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Research article

On total edge irregularity strength for some special types of uniform theta snake graphs

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Abstract: A labeling of a connected, simple and undirected graph $G(V,E)$ is a map that assigns the elements of a graph G with positive numbers. Many types of labeling for graph are found and one of them is a total edge irregularity strength (TEIS) of G, which denoted by $tes(G)$. In the current paper, we defined a new type of family of graph called uniform theta snake graph, $\theta_n(t,m)$. Also, the exact values of total edge irregularity strengths for some special types of the new family have been determined.

Keywords: irregular labelling; total edge irregularity strength; edge irregular total labeling; uniform theta snake graph
Mathematics Subject Classification: 05C78, 05C38

1. Introduction

In graph theory, graph labeling is an assignment of labels or weights to the vertices and edges of a graph. Graph labeling plays an important role in many fields such as computer science, coding theory and physics [32]. Baca et al. [10] have introduced the definition of an edge irregular total ℓ-labeling of any graph as a labeling $\mathcal{L}: V \cup E \to \{1,2,3,\dots,\ell\}$ in which every two distinct edges fh and f^*h^* of a graph G have distinct weights, this means that $w_{\mathcal{L}}(fh) \neq w_{\mathcal{L}}(f^*h^*)$ where $w_{\mathcal{L}}(fh) = \mathcal{L}(f) +$

$\mathcal{L}(h) + \mathcal{L}(fh)$. They have deduced inequality which gives a lower bound of $tes(G)$ for a graph G,

$$tes(G) \geq max\left\{\left\lceil\frac{|E(G)|+2}{3}\right\rceil, \left\lceil\frac{\Delta G+1}{2}\right\rceil\right\} \tag{1}$$

Also, they have introduced the exact value of TEIS, $tes(G)$ for some families of graphs like fan graph F_n and wheel graph W_n,

$$tes(F_n) = \left\lceil\frac{3n+2}{3}\right\rceil,$$

$$tes(W_n) = \left\lceil\frac{2n+2}{3}\right\rceil.$$

In [15] authors have proved that for any tree T

$$tes(T) = max\left\{\left\lceil\frac{k+1}{3}\right\rceil, \left\lceil\frac{\Delta G+1}{2}\right\rceil\right\},$$

where ΔG is maximum degree on k vertices. In addition, Salama [26] investigated the exact value of TEIS for a polar grid graph,

$$tes(P_{m,n}) = \left\lceil\frac{2mn+2}{3}\right\rceil.$$

Authors in [1] determined TEIS for zigzag graphs. Also, the exact value of TEIS of the generalized web graph $W_{n,m}$and some families has been determined, see [14]. Tilukay et al. [31] have investigated total irregularity strength for a wheel graph, a fan graph, a triangular Book graph and a friendship graph. On the other hand, in [2,3,8,17,20,24,29] the total edge irregularity strengths for hexagonal grid graphs, centralized uniform theta graphs, generalized helm graph, series parallel graphs, disjoint union of isomorphic copies of generalized Petersen graph, disjoint union of wheel graphs, subdivision of star S_n and categorical product of two cycles have been investigated. For more details, see [4–7,9,11–13,16,18,19,21,23,25,27,28,30].

A generalized theta graph $\theta(t_1, t_2, \ldots, t_n)$ is a pair of n internal disjoint paths with lengths at least two joined by end vertices where the end vertices are named south pole S and north pole N and t_i is the number of vertices in the nth path. Uniform theta graph $\theta(t,m)$ is a generalized theta graph in which all paths have the same numbers of internal vertices, for more details see [22].

In this paper, we have defined a new type of family of graph called uniform theta snake graph, $\theta_n(t,m)$. Also, the exact value of TEIS for some special types of the new family has been determined.

2. Main results

In the following, we define a new type of graph which is called uniform theta snake graph.

Definition 1. If we replace each edge of a path $P_n$by a uniform theta graph $\theta(t,m)$, we have a uniform theta snake graph $\theta_n(t,m)$. See Figure 1.

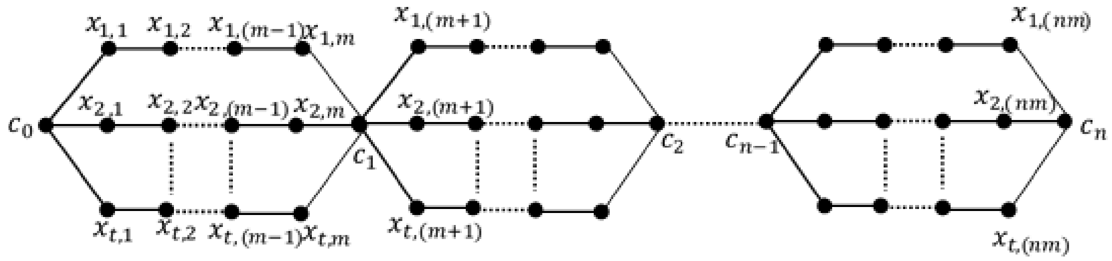

Figure 1. Uniform theta snake graph $\theta(t,m)$.

It is clear that for a uniform theta snake graph $|E(\theta_n(t,m))| = t(m+1)n$ and $|V(\theta_n(t,m))| = (tm+1)n+1$. In this section, we determine the exact value of TEIS for uniform theta snake graph $\theta_n(3,3)$, $\theta_n(3,m)$, $\theta_n(t,3)$, $\theta_n(4,m)$, and $\theta_n(t,4)$.

Theorem 1. For a uniform theta snake graph $\theta_n(3,3)$ with $10n+1$ vertices and $12n$ edges, we have

$$tes(\theta_n(3,3)) = 4n+1.$$

Proof. Since a uniform theta snake graph $\theta_n(3,3)$ has $12n$ edges and $\Delta(\theta_n(3,3)) = 6$, then from (1) we have:

$$tes(\theta_n(3,3)) \geq 4n+1.$$

To prove the invers inequality, we show that $\hbar$ −labeling is an edge irregular total for $\theta_n(3,3)$, see Figure 2, and $\hbar = 4n+1$. Let $\hbar = 4n+1$ and a total $\hbar$ −labeling $\alpha: V(\theta_n(3,3)) \cup E(\theta_n(3,3)) \to \{1,2,3,\dots,\hbar\}$ is defined as:

$$\alpha(c_0) = 1,$$

$$\alpha(c_s) = 4s \qquad \text{for } 1 \leq s \leq n-1$$

$$\alpha(c_n) = \hbar,$$

$$\alpha(x_{i,j}) = \begin{cases} j & for\ 1 \leq j \leq 3 \\ j+1 & for\ 4 \leq j \leq 6 \\ . & . \\ . & . \\ j+n-1 & for\ 3n-2 \leq j \leq 3n-1 \end{cases}, \quad i = 1,2,3,$$

$$\alpha(x_{i,3n}) = \hbar - 1 \qquad for\ i = 1,2,3$$

$$\alpha(c_0 x_{i,1}) = i \qquad for\ i = i1,2,3$$

$$\alpha(c_S x_{i,3S}) = 4S + i \qquad for\ 1 \leq S \leq n-1, \qquad i = 1,2,3$$

$$\alpha(c_S x_{i,3S+1}) = 4S + i + 1 \qquad for\ 1 \leq S \leq n-1, \qquad i = 1,2,3$$

$$\alpha(c_n x_{i,3n}) = \begin{cases} \hbar - 2 & for\ i = 1 \\ \hbar - 1 & for\ i = 2 \\ \hbar & for\ i = 3 \end{cases},$$

$$\alpha(x_{i,j} x_{i,j+1i}) = \begin{cases} j + i + 1 & for\ 1 \le j \le 2 \\ j + i + 2 & for\ 4 \le j \le 5 \\ . & . \\ . & . \\ j + i + n - I1 & for\ 3n - 5 \le j \le 3n - 4 \\ \hbar + i - 3 & for\ 3n - 2 \le j \le 3n - 1 \end{cases}, \qquad i = 1, 2, 3.$$

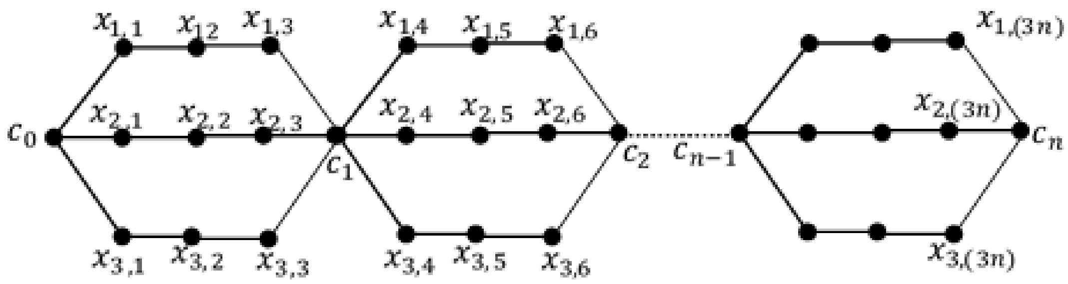

Figure 2. Uniform theta snake graph $\theta(3,3)$.

