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Research article

Subordination problems for a new class of Bazilevič functions associated with k-symmetric points and fractional q-calculus operators

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Abstract: In this article, we introduce and investigate a new class of Bazilevič functions with respect to k-symmetric points defined by using fractional q-calculus operators that are analytic in the open unit disk $\mathbb{D}$. Several interesting subordination problems are also derived for the functions belonging to this new class.

Keywords: univalent functions; starlike with respect to symmetric points; Bazilevič functions; fractional q-calculus operators; subordination
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1. Introduction and preliminaries

The hypothesis of fractional calculus, which is primarily due to its proven developments in many branches of science and engineering over the past three decades or so, has gained considerable prominence and recognition in recent times. Most of the theory of fractional calculus is specifically based upon the familiar Riemann-Liouville fractional derivative (or integral). The fractional q-calculus is the extension of the ordinary fractional calculus in the q-theory. Recently, there was notable increase of articles written in the area of the q-calculus due to significant applications of the q-calculus in mathematics, statistics and physics. For more details, interested readers may refer to the books of [1, 2, 4–6, 8, 9, 14] on the subject.

Indeed the fourth author of this article with Raina in [11] have used the fractional q-calculus operators in investigating certain classes of functions which are analytic in the open disk $\mathbb{D}$. Recently, many authors have introduced new classes of analytic functions using q-calculus operators. For some recent investigations on the classes of analytic functions defined by using q-calculus operators and related topics, we refer the reader to [12, 13, 17, 18] and the references cited therein. In the present paper, we aim at introducing a new class of Bazilevič functions involving the fractional q-calculus operators, which is analytic in the open unit disk. Certain interesting subordination results are also derived for the functions belonging to this class.

We first give various definitions and notations in q-calculus which are useful to understand the subject of this paper.

For any complex number α, the q-shifted factorials are defined as

$$(\alpha; q)_0 = 1, \qquad (\alpha; q)_n = \prod_{k=0}^{n-1}(1 - \alpha q^k), \qquad n \in \mathbb{N}, \tag{1.1}$$

and in terms of the basic analogue of the gamma function

$$(q^\alpha; q)_n = \frac{\Gamma_q(\alpha + n)(1 - q)^n}{\Gamma_q(\alpha)}, \qquad (n > 0),$$

where the q-gamma function is defined by

$$\Gamma_q(x) = \frac{(q, q)_\infty (1 - q)^{1-x}}{(q^x; q)_\infty}, \qquad (0 < q < 1).$$

If $|q| < 1$, the definition (1.1) remains meaningful for $n = \infty$ as a convergent infinite product

$$(\alpha; q)_\infty = \prod_{j=0}^{\infty}(1 - \alpha q^j).$$

In view of the relation

$$\lim_{q \to 1^-} \frac{(q^\alpha; q)_n}{(1 - q)^n} = (\alpha)_n,$$

we observe that the q-shifted factorial (1.1) reduces to the familiar Pochhammer symbol $(\alpha)_n$, where $(\alpha)_n = \alpha(\alpha + 1) \cdots (\alpha + n - 1)$.

Also, the q-derivative and q-integral of a function on a subset of $\mathbb{C}$ are, respectively, given by (see [6] pp.19–22)

$$D_q f(z) = \frac{f(z) - f(zq)}{(1 - q)z}, \qquad (z \neq 0, q \neq 0) \tag{1.2}$$

and

$$\int_0^z f(t) d_q t = z(1 - q) \sum_{k=0}^{\infty} q^k f(zq^k). \tag{1.3}$$

Therefore, the q-derivative of $f(z) = z^n$, where n is a positive integer is given by

$$D_q z^n = \frac{z^n - (zq)^n}{(1-q)z} = [n]_q \, z^{n-1},$$

where

$$[n]_q = \frac{1-q^n}{1-q} = q^{n-1} + \cdots + 1$$

and is called the q-*analogue* of n. As $q \to 1$, we have $[n]_q = q^{n-1} + \cdots + 1 \to 1 + \cdots + 1 = n$. The q-derivative of $f(z) = \ln z$ is given by

$$D_q \ln z = \frac{\ln q^{-1}}{(1-q)z}.$$

We now define the fractional q-calculus operators of a complex-valued function $f(z)$, which were recently studied by Purohit and Raina [11].

