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Research article

Some results on transcendental entire solutions to certain nonlinear differential-difference equations

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Abstract: In this paper, we study the transcendental entire solutions for the nonlinear differential-difference equations of the forms:

$$f^2(z) + \widetilde{\omega} f(z) f'(z) + q(z)e^{Q(z)} f(z+c) = u(z)e^{v(z)},$$

and

$$f^n(z) + \omega f^{n-1}(z) f'(z) + q(z)e^{Q(z)} f(z+c) = p_1 e^{\lambda_1 z} + p_2 e^{\lambda_2 z}, \quad n \geq 3,$$

where ω is a constant, $\widetilde{\omega}, c, \lambda_1, \lambda_2, p_1, p_2$ are non-zero constants, q, Q, u, v are polynomials such that Q, v are not constants and $q, u \not\equiv 0$. Our results are improvements and complements of some previous results.

Keywords: entire solution; nonlinear differential-difference equations; Nevanlinna theory
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1. Introduction

Let $f(z)$ be a nonconstant function meromorphic on the complex plane $\mathbb{C}$. We assume that the reader is familiar with the fundamental results and standard notations of Nevanlinna theory (see [10, 12, 21]). For simplicity, we denote by $S(r, f)$ any quantity satisfying $S(r, f) = o(T(r, f))$ as $r \to \infty$, outside of a possible exceptional set of finite logarithmic measure.

Nevanlinna theory and its difference analogues have been used to study the growth, oscillation, solvability and existence of entire or meromorphic solutions to differential equations, difference equations and differential-difference equations, see, e.g.,[1, 2, 4, 6, 12–16, 18–20].

In 1964, Hayman [10] considered the following non-linear differential equation

$$f^n(z) + Q_d(z, f) = g(z), \tag{1.1}$$

where $Q_d(z, f)$ is a differential polynomial in f with degree d and obtained the following result which is an extension of Tumura–Clunie theory.

Theorem A. *Suppose that $f(z)$ is a nonconstant meromorphic function, $d \leq n-1$, and f, g satisfy $N(r, f) + N(r, 1/g) = S(r, f)$ in* (1.1)*. Then we have $g(z) = (f(z) + \gamma(z))^n$, where $\gamma(z)$ is meromorphic and a small function of $f(z)$.*

Since then, the non-linear differential equation (1.1) has been studied extensively over the years, see [16, 19, 20] etc.

In 2006, Li and Yang [13] studied the particular case that $g(z)$ has the form $p_1e^{\alpha_1 z} + p_2e^{\alpha_2 z}$ in Eq (1.1) and proved the following result.

Theorem B. *Let $n \geq 4$ be an integer and $Q_d(z, f)$ denote an algebraic differential polynomial in f of degree $d \leq n-3$. Let p_1, p_2 be two nonzero polynomials, α_1 and α_2 be two nonzero constants with $\alpha_1/\alpha_2 \neq$ rational. Then the differential equation*

$$f^n(z) + Q_d(z, f) = p_1e^{\alpha_1 z} + p_2e^{\alpha_2 z} \tag{1.2}$$

has no transcendental entire solutions.

Moreover, Yang and Li [13] also studied the case when $n = 3$, and found the exact forms of solutions to Eq (1.2) under some related conditions.

In 2014, Liao and Ye [14] investigated the structure of solutions to the following differential equation

$$f^n f' + Q_d(z, f) = u(z)e^{v(z)} \tag{1.3}$$

with non-zero rational function u and non-constant polynomial v and obtained the following result.

Theorem C. *Suppose that f is a meromorphic solution of* (1.3) *which has finitely many poles. Then*

$$Q_d(z, f) \equiv 0, \;\; f(z) = s(z)e^{v(z)/(n+1)}$$

for $n \geq d+1$ and s is a rational function satisfying $s^n[(n+1)s' + v's] = (n+1)u$.

In 2012, Wen et al. [18] classified the finite order entire solutions of the equation

$$f^n(z) + q(z)e^{Q(z)}f(z+c) = P(z), \tag{1.4}$$

where q, Q, P are polynomials, $n \geq 2$ is an integer, and $c \in \mathbb{C} \setminus \{0\}$. In 2019, Chen et al. [2] derived some conclusions when the term $P(z)$ on the right-hand side of Eq (1.4) is replaced by $p_1e^{\lambda z} + p_2e^{-\lambda z}$, where p_1, p_2, λ are non-zero constants.

Based on the above results, one can see that there exists only one dominant term f^n or f^nf' on the left-hand side of these equations. In 2020, Chen, Hu and Wang [4] investigated the following non-linear differential-difference equation which has two dominated terms

$$f^n(z) + \omega f^{n-1}(z)f'(z) + q(z)e^{Q(z)}f(z+c) = u(z)e^{v(z)}, \tag{1.5}$$

where n is a positive integer, $c \neq 0$, ω are constants, q, Q, u, v are polynomials such that Q, v are not constants and $q, u \not\equiv 0$, and obtained the following result.

Theorem D. *Let n be an integer satisfying $n \geq 3$ for $\omega \neq 0$ and $n \geq 2$ for $\omega = 0$. Suppose that f is a non-vanishing transcendental entire solution of finite order of* (1.5). *Then every solution f satisfies one of the following results:*

(1) $\rho(f) < \deg v = \deg Q$ and $f = Ce^{-z/\omega}$, where C is a constant.
(2) $\rho(f) = \deg Q \geq \deg v$.