It is clear that $\hbar$ is the greatest used label. The weights of edges of $\theta_n(3,3)$ are given by:

$$w_\alpha(c_0 x_{i,1}) = i + 2 \qquad for\ i = 1, 2, 3,$$

$$w_\alpha(c_S x_{i,3S}) = 12S + i - 1 \qquad for\ 1 \le S \le n - 1\ , i = 1, 2, 3$$

$$w_\alpha(c_S x_{i,3S+1}) = 12S + i + 2 \quad for\ 1 \le S \le n - 1, i = 1, 2, 3,$$

$$w_\alpha(c_n x_{i,3n}) = \begin{cases} 3(\hbar - 1) & for\ i = 1 \\ 3\hbar - 2 & for\ i = 2 \\ 3\hbar - 1 & for\ i = 3 \end{cases},$$

$$w_\alpha(x_{i,j} x_{i,j+1}) = \begin{cases} 3j + i + 2 & for\ 1 \le j \le 2 \\ 3j + i + 5 & for\ 4 \le j \le 5 \\ . & . \\ . & . \\ 3j + i + 3n - 4 & for\ 3n - 5 \le j \le 3n - 4 \\ 3\hbar + i - 10 & for\ j = 3n - 2 \\ 3\hbar + i - 7 & for\ j = 3n - 1 \end{cases}, \qquad i = 1, 2, 3$$

Obviously, the weights of edges are distinct. So α is an edge irregular total $\hbar$ −labeling. Hence

$$tes(\theta_n(3,3)) = 4n + 1.$$

Theorem 2. For $\theta_n(3, m), m > 3$ be a uniform theta snake graph. Then

$$tes(\theta_n(3,m)) = (m+1)n+1.$$

Proof. Since $|E(\theta_n(3,m))| = 3(m+1)n$ and $\Delta(\theta_n(3,m)) = 6$. Substituting in (1), we find

$$tes(\theta_n(3,m)) \geq (m+1)n+1.$$

The existence of an edge irregular total λ −labeling for $\theta_n(3,m)$, See Figure 3, $m > 3$ will be shown, with $\lambda = (m+1)n+1$. Define a total λ −labeling $\beta: V(\theta_n(3,m)) \cup E(\theta_n(3,m)) \to \{1,2,3,\ldots,\lambda\}$ for $\theta_n(3,m)$ as:

$$\beta(c_0) = 1,$$

$$\beta(c_s) = (m+1)s \qquad for\ 1 \leq s \leq n-1,$$

$$\beta(c_n) = \lambda$$

$$\beta(x_{i,j}) = \begin{cases} j & for\ 1 \leq j \leq m \\ j+1 & for\ m+1 \leq j \leq 2m \\ . & . \\ . & . \\ j+n-1 & for\ m(n-1)+1 \leq j \leq mn-1 \end{cases},$$

$$\beta(x_{i,mn}) = \lambda - 1 \qquad for\ i = 1,2,3$$

$$\beta(c_0 x_{i,1}) = 1 \qquad for\ i = 1,2,3$$

$$\beta(c_S x_{i,mS}) = (m+1)S + i \qquad for\ 1 \leq S \leq\ n-1, \qquad i = 1,2,3$$

$$\beta(c_S x_{i,mS+1}) = (m+1)S + i + 1 \qquad for\ 1 \leq S \leq n-1, \qquad i = 1,2,3$$

$$\beta(c_n x_{i,mn}) = \begin{cases} \lambda - 2 & for\ i = 1 \\ \lambda - 1 & for\ i = 2 \\ \lambda & for\ i = 3 \end{cases},$$

$$\beta(x_{i,j} x_{i,j+1}) = \begin{cases} j+i+1 & for\ 1 \leq j \leq m-1 \\ j+i+2 & for\ m+1 \leq j \leq 2m-1 \\ . & . \\ . & . \\ j+i+n & for\ m(n-1)+1 \leq j \leq mn-2 \\ j+i+n-1 & for\ j = mn-1 \end{cases}.$$

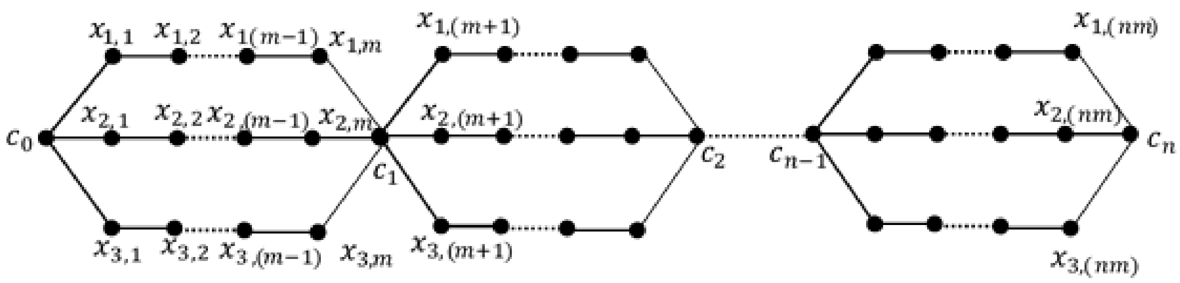

Figure 3. Uniform theta snake graph $\theta(3,m)$.

Clearly, λ is the most label of edges and vertices. The edges weights are given as follows:

$$w_\beta(c_0x_{i,1}) = i+2 \quad for\ i = 1,2,3,$$

$$w_\beta(c_Sx_{i,mS}) = 3(m+1)S + i - 1 \qquad for\ 1 \le S \le n-1, i = 1,2,3$$

$$w_\beta(c_Sx_{i,mS+1}) = 3(m+1)S + i + 2 \qquad for\ 1 \le S \le n-1,\ i = 1,2,3,$$

$$w_\beta(c_nx_{i,mn}) = \begin{cases} 3ƛ - 3 & for\ i = 1 \\ 3ƛ - 2 & for\ i = 2 \\ 3ƛ - 1 & for\ i = 3 \end{cases},$$

$$w_\beta(x_{i,j}x_{i,j+1}) = \begin{cases} 3j + i + 2 & for\ 1 \le j \le m-1 \\ 3j + i + 5 & for\ m+1 \le j \le 2m-1 \\ . & . \\ . & . \\ 3j\mathrm{I} + i + 3n - 1 & for\ m(n-1)+1 \le j \le mn-2 \\ 3ƛ + i - 7 & for\ j = mn-1 \end{cases},$$

It is obvious that the weights of edges are different, thus β is an edge irregular total $ƛ$ −labeling of $\theta_n(3,m)$. Hence

$$tes(\theta_n(3,m)) = (m+1)n + 1.$$

Theorem 3. Let $\theta_n(t,3)$ be a theta snake graph for $t > 3$. Then

$$tes(\theta_n(t,3)) = \left\lceil \frac{4tn+2}{3} \right\rceil.$$

Proof. A size of the graph $\theta_n(t,3)$ equals $4tn$ and $\Delta(\theta_n(t,3)) = 2t$, then from (1) we have

$$tes(\theta_n(t,3)) \ge \left\lceil \frac{4tn+2}{3} \right\rceil.$$

We define an edge irregular total ħ− labeling for $\theta_n(t,3)$ to get upper bound. So, let $ħ = \left\lceil \frac{4tn+2}{3} \right\rceil$ and a total ħ− labeling $\gamma: V(\theta_n(t,3)) \cup E(\theta_n(t,3)) \to \{1,2,3,\dots,ħ\}$ is defined in the following three cases:

Case 1. $4tn + 2 \equiv 0 (mod\ 3)$

γ is defined as:

$$\gamma(c_0) = 1,$$

$$\gamma(c_S) = (t+1)S \qquad for\ 1 \le S \le n-1,$$

$$\gamma(c_n) = ħ$$

$$\gamma(x_{i,j})\mathrm{I} = \begin{cases} i & for\ 1 \le j \le 3,\ i = 1,2,\dots,t \\ i + t + 1 & for\ 4 \le j \le 6, i = 1,2,\dots,t \\ i + 2(t+1) & for\ 7 \le j \le 9, i = 1,2,\dots,t \\ . & . \\ . & . \\ . & . \\ i + (n-1)(t+1) & for\ 3n-5 \le j \le 3n-3, i = 1,2,\dots,t \\ ħ - 1 & for\ 3n-2 \le j \le 3n, i = 1 \\ ħ & for\ 3n-2 \le j \le 3n, i = 2,3,\ \dots,t \end{cases},$$

$$\gamma(c_0x_{i,1}) = 1 \qquad for\ i = 1,2,\dots,t$$

$$\gamma(c_Sx_{i,3S}) = 2St - 2S + 3 \qquad for\ 1 \le S \le n-1, \quad i = 1,2,\dots,t$$

$$\gamma(c_nx_{i,3n}) = \begin{cases} ħ - t + 2 & for\ i = 1 \\ ħ - t + i & for\ i = 2,3,\ \dots,t \end{cases},$$

$$\gamma(c_Sx_{i,3S+1}) = 2St - 2S + 2 \qquad for\ 1 \le S \le n-1, \quad i = 1,2,\dots,t$$

$$\gamma(c_{n-1}x_{i,3n-2}) = \begin{cases} \begin{cases} (t+2)n - t - 5 & for\ i = 1 \\ (t+2)n - t + i - 7 & for\ i = 2,3,\dots,t \end{cases} & ,n = 2,3 \\ \begin{cases} (t+1)n - t - 1 & for\ i = 1 \\ (t+1)n - t + i - 3 & for\ i = 2,3,\dots,t \end{cases} & ,n \ne 2,3 \end{cases}$$

$$\gamma(x_{i,j}x_{i,j+1}) = \begin{cases} \begin{cases} t + j & for\ 1 \le j \le 2 \\ 3t + j - 5 & for\ 4 \le j \le 5 \\ 5t + j - 10 & for\ 7 \le j \le 8 \\ . & . \\ . & . \\ . & . \\ (2n-3)t + j - 5(n-2) & for\ 3n-5 \le j \le 3n-4 \end{cases} & ,i = 1,2,\dots,t \\ ħ - 3(t+n) + j + 5 & for\ 3n-2 \le j \le 3n-1, \quad i = 1 \\ ħ - 3(t+n) + j + 5 + 2(i-2) & for\ 3n-2 \le j \le 3n-1, i = 2,3,\dots,t \end{cases}$$

Obviously, ħ is the greatest label. The edges weights of $\theta_n(t,3)$ can be expressed as:

$$w_\gamma(c_0x_{i,1}) = i + 2 \qquad for\ i = 1,2,\dots,t$$

$$w_\gamma(c_Sx_{i,3S}) = t(4S-1) + i + 2 \qquad for\ 1 \le S \le n-1, i = 1,2,\dots,t$$

$$w_\gamma(c_nx_{i,3n}) = 3ħ - t + i \qquad for\ i = 1,2,\dots,t$$

$$w_\gamma(c_Sx_{i,3S+1}) = 4St + i + 2 \qquad for\ 1 \le S \le n-1, i = 1,2,\dots,t,$$

$$w_\gamma(c_{n-1}x_{i,3n-2}) = \begin{cases} 2nt + 3n - 2t + ħ + i - 8 & for\ n = 2,3 \\ 2nt + 2n - 2t + ħ + i - 4 & for\ n \ne 2,3 \end{cases}, i = 1,2,\dots,t$$

$$w_\gamma(x_{i,j}x_{i,j+1}) = \begin{cases} \begin{cases} t + j + 2i & for\ 1 \le j \le 2 \\ 5t + j + 2i - 4 & for\ 4 \le j \le 5 \\ 9t + j + 2i - 6 & for\ 7 \le j \le 8 \\ . & . \\ . & . \\ . & . \\ (4n-5)t + j + 2i - 3n + 8 & for\ 3ni - 5 \le j \le 3n-4 \end{cases} & ,i = 1,2,\dots,t \\ 3ħ - 3(t+in) + j + 3 & for\ 3n-2 \le j \le 3n-1, i = 1 \\ 3ħ - 3(t+in) + j + 2i + 3 & for\ 3n-2 \le j \le 3n-1, i = 2,3,\dots,t \end{cases}$$

It implies that the edges weights have distinct values. So γ is the desired edge irregular total ħ $-$labeling, $ħ = \left\lceil\frac{4tn+2}{3}\right\rceil$. Hence

$$tes(\theta_n(t,3)) = \left\lceil\frac{4tn+2}{3}\right\rceil.$$

Case 2. $4tn + 2 \equiv 1(mod\ 3)$

Define γ as:

$$\gamma(c_0) = 1,$$

$$\gamma(c_S) = (t+1)S \qquad for\ 1 \le S \le n-1,$$

$$\gamma(c_n) = \hbar$$

$$\gamma(x_{i,j}) = \begin{cases} i & for\ 1 \le j \le 3,\ i = 1,2,\dots,t \\ i+t+1 & for\ 4 \le j \le 6, i = 1,2,\dots,t \\ i+2(t+1) & for\ 7 \le j \le 9, i = 1,2,\dots,t \\ . & . \\ . & . \\ . & . \\ i+(n+1)(t+1) & for\ 3n-5 \le j \le 3n-3, i = 1,2,\dots,t \\ \hbar - 1 & for\ 3n-2 \le j \le 3n, i = 1 \\ \hbar & for\ 3n-2 \le j \le 3n, i = 2,3,\ \dots,t \end{cases},$$

$$\gamma(c_0 x_{i,1}) = 1 \qquad for\ i = 1,2,\dots,t$$

$$\gamma(c_S x_{i,3S}) = 2St - 2S + 3 \qquad for\ 1 \le S \le n-1,\ i = 1,2,\dots,\mathrm{t}$$

$$\gamma(c_n x_{i,3n}) = \begin{cases} \hbar - t & for\ i = 1 \\ \hbar - t + i - 2 & for\ i = 2,3,\ \dots,t \end{cases},$$

$$\gamma(c_S x_{i,3S+1}) = 2St - 2S + 2 \qquad for\ 1 \le \mathrm{I}S \le n-1, i = 1,2,\dots t$$

$$\gamma(c_{n-1} x_{i3n-2}) = \begin{cases} \begin{cases} (t+2)n - t - 5 & for\ i = 1 \\ (t+2)n - t + i - 7 & for\ i = 2,3,\dots,t \end{cases} & ,n = 2,3 \\ \begin{cases} (t+1)n - t - 1 & for\ i = 1 \\ (\mathrm{I}t + \mathrm{I}1)n - t + i - 3 & for\ i = 2,3,\dots,t \end{cases} & ,n \ne 2,3 \end{cases}$$

$$\gamma(x_{i,j} x_{i,j+1}) = \begin{cases} \begin{cases} t+j & for\ 1 \le j \le 2 \\ 3t+j-5 & for\ 4 \le j \le 5 \\ 5t+j-10 & for\ 7 \le j \le 8 \\ . & . \\ . & . \\ . & . \\ (2n-3)t+j-5(n-2) & for\ 3n-5 \le j \le 3n-4 \end{cases} & ,i = 1,2,\dots,t \\ \hbar - 3(t+n) + j + 3 & for\ 3n-2 \le j \le 3n-1, i = 1 \\ \hbar - 3(t+n) + j + 2(i-2) & for\ 3n-2 \le j \le 3n-1, i = 2,3,\dots,t \end{cases}$$