Definition 1.1. (**Fractional q-integral operator**) The fractional q-integral operator $I^{\delta}_{q,z}$ of a function $f(z)$ of order δ is defined by

$$I^{\delta}_{q,z} f(z) \equiv D^{-\delta}_{q,z} f(z) = \frac{1}{\Gamma_q(\delta)} \int_0^z (z - tq)_{\delta-1} f(t) \, d_q t, \qquad (\delta > 0), \tag{1.4}$$

where $f(z)$ is analytic in a simply connected region of the z-plane containing the origin and the q-binomial function $(z - tq)_{\delta-1}$ is given by

$$(z - tq)_{\delta-1} = z^{\delta-1} \, {}_1\Phi_0[q^{-\delta+1}; -; q, tq^{\delta}/z].$$

The series ${}_1\Phi_0[\delta; -; q, z]$ is single valued when $|\arg(z)| < \pi$ and $|z| < 1$ (see for details [6], pp.104–106), therefore, the function $(z - tq)_{\delta-1}$ in (1.4) is single valued when $|\arg(-tq^{\delta}/z)| < \pi$, $|tq^{\delta}/z| < 1$ and $|\arg(z)| < \pi$.

Definition 1.2. (**Fractional q-derivative operator**) The fractional q-derivative operator $D^{\delta}_{q,z}$ of a function $f(z)$ of order δ is defined by

$$D^{\delta}_{q,z} f(z) \equiv D_{q,z} I^{1-\delta}_{q,z} f(z) = \frac{1}{\Gamma_q(1-\delta)} D_{q,z} \int_0^z (z - tq)_{-\delta} f(t) \, d_q t, \qquad (0 \leqslant \delta < 1), \tag{1.5}$$

where $f(z)$ is suitably constrained and the multiplicity of $(z - tq)_{-\delta}$ is removed as in Definition 1.1.

Definition 1.3. (**Extended fractional q-derivative operator**) Under the hypotheses of Definition 1.2, the fractional q-derivative for a function $f(z)$ of order δ is defined by

$$D^{\delta}_{q,z} f(z) = D^{m}_{q,z} I^{m-\delta}_{q,z} f(z), \tag{1.6}$$

where $m - 1 \leqslant \delta < 1$, $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and $\mathbb{N}$ denotes the set of natural numbers.

Remark 1.1. It can be easily seen from Definition 1.2 that

$$D^{\delta}_{q,z} z^n = \frac{\Gamma_q(n+1)}{\Gamma_q(n+1-\delta)} z^{n-\delta} \quad (\delta \geqslant 0, \text{ and } n > -1).$$

2. The class $\mathcal{B}^m_{q,k}(\lambda,\delta,\gamma;\phi)$

Let $\mathcal{H}(a,n)$ denote the class of functions $f(z)$ of the form

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots, \quad (z \in \mathbb{D}), \tag{2.1}$$

which are analytic in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. In particular, let $\mathcal{A}$ be the subclass of $\mathcal{H}(0,1)$ containing functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{2.2}$$

We denote by $\mathcal{S}$, $\mathcal{S}^*$, C and $\mathcal{K}$, the classes of all functions in $\mathcal{A}$ which are, respectively, univalent, starlike, convex and close-to-convex in $\mathbb{D}$. Let $f(z)$ and $g(z)$ be analytic in $\mathbb{D}$. A function f is subordinate to g in $\mathbb{D}$, if there exists an analytic function $w(z)$ in $\mathbb{D}$ with

$$w(0) = 0, \quad |w(z)| < 1 \quad (z \in \mathbb{D}),$$

such that

$$f(z) = g(w(z)) \qquad (z \in \mathbb{D}).$$

We denote this subordination by $f(z) \prec g(z)$. Further, if the function $g(z)$ is univalent in $\mathbb{D}$, then $f(z) \prec g(z) \quad (z \in \mathbb{D}) \Longleftrightarrow f(0) = g(0)$ and $f(\mathbb{D}) \subset g(\mathbb{D})$.
Let k be a positive integer and let $\varepsilon_k = \exp(\frac{2\pi i}{k})$. For $f \in \mathcal{A}$, let

$$f_k(z) = \frac{1}{k} \sum_{j=0}^{k-1} \varepsilon_k^{-j} f(\varepsilon_k^j z). \tag{2.3}$$