In the meantime, Chen et al. [4] proposed a conjecture that the conclusions of Theorem D are still valid when $n = 2$ and $\omega \neq 0$.

In this paper, we consider the above conjecture and obtain the following result, which is a complement of Theorem D.

Theorem 1.1. *Let $c, \widetilde{\omega} \neq 0$ be constants, q, Q, u, v be polynomials such that Q, v are not constants and $q, u \not\equiv 0$. Suppose that f is a transcendental entire solution with finite order to*

$$f^2(z) + \widetilde{\omega} f(z)f'(z) + q(z)e^{Q(z)}f(z+c) = u(z)e^{v(z)}, \tag{1.6}$$

satisfying $\lambda(f) < \rho(f)$, then $\deg Q = \deg v$, and one of the following relations holds:

(1) $\sigma(f) < \deg Q = \deg v$, and $f = Ce^{-z/\widetilde{\omega}}$
(2) $\sigma(f) = \deg Q = \deg v$.

The following two examples given by Chen et al. [4] can illustrate the Conclusions (1) and (2) of Theorem 1.1, respectively.

Example 1.2. *$f_0(z) = 2e^{-z}$ is a transcendental entire solution to the following differential-difference equation*

$$f^2 + ff' + ze^{z^2+z+1}f(z+1) = 2ze^{z^2},$$

where $\widetilde{\omega} = 1 \neq 0$, $Q = z^2 + z + 1$, $v = z^2$ and $0 = \lambda(f_0) < \sigma(f_0) = 1$. Then we have $\sigma(f_0) = 1 < 2 = \deg Q = \deg v$, and $f_0 = Ce^{-z/\widetilde{\omega}}$, where $C = 2$. This shows that the Conclusion (1) of Theorem 1.1 occurs.

Example 1.3. *$f_1(z) = e^{z^2}$ is a transcendental entire solution to the following differential-difference equation*

$$f^2 + ff' + e^{z^2-2z-1}f(z+1) = 2(z+1)e^{2z^2},$$

where $\widetilde{\omega} = 1 \neq 0$, $Q = z^2 - 2z - 1$, $v = 2z^2$ and $0 = \lambda(f_1) < \sigma(f_1) = 2$. Then we have $\sigma(f_1) = 2 = \deg Q = \deg v$. This illustrates that the Conclusion (2) of Theorem 1.1 also exits.

In [4], Chen, Hu and Wang also investigated the entire solutions with finite order to the following differential-difference equation

$$f^n(z) + \omega f^{n-1}(z)f'(z) + q(z)e^{Q(z)}f(z+c) = p_1e^{\lambda z} + p_2e^{-\lambda z}, \tag{1.7}$$

where n is an integer, c, λ, p_1, p_2 are non-zero constants and ω is a constant, and $q \not\equiv 0$, Q are polynomials such that Q is not a constant. They obtained the following result.

Theorem E. *If f is a transcendental entire solution with finite order to Eq.* (1.7)*, then the following conclusions hold:*

(i) If $n \geq 4$ for $\omega \neq 0$ and $n \geq 3$ for $\omega = 0$, then every solution f satisfies $\rho(f) = \deg Q = 1$.
(ii) If $n \geq 1$ and f is a solution to Eq.(1.7) *which belongs to Γ_0, then*

$$f(z) = e^{\lambda z/n+B},\ Q(z) = -\frac{n+1}{n}\lambda z + b$$

or

$$f(z) = e^{-\lambda z/n+B},\ Q(z) = \frac{n+1}{n}\lambda z + b$$

where $b, B \in \mathbb{C}$, and $\Gamma_0 = \{e^{\alpha(z)} : \alpha(z)$ is a non-constant polynomial$\}$.

Given Theorem E, it is natural to ask: how about the solutions to the following more general form

$$f^n(z) + \omega f^{n-1}(z)f'(z) + q(z)e^{Q(z)}f(z+c) = p_1e^{\lambda_1 z} + p_2e^{\lambda_2 z}, \tag{1.8}$$

where n is a positive integer, ω is a constant and $c, \lambda_1, \lambda_2, p_1, p_2$ are non-zero constants, q, Q are polynomials such that Q is not a constant and $q \not\equiv 0$.

In this paper, we study this problem and derive the following result.

Theorem 1.4. *If f is a transcendental entire solution with finite order to Eq* (1.8)*, then the following conclusions hold:*

(1) If $n \geq 4$ for $\omega \neq 0$ and $n \geq 3$ for $\omega = 0$, then every solution f satisfies $\sigma(f) = \deg Q = 1$.
(2) If $n \geq 1$ and f is a solution to Eq (1.8) *with $\lambda(f) < \sigma(f)$, then*

$$f(z) = \left(\frac{p_2 n}{n+\omega\lambda_2}\right)^{\frac{1}{n}} e^{\frac{\lambda_2 z}{n}},\ Q(z) = \left(\lambda_1 - \frac{\lambda_2}{n}\right)z + b_1,$$

or

$$f(z) = \left(\frac{p_1 n}{n+\omega\lambda_1}\right)^{\frac{1}{n}} e^{\frac{\lambda_1 z}{n}},\ Q(z) = \left(\lambda_2 - \frac{\lambda_1}{n}\right)z + b_2,$$

where $b_1, b_2 \in \mathbb{C}$ satisfy $p_1 = q\left(\frac{p_2 n}{n+\omega\lambda_2}\right)^{\frac{1}{n}} e^{\frac{\lambda_2 c}{n}+b_1}$ and $p_2 = q\left(\frac{p_1 n}{n+\omega\lambda_1}\right)^{\frac{1}{n}} e^{\frac{\lambda_1 c}{n}+b_2}$, respectively.

