Research article

Fixed point results of a generalized reversed F-contraction mapping and its application

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Abstract: In this paper, we introduce the reversal of generalized Banach contraction principle and mean Lipschitzian mapping respectively. Secondly, we prove the existence and uniqueness of fixed points for these expanding type mappings. Further, we extend Wardowski's idea of F-contraction by introducing the reversed generalized F-contraction mapping and use our obtained result to prove the existence and uniqueness of its fixed point. Finally, we apply our results to prove the existence of a unique solution of a non-linear integral equation.

1. Introduction

The Banach contraction principle [2] was established in 1922, and due to its effectiveness and coherence, it has turned out to be an exceptionally popular tool in numerous branches of mathematical analysis (for details, see [4, 15–17]). Several researchers studied the Banach contraction principle in various directions and established the generalizations, extensions, and applications of their findings (for details, see [3, 5, 8, 10, 14, 19]). Among them, Goebel and Japón Pineda [7] introduced the idea of mean non-expansive mapping that further extended by Goebel and Sims [9] to the class of mean lipschitzian mapping. Such mapping restricts the distance of iterates to expand more than a certain limit. We modify this idea by introducing generalized reversed contraction and reversed mean lipschitzian mapping. We prove the existence and uniqueness of the fixed point for such mappings. Such mappings allow the distance of iterate to expand without any limit. Further, the conditions in the definition of these mappings also allows the contraction of its iterates, which makes our result more interesting and significant.

In 2012, Wardowski [11] provided a very interesting extension of Banach's fixed point theorem by introducing F-contraction and proved a new fixed point theorem concerning F-contraction. Later, Gornicki [6] presented some fixed point results for F-expanding mapping. In our research, we generalized the idea of F-expanding mapping by introducing generalized reversed F-contraction mapping and replacing the conditions $(F2)$, $(F3)$ of F-expanding mapping with certain simple conditions.

2. Preliminaries

In this article, we represent the set of natural numbers by $\mathbb{N}$, set of whole numbers by $\mathbb{N}_0$, and set of positive real numbers by $\mathbb{R}_+$.

Definition 2.1. *[9]* Let $(M, \mathcal{D})$ be a metric space. A mapping $\mathfrak{M} : M \mapsto M$ is said to be a mean lipschitzian, if for all $x, y \in M$ and $k > 0$, we have

$$\sum_{i=1}^{n} \upsilon_i \mathcal{D}\left(\mathfrak{M}^i x, \mathfrak{M}^i y\right) \leq k\mathcal{D}(x, y),$$

where, $\upsilon_1, \upsilon_n > 0$, $\upsilon_i \geq 0$, and $\sum_{i=1}^{n} \upsilon_i = 1$.

In 2002, James Merryfield [12] established following fixed point theorem as a generalization of Banach contraction principle (see also, [1]).

Theorem 2.1. *[12]* Let $\mathfrak{M}$ be a self mapping on a complete metric space $(M, \mathcal{D})$, and let $k \in (0, 1)$ and suppose that p be an integer. Assume that mapping $\mathfrak{M}$ satisfy the following:

$$\min\left\{\mathcal{D}\left(\mathfrak{M}^i x, \mathfrak{M}^i y\right) : i = 1, ..., p\right\} \leq k\mathcal{D}(x, y),$$

for all $x, y \in M$. Then, $\mathfrak{M}$ has a unique fixed point.

In 2012, Wardowski [18], defined the concept of F-contraction in the following way.

Definition 2.2. Let $(M, \mathcal{D})$ be a metric space. A mapping $\mathfrak{M} : M \mapsto M$ is said to be a F-contraction, if there exists $\tau > 0$, such that,

$$\left[\mathcal{D}(\mathfrak{M}x, \mathfrak{M}y) > 0 \Rightarrow \tau + F(\mathcal{D}(\mathfrak{M}x, \mathfrak{M}y)) \leq F(\mathcal{D}(x, y))\right],$$

for all $x, y \in M$, where $F : \mathbb{R}_+ \mapsto \mathbb{R}$ is a mapping satisfying the following conditions:

$(F1)$ F is strictly increasing, that is, for all $x, y \in \mathbb{R}_+$, $x < y$, $F(x) < F(y)$;
$(F2)$ For each sequence $\{\alpha_n\}_{n=1}^{\infty}$ of positive numbers, $\lim_{n \to \infty} \alpha_n = 0$, if and only if $\lim_{n \to \infty} F(\alpha_n) = -\infty$;
$(F3)$ There exists $k \in (0, 1)$ such that $\lim_{\alpha \to 0^+} \alpha^k F(\alpha) = 0$.

We denote by $\mathcal{F}$, the set of all functions satisfying the conditions $(F1) - (F3)$.

Example 2.1. Let $F_i : \mathbb{R}_+ \mapsto \mathbb{R}$, $i = 1, 2, 3, 4$, are defined as:

  i. $F_1(t) = \ln t$.

  ii. $F_2(t) = t + \ln t$.

  iii. $F_3(t) = -\frac{1}{\sqrt{t}}$.

  iv. $F_4(t) = \ln\left(t^2 + t\right)$.

Then, $F_1, F_2, F_3, F_4 \in \mathcal{F}$.

Remark. From $(F1)$, and the definition 2.2, it is easy to conclude that every F-contraction is necessarily continuous.

3. Main results

In this section, we will define generalized reversed contraction, reversed mean lipschitzian or υ-lipschitzian mapping and related results. At the end of this section, we will provide an application of our main result to prove the existence of unique solution of non-linear integral equation.

We begin with the following main definitions of contractive mappings.