The function f is said to be starlike with respect to k-symmetric points if it satisfies

$$\Re\left(\frac{zf'(z)}{f_k(z)}\right) > 0, \quad z \in \mathbb{D}. \tag{2.4}$$

We denote by $S_s^{(k)}$ the subclass of $\mathcal{A}$ consisting of all functions starlike with respect to k-symmetric points in $\mathbb{D}$. The class $S_s^{(2)}$ was introduced and studied by Sakaguchi [15]. We also note that different subclasses of $S_s^{(k)}$ can be obtained by replacing condition (2.4) by

$$\frac{zf'(z)}{f_k(z)} \prec h(z),$$

where $h(z)$ is a given convex function, with $h(0) = 1$ and $\Re\{h(z)\} > 0$.
Using $D^\delta_{q,z}$, we define a fractional q-differintegral operator $\Omega^\delta_{q,z} : \mathcal{A} \longrightarrow \mathcal{A}$, as follows:

$$\Omega^\delta_{q,z} f(z) = \frac{\Gamma_q(2-\delta)}{\Gamma_q(2)} z^\delta D^\delta_{q,z} f(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma_q(2-\delta)\Gamma_q(n+1)}{\Gamma_q(2)\Gamma_q(n+1-\delta)} a_n z^n, \tag{2.5}$$
$$(\delta < 2;\ n \in \mathbb{N};\ 0 < q < 1;\ z \in \mathbb{D}),$$

where $D^{\delta}_{q,z} f(z)$ in (2.5) represents, respectively, a fractional q-integral of $f(z)$ of order δ when $-\infty < \delta < 0$ and a fractional q-derivative of $f(z)$ of order δ when $0 \leqslant \delta < 2$. Here we note that $\Omega^{0}_{q,z} f(z) = f(z)$.

We now define a linear multiplier fractional q-differintegral operator $\mathscr{D}^{\delta,m}_{q,\lambda}$ as follows:

$$
\begin{aligned}
\mathscr{D}^{\delta,0}_{q,\lambda} f(z) &= f(z),\\
\mathscr{D}^{\delta,1}_{q,\lambda} f(z) &= (1-\lambda)\Omega^{\delta}_{q,z} f(z) + \lambda\, z\,(\Omega^{\delta}_{q,z} f(z))', \qquad (\lambda \geqslant 0),\\
\mathscr{D}^{\delta,2}_{q,\lambda} f(z) &= \mathscr{D}^{\delta,1}_{q,\lambda}(\mathscr{D}^{\delta,1}_{q,\lambda} f(z)),\\
&\ \ \vdots\\
\mathscr{D}^{\delta,m}_{q,\lambda} f(z) &= \mathscr{D}^{\delta,1}_{q,\lambda}(\mathscr{D}^{\delta,m-1}_{q,\lambda} f(z)), \qquad m \in \mathbb{N}.
\end{aligned}
\tag{2.6}
$$

If $f(z)$ is given by (2.2), then by (2.6), we have

$$
\mathscr{D}^{\delta,m}_{q,\lambda} f(z) = z + \sum_{n=2}^{\infty} \left(\frac{\Gamma_q(2-\delta)\Gamma_q(n+1)}{\Gamma_q(2)\Gamma_q(n+1-\delta)} \left[1-\lambda+n\lambda\right] \right)^{m} a_n z^n.
$$

It can be seen that, by specializing the parameters the operator $\mathscr{D}^{\delta,m}_{q,\lambda}$ reduces to many known and new integral and differential operators. In particular, when $\delta = 0$ the operator $\mathscr{D}^{\delta,m}_{q,\lambda}$ reduces to the operator introduced by Al-Oboudi [3] and for $\delta = 0$, $\lambda = 1$ it reduces to the operator introduced by Sălăgean [16].

Throughout this paper, we assume that

$$
f^{m}_{q,k}(\lambda,\delta;z) = \frac{1}{k} \sum_{j=0}^{k-1} \varepsilon_k^{-j} (\mathscr{D}^{\delta,m}_{q,\lambda} f(\varepsilon_k^{j} z)) = z + \cdots, \qquad (f \in \mathcal{A}).
\tag{2.7}
$$

Clearly, for $k = 1$, we have

$$
f^{m}_{q,1}(\lambda,\delta;z) = \mathscr{D}^{\delta,m}_{q,\lambda} f(z).
$$

Let $\mathcal{P}$ denote the class of analytic functions $h(z)$ with $h(0) = 1$, which are convex and univalent in $\mathbb{D}$ and for which $\Re\{h(z)\} > 0, \quad (z \in \mathbb{D})$.

In the present article, a new subclass $\mathcal{B}^{m}_{q,k}(\lambda,\delta,\gamma;\phi)$ of analytic functions, using the linear multiplier q-fractional differintegral operator $\mathscr{D}^{\delta,m}_{q,\lambda}$ defined by (2.6) is introduced. It is interesting to note that the class $\mathcal{B}^{m}_{q,k}(\lambda,\delta,\gamma;\phi)$ is a generalization of the class of Bazilevič functions.