Remark 1. *It is easy to see that Theorem 1.4 generalizes and improves the Theorem E of Chen et al. [4]. In addition, we conjecture that the Conclusion (1) is still true for $n = 2$, and $n = 3$ when $\omega \neq 0$ in Eq* (1.8)*.*

For the case when $n = 3$ and $\omega \neq 0$ in Eq (1.8), we prove the following result under certain assumption.

Theorem 1.5. *Let $\omega, c, \lambda_1, \lambda_2, p_1, p_2$ be non-zero constants, q, Q be polynomials such that Q is not a constant and $q \not\equiv 0$. If f is a transcendental entire solution with finite order to*

$$f^3(z) + \omega f^2(z)f'(z) + q(z)e^{Q(z)}f(z+c) = p_1e^{\lambda_1 z} + p_2e^{\lambda_2 z}, \tag{1.9}$$

satisfying $N_{1)}(r, 1/f) < (\kappa + o(1))T(r, f)$, where $0 \leq \kappa < 1$ and $N_{1)}(r, 1/f)$ denotes the counting functions corresponding to simple zeros of f, then $\sigma(f) = \deg Q = 1$.

The two examples below exhibit the occurence of Theorem 1.5.

Example 1.6. *[4] $f_2(z) = e^z$ is a transcendental entire solution to the nonlinear differential-difference equation*

$$f^3 + f^2 f' + \frac{1}{2}e^{-4z}f(z + \log 2) = 2e^{3z} + e^{-3z},$$

where $\omega = 1 \neq 0$, $Q = -4z$, $N_{1)}(r, 1/f_2) = 0$ from the fact that 0 is a Picard exceptional value of f_2. Thus, the conclusion $\sigma(f_2) = 1 = \deg Q$ holds.

Example 1.7. *$f_3(z) = e^{2z} - e^z$ is a transcendental entire solution to the nonlinear differential-difference equation*

$$f^3 - f^2 f' - \frac{1}{5}e^{3z}f(z + \log 5) = -e^{6z} - 3e^{5z},$$

where $\omega = -1 \neq 0$, $Q = 3z$, $\kappa = 1/2$ since $N_{1)}(r, 1/f_3) = N(r, 1/f_3) = r/\pi + o(r)$ and $T(r, f_3) = 2r/\pi + o(r)$ by using the following Lemma 2.5. Thus, the conclusion $\sigma(f_3) = 1 = \deg Q$ holds.

2. Preliminary Lemmas

In order to prove our results, we need the following lemmas. Lemmas 2.1 and 2.2 are useful tools to solve differential-difference equations.

Lemma 2.1 ([21]). *Let $f_j(z)\,(j = 1, \ldots, n)\,(n \geq 2)$ be meromorphic functions, and let $g_j(z)\,(j = 1, \ldots, n)$ be entire functions satisfying*

(i) $\sum_{j=1}^{n} f_j(z)e^{g_j(z)} \equiv 0$;
(ii) *when $1 \leq j < k \leq n$, then $g_j(z) - g_k(z)$ is not a constant;*
(iii) *when $1 \leq j \leq n, 1 \leq h < k \leq n$, then*

$$T(r, f_j) = o\{T(r, e^{g_h - g_k})\} \quad (r \to \infty, r \notin E),$$

where $E \subset (1, \infty)$ is of finite linear measure or logarithmic measure.

Then, $f_j(z) \equiv 0\,(j = 1, \ldots, n)$.

Lemma 2.2 ([21]). *Let* $f_j(z),\ j = 1, 2, 3$ *be meromorphic functions and* $f_1(z)$ *is not a constant. If*

$$\sum_{j=1}^{3} f_j(z) \equiv 1,$$

and

$$\sum_{j=1}^{3} N\left(r, \frac{1}{f_j}\right) + 2\sum_{j=1}^{3} \overline{N}(r, f_j) < (\lambda + o(1))T(r), \quad r \in I,$$

where $\lambda < 1$, $T(r) = \max_{1\le j\le 3}\{T(r, f_j)\}$ *and* I *represents a set of* $r \in (0, \infty)$ *with infinite linear measure. Then* $f_2 \equiv 1$ *or* $f_3 \equiv 1$.

The difference analogues of the logarithmic derivative lemma (see [3, 7–9, 11]) are of great importance in the study of complex difference equations. The following version is a particular case of [11, Lemma 2.2].

Lemma 2.3 ([11]). *Let* f *be a non-constant meromorphic function, let* c, h *be two complex numbers such that* $c \neq h$. *If he hyper-order of* $T(r, f)$ *i.e.* $\sigma_2(f) < 1$, *then*

$$m\left(r, \frac{f(z+h)}{f(z+c)}\right) = S(r, f),$$

for all r outside of a set of finite logarithmic measure.