Definition 3.1. Let $(M, \mathcal{D})$ be a metric space. A mapping $\mathfrak{M} : M \mapsto M$ is said to be a generalized reversed contraction, if for all $x, y \in M$, there exists a real number $k > 1$, such that

$$Min\left\{\mathcal{D}\left(\mathfrak{M}^i x, \mathfrak{M}^i y\right); i = 1, ..., p\right\} \geq k\mathcal{D}(x, y). \tag{1}$$

Definition 3.2. Let $(M, \mathcal{D})$ be a metric space. A mapping $\mathfrak{M} : M \mapsto M$ is said to be a reversed mean lipschitzian or υ-lipschitzian, if for all $x, y \in M$, we have

$$\sum_{i=1}^{n} \upsilon_i \mathcal{D}\left(\mathfrak{M}^i x, \mathfrak{M}^i y\right) \geq k\mathcal{D}(x, y), \tag{2}$$

where, $\upsilon_1 > 0, \upsilon_n > 0, \upsilon_i \geq 0$ and $\sum_{i=1}^{n} \upsilon_i = 1$.

We start our results with the following lemma, which will be required to establish our main result.

Lemma 3.1. Let $(M, \mathcal{D})$ be a metric space and $\mathfrak{M} : M \mapsto M$ be a surjective generalized reversed contraction with $p = 2$ on M. Then for any $x \in M$ and $r \in \mathbb{N}_0$, we have

$$\mathfrak{M}^r x = x \quad \text{if and only if} \quad r = 0.$$

Proof. Let $\mathfrak{M}x \neq x$ and $\mathfrak{M}^2 x = x$. Then, for some $x, \mathfrak{M}x \in M$, the condition (1) yields,

$$Min\left\{\mathcal{D}(\mathfrak{M}x, \mathfrak{M}^2 x), \mathcal{D}(\mathfrak{M}^2 x, \mathfrak{M}^3 x)\right\} \geq k\mathcal{D}(x, \mathfrak{M}x).$$

Using the assumption that $\mathfrak{M}^2 x = x$, then the above inequality becomes

$$Min\left\{\mathcal{D}(\mathfrak{M}x, x), \mathcal{D}(x, \mathfrak{M}x)\right\} \geq k\mathcal{D}(x, \mathfrak{M}x),$$

which is a contradiction.

Now, suppose that $r > 2$, and suppose that $r > 2$ is the least number such that $\mathfrak{M}^r x = x$. Then for, $\mathfrak{M}^{r-2} x, \mathfrak{M}^{r-1} x \in M$, the inequality (1) implies,

$$Min\left\{\mathcal{D}(\mathfrak{M}^{r-1} x, \mathfrak{M}^r x), \mathcal{D}(\mathfrak{M}^r x, \mathfrak{M}^{r+1} x)\right\} \geq k\mathcal{D}(\mathfrak{M}^{r-2} x, \mathfrak{M}^{r-1} x).$$

Using the assumption $\mathfrak{M}^r x = x$, we have

$$Min\left\{\mathcal{D}(\mathfrak{M}^{r-1} x, x), \mathcal{D}(x, \mathfrak{M}x)\right\} \geq k\mathcal{D}(\mathfrak{M}^{r-2} x, \mathfrak{M}^{r-1} x). \tag{3}$$

Similarly, for $\mathfrak{M}^{r-3}x, \mathfrak{M}^{r-2}x \in M$, using the condition (1), we have

$$Min\left\{\mathcal{D}(\mathfrak{M}^{r-2}x, \mathfrak{M}^{r-1}x), \mathcal{D}(\mathfrak{M}^{r-1}x, \mathfrak{M}^r x)\right\} \geq k\mathcal{D}(\mathfrak{M}^{r-3}x, \mathfrak{M}^{r-2}x).$$

Then either,

$$Min\left\{\mathcal{D}(\mathfrak{M}^{r-2}x, \mathfrak{M}^{r-1}x), \mathcal{D}(\mathfrak{M}^{r-1}x, \mathfrak{M}^r x)\right\} = \mathcal{D}(\mathfrak{M}^{r-2}x, \mathfrak{M}^{r-1}x), \tag{4}$$

or,

$$Min\left\{\mathcal{D}(\mathfrak{M}^{r-2}x, \mathfrak{M}^{r-1}x), \mathcal{D}(\mathfrak{M}^{r-1}x, \mathfrak{M}^r x)\right\} = \mathcal{D}(\mathfrak{M}^{r-1}x, \mathfrak{M}^r x). \tag{5}$$

If inequality (4) holds, then we have

$$\mathcal{D}(\mathfrak{M}^{r-2}x, \mathfrak{M}^{r-1}x) \geq k\mathcal{D}(\mathfrak{M}^{r-3}x, \mathfrak{M}^{r-2}x). \tag{6}$$

If inequality (5) holds, then we can write

$$\mathcal{D}(\mathfrak{M}^{r-2}x, \mathfrak{M}^{r-1}x) \geq \mathcal{D}(\mathfrak{M}^{r-1}x, \mathfrak{M}^r x) \geq k\mathcal{D}(\mathfrak{M}^{r-3}x, \mathfrak{M}^{r-2}x). \tag{7}$$

From inequalities (6) and (7), the relation (3) implies that

$$Min\left\{\mathcal{D}(\mathfrak{M}^{r-1}x, x), \mathcal{D}(x, \mathfrak{M}x)\right\} \geq k^2\mathcal{D}(\mathfrak{M}^{r-3}x, \mathfrak{M}^{r-2}x).$$

Continuing this process, we will have

$$Min\left\{\mathcal{D}(\mathfrak{M}^{r-1}x, x), \mathcal{D}(x, \mathfrak{M}x)\right\} \geq k^r\mathcal{D}(\mathfrak{M}x, x). \tag{8}$$

Inequality (8) give the rise to following two possible cases

$$Min\left\{\mathcal{D}(\mathfrak{M}^{r-1}x, x), \mathcal{D}(x, \mathfrak{M}x)\right\} = \mathcal{D}(\mathfrak{M}x, x), \tag{9}$$

or,

$$Min\left\{\mathcal{D}(\mathfrak{M}^{r-1}x, x), \mathcal{D}(x, \mathfrak{M}x)\right\} = \mathcal{D}(\mathfrak{M}^{r-1}x, x). \tag{10}$$