Definition 2.1. A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{B}^{m}_{q,k}(\lambda,\delta,\gamma;\phi)$ if and only if

$$
\frac{z\left(\mathscr{D}^{\delta,m}_{q,\lambda} f(z)\right)'}{[f^{m}_{q,k}(\lambda,\delta;z)]^{1-\gamma}[g^{m}_{q,k}(\lambda,\delta;z)]^{\gamma}} \prec \phi(z), \qquad (g \in \mathcal{S}^*, \quad z \in \mathbb{D}),
$$

where $\gamma \in [0,\, \infty)$, $\phi \in \mathcal{P}$, $f^{m}_{q,k}(\lambda,\delta;z)$ and $g^{m}_{q,k}(\lambda,\delta;z) \neq 0$ for all $z \in \mathbb{D}\backslash\{0\}$ are defined as in (2.7).

It is easy to see to verify the following relations: If $q \to 1^-$, then

1. $\mathcal{B}^{0}_{q,1}(\lambda,\delta,0;\frac{1+z}{1-z}) = \mathcal{S}^*$;

2. $\mathcal{B}_{q,k}^{0}(\lambda, \delta, 0; \frac{1+z}{1-z}) = \mathcal{S}_s^{(k)}$;

3. $\mathcal{B}_{q,1}^{0}(\lambda, \delta, 1; \frac{1+z}{1-z}) = \mathcal{K}$;

4. $\mathcal{B}_{q,1}^{0}(\lambda, \delta, \gamma; \frac{1+z}{1-z}) = \mathcal{B}(\gamma)$, the class of Bazilevič functions of type γ.

Now, we derive some sufficient conditions for functions belonging to the class $\mathcal{B}_{q,k}^{m}(\lambda, \delta, \gamma; \phi)$. In order to prove our results we need the following results.

Lemma 2.1. (*[10], see also [7]) Let h be convex in $\mathbb{D}$, with $h(0) = a, \beta \neq 0$ and $\Re(\beta) \geqslant 0$. If $g \in \mathcal{H}(a, n)$ and*

$$g(z) + \frac{zg'(z)}{\beta} \prec h(z),$$

then

$$g(z) \prec \phi(z) \prec h(z),$$

where

$$\phi(z) = \frac{\beta}{n\, z^{\beta/n}} \int_0^z h(t)\, t^{(\beta/n)-1}\, dt.$$

The function ϕ is convex and is the best (a, n)-dominant.

Lemma 2.2. (*[10], see also [19]) Let h be starlike in $\mathbb{D}$, with $h(0) = 0$. If $g \in \mathcal{H}(a, n)$ satisfies*

$$zg'(z) \prec h(z),$$

then

$$g(z) \prec \phi(z) = a + n^{-1} \int_0^z h(t)\, t^{-1}\, dt.$$

The function ϕ is convex and is the best (a, n)-dominant.

3. Subordination problems for the class $\mathcal{B}_{q,k}^{m}(\lambda, \delta, \gamma; \phi)$

In this section, we establish some sufficient conditions of subordination for analytic functions defined above involving the q-differential operator.

Theorem 3.1. *Let f, $g \in \mathcal{A}$ with $f_{q,k}^{m}(\lambda, \delta; z)$ and $g_{q,k}^{m}(\lambda, \delta; z) \neq 0$ for all $z \in \mathbb{D}\backslash\{0\}$ and let h be convex in $\mathbb{D}$, with $h(0) = 1$ and $\Re\{h(z)\} > 0$. Further suppose that $g \in \mathcal{S}^*$ and*

$$\begin{aligned}\left\{\frac{z\left(\mathscr{D}_{q,\lambda}^{\delta,m} f(z)\right)'}{[f_{q,k}^{m}(\lambda,\delta;z)]^{1-\gamma}[g_{q,k}^{m}(\lambda,\delta;z)]^{\gamma}}\right\}^2 \Bigg[3 + \frac{2z\left(\mathscr{D}_{q,\lambda}^{\delta,m} f(z)\right)''}{\left(\mathscr{D}_{q,\lambda}^{\delta,m} f(z)\right)'} - 2(1-\gamma)\frac{z\left(f_{q,k}^{m}(\lambda,\delta;z)\right)'}{f_{q,k}^{m}(\lambda,\delta;z)} \\ - 2\gamma\frac{z\left(g_{q,k}^{m}(\lambda,\delta;z)\right)'}{g_{q,k}^{m}(\lambda,\delta;z)}\Bigg] \prec h(z).\end{aligned} \tag{3.1}$$

Then

$$\frac{z\left(\mathscr{D}_{q,\lambda}^{\delta,m} f(z)\right)'}{[f_{q,k}^{m}(\lambda,\delta;z)]^{1-\gamma}[g_{q,k}^{m}(\lambda,\delta;z)]^{\gamma}} \prec q(z) = \sqrt{Q(z)}, \tag{3.2}$$

where

$$Q(z) = \frac{1}{z}\int_0^z h(t)dt, \tag{3.3}$$

and the function $q(z)$ is the best dominant.