The following lemma, which is a special case of [11, Theorem 3.1], gives a relationship for the Nevanlinna characteristic of a meromorphic function with its shift.

Lemma 2.4 ([12]). *Let* $f(z)$ *be a meromorphic function with the hyper-order less that one, and* $c \in \mathbb{C} \setminus \{0\}$. *Then we have*

$$T(r, f(z+c)) = T(r, f(z)) + S(r, f).$$

The following lemma gives the Nevanlinna characteristic function, proximity function and counting function of a given exponential polynomial. For convenience of the readers, we give the definition of an exponential polynomial with the following form:

$$f(z) = P_1(z)e^{Q_1(z)} + \cdots + P_k(z)e^{Q_k(z)}, \tag{2.1}$$

where P_j and Q_j are polynomials in z for $1 \le j \le k$. Denote $q = \max\{\deg Q_j : Q_j \not\equiv 0\}$, and let $\omega_1, \ldots, \omega_m$ be pairwise distinct leading coefficients of polynomials that attain the maximum degree q. Thus f in (2.1) can be written in the normalized form

$$f(z) = H_0(z) + H_1(z)e^{\omega_1 z^q} + \cdots + H_m(z)e^{\omega_m z^q}, \tag{2.2}$$

where H_j are either exponential polynomials of order $< q$ or ordinary polynomials in z, and $m \le k$. In addition, we denote the convex hull of a finite set $W(\subset \mathbb{C})$ by $co(W)$, and the circumference of $co(W)$ by $C(co(W))$, refer to [17, 18] for more details.

Lemma 2.5 ([17]). *Let $f(z)$ be given by (2.2), $W = \{\overline{\omega_1}, \ldots, \overline{\omega_m}\}$ and $W_0 = W \cup \{0\}$. Then*

$$T(r, f) = C(co(W_0))\frac{r^q}{2\pi} + o(r^q).$$

If $H_0(z) \not\equiv 0$, then

$$m\left(r, \frac{1}{f}\right) = o(r^q),$$

while if $H_0(z) \equiv 0$, then

$$N\left(r, \frac{1}{f}\right) = C(co(W))\frac{r^q}{2\pi} + o(r^q).$$

The following lemma is a revised version of [12, Lemma 2.4.2].

Lemma 2.6. *Let $f(z)$ be a transcendental meromorphic solution to the equation:*

$$f^n P(z, f) = Q(z, f),$$

where $P(z, f)$ and $Q(z, f)$ are polynomials in f and its derivatives with meromorhphic coefficients, say $\{a_\lambda | \lambda \in I\}$, n be a positive integer. If the total degree of $Q(z, f)$ as a polynomial in f and its derivatives is at most n, then

$$m(r, P(z, f)) \leq \sum_{\lambda \in I} m(r, a_\lambda) + S(r, f).$$

Lemma 2.7. *(Hadamard factorization theorem [21, Theorem 2.7] or [5, Theorem 1.9]) Let f be a meromorphic function of finite order $\sigma(f)$. Write*

$$f(z) = c_k z^k + c_{k+1} z^{k+1} + \cdots \ (c_k \neq 0)$$

near $z = 0$ and let $\{a_1, a_2, \ldots\}$ and $\{b_1, b_2, \ldots\}$ be the zeros and poles of f in $\mathbb{C}\backslash\{0\}$, respectively. Then

$$f(z) = z^k e^{Q(z)} \frac{P_1(z)}{P_2(z)},$$

where $P_1(z)$ and $P_2(z)$ are the canonical products of f formed with the non-null zeros and poles of $f(z)$, respectively, and $Q(z)$ is a polynomial of degree $\leq \sigma(f)$.

Remark 2. *A well known fact about Lemma 2.7 asserts that $\lambda(f) = \lambda(z^k P_1) = \sigma(z^k P_1) \leq \sigma(f)$, $\lambda(1/f) = \lambda(P_2) = \sigma(P_2) \leq \sigma(f)$ if $k \geq 0$; and $\lambda(f) = \lambda(P_1) = \sigma(P_1) \leq \sigma(f)$, $\lambda(1/f) = \lambda(z^{-k} P_2) = \sigma(z^{-k} P_2) \leq \sigma(f)$ if $k < 0$. So we have $\sigma(f) = \sigma(e^Q)$ when* $\max\{\lambda(f), \lambda(1/f)\} < \sigma(f)$.

By combinig [21, Theorem1.42] and [21, Theorem1.44], we have the following lemma.

Lemma 2.8 ([21]). *Let $f(z)$ be a non-constant meromorphic function in the complex plane. If $0, \infty$ are Picard exceptional values of $f(z)$, then $f(z) = e^{h(z)}$, where $h(z)$ is a non-constant entire function. Moreover, $f(z)$ is of normal growth, and*

(i) if $h(z)$ is a polynomial of degree p, then $\sigma(f) = p$;
(ii) if $h(z)$ is a transcendental entire function, then $\sigma(f) = \infty$.

The following lemma gives a relationship between the growth order of a meromorphic function with its derivative.