If the relation (9) holds then the relation (8) implies

$$\mathcal{D}(x, \mathfrak{M}x) \geq k^r\mathcal{D}(x, \mathfrak{M}x),$$

which is a contradiction. If the inequality (10) holds then the inequality (8) implies

$$\mathcal{D}(\mathfrak{M}^{r-1}x, x) \geq k^r\mathcal{D}(\mathfrak{M}x, x). \tag{11}$$

Further, inequality (10) also implies that

$$\mathcal{D}(x, \mathfrak{M}x) \geq \mathcal{D}(\mathfrak{M}^{r-1}x, x). \tag{12}$$

Both the inequalities (11), (12) yields a contradiction as follows

$$\mathcal{D}(\mathfrak{M}^{r-1}x, x) \geq k^r\mathcal{D}(\mathfrak{M}^{r-1}x, x).$$

Therefore, we have

$$\mathfrak{M}^r x = x \quad \text{if and only if} \quad r = 0. \tag{13}$$

Lemma 3.2. Let $(M, \mathcal{D})$ be a metric space and for some $x \in M$, consider a set $C = C_1 \cup C_2 \subseteq M$, such that, $C_1 = \{x_n = \mathfrak{M}^n x, \quad \forall n \in \mathbb{N}_0\}$, *and* $C_2 = \{y_n \,|\, \mathfrak{M}^n y_n = x, \quad \forall n \in \mathbb{N}\}$. Suppose a mapping $\mathfrak{M} : M \mapsto M$ is a generalized reversed contraction mapping with $p = 2$. Then $\mathfrak{M} : C \mapsto C$ is one to one mapping.

Proof. Consider the following three possibilities

A. For all $x_m, x_n \in C$, $\mathfrak{M}x_m = \mathfrak{M}x_n \quad \leftrightarrow \quad x_m = x_n$, where $m, n \in \mathbb{N}_0$.

B. For all $y_m, y_n \in C$, $\mathfrak{M}y_m = \mathfrak{M}y_n \quad \leftrightarrow \quad y_m = y_n$, where $m, n \in \mathbb{N}$.

C. For all $x_m, y_n \in C$, $\mathfrak{M}x_m = \mathfrak{M}y_n \quad \leftrightarrow \quad x_m = y_n$, where $m \in \mathbb{N}_0, n \in \mathbb{N}$.

To prove (A), suppose $\mathfrak{M}x_m = \mathfrak{M}x_n$. Therefore, $\mathfrak{M}^{m+1}x = \mathfrak{M}^{n+1}x$.

Let m is greater than n by r i.e. $m = n + r$. So, we have

$$\mathfrak{M}^{n+r+1}x = \mathfrak{M}^{n+1}x.$$

That can be written as

$$\mathfrak{M}^r\left(\mathfrak{M}^{n+1}x\right) = \mathfrak{M}^{n+1}x.$$

As, $\mathfrak{M}^{n+1}x \neq 0$, lemma 3.1 implies that, $r = 0$ and $m = n$. Therefore, we have $x_m = x_n$. For converse, if $x_m = x_n$, then we have, $\mathfrak{M}x_m = \mathfrak{M}x_n$.

Now, to prove condition (B), we take, $y_m, y_n \in C$ such that $\mathfrak{M}^m y_m = x$, $\mathfrak{M}^n y_n = x$ and $m = n + r$.

Therefore, if

$$\mathfrak{M}y_m = \mathfrak{M}y_n,$$

then, we can write

$$\mathfrak{M}^n y_m = \mathfrak{M}^n y_n,$$

further, we have

$$\mathfrak{M}^n y_m = x.$$

So that,

$$\mathfrak{M}^{n+r} y_m = \mathfrak{M}^r x,$$

or,

$$\mathfrak{M}^m y_m = \mathfrak{M}^r x.$$

Therefore,

$$x = \mathfrak{M}^r x.$$

By lemma 3.1 we have $r = 0$ and $y_m = y_n$.

Now, conversely, if $y_m = y_n$, then we have, $\mathfrak{M}y_m = \mathfrak{M}y_n$.

Similarly, to prove condition (C), we assume that $x_m, y_n \in C$ such that, $x_m = \mathfrak{M}^m x$ and $\mathfrak{M}^n y_n = x$.

If, $\qquad \mathfrak{M}x_m = \mathfrak{M}y_n.$

we have, $\qquad \mathfrak{M}^{m+1}x = \mathfrak{M}y_n,$

so that, $\qquad \mathfrak{M}^{m+n}x = \mathfrak{M}^n y_n = x.$

Now, according to lemma 3.1, we have $m + n = 0$ and $m = n = 0$. That means $x_m = x$ and $y_n = x$. Hence, we have, $x_m = y_n$.

For converse, if $x_m = y_n$, then we have, $\mathfrak{M}x_m = \mathfrak{M}y_n$.

Therefore, the possible existence of (A), (B) or, (C) proves that $\mathfrak{M} : C \mapsto C$ is one-to-one.

Now, we are going to state and prove our first main result for generalized reversed contraction mapping.

Theorem 3.1. Let $(M, \mathcal{D})$ be a complete metric space. Every surjective generalized reversed contraction mapping $\mathfrak{M} : M \mapsto M$ with $p = 2$ has a unique fixed point.

Proof. Since, $\mathfrak{M} : M \mapsto M$ is a surjective generalized reversed contraction mapping therefore, by using lemma 3.2 holds for every $C = C_1 \cup C_2 \subset M$ with

$$C_1 = \{x_n = \mathfrak{M}^n x, \quad \forall n \in \mathbb{N}_0\}, \quad and \quad C_2 = \{y_n \mid \mathfrak{M}^n y_n = x, \quad \forall n \in \mathbb{N}\},$$

since mapping $\mathfrak{M} : C \mapsto C$ is one to one, hence invertible.