It is clear that the greatest label is ħ. We define the weights of edges of $\theta_n(t,3)$ as:

$$w_\gamma(c_0 x_{i,1}) = i + 2 \quad for\ i = 1,2,\dots,t$$

$$w_\gamma(c_S x_{i,3S}) = t(4S-1) + i + 2 \qquad for\ 1 \le S \le n-1, i = 1,2,\dots,t$$

$$w_\gamma(c_n x_{i,3n}) = 3\hbar - t + i - 2 \qquad for\ 1 \le S \le n-1, \qquad i = 1,2,\dots,t$$

$$w_\gamma(c_S x_{i,3S+1}) = 4St + i + 2 \quad for\ 1 \le S \le n-1, i = 1,2,\dots,t,$$

$$w_\gamma(c_{n-1}x_{i,3n-2}) = \begin{cases} 2nt+3n-2t+\hbar+i-8 & for\ n=2,3 \\ 2nt+2n-2t+\hbar+i-4 & for\ n\neq 2,3 \end{cases}, \quad i=1,2,\dots,t$$

$$w_\gamma(x_{i,j}x_{i,j+1}) = \begin{cases} \begin{cases} t+j+2i & for\ 1\leq j\leq 2 \\ 5t+j+2i-4 & for\ 4\leq j\leq 5 \\ 9t+j+2i-6 & for\ 7\leq j\leq 8 \\ . & . \\ . & . \\ . & . \\ (4n-5)t+j+2i-3n+8 & for\ 3n-5\leq j\leq 3n-4 \end{cases}, i=1,2,\dots,t \\ 3\hbar-3(t+n)+j+1 \quad for\ 3n-2\leq j\leq 3n-1, \quad i=1 \\ 3\hbar-3(t+n)+j+2(i-2) \quad for\ 3n-2\leq j\leq 3n-1, i=2,3,\dots,t \end{cases}$$

It is obvious that the edges weights are different. Then

$$tes(\theta_n(t,3)) = \left\lceil \frac{4tn+2}{3} \right\rceil.$$

Case 3. $4tn+2\equiv 2(mod\ 3)$

γ is defined as follows:

$$\gamma(c_0)=1,$$

$$\gamma(c_S)=(t+1)S \qquad for\ 1\leq S\leq n-1,$$

$$\gamma(c_n)=\hbar$$

$$\gamma(x_{i,j}) = \begin{cases} i & for\ 1\leq j\leq 3, \quad i=1,2,\dots,t \\ i+t+1 & for\ 4\leq j\leq 6, i=1,2,\dots,t \\ i+2(t+1) & for\ 7\leq j\leq 9, i=1,2,\dots,t \\ . & . \\ . & . \\ . & . \\ i+(n-1)(t+1) & for\ 3n-5\leq j\leq 3n-3, i=1,2,\dots,t \\ \hbar-1 & for\ 3n-2\leq j\leq 3n, i=1 \\ \hbar & for\ 3n-2\leq j\leq 3n, i=2,3,\dots,t \end{cases},$$

$$\gamma(c_0x_{i,1})=1 \qquad for\ i=1,2,\dots,t$$

$$\gamma(c_Sx_{i,3S})=2St-2S+3 \qquad for\ 1\leq S\leq n-1, \quad i=1,2,\dots,t$$

$$\gamma(c_nx_{i,3n}) = \begin{cases} \hbar-t+1 & for\ i=1 \\ \hbar-t+i-1 & for\ i=2,3,\dots,t \end{cases},$$

$$\gamma(c_Sx_{i,3S+1})=2St-2S+2 \qquad for\ 1\leq S\leq n-2, \quad i=1,2,\dots,t$$

$$\gamma(c_{n-1}x_{i,3n-2}) = \begin{cases} \begin{cases} (t+2)n-t-5 & for\ i=1 \\ (t+2)n-t+i-7 & for\ i=2,3,\dots,t \end{cases}, n=2,3 \\ \begin{cases} (t+1)n-t-1 & for\ i=1 \\ (t+1)n-t+i-3 & for\ i=2,3,\dots,t \end{cases}, n\neq 2,3 \end{cases}$$

$$\gamma(x_{i,j}x_{i,j+1}) = \begin{cases} \begin{cases} t+j & for\ 1 \le j \le 2 \\ 3t+j-5 & for\ 4 \le j \le 5 \\ 5t+j-10 & for\ 7 \le j \le 8 \\ . & . \\ . & . \\ . & . \\ (2n-3)t+j-5(n-2) & for\ 3n-5 \le j \le 3n-4 \end{cases}, i = 1,2,\dots t \\ ħ-3(t+i)+j+4 \quad for\ 3n-2 \le j \le 3ni-1, i=1 \\ ħ-3(t+n)+j+2i \quad for\ 3n-2 \le j \le 3n-1, i=2,3,\dots,t \end{cases}$$

We can see that ħ is the greatest label. For edges weights of $\theta_n(t,3)$, we have

$$w_\gamma(c_0x_{i,1}) = i+2 \qquad for\ i = 1,2,\dots,t$$

$$w_\gamma(c_0x_{i,3S}) = t(4S-1)+i+2 \qquad for\ 1 \le S \le n-1, i = 1,2,\dots,t$$
$$w_\gamma(c_nx_{i,3n}) = 3ħ-t+i-1 \qquad for\ 1 \le S \le n-1, i = 1,2,\dots,t$$

$$w_\gamma(c_Sx_{i,3S+1}) = 4St+i+2 \qquad for\ 1 \le S \le n-1, i = 1,2,\dots,t,$$

$$w_\gamma(c_nx_{i,3n-2}) = \begin{cases} 2nt-3n-2t+ħ+i-8 & for\ n=2,3 \\ 2nt+2n-2t+ħ+i-4 & for\ n \ne 2,3 \end{cases}, i = 1,2,\dots,t$$

$$w_\gamma(x_{i,j}x_{i,j+1}) = \begin{cases} \begin{cases} t+j+2i & for\ 1 \le j \le 2 \\ 5t+j+2i-4 & for\ 4 \le j \le 5 \\ 9t+j+2i-6 & for\ 7 \le j \le 8 \\ . & . \\ . & . \\ . & . \\ (4n-5)t+j+2i-3n+8 & for\ 3n-5 \le j \le 3n-4 \end{cases}, i = 1,2,\dots,t \\ 3ħ-3(t+n)+j+2 \quad for\ 3n-2 \le j \le 3n-1, \quad i=1 \\ 3ħ-3(t+n)+j+2i \quad for\ 3n-2 \le j \le 3n-1, i=2,3,\dots,t \end{cases}$$

It clears that the edges weights are i distinct. So γ is the desired edge irregular total ħ −labeling, $ħ = \left\lceil\frac{4tn+2}{3}\right\rceil$. Hence

$$tes(\theta_n(t,3)) = \left\lceil\frac{4tn+2}{3}\right\rceil.$$

Theorem 4. For $\theta_n(4,m)$ be a theta snake graph for $t > 3$. Then

$$tes(\theta_n(4,m)) = \left\lceil\frac{4(m+1)n+2}{3}\right\rceil.$$

Proof. Since $|E(\theta_n(4,m))| = 4(m+1)n$ and $\Delta(\theta_n(4,m)) = 8$, then from (1) we have

$$tes(\theta_n(4,m)) \ge \left\lceil\frac{4(m+1)n+2}{3}\right\rceil.$$

The existence of an edge irregular total ƛ −labeling for $\theta_n(4,m)$, $m > 3$ will be shown, with $ƛ = \left\lceil\frac{4(m+1)n+2}{3}\right\rceil$. Define a total ƛ − labeling $\beta: V(\theta_n(4,m)) \cup E(\theta_n(4,m)) \to \{1,2,3,\dots,ƛ\}$ for $\theta_n(4,m)$ in the following three cases as:

Case 1. $4(m+1)n+2 \equiv 0(mod\ 3)$, $i = 1,2,3,4$

β is defined as:

$$\beta(c_s)=\begin{cases}1 & for\ s=0\\ (m+1)s & for\ 1\le s\le\left\lceil\frac{n}{2}\right\rceil,\\ \lambda+s-n & for\ \left\lceil\frac{n}{2}\right\rceil\le s\le n\end{cases}$$

$$\beta(x_{i,j})=\begin{cases}j & for\ 1\le j\le m\\ j+1 & for\ m+1\le j\le 2m\\ . & .\\ . & .\\ j+\left\lceil\frac{n}{2}\right\rceil-1 & for\ m\left(\left\lceil\frac{n}{2}\right\rceil-1\right)+1\le j\le m\left\lceil\frac{n}{2}\right\rceil+1\\ \lambda-j+22 & for\ m\left\lceil\frac{n}{2}\right\rceil+2\le j\le m(n-1)\\ \lambda & for\ m(n-1)+1\le j\le mn-1\end{cases},$$

$$\beta(c_0x_{i,1})=1 \qquad for\ i=1,2,3,4$$

$$\beta(c_Sx_{i,mS})=\begin{cases}2c_S+i-1 & for\ 1\le S\le\left\lceil\frac{n}{2}\right\rceil-1\\ c_S+i-4(m+1) & for\ \left\lceil\frac{n}{2}\right\rceil\le s\le n-1\\ \lambda-4+i & for\ s=n\end{cases},i=1,2,3,4$$

$$\beta(c_Sx_{i,mS+1})=\begin{cases}2c_S+i+1 & for\ 1\le S\le\left\lceil\frac{n}{2}\right\rceil\\ c_S+i-4(m+1)+2 & for\ \left\lceil\frac{n}{2}\right\rceil+1\le s\le n-1\end{cases},i=1,2,3,4$$

$$\beta(c_nx_{i,mn})=\begin{cases}\lambda-3 & for\ i=1\\ \lambda-2 & for\ i=2\\ \lambda-1 & for\ i=3\\ \lambda & for\ i=4\end{cases},$$

$$\beta(x_{i,j}x_{i,j+1})=\begin{cases}j+i+1 & for\ 1\le j\le m-1\\ j+i+2 & for\ m+1\le j\le 2m-1\\ . & .\\ . & .\\ j+i+\left\lceil\frac{n}{2}\right\rceil & for\ j=m\left(\left\lceil\frac{n}{2}\right\rceil-1\right)+1\\ 2j+i-2\left[nm\left(\left\lceil\frac{n}{2}\right\rceil-1\right)+1\right] & for\ m\left(\left\lceil\frac{n}{2}\right\rceil-1\right)+2\le j\le mn-1\end{cases}.$$

It is clear that λ is the greatest used label. The weights of edges of $\theta_n(4,m)$ are given by:

$$w_\beta(c_0x_{i,1})=i+2 \qquad for\ i=1,2,3,4,$$

$$w_\beta(c_Sx_{i,mS})=\begin{cases}2ms+s+2c_S+i-1 & for\ 1\le S\le\left\lceil\frac{n}{2}\right\rceil-1,\\ c_S+i+\lambda+(s-4)(m+1)-n+\left\lceil\frac{n}{2}\right\rceil-1 & for\ \left\lceil\frac{n}{2}\right\rceil\le s\le n-1\\ 3\lambda-4+i+s-n & for\ s=n\end{cases},i=1,2,3,4$$

$$w_\beta(c_Sx_{i,mS+1})=\begin{cases}(2m+1)s+2c_S+i+1 & for\ 1\le S\le\left\lceil\frac{n}{2}\right\rceil,\\ 2\lambda+s-n+c_S+i-4(m+1)+2 & for\ \left\lceil\frac{n}{2}\right\rceil\le s\le n-1\end{cases}\quad i=1,2,3,4,$$

$$w_\beta(c_n x_{i,mn}) = \begin{cases} 3\lambda + s - n - 3 & for\ i = 1 \\ 3\lambda + s - n - 2 & for\ i = 2 \\ 3\lambda + s - n - 1 & for\ i = 3 \\ 3\lambda + s - n & for\ i = 4 \end{cases},$$

$$w_\beta(x_{i,j} x_{i,j+1}) = \begin{cases} 3j + i + 2 & for\ 1 \le j \le m - 1 \\ 3j + i + 4 & for\ m + 1 \le j \le 2m - 1 \\ . & . \\ . & . \\ 3j + i + 3\left\lceil\frac{n}{2}\right\rceil - 1 & for\ j = m\left(\left\lceil\frac{n}{2}\right\rceil - 1\right) + 1 \\ 4j + 2\lambda + 45 + i - 2[nm\left(\left\lceil\frac{n}{2}\right\rceil - 1\right) + 1] & for\ m\left\lceil\frac{n}{2}\right\rceil + 2 \le j \le m(n-1) \\ 2j + 2\lambda + i - 2[nm\left(\left\lceil\frac{n}{2}\right\rceil - 1\right) + 1] & for\ m(n-1) + 1 \le j \le mn - 1 \end{cases},$$

It is obvious that the weights of edges are different, thus β is an edge irregular total λ −labeling of $\theta_n(4, m)$. Hence

$$tes(\theta_n(4, m)) = \left\lceil\frac{4(m+1)n+2}{3}\right\rceil.$$

Case 2. $4(m+1)n + 2 \equiv 1(mod\ 3),\ i = 1, 2, 3, 4$

β is defined as:

$$\beta(c_s) = \begin{cases} 1 & for\ s = 0 \\ (m+1)s & for\ 1 \le s \le \left\lceil\frac{n}{2}\right\rceil, \\ \lambda + s - n & for\ \left\lceil\frac{n}{2}\right\rceil \le s \le n \end{cases}$$

$$\beta(x_{i,j}) = \begin{cases} j & for\ 1 \le j \le m \\ j + 1 & for\ m + 1 \le j \le 2m \\ . & . \\ . & . \\ j + \left\lceil\frac{n}{2}\right\rceil - 1 & for\ m\left(\left\lceil\frac{n}{2}\right\rceil - 1\right) + 1 \le j \le m\left\lceil\frac{n}{2}\right\rceil, \\ \lambda - j + 22 & for\ m\left\lceil\frac{n}{2}\right\rceil + 1 \le j \le m(n-1) \\ \lambda & for\ m(n-1) + 1 \le j \le mn - 1 \end{cases}$$

$$\beta(c_0 x_{i,1}) = 1 \qquad for\ i = 1, 2, 3, 4$$

$$\beta(c_S x_{i,mS}) = \begin{cases} 2c_S + i - 1 & for\ 1 \le S \le \left\lceil\frac{n}{2}\right\rceil - 1, \\ \lambda - 7 + i & for\ s = \left\lceil\frac{n}{2}\right\rceil \\ c_S + i - 4m - 2 & for\ \left\lceil\frac{n}{2}\right\rceil + 1 \le s \le n - 1 \\ \lambda - 6 + i & for\ s = n \end{cases},\ i = 1, 2, 3, 4$$

$$\beta(c_S x_{i,mS+1}) = \begin{cases} 2c_S + i + 1 & for\ 1 \le S \le \left\lceil\frac{n}{2}\right\rceil \\ c_S + i - 4m & for\ \left\lceil\frac{n}{2}\right\rceil \le s \le n - 1 \end{cases},\ i = 1, 2, 3, 4$$

$$\beta(c_n x_{i,mn}) = \begin{cases} \lambda - 5 & for\ i = 1 \\ \lambda - 4 & for\ i = 2 \\ \lambda - 3 & for\ i = 3 \\ \lambda - 2 & for\ i = 4 \end{cases},$$

$$\beta(x_{i,j} x_{i,j+1}) = \begin{cases} j + i + 1 & for\ 1 \le j \le m - 1 \\ j + i + 2 & for\ m + 1 \le j \le 2m - 1 \\ \cdot & \cdot \\ \cdot & \cdot \\ j + i + \left\lceil \frac{n}{2} \right\rceil & for\ j = m\left(\left\lceil \frac{n}{2} \right\rceil - 1\right) + 1 \\ 2j + i - 2[nm\left(\left\lceil \frac{n}{2} \right\rceil - 1\right) + 1] & for\ m\left(\left\lceil \frac{n}{2} \right\rceil - 1\right) + 2 \le j \le mn - 1 \end{cases}.$$