Lemma 2.9 ([21]). *Suppose that $f(z)$ is meromorphic in the complex plane and n is a positive integer. Then $f(z)$ and $f^{(n)}(z)$ have the same order.*

3. Proof of Theorem 1.1.

Suppose that $f(z)$ is a finite-order transcendental entire solution to Eq (1.6) and satisfies $\lambda(f) < \sigma(f)$. Then, by Lemma 2.7 and Remark 2, we have

$$f(z) = d(z)e^{h(z)}, \tag{3.1}$$

where d is the canonical product formed by zeros of f such that $\sigma(d) = \lambda(f) < \sigma(f)$, and $h = a_m z^m + a_{m-1}z^{m-1} + \cdots + a_0$ is a non-constant polynomial, where $a_m(\neq 0), \ldots, a_0$ are constants, $\deg h = m = \sigma(f)$ is a positive integer.

Set $f_c = f(z + c)$, we rewrite (1.6) as

$$f^2 + \widetilde{\omega} f f' + qe^Q f_c = ue^v. \tag{3.2}$$

By Lemmas 2.4 and 2.9, we have $\sigma(f_c) = \sigma(f) = \sigma(f')$. From (3.2), by the order property, we get

$$\begin{aligned} \deg v = \sigma(ue^v) &\leq \max\{\sigma(f') = \sigma(f) = \sigma(f_c), \sigma(e^Q), \sigma(q)\} \\ &= \max\{\deg h, \deg Q\}. \end{aligned} \tag{3.3}$$

By substituting (3.1) into (3.2), we obtain that

$$d\left(d + \widetilde{\omega}(d' + dh')\right) e^{2h} + qd_c e^{Q+h_c} = ue^v. \tag{3.4}$$

Next, we consider the following three cases.

Case 1. $\sigma(f) > \deg Q$. Then $\deg h > \deg Q \geq 1$, and by (3.3) we have $\deg v \leq \deg h$.

Subcase 1.1. $\deg h > \deg v$. From (3.4) we have

$$d\left(d + \widetilde{\omega}(d' + dh')\right) e^{h_1} e^{2a_m z^m} + qd_c e^{h_2} e^{a_m z^m} = ue^v, \tag{3.5}$$

where $h_1 = 2a_{m-1}z^{m-1} + \cdots$ and $h_2 = Q + (a_m mc + a_{m-1})z^{m-1} + \cdots$ are polynomials with degree $\leq m - 1$. So, noting $\sigma(d') = \sigma(d) = \sigma(d_c) < m$, by applying Lemma 2.1 to (3.5), it follows that

$$qd_c \equiv 0.$$

which yields a contradiction.

Subcase 1.2. $\deg h = \deg v$. Denote $v(z) = v_m z^m + v_{m-1}z^{m-1} + \cdots + v_0$, where $v_m(\neq 0), \ldots, v_0$ are constants. From (3.4) we have

$$d\left(d + \widetilde{\omega}(d' + dh')\right) e^{h_1} e^{2a_m z^m} + qd_c e^{h_2} e^{a_m z^m} = ue^{h_3} e^{v_m z^m}, \tag{3.6}$$

where h_1 and h_2 are defined as in Subcase 1.1, and $h_3 = v_{m-1}z^{m-1} + \cdots + v_0$ is a polynomial with $\deg h_3 \leq m - 1$.

If $v_m \neq 2a_m$ and $v_m \neq a_m$, since $\sigma(d') = \sigma(d) = \sigma(d_c) < m$, by applying Lemma 2.1 to (3.6), we get $u \equiv 0$, which is a contradiction.

If $v_m = 2a_m$, then (3.6) can be written as

$$\left(d\left(d + \widetilde{\omega}(d' + dh')\right) e^{h_1} - ue^{h_3}\right) e^{2a_m z^m} + qd_c e^{h_2} e^{a_m z^m} = 0.$$

By applying Lemma 2.1, we have $qd_c \equiv 0$, which implies a contradiction.

If $v_m = a_m$, then (3.6) can be written as

$$d\,(d + \widetilde{\omega}(d' + dh'))\, e^{h_1} e^{2a_m z^m} + (qd_c e^{h_2} - ue^{h_3})e^{a_m z^m} = 0.$$

Similarly as above, by Lemma 2.1, we get

$$d\,(d + \widetilde{\omega}(d' + dh')) \equiv 0.$$

Noting $d \not\equiv 0$, it follows that

$$1 + \widetilde{\omega}\left(\frac{d'}{d} + h'\right) \equiv 0.$$

By integrating the above equation, we have

$$d = c_1 e^{-\frac{1}{\widetilde{\omega}}z - h}, \quad c_1 \in \mathbb{C} \setminus \{0\}.$$

Noting that $\deg h > \deg Q \geq 1$, we obtain $\sigma(d) = \deg h = \sigma(f)$, which contradicts with $\sigma(d) < \sigma(f)$.