Define $\mathfrak{S} : C \mapsto C$, such that, $\mathfrak{M}\mathfrak{S} = \mathfrak{S}\mathfrak{M} = I$. For $x, \mathfrak{M}x \in C \subset M$ inequality (1) yields,

$$Min\left\{\mathcal{D}(\mathfrak{M}x, \mathfrak{M}^2 x), \mathcal{D}(\mathfrak{M}^2 x, \mathfrak{M}^3 x)\right\} \geq k\mathcal{D}(x, \mathfrak{M}x).$$

Above inequality give rise to the following two possible cases

$$\mathcal{D}(\mathfrak{M}x, \mathfrak{M}^2 x) \geq k\mathcal{D}(x, \mathfrak{M}x), \tag{14}$$

or,

$$\mathcal{D}(\mathfrak{M}^2 x, \mathfrak{M}^3 x) \geq k\mathcal{D}(x, \mathfrak{M}x). \tag{15}$$

Let, $\mathfrak{M}^3 x = u$, so that, $\mathfrak{S}(u) = \mathfrak{M}^2 x$, $\mathfrak{S}^2 u = \mathfrak{M}x$, and $\mathfrak{S}^3 u = x$.

Then, by inequalities (14) and (15), we have,

$$\mathcal{D}(\mathfrak{S}^2 u, \mathfrak{S}u) \geq k\mathcal{D}(\mathfrak{S}^3 u, \mathfrak{S}^2 u),$$

or,

$$\mathcal{D}(\mathfrak{S}u, u) \geq k\mathcal{D}(\mathfrak{S}^3 u, \mathfrak{S}^2 u).$$

Therefore,

$$Min\left\{\mathcal{D}(\mathfrak{S}^2 u, S u), \mathcal{D}(\mathfrak{S}^3 u, \mathfrak{S}^2 u)\right\} = \mathcal{D}(\mathfrak{S}^3 u, \mathfrak{S}^2 u) \leq \frac{1}{k}\mathcal{D}(\mathfrak{S}u, u).$$

That is,

$$Min\left\{\mathcal{D}(\mathfrak{S}^2 u, S u), \mathcal{D}(\mathfrak{S}^3 u, \mathfrak{S}^2 u)\right\} \leq \gamma\mathcal{D}(\mathfrak{S}u, u) \quad \gamma = \frac{1}{k} < 1.$$

Which shows that $\mathfrak{S} : C \mapsto C$ is a generalized Banach contraction mapping with $C \subset M$. Theorem 2.1 assures that there exist a unique fixed point $a \in M$, such that $\mathfrak{S}(a) = a$ or $\mathfrak{M}(a) = a$.

We begin our next result by introducing a new modification of F-expanding mapping named as generalized reversed $\mathfrak{F}$-contraction mapping.

Definition 3.2. Let $\mathfrak{F} : \mathbb{R}_+ \mapsto \mathbb{R}$ is such that:

$(F1')$ $\mathfrak{F}(\alpha\beta) \geq \mathfrak{F}(\alpha) + \mathfrak{F}(\beta)$

$(F2')$ $\mathfrak{F}$ is continuous and strictly increasing on $(0, \infty)$.

Let F be the set of all functions $\mathfrak{F} : (0, \infty) \mapsto \mathbb{R}$ satisfying $(F1'), (F2')$.

Example 3.1. Let $\mathfrak{F}_i : \mathbb{R}_+ \mapsto \mathbb{R}, \quad i = 1, 2, 3, 4$ defined by,

i. $\mathfrak{F}_1(t) = \ln t$

ii. $\mathfrak{F}_2(t) = e^t$

iii. $\mathfrak{F}_3(t) = -\frac{1}{t}$

iv. $\mathfrak{F}_4(t) = \ln t - \frac{1}{t} + \sqrt{t}$

v. $\mathfrak{F}_5(t) = -c + \ln(t)$, where $c > 0$ is a constant.

Then, $\mathfrak{F}_1, \mathfrak{F}_2, \mathfrak{F}_3, \mathfrak{F}_4, \mathfrak{F}_5 \in F$.

Definition 3.2. Let $(M, \mathcal{D})$ be a metric space. A mapping $\mathfrak{M} : M \mapsto M$ is said to be a generalized reversed F-contraction, if there exists a $\mathfrak{F} \in F$ such that

$$\mathfrak{F}\left(Min\left\{\mathcal{D}(\mathfrak{M}x, \mathfrak{M}y), \mathcal{D}\left(\mathfrak{M}^2 x, \mathfrak{M}^2 y\right)\right\}\right) \geq \tau + \mathfrak{F}(\mathcal{D}(x, y)),$$

for all $x, y \in X$.

Next, we prove a fixed point theorem for a generalized reversed F-contraction mapping by using the obtained result of theorem 3.1.

Theorem 3.2. Let $(M, \mathcal{D})$ be a complete metric space. Suppose a surjective mapping $\mathfrak{M} : M \mapsto M$ be a generalized reversed F-contraction such that $\tau \geq \mathfrak{F}(k)$, $k > 1$. Then, $\mathfrak{M}$ has a unique fixed point for all $\mathfrak{F} \in F$.

Proof. Since $\mathfrak{M} : M \mapsto M$ is a reversed generalized $\mathfrak{F}$-contraction mapping. Therefore, $\mathcal{D}(\mathfrak{M}x, \mathfrak{M}y) > 0, \mathcal{D}\left(\mathfrak{M}^2 x, \mathfrak{M}^2 y\right) > 0$, which implies that

$$\mathfrak{F}\left(Min\left\{\mathcal{D}(\mathfrak{M}x, \mathfrak{M}y), \mathcal{D}\left(\mathfrak{M}^2 x, \mathfrak{M}^2 y\right)\right\}\right) \geq \tau + \mathfrak{F}(\mathcal{D}(x, y)).$$

As, $\tau \geq \mathfrak{F}(k)$, above inequality can be written as,

$$\mathfrak{F}\left(Min\left\{\mathcal{D}(\mathfrak{M}x, \mathfrak{M}y), \mathcal{D}\left(\mathfrak{M}^2 x, \mathfrak{M}^2 y\right)\right\}\right) \geq \mathfrak{F}(k) + \mathfrak{F}(\mathcal{D}(x, y)).$$

Condition $(F1')$ yields,

$$\mathfrak{F}\left(Min\left\{\mathcal{D}(\mathfrak{M}x, \mathfrak{M}y), \mathcal{D}\left(\mathfrak{M}^2 x, \mathfrak{M}^2 y\right)\right\}\right) \geq \mathfrak{F}(k\mathcal{D}(x, y)).$$

As $\mathfrak{F}$ is increasing so, we will have

$$Min\left\{\mathcal{D}(\mathfrak{M}x, \mathfrak{M}y), \mathcal{D}\left(\mathfrak{M}^2 x, \mathfrak{M}^2 y\right)\right\} \geq k\mathcal{D}(x, y).$$

Therefore, theorem 3.1 proves the existence of unique fixed point for $\mathfrak{M}$.