It is clear that λ is the greatest used label. The weights of edges of $\theta_n(4, m)$ are given by:

$$w_\beta(c_0 x_{i,1}) = i + 2 \quad for\ i = 1, 2, 3, 4,$$

$$w_\beta(c_S x_{i,mS}) = \begin{cases} 2ms + s + 2c_S + i - 1 & for\ 1 \le S \le \left\lceil \frac{n}{2} \right\rceil - 1, \\ 2\lambda - m\left\lceil \frac{n}{2} \right\rceil + (m+1)s + i + 15 & for\ s = \left\lceil \frac{n}{2} \right\rceil \\ c_S + i + \lambda + (s-4)(m+1) - n + \left\lceil \frac{n}{2} \right\rceil - 1 & for\ \left\lceil \frac{n}{2} \right\rceil \le s \le n - 1 \\ 3\lambda - 4 + i + s - n & for\ s = n \end{cases}$$

$$w_\beta(c_S x_{i,mS+1}) = \begin{cases} (2m+1)s + 2c_S + i + 1 & for\ 1 \le S \le \left\lceil \frac{n}{2} \right\rceil, i = 1, 2, 3, 4 \\ 2\lambda + s - n + c_S + i - 4m & for\ \left\lceil \frac{n}{2} \right\rceil \le s \le n - 1 \end{cases} \quad i = 1, 2, 3, 4,$$

$$w_\beta(c_n x_{i,mn}) = \begin{cases} 3\lambda + s - n - 5 & for\ i = 1 \\ 3\lambda + s - n - 4 & for\ i = 2 \\ 3\lambda + s - n - 3 & for\ i = 3 \\ 3\lambda + s - n - 2 & for\ i = 4 \end{cases},$$

$$w_\beta(x_{i,j} x_{i,j+1}) = \begin{cases} 3j + i + 2 & for\ 1 \le j \le m - 1 \\ 3j + i + 4 & for\ m + 1 \le j \le 2m - 1 \\ \cdot & \cdot \\ \cdot & \cdot \\ 3j + i + 3\left\lceil \frac{n}{2} \right\rceil - 1 & for\ j = m\left(\left\lceil \frac{n}{2} \right\rceil - 1\right) + 1 \\ 4j + 2\lambda + 45 + i - 2[nm\left(\left\lceil \frac{n}{2} \right\rceil - 1\right) + 1] & for\ m\left\lceil \frac{n}{2} \right\rceil + 2 \le j \le m(n-1) \\ 2j + 2\lambda + i - 2[nm\left(\left\lceil \frac{n}{2} \right\rceil - 1\right) + 1] & for\ m(n-1) + 1 \le j \le mn - 1 \end{cases},$$

It is obvious that the weights of edges are different, thus β is an edge irregular total λ −labeling of $\theta_n(4, m)$. Hence

$$tes(\theta_n(4, m)) = \left\lceil \frac{4(m+1)n+2}{3} \right\rceil.$$

Case 3. $4(m + 1)n + 2 \equiv 2(mod\ 3), i = 1, 2, 3, 4$
β is defined as:

$$\beta(c_s)=\begin{cases}1 & for\ s=0\\ (m+1)s & for\ 1\le s\le\left\lceil\frac{n}{2}\right\rceil,\\ \lambda+s-n & for\ \left\lceil\frac{n}{2}\right\rceil\le s\le n\end{cases}$$

$$\beta(x_{i,j})=\begin{cases}j & for\ 1\le j\le m\\ j+1 & for\ m+1\le j\le 2m\\ . & .\\ . & .\\ j+\left\lceil\frac{n}{2}\right\rceil-1 & for\ m\left(\left\lceil\frac{n}{2}\right\rceil-2\right)+1\le j\le m\left(\left\lceil\frac{n}{2}\right\rceil-1\right)\\ \lambda-j+22 & for\ m\left(\left\lceil\frac{n}{2}\right\rceil-1\right)+1\le j\le m(n-1)\\ \lambda & for\ m(n-1)+1\le j\le mn-1\end{cases},$$

$$\beta(c_0x_{i,1})=1 \qquad for\ i=1,2,3,4$$

$$\beta(c_Sx_{i,mS})=\begin{cases}2c_S+i-1 & for\ 1\le S\le\left\lceil\frac{n}{2}\right\rceil-1, i=1,2,3,4\\ \lambda-7+i & for\ s=\left\lceil\frac{n}{2}\right\rceil\\ c_S+i-4m-2 & for\ \left\lceil\frac{n}{2}\right\rceil+1\le s\le n-1\\ \lambda-5+i & for\ s=n\end{cases}$$

$$\beta(c_Sx_{i,mS+1})=\begin{cases}2c_S+i+1 & for\ 1\le S\le\left\lceil\frac{n}{2}\right\rceil-1, i=1,2,3,4\\ c_S+1+i & for\ s=\left\lceil\frac{n}{2}\right\rceil\\ c_S+i-4m+1 & for\ \left\lceil\frac{n}{2}\right\rceil+1\le s\le n-1\end{cases}$$

$$\beta(c_nx_{i,mn})=\begin{cases}\lambda-4 & for\ i=1\\ \lambda-3 & for\ i=2\\ \lambda-2 & for\ i=3\\ \lambda-1 & for\ i=4\end{cases},$$

$$\beta(x_{i,j}x_{i,j+1})=\begin{cases}j+i+1 & for\ 1\le j\le m-1\\ j+i+2 & for\ m+1\le j\le 2m-1\\ . & .\\ . & .\\ j+i+\left\lceil\frac{n}{2}\right\rceil & for\ j=m\left(\left\lceil\frac{n}{2}\right\rceil-1\right)+1\\ 2j+i-2\left[nm\left(\left\lceil\frac{n}{2}\right\rceil-1\right)+1\right]+1 & for\ m\left(\left\lceil\frac{n}{2}\right\rceil-1\right)+2\le j\le mn-1\end{cases}.$$

It is clear that λ is the greatest used label. The weights of edges of $\theta_n(4,m)$ are given by:

$$w_\beta(c_0x_{i,1})=i+2 \qquad for\ i=1,2,3,4,$$

$$w_\beta(c_S x_{i,mS}) = \begin{cases} 2ms + s + 2c_S + i - 1 & for\ 1 \le S \le \left\lceil \frac{n}{2} \right\rceil - 1, \\ 2\lambda - m\left\lceil \frac{n}{2} \right\rceil + (m+1)s + i + 15 & for\ s = \left\lceil \frac{n}{2} \right\rceil \\ c_S + i + \lambda + (s-4)(m+1) - n + \left\lceil \frac{n}{2} \right\rceil - 1 & for\ \left\lceil \frac{n}{2} \right\rceil \le s \le n-1 \\ 3\lambda - 3 + i + s - n & for\ s = n \end{cases}$$

$$w_\beta(c_S x_{i,mS+1}) = \begin{cases} (2m+1)s + 2c_S + i + 1 & for\ 1 \le S \le \left\lceil \frac{n}{2} \right\rceil, i = 1,2,3,4 \\ 2\lambda + s - n + c_S + i - 4m + 1 & for\ \left\lceil \frac{n}{2} \right\rceil \le s \le n-1 \end{cases},$$

$$w_\beta(c_n x_{i,mn}) = \begin{cases} 3\lambda + s - n - 3 & for\ i = 1 \\ 3\lambda + s - n - 2 & for\ i = 2 \\ 3\lambda + s - n - 1 & for\ i = 3 \\ 3\lambda + s - n & for\ i = 4 \end{cases},$$

$$w_\beta(x_{i,j} x_{i,j+1}) = \begin{cases} 3j + i + 2 & for\ 1 \le j \le m-1 \\ 3j + i + 4 & for\ m+1 \le j \le 2m-1 \\ \vdots & \vdots \\ 3j + i + 3\left\lceil \frac{n}{2} \right\rceil - 1 & for\ j = m\left(\left\lceil \frac{n}{2} \right\rceil - 1\right) + 1 \\ 4j + 2\lambda + 45 + i - 2[nm\left(\left\lceil \frac{n}{2} \right\rceil - 1\right) + 1] & for\ m\left\lceil \frac{n}{2} \right\rceil + 2 \le j \le m(n-1) \\ 2j + 2\lambda + i - 2[nm\left(\left\lceil \frac{n}{2} \right\rceil - 1\right) + 1] & for\ m(n-1) + 1 \le j \le mn - 1 \end{cases},$$

It is obvious that the weights of edges are different, thus β is an edge irregular total λ −labeling of $\theta_n(4,m)$. Hence

$$tes(\theta_n(4,m)) = \left\lceil \frac{4(m+1)n + 2}{3} \right\rceil$$

Theorem 5. If $\theta_n(t,4)$ is theta snake graph for $t > 3$. Then

$$tes(\theta_n(t,4)) = \left\lceil \frac{5tn+2}{3} \right\rceil.$$

Proof. Since $|E(\theta_n(t,4))| = 5tn$ and $\Delta(\theta_n(t,4)) = 2t$. Substituting in (1), we have

$$tes(\theta_n(t,4)) \ge \left\lceil \frac{5tn+2}{3} \right\rceil.$$

We define an edge irregular total $\hbar$ −labeling for $\theta_n(t,4)$ to get upper bound. Let $\hbar = \left\lceil \frac{5tn+2}{3} \right\rceil$ and a total $\hbar$ −labeling $\gamma: V(\theta_n(t,4)) \cup E(\theta_n(t,4)) \to \{1,2,3,\dots,\hbar\}$ is defined in the following three cases:

Case 1. $5tn + 2 \equiv 0 (mod\ 3)$

Define γ as:

$$\gamma(c_0) = 1,$$

$$\gamma(c_S) = (t+1)S \qquad for\ 1 \le S \le n-1,$$

$$\gamma(c_n) = \hbar$$

$$\gamma(x_{i,j}) = \begin{cases} i & for\ 1 \le j \le 4,\ i = 1, 2, \dots, t \\ i + t + 1 & for\ 5 \le j \le 8, i = 1, 2, \dots, t \\ i + 2(t+1) & for\ 9 \le j \le 12, i = 1, 2, \dots, t \\ . & . \\ . & . \\ . & . \\ i + (n-1)(t+1) & for\ 4n - 7 \le j \le 4n - 4, i = 1, 2, \dots, t \\ ħ - 1 & for\ 4n - 3 \le j \le 4n, i = 1 \\ ħ & for\ 4n - 3 \le j \le 4n, i = 2, 3, \ \dots, t \end{cases},$$

$$\gamma(c_0 x_{i,1}) = 1 \qquad for\ i = 1, 2, \dots, t$$

$$\gamma(c_S x_{i,4S}) = 3St - 2S + 3 \qquad for\ 1 \le S \le n - 1, \qquad i = 1, 2, \dots, t$$

$$\gamma(c_n x_{i,4n}) = \begin{cases} ħ - t + 2 & for\ i = 1 \\ ħ - t + i & for\ i = 2, 3, \ \dots, t \end{cases},$$

$$\gamma(c_S x_{i,4S+1}) = 3St - 2S + 2 \qquad for\ 1 \le S \le n - 1, \qquad i = 1, 2, \dots, t$$

$$\gamma(c_{n-1} x_{i,4n-3}) = \begin{cases} \begin{cases} (t+2)n - t - 5 & for\ i = 1 \\ (t+2)n - t + i - 7 & for\ i = 2, 3, \dots, t \end{cases} & , n = 2, 3 \\ (t+1)n - t + i - 3 & for\ i = 2, 3, \dots, t \quad , n \ne 2, 3 \end{cases}$$

$$\gamma(x_{i,j} x_{i,j+1}) = \begin{cases} \begin{cases} t + j & for\ 1 \le j \le 2 \\ 3t + j - 5 & for\ 4 \le j \le 5 \\ 5t + j - 10 & for\ 7 \le j \le 8 \\ . & . \\ . & . \\ . & . \\ (2n-3)t + j - 5(n-2) & for\ 4n - 5 \le j \le 4n - 4 \end{cases} & , i = 1, 2, \dots, t \\ ħ - 3(t+n) + j + 5 & for\ 4n - 2 \le j \le 4n, \quad i = 1 \\ ħ - 3(t+n) + j + 5 + 2(i-2) & for\ 4n - 2 \le j \le 4n, i = 2, 3, \dots, t \end{cases}$$

It is clear that, $ħ$ is the greatest label. The edges weights of $\theta_n(t, 4)$ can be expressed as:

$$w_\gamma(c_0 x_{i,1}) = i + 2 \qquad for\ i = 1, 2, \dots, t$$

$$w_\gamma(c_S x_{i,4S}) = t(5S - 1) + i + 2 \qquad for\ 1 \le S \le n - 1, i = 1, 2, \dots, t$$

$$w_\gamma(c_n x_{i,4n}) = 3ħ - t + i \qquad for\ i = 1, 2, \dots, t$$

$$w_\gamma(c_S x_{i,4S+1}) = 5St + i + 2 \qquad for\ 1 \le S \le n - 1, i = 1, 2, \dots, t,$$

$$w_\gamma(c_{n-1} x_{i,4n-2}) = \begin{cases} 2nt + 3n - 2t + ħ + i - 8 & for\ n = 2, 3 \\ 2nt + 2n - 2t + ħ + i - 6 & for\ n \ne 2, 3 \end{cases}, i = 1, 2, \dots, t$$

$$w_\gamma(x_{i,j}x_{i,j+1}) = \begin{cases} \begin{cases} t+j+2i & for\ 1 \le j \le 2 \\ 5t+j+2i-4 & for\ 4 \le j \le 5 \\ 9t+j+2i-6 & for\ 7 \le j \le 8 \\ \vdots & \vdots \\ (4n-5)t+j+2i-3n+8 & for\ 4n-5 \le j \le 4n-4 \end{cases} & , i = 1,2,\dots,t \\ 3ħ-3(t+n)+j+3 & for\ 4n-2 \le j \le 4n-1, i = 1 \\ 3ħ-3(t+n)+j+2i+3 & for\ 4n-2 \le j \le 4n-1, i = 2,3,\dots,t \end{cases}$$

It implies that the edges weights have distinct values. So γ is the desired edge irregular total $ħ$ −labeling, $ħ = \left\lceil \frac{5tn+2}{3} \right\rceil$. Hence

$$tes(\theta_n(t,4)) = \left\lceil \frac{5tn+2}{3} \right\rceil.$$

Case 2. $5tn + 2 \equiv 1(mod\ 3)$
Define γ as:

$$\gamma(c_0) = 1,$$

$$\gamma(c_S) = (t+1)S \qquad for\ 1 \le S \le n-1,$$

$$\gamma(c_n) = ħ$$

$$\gamma(x_{i,j}) = \begin{cases} i & for\ 1 \le j \le 4,\ i = 1,2,\dots,t \\ i+t+1 & for\ 5 \le j \le 8, i = 1,2,\dots,t \\ i+2(t+1) & for\ 9 \le j \le 12, i = 1,2,\dots,t \\ \vdots & \vdots \\ i+(n+1)(t+1) & for\ 4n-7 \le j \le 4n-4, i = 1,2,\dots,t \\ ħ-1 & for\ 4n-3 \le j \le 4n, i = 1 \\ ħ & for\ 4n-3 \le j \le 4n, i = 2,3,\ \dots,t \end{cases},$$

$$\gamma(c_0x_{i,1}) = 1 \qquad for\ i = 1,2,\dots,t$$

$$\gamma(c_Sx_{i,4S}) = 3St-2S+3 \qquad for\ 1 \le S \le n-1,\ i = 1,2,\dots,t$$

$$\gamma(c_nx_{i,4n}) = \begin{cases} ħ-t & for\ i = 1 \\ ħ-t+i-2 & for\ i = 2,3,\ \dots,t \end{cases},$$

$$\gamma(c_Sx_{i,4S+1}) = 3St-2S+2 \quad for\ 1 \le S \le n-1, i = 1,2,\dots t$$

$$\gamma(c_{n-1}x_{i,4n-3}) = \begin{cases} \begin{cases} (t+2)n-t-5 & for\ i = 1 \\ (t+2)n-t+i-7 & for\ i = 2,3,\dots,t \end{cases} & , n = 2,3 \\ \begin{cases} (t+1)n-t-1 & for\ i = 1 \\ (t+1)n-t+i-3 & for\ i = 2,3,\dots,t \end{cases} & , n \ne 2,3 \end{cases}$$

$$\gamma(x_{i,j}x_{i,j+1}) = \begin{cases} \begin{cases} t+j & for\ 1 \le j \le 3 \\ 3t+j-5 & for\ 5 \le j \le 7 \\ 5t+j-10 & for\ 9 \le j \le 11 \\ . & . \\ . & . \\ . & . \\ (2n-3)t+j-5(n-2) & for\ 4n-7 \le j \le 4n-5 \end{cases} , i = 1,2,\dots,t \\ ħ-4(t+n)+j+3 \quad for\ 4n-3 \le j \le 4n-1, i=1 \\ ħ-4(t+n)+j+2(i-2) \quad for\ 4n-3 \le j \le 4n-1, i=2,3,\dots,t \end{cases}$$