Proof. Let $p(z) = \dfrac{z\left(\mathscr{D}_{q,\lambda}^{\delta,m} f(z)\right)'}{[f_{q,k}^m(\lambda,\delta;z)]^{1-\gamma}[g_{q,k}^m(\lambda,\delta;z)]^\gamma} \qquad (z \in \mathbb{D}; z \neq 0)$.
Then $p(z) \neq 0$ in $\mathbb{D}$. Further, $p(z) \in \mathcal{H}(1,1)$. Differentiating (3.3) we obtain

$$Q(z) + z\,Q'(z) = h(z) \tag{3.4}$$

If we let

$$p_1(z) = 1 + \frac{z\,Q''(z)}{Q'(z)},$$

then p_1 is analytic and $p_1(0) = 1$. From (3.4) we obtain

$$Q'(z)\big[p_1(z) + 1\big] = h'(z)$$

and

$$p_1(z) + \frac{z\,p_1{}'(z)}{p_1(z)+1} = 1 + \frac{z\,h''(z)}{h'(z)}.$$

Since h is convex, we have

$$\Re\left(p_1(z) + \frac{z\,p_1'(z)}{p_1(z)+1}\right) > 0$$

which implies $\Re(p_1(z)) > 0$ ([10], Theorem 3.2a) and therefore, $Q(z)$ is convex and univalent. We now set $P(z) = p^2(z)$. Then $P(z) \in \mathcal{H}(1,1)$ with $P(z) \neq 0$ in $\mathbb{D}$. By logarithmic differentiation we have,

$$\frac{z\,P'(z)}{P(z)} = 2\left[1 + \frac{z\left(\mathscr{D}_{q,\lambda}^{\delta,m} f(z)\right)''}{\left(\mathscr{D}_{q,\lambda}^{\delta,m} f(z)\right)'} - (1-\gamma)\frac{z\left(f_{q,k}^m(\lambda,\delta;z)\right)'}{f_{q,k}^m(\lambda,\delta;z)} - \gamma\frac{z\left(g_{q,k}^m(\lambda,\delta;z)\right)'}{g_{q,k}^m(\lambda,\delta;z)}\right].$$

Therefore, by (3.1) we have

$$P(z) + z\,P'(z) \prec h(z). \tag{3.5}$$

Now, by Lemma 2.1 with $\beta = 1$, we deduce that

$$P(z) \prec Q(z) \prec h(z),$$

and Q is the best dominant of (3.5). Since $\Re\{h(z)\} > 0$ and $Q(z) \prec h(z)$ we also have $\Re\{Q(z)\} > 0$. Hence, the univalence of Q implies the univalence of $q(z) = \sqrt{Q(z)}$, and

$$p^2(z) = P(z) \prec Q(z) = q^2(z),$$

which implies that $p(z) \prec q(z)$. Since Q is the best dominant of (3.5), we deduce that $q(z)$ is the best dominant of (3.2). This completes the proof. □

Corollary 3.2. *Let $f,\ g \in \mathcal{A}$ with $f^m_{q,k}(\lambda,\delta;z)$ and $g^m_{q,k}(\lambda,\delta;z) \neq 0$ for all $z \in \mathbb{D}\backslash\{0\}$. If $g \in \mathcal{S}^*$ and $\Re\big(\Psi(z)\big) > \alpha, \quad (0 \leqslant \alpha < 1)$, where*

$$\Psi(z) = \left\{ \frac{z\left(\mathscr{D}^{\delta,m}_{q,\lambda} f(z)\right)'}{[f^m_{q,k}(\lambda,\delta;z)]^{1-\gamma}[g^m_{q,k}(\lambda,\delta;z)]^{\gamma}} \right\}^2 \times \left[3 + \frac{2z\left(\mathscr{D}^{\delta,m}_{q,\lambda} f(z)\right)''}{\left(\mathscr{D}^{\delta,m}_{q,\lambda} f(z)\right)'} - 2(1-\gamma)\frac{z\left(f^m_{q,k}(\lambda,\delta;z)\right)'}{f^m_{q,k}(\lambda,\delta;z)} - 2\gamma\frac{z\left(g^m_{q,k}(\lambda,\delta;z)\right)'}{g^m_{q,k}(\lambda,\delta;z)} \right],$$

then

$$\Re\left\{ \frac{z\left(\mathscr{D}^{\delta,m}_{q,\lambda} f(z)\right)'}{[f^m_{q,k}(\lambda,\delta;z)]^{1-\gamma}[g^m_{q,k}(\lambda,\delta;z)]^{\gamma}} \right\} > \mu(\alpha),$$

where $\mu(\alpha) = \big[2(1-\alpha)\ln 2 + (2\alpha-1)\big]^{\frac{1}{2}}$, and this result is sharp.