Case 2. $\sigma(f) = \deg Q$. Then from (3.3), we have $\deg v \leq \deg h = \deg Q$. Next, we deduce that $\deg v = \deg h = \deg Q$. Otherwise, if $\deg v < \deg h = \deg Q$, we will get a contradiction by the following. We suppose that $Q(z) = b_m z^m + b_{m-1} z^{m-1} + \cdots + b_0$, where $b_m (\neq 0), \ldots, b_0$ are constants. Then Eq (3.4) can be written as

$$d\,(d + \widetilde{\omega}(d' + dh'))\, e^{h_1} e^{2a_m z^m} + (qd_c e^{\widetilde{h_2}})e^{(a_m + b_m)z^m} = ue^v, \tag{3.7}$$

where h_1 is defined as in Subcase 1.1, and $\widetilde{h_2} = (a_m mc + a_{m-1} + b_{m-1})z^{m-1} + \cdots$ with $\deg \widetilde{h_2} \leq m - 1$.

If $b_m \neq \pm a_m$, since $\sigma(d') = \sigma(d) = \sigma(d_c) < m$, by applying Lemma 2.1 to (3.7), we get $u \equiv 0$, which is a contradiction.

If $b_m = a_m$, then Eq (3.7) can be rewritten as

$$\left(d\,(d + \widetilde{\omega}(d' + dh'))\, e^{h_1} + qd_c e^{\widetilde{h_2}}\right) e^{2a_m z^m} = ue^v.$$

Thus, by using Lemma 2.1, we have $u \equiv 0$, a contradiction.

If $b_m = -a_m$, then Eq (3.7) can be rewritten as

$$d\,(d + \widetilde{\omega}(d' + dh'))\, e^{h_1} e^{2a_m z^m} = ue^v - qd_c e^{\widetilde{h_2}}.$$

So by Lemma 2.1, we get

$$d\,(d + \widetilde{\omega}(d' + dh')) \equiv 0,$$

which implies

$$d = c_2 e^{-\frac{1}{\widetilde{\omega}}z - h},\ c_2 \in \mathbb{C} \setminus \{0\}.$$

Noting that $\deg h > \deg v \geq 1$, we have $\sigma(d) = \deg h = \sigma(f)$, which contradicts with $\sigma(d) < \sigma(f)$. Therefore, $\deg v = \deg h = \deg Q$, which implies that the Conclusion (2) holds.

Case 3. $\sigma(f) < \deg Q$. Then we have $T(r, f) = S(r, e^Q)$. By Milloux's theorem and lemma 2.4, we have $T(r, f') = S(r, e^Q)$ and $T(r, f_c) = S(r, e^Q)$. Therefore, it follows from (3.2) that

$$\begin{aligned} T(r, e^Q) + S(r, e^Q) &= T(r, f^2 + \widetilde{\omega} f f' + q f_c e^Q) \\ &= T(r, ue^v) = T(r, e^v) + S(r, e^v). \end{aligned}$$

Thus,

$$\deg Q = \deg v.$$

Differentiating (3.2) yields

$$2ff' + \widetilde{\omega}(f')^2 + \widetilde{\omega} f f'' + Ae^Q = (u' + uv')e^v, \tag{3.8}$$

where $A = q' f_c + q f_c' + q f_c Q'$.

Eliminating e^v from (3.2) and (3.8), we have

$$B_1 e^Q + B_2 = 0, \tag{3.9}$$

where

$$B_1 = uA - q f_c(u' + uv'),$$

$$B_2 = u[2ff' + \widetilde{\omega}(f')^2 + \widetilde{\omega} f f''] - (f^2 + \widetilde{\omega} f f')(u' + uv').$$

Note that $\sigma(f'') = \sigma(f') = \sigma(f) = \sigma(f_c) < \deg Q$, then by applying Lemma 2.1 to (3.9), we get $B_1 \equiv B_2 \equiv 0$. It follows from $B_1 \equiv 0$ that

$$\frac{q'}{q} + \frac{f_c'}{f_c} + Q' = \frac{u'}{u} + v',$$

by integration, we have $q f_c e^Q = c_3 u e^v$, where c_3 is a non-zero constant.

If $c_3 = 1$, then substituting $q f_c e^Q = u e^v$ into (3.2), we see that $f^2 + \widetilde{\omega} f f' = 0$. Thus we can easily get $f = c_4 e^{-z/\widetilde{\omega}}$, where $c_4 \in \mathbb{C} \setminus \{0\}$, which implies that the Conclusion (1) holds.

If $c_3 \neq 1$, we have $f = c_3 u_{-c}/q_{-c} e^{v_{-c} - Q_{-c}}$. By substituting this expression into (3.2), we obtain

$$\frac{c_3 u_{-c}}{q_{-c}} \left(\frac{c_3 u_{-c}}{q_{-c}} + \widetilde{\omega} \left(\left(\frac{c_3 u_{-c}}{q_{-c}} \right)' + \frac{c_3 u_{-c}}{q_{-c}} (v_{-c} - Q_{-c})' \right) \right) e^{2(v_{-c} - Q_{-c})} = (1 - c_3) u e^v.$$

Since $1 \leq \sigma(f) = \deg(v_{-c} - Q_{-c}) < \deg Q = \deg v$, then by Lemma 2.1 and $(1 - c_3)u \not\equiv 0$ we can easily deduce a contradiction.