It is clear that the i greatest label is ħ. We define the weights of edges of $\theta_n(t,4)$ as:

$$w_\gamma(c_0x_{i,1}) = i+2 \quad for\ i=1,2,\dots,t$$

$$w_\gamma(c_Sx_{i,4S}) = t(5S-1)+i+2 \quad for\ 1 \le S \le n-1, i=1,2,\dots,t$$

$$w_\gamma(c_nx_{i,4n}) = 3ħ-t+i-2 \quad for\ 1 \le S \le n-1, i=1,2,\dots,t$$

$$w_\gamma(c_Sx_{i,4S+1}) = 5St+i+2 \quad for\ 1 \le S \le n-1, i=1,2,\dots,t,$$

$$w_\gamma(c_{n-1}x_{i,4n-3}) = \begin{cases} 3nt+3n-2t+ħ+i-8 & for\ n=2,3 \\ 3nt+2n-2t+ħ+i-6 & for\ n \ne 2,3 \end{cases}, i=1,2,\dots,t$$

$$w_\gamma(x_{i,j}x_{i,j+1}) = \begin{cases} \begin{cases} t+j+2i & for\ 1 \le j \le 3 \\ 5t+j+2i-4 & fori\ 5 \le j \le 7 \\ 9t+j+2i-6 & for\ 9 \le j \le 11 \\ . & . \\ . & . \\ . & . \\ (4n-5)t+j+2i-3n+8 & for\ 4n-7 \le j \le 4n-5 \end{cases} , i=1,2,\dots,t \\ 3ħ-4(t+n)+j+1 \quad for\ 4n-3 \le j \le 4n-1, \quad i=1 \\ 3ħ-4(t+n)+j+2(i-2) \quad for\ 4n-3 \le j \le 4n-1, i=2,3,\dots,t \end{cases}$$

It is obvious that the edges weights are different. Then

$$tes(\theta_n(t,4)) = \left\lceil \frac{5tn+2}{3} \right\rceil.$$

Case 3. $5tn+2 \equiv 2(mod\ 3)$

Define γ as:

$$\gamma(c_0) = 1,$$

$$\gamma(c_S) = (t+1)S \quad for\ 1 \le S \le n-1,$$

$$\gamma(c_n) = ħ$$

$$\gamma(x_{i,j}) = \begin{cases} i & for\ 1 \le j \le 4, \quad i = 1,2,\dots,t \\ i+t+1 & for\ 5 \le j \le 8, \quad i = 1,2,\dots,t \\ i+2(t+1) & for\ 9 \le j \le 12, \quad i = 1,2,\dots,t \\ . & . \\ . & . \\ . & . \\ i+(in-1)(t+1) & for\ 4n-7 \le j \le 4n-4, \quad i = 1,2,\dots,t \\ \hbar - 1 & for\ 4n-3 \le j \le 4n, \quad i = 1 \\ \hbar & for\ 4n-3 \le j \le 4n, \quad i = 2,3,\dots,t \end{cases},$$

$$\gamma(c_0 x_{i,1}) = 1 \qquad for\ i = 1,2,\dots,t$$

$$\gamma(c_S x_{i,4S}) = 3St - 2S + 3 \qquad for\ 1 \le S \le n-1, \quad i = 1,2,\dots,t$$

$$\gamma(c_n x_{i,4n}) = \begin{cases} \hbar - t + 1 & for\ i = 1 \\ \hbar - t + i - 1 & for\ i = 2,3,\ \dots,t \end{cases},$$

$$\gamma(c_S x_{i,4S+1}) = 3St - 2S + 2 \qquad for\ 1 \le S \le n-2, \quad i = 1,2,\dots,t$$

$$\gamma(c_{n-1} x_{i,4n-3}) = \begin{cases} \begin{cases} (t+2)n - t - 5 & for\ i = 1 \\ (t+2)n - t + i - 7 & for\ i = 2,3,\dots,t \end{cases} & ,n = 2,3 \\ \begin{cases} (t+1)n - t - 1 & for\ i = 1 \\ (t+1)n - t + i - 3 & for\ i = 2,3,\dots,t \end{cases} & ,n \ne 2,3 \end{cases}$$

$$\gamma(x_{i,j} x_{i,j+1}) = \begin{cases} \begin{cases} t+j & for\ 1 \le j \le 3 \\ 3t+j-5 & for\ 5 \le j \le 7 \\ 5t+j-10 & for\ 9 \le j \le 11 \\ . & . \\ . & . \\ . & . \\ (2n-3)t+j-5(n-2) & for\ 4n-7 \le j \le 4n-5 \end{cases} & ,i = 1,2,\dots t \\ \hbar - 4(t+n) + j + 4 \quad for\ 4n-3 \le j \le 4n-1, & i = 1 \\ \hbar - 4(t+n) + j + 2i \quad for\ 4n-3 \le j \le 4n-1, & i = 2,3,\dots,t \end{cases}$$

We can see that $\hbar$ is the greatest label. For edges weights of $\theta_n(t,4)$, we have:

$$w_\gamma(c_0 x_{i,1}) = i + 2 \qquad for\ i = 1,2,\dots,t$$

$$w_\gamma(c_0 x_{i,4S}) = t(5S-1) + i + 2 \quad for\ 1 \le S \le n-1, \quad i = 1,2,\dots,t$$

$$w_\gamma(c_n x_{i,4n}) = 3\hbar - t + i - 1 \quad for\ 1 \le S \le in-1, \quad i = 1,2,\dots,t$$

$$w_\gamma(c_S x_{i,4S+1}) = 5St + i + 2 \qquad for\ 1 \le S \le n-1,\ i = 1,2,\dots,t,$$

$$w_\gamma(c_n x_{i,4n-3}) = \begin{cases} 2nt - 3n - 2t + \hbar + i - 8 & for\ n = 2,3 \\ 2nt + 2n - 2t + \hbar + i - 6 & for\ n \ne 2,3 \end{cases} ,i = 1,2,\dots,t$$

$$w_\gamma(x_{i,j} x_{i,j+1}) = \begin{cases} \begin{cases} t+j+2i & for\ 1 \le j \le 3 \\ 5t+j+2i-4 & for\ 5\ \le j \le 7 \\ 9t+j+2i-6 & for\ 9 \le j \le 11 \\ . & . \\ . & . \\ . & . \\ (4n-5)t+j+2i-3n+8 & for\ 4n-7 \le j \le 4n-5 \end{cases} & ,i = 1,2,\dots,t \\ 3\hbar - 4(t+n) + j + 2 \quad for\ 4n-3 \le j \le 3n-1, & i = 1 \\ 3\hbar - 4(t\ +n\ ) + j + 2i \quad for\ 4n-3 \le j \le 4n-1, & i = 2,3,\dots,t \end{cases}$$

It is obvious that the edges weights are distinct. So γ is the desired edge irregular total ħ −labeling, $ħ = \left\lceil\frac{5tn+2}{3}\right\rceil$. Hence

$$tes(\theta_n(t,4)) = \left\lceil\frac{5tn+2}{3}\right\rceil.$$

The previous results lead us to introduce the following conjecture for a general case of a uniform theta snake graph $\theta_n(t,m)$.

The previous results lead us to introduce the following conjecture for a general case of a uniform theta snake graph $\theta_n(t,m)$.

Conjecture. For uniform theta snake graph $\theta_n(t,m), n \geq 2, t \geq 3, and\ m \geq 3$ we have

$$tes(\theta_n(t,m)) = \left\lceil\frac{(m+1)tn+2}{3}\right\rceil.$$

3. Conclusions

In the current paper, we have defined a new type of a family of graph called uniform theta snake graph, $\theta_n(t,m)$. Also, the exact i value of TEISs for $\theta_n(3,3)$, $\theta_n(3,m)$ and $\theta_n(t,3)$ has been determined. Finally, we have generalized for t, m and found TEIS of a uniform theta snake graph $\theta_n(t,m)$ for $m \geq 3,\ t \geq 3$.

$$tes(\theta_n(3,3)) = 4n + 1.$$

$$tes(\theta_n(3,im)) = (im + 1)in + 1.$$

$$tes(\theta_n(t,3)) = \left\lceil\frac{4tn+2}{3}\right\rceil$$

$$tes(\theta_n(t,m)) = \left\lceil\frac{(m+1)tn+2}{3}\right\rceil.$$

Conflict of interest

All authors declare no conflict of interest in this paper.

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