Proof. Let $h(z) = \dfrac{1+(2\alpha-1)z}{1+z}$ with $0 \leqslant \alpha < 1$. Then from Theorem 3.1, it follows that $Q(z)$ is convex and $\Re\{Q(z)\} > 0$. Also we have,

$$\min_{|z|\leqslant 1} \Re\, q(z) = \min_{|z|\leqslant 1} \Re\, \sqrt{Q(z)} = \sqrt{Q(1)} = \Big[2(1-\alpha)\ln 2 + (2\alpha-1)\Big]^{\frac{1}{2}}.$$

This completes the proof the corollary. □

By setting $m = 0$ in Corollary 3.2, we have the following corollary.

Corollary 3.3. *Let $f,\ g \in \mathcal{A}$ with $f'(z),\ f_k(z)$ and $g_k(z) \neq 0$ for all $z \in \mathbb{D}\backslash\{0\}$. If $g \in \mathcal{S}^*$ and*

$$\Re\left\{ \left(\frac{zf'(z)}{\big(f_k(z)\big)^{1-\gamma}\big(g_k(z)\big)^{\gamma}} \right)^2 \left[3+2\frac{zf''(z)}{f'(z)} - 2(1-\gamma)\frac{zf_k'(z)}{f_k(z)} - 2\gamma\frac{zg_k'(z)}{g_k(z)} \right] \right\} > \alpha,$$
$$(0 \leqslant \alpha < 1),$$

then

$$\Re\left\{ \frac{zf'(z)}{\big(f_k(z)\big)^{1-\gamma}\big(g_k(z)\big)^{\gamma}} \right\} > \mu(\alpha),$$

where $\mu(\alpha) = \big[2(1-\alpha)\ \ln 2 + (2\alpha-1)\big]^{\frac{1}{2}}$, and this result is sharp.

Also, taking $\gamma = 0$ in Corollary 3.3, we obtain

Corollary 3.4. *Let $f \in \mathcal{A}$ with $f'(z)$ and $f_k(z) \neq 0$ for all $z \in \mathbb{D}\backslash\{0\}$. If*

$$\Re\left\{ \left(\frac{zf'(z)}{f_k(z)} \right)^2 \left[3 + 2\frac{zf''(z)}{f'(z)} - 2\frac{zf_k'(z)}{f_k(z)} \right] \right\} > \alpha, \quad (0 \leqslant \alpha < 1),$$

then

$$\Re\left\{ \frac{zf'(z)}{f_k(z)} \right\} > \mu(\alpha),$$

where $\mu(\alpha) = \big[2(1-\alpha)\ \ln 2 + (2\alpha-1)\big]^{\frac{1}{2}}$, and this result is sharp.

Further, setting $\gamma = 1$ in Corollary 3.3, we obtain the following corollary.

Corollary 3.5. *Let $f,\ g \in \mathcal{A}$ with $f'(z)$ and $g_k(z) \neq 0$ for all $z \in \mathbb{D}\backslash\{0\}$. If $g \in \mathcal{S}^*$ and*

$$\Re\left\{\left(\frac{zf'(z)}{g_k(z)}\right)^2\left[3+2\frac{zf''(z)}{f'(z)}-2\frac{zg_k'(z)}{g_k(z)}\right]\right\} > \alpha, \quad (0 \leqslant \alpha < 1),$$

then

$$\Re\left\{\frac{zf'(z)}{g_k(z)}\right\} > \mu(\alpha),$$

where $\mu(\alpha) = \left[2(1-\alpha)\ln 2 + (2\alpha - 1)\right]^{\frac{1}{2}}$, and this result is sharp.

Next, we prove the following result.