Remark 3.1. Fixed point result of the generalized reversed F-contraction mapping presented in theorem 3.2 is interesting and significant in comparison to Wardowski's fixed point theorem because, the real constant τ can assume negative values as well. For example, $\tau \geq \mathfrak{F}_5(k) = -c + \ln(t)$, where $k > 1, c > 0$ is a constant.

Theorem 3.3. Let $(M, \mathcal{D})$ be a complete metric space. Suppose a surjective mapping $\mathfrak{M} : M \mapsto M$, which satisfy the following

$$\mathcal{D}(\mathfrak{M}x, \mathfrak{M}y) + \mathcal{D}(\mathfrak{M}^2 x, \mathfrak{M}^2 y) \geq k\mathcal{D}(x, y), \tag{16}$$

and

$$l\mathcal{D}(x, y) \leq \mathcal{D}(\mathfrak{M}x, \mathfrak{M}y) \leq (k - l)\mathcal{D}(x, y), \tag{17}$$

where, $k, l \in (1, \infty), k > 2l$, and for all $x, y \in M$. Then, $\mathfrak{M}$ has a unique fixed point.

Proof. If for any $x, y \in M$, inequality (16) implies

$$Min\left\{\mathcal{D}(\mathfrak{M}x, \mathfrak{M}y), \mathcal{D}(\mathfrak{M}^2x, \mathfrak{M}^2y)\right\} = \mathcal{D}(\mathfrak{M}x, \mathfrak{M}y),$$

then the above equation along with the condition (17) yields,

$$Min\left\{\mathcal{D}(\mathfrak{M}x, \mathfrak{M}y), \mathcal{D}(\mathfrak{M}^2x, \mathfrak{M}^2y)\right\} \geq l\mathcal{D}(x, y). \tag{18}$$

Moreover, if for any $x, y \in M$, we have,

$$Min\left\{\mathcal{D}(\mathfrak{M}x, \mathfrak{M}y), \mathcal{D}(\mathfrak{M}^2x, \mathfrak{M}^2y)\right\} = \mathcal{D}(\mathfrak{M}^2x, \mathfrak{M}^2y),$$

then, the above equation along with condition (16) yields,

$$k\mathcal{D}(x, y) - Min\left\{\mathcal{D}(\mathfrak{M}x, \mathfrak{M}y), \mathcal{D}(\mathfrak{M}^2x, \mathfrak{M}^2y)\right\} = \mathcal{D}(\mathfrak{M}x, \mathfrak{M}y). \tag{19}$$

Eq (19) along with the inequality (17) takes the following form.

$$Min\left\{\mathcal{D}(\mathfrak{M}x, \mathfrak{M}y), \mathcal{D}(\mathfrak{M}^2x, \mathfrak{M}^2y)\right\} \geq l\mathcal{D}(x, y).$$

Which is a generalized reversed contraction mapping, hence theorem 3.1 guarantees a unique fixed point. One can easily observe that condition (16) along with the restriction of condition (17) allows the distances $\mathcal{D}(\mathfrak{M}x, \mathfrak{M}y)$, $\mathcal{D}(\mathfrak{M}^2x, \mathfrak{M}^2y)$ to expand without any limit, hence represents the generalization of generalized reversed contraction mapping.

It is a well-known fact that the mean lipschitzian mapping takes into account not only the mapping itself but also the behavior of its iterates. Next, we will establish a fixed theorem for the reversal of mean lipschitzian mapping under certain conditions that have a significant impact not only on the behavior of the sequence of Lipschitz constants but also has a serious influence on the asymptotic behavior of iterates expressed in terms of certain Lipschitz constants.

Theorem 3.4. Let $(M, \mathcal{D})$ be a complete metric space. Suppose a surjective reversed (υ_1, υ_2)-lipschitzian mapping $\mathfrak{M} : M \mapsto M$ is such that, for $k \in (1, \infty)$, and for all $x, y \in M$, we have,

$$\mathcal{D}(\mathfrak{M}^2x, \mathfrak{M}^2y) \geq k\mathcal{D}(x, y). \tag{20}$$

Then, $\mathfrak{M}$ has a unique fixed point.

Proof. Since, $\mathfrak{M} : M \mapsto M$ is a reversed mean lipschitzian mapping, so that we can write,

$$\upsilon_1\mathcal{D}(\mathfrak{M}x, \mathfrak{M}y) + \upsilon_2\mathcal{D}(\mathfrak{M}^2x, \mathfrak{M}^2y) \geq k\mathcal{D}(x, y). \tag{21}$$

Now for any $x, \mathfrak{M}x \in M$, inequality (21) yields,

$$\upsilon_1\mathcal{D}(\mathfrak{M}x, \mathfrak{M}^2x) + \upsilon_2\mathcal{D}(\mathfrak{M}^2x, \mathfrak{M}^3x) \geq k\mathcal{D}(x, \mathfrak{M}x). \tag{22}$$

Firstly, we will prove that $\mathfrak{M} : M \mapsto M$ is one to one. For this purpose, we suppose that for all $x, y \in M$, $\mathfrak{M}x = \mathfrak{M}y$. Hence $\mathcal{D}(\mathfrak{M}x, \mathfrak{M}y) = \mathcal{D}\left(\mathfrak{M}^2x, \mathfrak{M}^2y\right) = 0$.

Using this information in (21), which yields (as $k > 0$), $\mathcal{D}(x, y) = 0$ which implies $x = y$. For converse, if $x = y$, then we have, $\mathfrak{M}x = \mathfrak{M}y$.

