear Eq (2.19) has a unique solution. Whereupon, we have the following result for the unique solution of the AVE (1.4).

Theorem 2.5. *Let* $\det(A) \neq 0$ *in* (1.4). *When*

$$\rho(CA^{-1}B\bar{E}) < 1 \tag{2.21}$$

for any diagonal matrix $\bar{E} = diag(\bar{e}_i)$ *with* $\bar{e}_i \in [-1, 1]$, *or*

$$\sigma_1(CA^{-1}B) < 1, \tag{2.22}$$

or

$$\rho(|CA^{-1}B|) < 1, \tag{2.23}$$

the AVE (1.4) *has a unique solution for any* $d \in \mathbb{R}^n$.

Since

$$\sigma_1(CA^{-1}B) \leq \sigma_1(C)\sigma_1(A^{-1})\sigma_1(B),$$

then Corollary 2.2 can be obtained.

Corollary 2.2. *Let* $\det(A) \neq 0$ *in* (1.4). *If*

$$\sigma_1(C)\sigma_1(B) < \sigma_n(A), \tag{2.24}$$

the AVE (1.4) *has a unique solution for any* $d \in \mathbb{R}^n$.

In addition, when the set of all the eigenvalues of matrix $I - CA^{-1}B\bar{E}$ does not contain 1, matrix $I - CA^{-1}B\bar{E}$ is nonsingular as well. Further, we have Theorem 2.6.

Theorem 2.6. *Let* $\det(A) \neq 0$ *in* (1.4). *If the set of all the eigenvalues of matrix* $I - CA^{-1}B\bar{E}$ *does not contain 1 for any diagonal matrix* $\bar{E} = diag(\bar{e}_i)$ *with* $\bar{e}_i \in [-1, 1]$, *then the AVE* (1.4) *has a unique solution for any* $d \in \mathbb{R}^n$.

Finally, we give a sufficient condition for the non-existence of solution of the AVE (1.4), which is an extension of a non-existence result in Proposition 9 in [17].

Theorem 2.7. *Let B and C be nonsingular in* (1.4), $0 \neq B^{-1}d \geq 0$, *and*

$$\|B^{-1}AC^{-1}\|_2 < 1.$$

Then the AVE (1.4) *has no solution.*

Proof. We prove by contradiction. Here, we assume that a nonzero solution x exists. Since matrices B and C are nonsingular in (1.4), we set $y = Cx$, then the AVE (1.4) is equal to

$$B^{-1}AC^{-1}y - |y| = B^{-1}d.$$

Further, we have

$$|y| = B^{-1}AC^{-1}y - B^{-1}d \leq B^{-1}AC^{-1}y.$$

So,

$$\|y\|_2 \leq \|B^{-1}AC^{-1}y\|_2 \leq \|B^{-1}AC^{-1}\|_2\|y\|_2 < \|y\|_2.$$

This is a contradiction result. This implies that the result in Theorem 2.7 holds. □

Based on Theorem 2.7, we have Corollary 2.3.

Corollary 2.3. *Let B and C be nonsingular in* (1.4)*,* $0 \neq B^{-1}d \geq 0$*, and*

$$\sigma_1(A) < \sigma_n(B)\sigma_n(C).$$

Then the AVE (1.4) *has no solution.*

Clearly, Corollary 2.3 is a generalization of form of Proposition 9 in [17]. When $B = C = I$, Corollary 2.3 becomes Proposition 9 in [17].

When $C = I$ or $B = I$, naturally, we can get some sufficient conditions for the non-existence of solution of the AVE (1.2) or (1.3) by Theorem 2.7 and Corollary 2.3. Here is omitted.

3. Conclusions

In this paper, for $A, B, C \in \mathbb{R}^{n\times n}$, we have gained some conditions to guarantee the unique solution of the absolute value equation (AVE) $Ax - B|Cx| = d$. The previous published works in [1–3] are generalized.

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Conflict of interest

The authors declare that they have no competing interests.

References

1. S. L. Wu, The unique solution of a class of the new generalized absolute value equation, *Appl. Math. Lett.,* **116** (2021), 107029.
2. S. L. Wu, S. Q. Shen, On the unique solution of the generalized absolute value equation, *Optim. Lett.,* (2020).
3. F. Mezzadri, On the solution of general absolute value equations, *Appl. Math. Lett.,* **107** (2020) 106462.
4. L. Caccetta, B. Qu, G. L. Zhou, A globally and quadratically convergent method for absolute value equations, *Comput. Optim. Appl.,* **48** (2011), 45–58.
5. O. L. Mangasarian, A generalized Newton method for absolute value equations, *Optim. Lett.,* **3** (2009), 101–108.
6. J. Rohn, An algorithm for solving the absolute value equations, *Electron. J. Linear Algebra,* **18** (2009), 589–599.
7. D. K. Salkuyeh, The Picard-HSS iteration method for absolute value equations, *Optim. Lett.,* **8** (2014), 2191–2202.

8. S. L. Wu, C. X. Li, A special Shift-splitting iterative method for the absolute value equations, *AIMS Math.*, **5** (2020), 5171–5183.
9. C. X. Li, S. L. Wu, Modified SOR-like iteration method for absolute value equations, *Math. Probl. Eng.,* **2020** (2020), 9231639.
10. P. Guo, S. L. Wu, C. X. Li, On SOR-like iteration method for solving absolute value equations, *Appl. Math. Lett.,* **97** (2019), 107–113.
11. Y. Y. Lian, C. X. Li, S. L. Wu, Weaker convergent results of the generalized Newton method for the generalized absolute value equations, *J. Comput. Appl. Math.,* **338** (2018), 221–226.
12. C. X. Li, A preconditioned AOR iterative method for the absolute value equations, *Int. J. Comput. Methods,* **14** (2017), 1750016.
13. C. X. Li, A modified generalized Newton method for the absolute value equations, *J. Optim. Theory Appl.,* **170** (2016), 1055–1059.
14. J. Rohn, Systems of linear interval equations, *Linear Algebra Appl.,* **126** (1989), 39–78.
15. J. Rohn, A theorem of the alternatives for the equation $Ax + B|x| = b$, *Linear Multilinear A.,* **52** (2004), 421–426.
16. S. L. Wu, C. X. Li, A note on unique solvability of the absolute value equation, *Optim. Lett.,* **14** (2020), 1957–1960.
17. O. L. Mangasarian, R. R. Meyer, Absolute value equations, *Linear Algebra Appl.,* **419** (2006), 359–367.
18. S. L. Wu, C. X. Li, The unique solution of the absolute value equations, *Appl. Math. Lett.,* **76** (2018), 195–200.
19. O. Prokopyev, On equivalent reformulations for absolute value equations, *Comput. Optim. Appl.,* **44** (2009), 363–372.
20. M. Hladík, Bounds for the solutions of absolute value equations, *Comput. Optim. Appl.,* **69** (2018), 243–266.

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