Theorem 3.6. *Let $f,\ g \in \mathcal{A}$ with $f^m_{q,k}(\lambda, \delta; z)$ and $g^m_{q,k}(\lambda, \delta; z) \neq 0$ for all $z \in \mathbb{D}\backslash\{0\}$. Further suppose that h is starlike with $h(0) = 0$, in the unit disk $\mathbb{D}$, $g \in \mathcal{S}^*$ and*

$$\left[1+\frac{z\left(\mathscr{D}^{\delta,m}_{q,\lambda}f(z)\right)''}{\left(\mathscr{D}^{\delta,m}_{q,\lambda}f(z)\right)'}-(1-\gamma)\frac{z\left(f^m_{q,k}(\lambda,\delta;z)\right)'}{f^m_{q,k}(\lambda,\delta;z)}-\gamma\frac{z\left(g^m_{q,k}(\lambda,\delta;z)\right)'}{g^m_{q,k}(\lambda,\delta;z)}\right] \prec h(z),$$

$$(z \in \mathbb{D};\ \gamma \geqslant 0).$$

Then

$$\frac{z\left(\mathscr{D}^{\delta,m}_{q,\lambda}f(z)\right)'}{[f^m_k(\lambda,\delta;z)]^{1-\gamma}[g^m_k(\lambda,\delta;z)]^{\gamma}} \prec \phi(z) = \exp\left(\int_0^z \frac{h(t)}{t}dt\right),$$

where ϕ is convex and is the best dominant.

Proof. Let $p(z) = \dfrac{z\left(\mathscr{D}^{\delta,m}_{q,\lambda}f(z)\right)'}{[f^m_{q,k}(\lambda,\delta;z)]^{1-\gamma}[g^m_{q,k}(\lambda,\delta;z)]^{\gamma}} \qquad (z \in \mathbb{D}; z \neq 0).$
Since $p(z) \in \mathcal{H}(1,1)$ with $p(z) \neq 0$ in $\mathbb{D}$, we can define the analytic function $P(z) \equiv \ln p(z)$. Clearly $P(z) \in \mathcal{H}(0,1)$. Now by logarithmic differentiation we have,

$$\frac{z\,p'(z)}{p(z)} = \left[1+\frac{z\left(\mathscr{D}^{\delta,m}_{q,\lambda}f(z)\right)''}{\left(\mathscr{D}^{\delta,m}_{q,\lambda}f(z)\right)'}-(1-\gamma)\frac{z\left(f^m_{q,k}(\lambda,\delta;z)\right)'}{f^m_{q,k}(\lambda,\delta;z)}-\gamma\frac{z\left(g^m_{q,k}(\lambda,\delta;z)\right)'}{g^m_{q,k}(\lambda,\delta;z)}\right].$$

Thus we have

$$z\,P'(z) \prec h(z) \qquad (z \in \mathbb{D}).$$

Now by Lemma 2.2, we deduce that

$$\ln p(z) \prec \int_0^z \frac{h(t)}{t}dt.$$

This completes the proof of Theorem 3.6. □

By putting $m = 0$ in Theorem 3.6, we arrive at the following corollary.

Corollary 3.7. *Let $f,\ g \in \mathcal{A}$ with $f'(z),\ \ f_k(z)$ and $g_k(z) \neq 0$ for all $z \in \mathbb{D}\backslash\{0\}$. Further suppose that h is starlike with $h(0) = 0$, in the unit disk $\mathbb{D}$, $\ g \in \mathcal{S}^*$ and*

$$1 + \frac{zf''(z)}{f'(z)} - (1-\gamma)\frac{zf_k'(z)}{f_k(z)} - \gamma\frac{zg_k'(z)}{g_k(z)} \prec h(z), \quad (z \in \mathbb{D};\ \gamma \geqslant 0).$$

Then

$$\frac{zf'(z)}{\big(f_k(z)\big)^{1-\gamma}\big(g_k(z)\big)^{\gamma}} \prec \phi(z) = \exp\left(\int_0^z \frac{h(t)}{t}dt\right),$$

where ϕ is convex and is the best dominant.

Also, specializing $\gamma = 0$ and $\gamma = 1$ in Corollary 3.7, we obtain the following interesting two corollaries as given below.

Corollary 3.8. *Let $f,\ g \in \mathcal{A}$ with $f'(z)$ and $f_k(z) \neq 0$ for all $z \in \mathbb{D}\backslash\{0\}$. Further suppose that h is starlike with $h(0) = 0$, in the unit disk $\mathbb{D}$ and*

$$1 + \frac{zf''(z)}{f'(z)} - \frac{zf_k'(z)}{f_k(z)} \prec h(z), \qquad (z \in \mathbb{D}).$$

Then

$$\frac{zf'(z)}{f_k(z)} \prec \phi(z) = \exp\left(\int_0^z \frac{h(t)}{t}dt\right),$$

where ϕ is convex and is the best dominant.