So that $\mathfrak{M}$ is invertible.

Consider a mapping $\mathfrak{S}$ such that $\mathfrak{M}\mathfrak{S} = \mathfrak{S}\mathfrak{M} = I$, where I is an identity mapping.

Since there exists, $z \in M$ such that $z = \mathfrak{M}^3 x$, so that $\mathfrak{S}z = \mathfrak{M}^2 x$, $\mathfrak{S}^2 z = \mathfrak{M}x$ and $\mathfrak{S}^3 z = x$.

By trichotomy property, for some $x, \mathfrak{M}x \in M$, we have the following three possibilities.

(D) $\quad \mathcal{D}(\mathfrak{M}x, \mathfrak{M}^2 x) = \mathcal{D}(\mathfrak{M}^2 x, \mathfrak{M}^3 x).$

(E) $\quad \mathcal{D}(\mathfrak{M}x, \mathfrak{M}^2 x) < \mathcal{D}(\mathfrak{M}^2 x, \mathfrak{M}^3 x).$

(F) $\quad \mathcal{D}(\mathfrak{M}x, \mathfrak{M}^2 x) > \mathcal{D}(\mathfrak{M}^2 x, \mathfrak{M}^3 x).$

If condition (D) holds, then using inequality (21), we have

$$\upsilon_1(\mathfrak{M}^2 x, \mathfrak{M}^3 x) + \upsilon_2 \mathcal{D}(\mathfrak{M}^2 x, \mathfrak{M}^3 x) \geq k\mathcal{D}(x, \mathfrak{M}x).$$

Equivalently,
$$(\mathfrak{M}^2 x, \mathfrak{M}^3 x) \geq k\mathcal{D}(x, \mathfrak{M}x). \tag{23}$$

Likewise,
$$(\mathfrak{M}x, \mathfrak{M}^2 x) \geq k\mathcal{D}(x, \mathfrak{M}x). \tag{24}$$

Inequalities (23), (24) among with (20) yields,

$$Min\left\{\mathcal{D}(x, \mathfrak{M}x), \mathcal{D}(\mathfrak{M}x, \mathfrak{M}^2 x)\right\} = \mathcal{D}(x, \mathfrak{M}x) \leq \frac{1}{k}\mathcal{D}(\mathfrak{M}^2 x, \mathfrak{M}^3 x),$$

so that,
$$Min\left\{\mathcal{D}(\mathfrak{S}^3 x, \mathfrak{S}^2 x), \mathcal{D}(\mathfrak{S}^2 x, \mathfrak{S}x)\right\} \leq \frac{1}{k}\mathcal{D}(\mathfrak{S}x, x). \tag{25}$$

If (E) holds, inequality (22) can be written as

$$\upsilon_1 \mathcal{D}(\mathfrak{M}^2 x, \mathfrak{M}^3 x) + \upsilon_2 \mathcal{D}(\mathfrak{M}^2 x, \mathfrak{M}^3 x) > k\mathcal{D}(x, \mathfrak{M}x),$$

or,
$$\mathcal{D}(\mathfrak{M}^2 x, \mathfrak{M}^3 x) > k\mathcal{D}(x, \mathfrak{M}x). \tag{26}$$

Then, either

$$\mathcal{D}(x, \mathfrak{M}x) < \mathcal{D}(\mathfrak{M}x, \mathfrak{M}^2 x) \text{ or } \mathcal{D}(x, \mathfrak{M}x) = \mathcal{D}(\mathfrak{M}x, \mathfrak{M}^2 x) \text{ or } \mathcal{D}(x, \mathfrak{M}x) > \mathcal{D}(\mathfrak{M}x, \mathfrak{M}^2 x).$$

Let, $\mathcal{D}(x, \mathfrak{M}x) = \mathcal{D}(\mathfrak{M}x, \mathfrak{M}^2 x)$, then using inequality (26), we have,

$$Min\left\{\mathcal{D}(x, \mathfrak{M}x), \mathcal{D}(\mathfrak{M}x, \mathfrak{M}^2 x)\right\} = \mathcal{D}(x, \mathfrak{M}x) = \mathcal{D}(\mathfrak{M}x, \mathfrak{M}^2 x) < \frac{1}{k}\mathcal{D}(\mathfrak{M}^2 x, \mathfrak{M}^3 x).$$

So that,
$$Min\left\{\mathcal{D}(\mathfrak{S}^3 x, \mathfrak{S}^2 x), \mathcal{D}(\mathfrak{S}^2 x, \mathfrak{S}x)\right\} < \frac{1}{k}\mathcal{D}(\mathfrak{S}x, x).$$

Now, if $\mathcal{D}(x, \mathfrak{M}x) < \mathcal{D}(\mathfrak{M}x, \mathfrak{M}^2 x)$ then, inequality (26) yields,

$$Min\left\{\mathcal{D}(x, \mathfrak{M}x), \mathcal{D}(\mathfrak{M}x, \mathfrak{M}^2 x)\right\} = \mathcal{D}(x, \mathfrak{M}x) < \frac{1}{k}\mathcal{D}(\mathfrak{M}^2 x, \mathfrak{M}^3 x).$$

So that,
$$Min\left\{\mathcal{D}(\mathfrak{S}^3 x, \mathfrak{S}^2 x), \mathcal{D}(\mathfrak{S}^2 x, \mathfrak{S}x)\right\} < \frac{1}{k}\mathcal{D}(\mathfrak{S}x, x).$$

Similarly, if $\mathcal{D}(x, \mathfrak{M}x) > \mathcal{D}(\mathfrak{M}x, \mathfrak{M}^2 x)$ then, using inequality (20), we can write

$$\mathcal{D}(x, \mathfrak{M}x) < \frac{1}{k}\mathcal{D}(\mathfrak{M}^2 x, \mathfrak{M}^3 x),$$

or,

$$\mathcal{D}(\mathfrak{M}x, \mathfrak{M}^2 x) < \frac{1}{k}\mathcal{D}(\mathfrak{M}^2 x, \mathfrak{M}^3 x).$$

So that,

$$Min\left\{\mathcal{D}(\mathfrak{S}^3 x, \mathfrak{S}^2 x), \mathcal{D}(\mathfrak{S}^2 x, \mathfrak{S}x)\right\} < \frac{1}{k}\mathcal{D}(\mathfrak{S}x, x).$$

Finally, we will consider the relation (F), that is, if $\mathcal{D}(\mathfrak{M}x, \mathfrak{M}^2 x) > \mathcal{D}(\mathfrak{M}^2 x, \mathfrak{M}^3 x)$.