4. Proof of Theorem 1.4.

Suppose that f is a finite-order transcendental entire solution to Eq (1.8). Denote $f_c = f(z + c)$, then Eq (1.8) can be written as

$$f^n + \omega f^{n-1} f' + q e^Q f_c = p_1 e^{\lambda_1 z} + p_2 e^{\lambda_2 z}. \tag{4.1}$$

By Lemma 2.4, we have $\sigma(f) = \sigma(f_c)$.

Differentiating both sides of (4.1), we have

$$nf^{n-1}f' + \omega(n-1)f^{n-2}(f')^2 + \omega f^{n-1}f'' + A_1 e^Q = p_1\lambda_1 e^{\lambda_1 z} + p_2\lambda_2 e^{\lambda_2 z}, \tag{4.2}$$

where $A_1 = q' f_c + q f_c' + q f_c Q'$.

Eliminating $e^{\lambda_2 z}$ from (4.1) and (4.2), we get

$$\begin{aligned}&\lambda_2 f^n + (\lambda_2\omega - n)f^{n-1}f' - \omega(n-1)f^{n-2}(f')^2 - \omega f^{n-1}f'' + A_2 e^Q\\ &= p_1(\lambda_2 - \lambda_1)e^{\lambda_1 z},\end{aligned} \tag{4.3}$$

where $A_2 = \lambda_2 q f_c - A_1$.

By taking the derivative of (4.3), we get

$$\begin{aligned}&\lambda_2 n f^{n-1}f' + (\lambda_2\omega - n)\left[(n-1)f^{n-2}(f')^2 + f^{n-1}f''\right]\\ &-\omega(n-1)\left[(n-2)f^{n-3}(f')^3 + f^{n-2}2f'f''\right] - \omega(n-1)f^{n-2}f'f'' - \omega f^{n-1}f'''\\ &+(A_2' + A_2 Q')e^Q = p_1(\lambda_2 - \lambda_1)\lambda_1 e^{\lambda_1 z}.\end{aligned} \tag{4.4}$$

Then eliminating $e^{\lambda_1 z}$ from (4.3) and (4.4) gives

$$\begin{aligned}&\lambda_1\lambda_2 f^n + (\lambda_1\lambda_2\omega - n\lambda_1 - \lambda_2 n)f^{n-1}f' - (n-1)\left[\lambda_1\omega + \lambda_2\omega - n\right]f^{n-2}(f')^2\\ &-(\lambda_1\omega + \lambda_2\omega - n)f^{n-1}f'' + \omega(n-1)(n-2)f^{n-3}(f')^3 + 3\omega(n-1)f^{n-2}f'f''\\ &+\omega f^{n-1}f''' + (\lambda_1 A_2 - A_2' - A_2 Q')e^Q = 0.\end{aligned} \tag{4.5}$$

Case 1. $\sigma(f) < 1$. By applying the logarithmic derivative lemma, Lemmas 2.3 and 2.5 to Eq (4.1) , we obtain

$$\begin{aligned}T\left(r, e^Q\right) &= m\left(r, e^Q\right) = m\left(r, \frac{p_1 e^{\lambda_1 z} + p_2 e^{\lambda_2 z} - f^n - \omega f^{n-1}f'}{q f_c}\right)\\ &\leq m\left(r, \frac{f}{q f_c}\right) + m\left(r, \frac{1}{f}\right) + m\left(r, p_1 e^{\lambda_1 z} + p_2 e^{\lambda_2 z}\right)\\ &\quad + m\left(r, \frac{f^n + \omega f^{n-1}f'}{f^n}\right) + m(r, f^n) + O(1)\\ &\leq (n+1)T(r,f) + C(co(W_0))\frac{r}{2\pi} + o(r) + S(r,f)\\ &\leq C(co(W_0))\frac{r}{2\pi} + o(r),\end{aligned}$$

where $W_0 = \{0, \overline{\lambda_1}, \overline{\lambda_2}\}$. Thus we have $\deg Q \leq 1$. Noting $\deg Q \geq 1$, we know that $\deg Q = 1$. Let $Q = az + b$, $a \in \mathbb{C}\setminus\{0\}$, $b \in \mathbb{C}$.

Thus, by applying Lemma 2.1 to (4.5), we have

$$\lambda_1 A_2 - A_2' - A_2 Q' = (\lambda_1 - a)A_2 - A_2' \equiv 0. \tag{4.6}$$

Subcase 1.1.$A_2 \equiv 0$. That is $\lambda_2 q f_c - q' f_c - q f_c' - q f_c a \equiv 0$. Noting $q f_c \not\equiv 0$, it follows that

$$\lambda_2 - \frac{q'}{q} - \frac{f_c'}{f_c} - a \equiv 0.$$

By integration, we have

$$qf_c = c_1 e^{(\lambda_2 - a)z}, \quad c_1 \in \mathbb{C} \setminus \{0\}.$$

If $a \neq \lambda_2$, then $\sigma(f) = \sigma(f_c) = 1$, which contradicts with $\sigma(f) < 1$. Thus we have $a = \lambda_2$, and $f_c = c_1/q$. Then $f(z) = c_1/q(z-c)$, which is impossible for a transcendental function f.