Corollary 3.9. *Let $f,\ g \in \mathcal{A}$ with $f'(z)$ and $g_k(z) \neq 0$ for all $z \in \mathbb{D}\backslash\{0\}$. Further suppose that h is starlike with $h(0) = 0$, in the unit disk $\mathbb{D}$, $\ g \in \mathcal{S}^*$ and*

$$1 + \frac{zf''(z)}{f'(z)} - \frac{zg_k'(z)}{g_k(z)} \prec h(z), \qquad (z \in \mathbb{D}).$$

Then

$$\frac{zf'(z)}{g_k(z)} \prec \phi(z) = \exp\left(\int_0^z \frac{h(t)}{t}dt\right),$$

where ϕ is convex and is the best dominant.

4. Conclusions

By suitably choosing the parameters, one can further easily obtain a number of inequalities from the main results stated in Theorem 3.1 and Theorem 3.6. Moreover, the class $\mathcal{B}^m_{q,k}(\lambda, \delta, \gamma; \phi)$ defined in this article can also be used in the investigation of various geometric properties like, the coefficient estimates, distortion bounds, radii of starlikeness, convexity and close to convexity etc. in the unit disk.

Conflict of interest

The authors declare that they have no competing interests.

Authors contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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References

1. M. H. Abu Risha, M. H. Annaby, M. E. H. Ismail, Z. S. Mansour, Linear q-difference equations, *Z. Anal. Anwend.*, **26** (2007), 481–494.
2. D. Albayrak, S. D. Purohit, F. Uçar, On q-analogues of Sumudu transforms, *An. Şt. Univ. Ovidius Constanţa*, **21** (2013), 239–260.
3. F. M. Al-Oboudi, On univalent functions defined by a generalized Sălăgean operator, *Int. J. Math. Math. Sci.*, **27** (2004), 1429–1436.
4. G. Bangerezako, Variational calculus on q-nonuniform lattices, *J. Math. Anal. Appl.*, **306** (2005), 161–179.
5. T. Ernst, *A Comprehensive Treatment of q-Calculus*, Basel: Birkhäuser, 2012.
6. G. Gasper, M. Rahman, *Basic hypergeometric series*, Encyclopedia of Mathematics and its Applications, Cambridge: Cambridge University Press, 1990.
7. D. J. Hallenbeck, S. Ruscheweyh, Subordination by convex functions, *Proc. Am. Math. Soc.*, **52** (1975), 191–195.
8. V. Kac, P. Cheung, *Quantum Calculus*, New York: Springer, 2002.
9. Z. S. I. Mansour, Linear sequential q-difference equations of fractional order, *Fractional Calculus Appl. Anal.*, **12** (2009), 159–178.
10. S. S. Miller, P. T. Mocanu, *Differential Subordinations: Theory and Applications*, Series in Pure and Applied Mathematics, New York, 2000.

11. S. D. Purohit, R. K. Raina, Certain subclasses of analytic functions associated with fractional q-calculus operators, *Math. Scand.*, **109** (2011), 55–70.
12. S. D. Purohit, A new class of multivalently analytic functions associated with fractional q-calculus operators, *Fractional Differ. Calculus*, **2** (2012), 129–138.
13. S. D. Purohit, K. A. Selvakumaran, On certain generalized q-integral operators of analytic functions, *Bull. Korean Math. Soc.*, **52** (2015), 1805–1818.
14. P. M. Rajković, S. D. Marinković, M. S. Stanković, Fractional integrals and derivatives in q-calculus, *Appl. Anal. Discrete Math.*, **1** (2007), 311–323.
15. K. Sakaguchi, On a certain univalent mapping, *J. Math. Soc. Japan*, **11** (1959), 72–75.
16. G. Ş. Sălăgean, Subclasses of univalent functions, In: C. A. Cazacu, N. Boboc, M. Jurchescu, I. Suciu, *Complex Analysis-Fifth Romanian-Finnish Seminar*, Lecture Notes in Mathematics, Berlin: Springer, 1983.
17. K. A. Selvakumaran, S. D. Purohit, A. Secer, M. Bayram, Convexity of certain q-integral operators of p-valent functions, *Abstr. Appl. Anal.*, **2014** (2014), 925902.
18. K. A. Selvakumaran, S. D. Purohit, A. Secer, Majorization for a class of analytic functions defined by q-differentiation, *Math. Probl. Eng.*, **2014** (2014), 653917.
19. T. J. Suffridge, Some remarks on convex maps of the unit disk, *Duke Math. J.*, **37** (1970), 775–777.