Then, by the use of inequality (22), we will have

$$v_1\mathcal{D}(\mathfrak{M}x, \mathfrak{M}^2 x) + v_2\mathcal{D}(\mathfrak{M}x, \mathfrak{M}^2 x) \geq k\mathcal{D}(x, \mathfrak{M}x).$$

Equivalently,

$$\mathcal{D}(\mathfrak{M}x, \mathfrak{M}^2 x) \geq k\mathcal{D}(x, \mathfrak{M}x).$$

So that,

$$Min\left\{\mathcal{D}(x, \mathfrak{M}x), \mathcal{D}(\mathfrak{M}x, \mathfrak{M}^2 x)\right\} = \mathcal{D}(x, \mathfrak{M}x) \leq \frac{1}{k}\mathcal{D}(\mathfrak{M}^2 x, \mathfrak{M}^3 x).$$

That is,

$$Min\left\{\mathcal{D}(\mathfrak{S}^3 x, \mathfrak{S}^2 x), \mathcal{D}(\mathfrak{S}^2 x, \mathfrak{S}x)\right\} \leq \frac{1}{k}\mathcal{D}(\mathfrak{S}x, x).$$

Therefore, for all possible cases, we have

$$Min\left\{\mathcal{D}(\mathfrak{S}^3 x, \mathfrak{S}^2 x), \mathcal{D}(\mathfrak{S}^2 x, \mathfrak{S}x)\right\} \leq k'\mathcal{D}(\mathfrak{S}x, x).$$

Where, $k' \leq \frac{1}{k}$. Therefore, $\mathfrak{S} : M \mapsto M$ being a generalized contraction mapping admits a unique fixed point, so does $\mathfrak{M} : M \mapsto M$.

In the following application we will prove the existence of a unique fixed point as a solution of an integral equation whose transformed model is a generalized reversed contraction and generalized reversed $\widetilde{\mathfrak{F}}$-contraction.

4. Application

As an application of our result, we consider an engineering problem in which the transformed mathematical model of a problem representing an activation of spring affected by an external force defines a non-linear integral equation (see [13]).

That is,

$$u(\mathfrak{r}) = \int_0^{\mathfrak{r}} H\left(\mathfrak{w}, u(\mathfrak{w})\right) G(\mathfrak{r}, \mathfrak{w}) d\mathfrak{w}, \quad \mathfrak{r} \in [0, I]. \tag{27}$$

Define a green function $G(\mathfrak{r}, \mathfrak{w})$ as:

$$
G(\mathfrak{r}, \mathfrak{w}) = \begin{cases} (\mathfrak{r} + \mathfrak{w})e^{\tau(\mathfrak{r}-\mathfrak{w})} & , if \quad 0 \le \mathfrak{w} \le \mathfrak{r} \le I \\ 0 & , if \quad 0 \le \mathfrak{r} \le \mathfrak{w} \le I, \end{cases}
$$

with constant $\tau(c, d) > 0$.

Let $H : [0, I] \mapsto \mathbb{R}_+$ and is defined as:

$$
\|H\|_\tau = \sup_{\mathfrak{r}\in[0,I]} \left\{ |H(\mathfrak{r})| \, e^{-2\tau\mathfrak{r}} \right\},
$$

further, $\mathcal{D} : X \times X \mapsto \mathbb{R}_+$ is defined as:

$$
\mathcal{D}(x, y) = \max \left\{ \|x\|_\tau, \|y\|_\tau \right\},
$$

for all $x, y \in X$, where, X is the set of continuous functions.

Now, in order to find the existence of solution to integral equation, we consider a function $g : X \mapsto X$ defined as;

$$
g\left(u(\mathfrak{r})\right) = \int_0^{\mathfrak{r}} H\left(\mathfrak{w}, u(\mathfrak{w})\right) G(\mathfrak{r}, \mathfrak{w})d\mathfrak{w}, \tag{28}
$$

for all $u \in X$ and $\mathfrak{r} \in [0, I]$.

Now, we will prove that there exists some $v \in X$ such that $g\left(v(\mathfrak{r})\right) = v(\mathfrak{r})$. That is, the fixed point of generalized reversed F-contraction mapping will represent the solution of integral Eq (27).

Theorem 4.1. The non-linear integral Eq (27) has a solution, if the following conditions hold,

a) $H\left(\mathfrak{w}, u(\mathfrak{w})\right)$ is an increasing function.

b) $|H(\mathfrak{w}, u)| \ge \tau^2 e^\tau u,$ such that, $\mathfrak{r}\tau \ge 1 + \frac{1}{2e^2}$ where, $\tau > 0$, $\mathfrak{r}, \mathfrak{w} \in [0, I]$ and $u \in \mathbb{R}_+$.

c) $g : X \mapsto X$ is a surjective mapping.

Proof. For all $v, w \in X$, using conditions (a) and (b), we can write,

$$
|g\left(v(\mathfrak{w})\right)| \ge \int_0^{\mathfrak{r}} \tau^2 e^\tau \, |v(\mathfrak{w})| G(\mathfrak{r}, \mathfrak{w})d\mathfrak{w}
$$

$$
= \int_0^{\mathfrak{r}} \tau^2 e^\tau \, |v(\mathfrak{w})| (\mathfrak{r} + \mathfrak{w})e^{\tau(\mathfrak{r}-\mathfrak{w})}d\mathfrak{w}
$$

$$
= \int_0^{\mathfrak{r}} \tau^2 e^\tau e^{2\tau\mathfrak{w}} e^{-2\tau\mathfrak{w}} \, |v(\mathfrak{w})| (\mathfrak{r} + \mathfrak{w})e^{\tau(\mathfrak{r}-\mathfrak{w})}d\mathfrak{w}
$$

$$
= \int_0^{\mathfrak{r}} \tau^2 e^\tau e^{2\tau\mathfrak{w}} \|v\|_\tau (\mathfrak{r} + \mathfrak{w})e^{\tau(\mathfrak{r}-\mathfrak{w})}d\mathfrak{w}
$$

$$
= \tau^2 e^\tau e^{\mathfrak{r}\tau} \|v\|_\tau \int_0^{\mathfrak{r}} e^{2\tau\mathfrak{w}} (\mathfrak{r} + \mathfrak{w})e^{\tau\mathfrak{w}}d\mathfrak{w}
$$

$$= \tau^2 e^{\mathfrak{r}\tau+\tau} \|v\|_\tau \left(2\frac{\mathfrak{r}e^{\mathfrak{r}\tau}}{\tau} - \frac{\mathfrak{r}}{\tau} - \frac{e^{\mathfrak{r}\tau}}{\tau^2} + \frac{1}{\tau^2} \right)$$

$$= e^{\mathfrak{r}\tau+\tau} \|v\|_\tau \left(2\mathfrak{r}\tau e^{\mathfrak{r}\tau} - \mathfrak{r}\tau - e^{\mathfrak{r}\tau} + 1 \right)$$
$$= e^{2\mathfrak{r}\tau} e^\tau \|v\|_\tau \left(2\mathfrak{r}\tau - \mathfrak{r}\tau e^{-\mathfrak{r}\tau} - 1 + e^{-\mathfrak{r}\tau} \right).$$

Therefore,

$$e^{-2\mathfrak{r}\tau} |g(v(\mathfrak{w}))| = e^\tau \|v\|_\tau \left(2t\tau - 1 + (1 - \mathfrak{r}\tau) e^{-\mathfrak{r}\tau} \right).$$

Using the condition (b), we will have, $2\mathfrak{r}\tau - 1 + (1 - \mathfrak{r}\tau) e^{-\mathfrak{r}\tau} \geq 1$, so that,

$$\|g(v(\mathfrak{w}))\|_\tau \geq e^\tau \|v\|_\tau.$$

Likewise, we have

$$\|g(w(\mathfrak{w}))\|_\tau \geq e^\tau \|w\|_\tau.$$

Since,

$$\max \{\|gv\|_\tau, \|gw\|_\tau\} \geq \max \{e^\tau \|v\|_\tau, e^\tau \|w\|_\tau\}.$$

Therefore,

$$\mathcal{D}(gv, gw) \geq e^\tau \max \{\|v\|_\tau, \|w\|_\tau\}.$$

Equivalently,

$$\mathcal{D}(gv, gw) \geq e^\tau \mathcal{D}(v, w).$$

Now, if

$$\min \left\{ \mathcal{D}(gv, gw), \mathrm{p}\left(g^2v, g^2w\right) \right\} = \mathcal{D}(gv, gw),$$

we can write,

$$\min \left\{ \mathcal{D}(gv, gw), \mathcal{D}\left(g^2v, g^2w\right) \right\} \geq e^\tau \mathcal{D}(v, w).$$

Further, if

$$\min \left\{ \mathcal{D}(gv, gw), \mathcal{D}\left(g^2v, g^2w\right) \right\} = \mathcal{D}\left(g^2v, g^2w\right),$$

we have

$$\min \left\{ \mathcal{D}(gv, gw), \mathcal{D}\left(g^2v, g^2w\right) \right\} \geq e^\tau \mathcal{D}(gv, gw)$$
$$\geq e^{2\tau} \mathcal{D}(v, w)$$
$$\geq e^\tau \mathcal{D}(v, w).$$

Therefore, in both cases, we have

$$\min \left\{ \mathcal{D}(gv, gw), \mathcal{D}\left(g^2v, g^2w\right) \right\} \geq e^\tau \mathcal{D}(v, w).$$

Using the property of logarithm, we can write

$$\ln \left(\min \left\{ \mathcal{D}(gv, gw), \mathcal{D}\left(g^2v, g^2w\right) \right\} \right) \geq \ln \left(e^\tau \mathcal{D}(v, w) \right),$$

that yields,

$$\ln\left(\min\left\{\mathcal{D}\left(gv, gw\right), \mathcal{D}\left(g^2v, g^2w\right)\right\}\right) \geq \tau + \ln \mathcal{D}\left(v, w\right).$$

Let $\mathfrak{F} \in F,$ such that $\mathfrak{F}(v_1) = \ln v_1.$
Then,

$$\mathfrak{F}\left(\min\left\{\mathcal{D}\left(gv, gw\right), \mathcal{D}\left(g^2v, g^2w\right)\right\}\right) \geq \tau + \mathfrak{F}\left(\mathcal{D}\left(v, w\right)\right).$$

Therefore, $g : X \mapsto X$ is generalized reversed $\mathfrak{F}$-contraction mapping and theorem 3.2 guarantees the existence of a unique fixed point for the integral equation (26).

5. Open questions

These new modifications of expanding type mappings may further provide some of the following results.

- One may obtain some fixed point results for the reversal of generalized F-contraction mapping by weakening the conditions $(F1), (F2), (F3)$.
- In the generalized reversed mean contraction mapping, we have $k \in (1, \infty)$. There may exist the possibility of obtaining some results for $\mathfrak{M}$, if $\min\left\{\left(\rho\left(\mathfrak{M}x, \mathfrak{M}y\right), \rho\left(\mathfrak{M}^2x, \mathfrak{M}^2y\right)\right)\right\} \geq \rho\left(x, y\right).$
- One may find results on generalized b-metric space and controlled metric space for the reversal of generalized Banach contraction principle.
- One may find the above results with multi-index $v = (v_1, ..., v_n)$ with $n > 2$.
- It will be a great idea to use the average of order $p > 1$ instead of arithmetical mean in the definition of generalized reversed mean contraction mapping.

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Conflict of interest

The authors declare that they have no competing interests.

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