Subcase 1.2.$A_2 \not\equiv 0$. From (4.6), we get

$$A_2 = c_2 e^{(\lambda_1 - a)z}, \quad c_2 \in \mathbb{C} \setminus \{0\}.$$

It follows that

$$(qf_c)' + (a - \lambda_2)(qf_c) = -c_2 e^{(\lambda_1 - a)z}, \quad c_2 \in \mathbb{C} \setminus \{0\}. \tag{4.7}$$

Since $\lambda_1 \neq \lambda_2$, we consider three subcases as follows.

Subcase 1.2.1. $a = \lambda_2$. Then (4.7) can be written as $(qf_c)' = -c_2 e^{(\lambda_1 - \lambda_2)z}$. By integration, we obtain $qf_c = \frac{c_2}{\lambda_2 - \lambda_1} e^{(\lambda_1 - \lambda_2)z} + c_3$, where $c_3 \in \mathbb{C}$. By Lemma 2.5 we have

$$\begin{aligned} T(r, qf_c) &= T\left(r, \frac{c_2}{\lambda_2 - \lambda_1} e^{(\lambda_1 - \lambda_2)z} + c_3\right) \\ &= C(co(W_1))\frac{r}{2\pi} + o(r), \quad W_1 = \{0, \overline{\lambda_1 - \lambda_2}\}. \end{aligned}$$

Since f is transcendental, by Lemma 2.4, it follows that

$$\begin{aligned} C(co(W_1))\frac{r}{2\pi} + o(r) = T(r, qf_c) &\leq T(r, f_c) + T(r, q) + O(1) \\ &= T(r, f) + S(r, f), \end{aligned}$$

which contradicts with $\sigma(f) < 1$.

Subcase 1.2.2. $a = \lambda_1$. Then (4.7) can be written as $(qf_c)' + (\lambda_1 - \lambda_2)(qf_c) = -c_2$. Thus, we obtain $qf_c = \frac{c_2}{\lambda_2 - \lambda_1} + c_4 e^{(\lambda_2 - \lambda_1)z}$, where $c_4 \in \mathbb{C}$. We assert that $c_4 \neq 0$. Otherwise, we have $f(z) = \frac{c_2}{\lambda_2 - \lambda_1} \frac{1}{q(z-c)}$, which contradicts with the assumption that f is transcendental. Therefore, $c_4 \neq 0$, similarly as in Subcase 1.2.1, by combining Lemmas 2.4, 2.5, and $\sigma(f) < 1$, we can get a contradiction.

Subcase 1.2.3. $a \neq \lambda_1$ and $a \neq \lambda_2$. Then by (4.7), we get that

$$qf_c = \frac{c_2}{\lambda_2 - \lambda_1} e^{(\lambda_1 - a)z} + c_5 e^{(\lambda_2 - a)z}, \quad c_5 \in \mathbb{C}.$$

Since $c_2 \neq 0$, $a \neq \lambda_1$, and $\lambda_1 \neq \lambda_2$, similarly as in Subcase 1.2.1, by combining Lemmas 2.4, 2.5, and $\sigma(f) < 1$, we also get a contradiction.

Case 2. $\sigma(f) > 1$. Denote $P = p_1 e^{\lambda_1 z} + p_2 e^{\lambda_2 z}$, then by Lemma 2.5 we have $\sigma(P) = 1$. We rewrite Eq (4.1) as

$$f^n + \omega f^{n-1} f' + (qf_c)e^Q = P. \tag{4.8}$$

Differentiating (4.8) yields

$$nf^{n-1}f' + \omega(n-1)f^{n-2}(f')^2 + \omega f^{n-1}f'' + Le^Q = P', \tag{4.9}$$

where $L = (qf_c)' + Q'(qf_c)$.

Subcase 2.1. $\omega \neq 0$ and $n \geq 4$. Eliminating e^Q from (4.8) and (4.9), we have

$$f^{n-2}H = PL - P'(qf_c), \tag{4.10}$$

where

$$H = Lf^2 + (\omega L - nqf_c)ff' - (n-1)\omega qf_c(f')^2 - \omega qf_c ff''.$$

Subcase 2.1.1. $H \not\equiv 0$. We rewrite $PL - P'(qf_c)$ as

$$\begin{aligned} &P\left[(qf_c)' + Q'(qf_c)\right] - P'(qf_c) \\ &= Pq\frac{(qf_c)'}{qf_c}\frac{f_c}{f} \cdot f + (PQ' - P')q\frac{f_c}{f} \cdot f \\ &= Pq\left(\frac{q'}{q} + \frac{f_c'}{f_c}\right)\frac{f_c}{f} \cdot f + (PQ' - P')q\frac{f_c}{f} \cdot f, \end{aligned}$$

and H as

$$\begin{aligned} &q\left(\frac{q'}{q} + \frac{f_c'}{f_c} + Q'\right)\frac{f_c}{f} \cdot f^3 + q\left(\omega\left(\frac{q'}{q} + \frac{f_c'}{f_c} + Q'\right) - n\right)\frac{f_c}{f} \cdot f^2 f' \\ &-(n-1)\omega q\frac{f_c}{f} \cdot f(f')^2 - \omega q\frac{f_c}{f} \cdot f^2 f'', \end{aligned}$$

thus both $PL - P'(qf_c)$ and H are differential polynomials with meromorphic coefficients. By the logarithmic derivative lemma and Lemma 2.4, we have $m(r, f_c'/f_c) = S(r, f_c) = S(r, f)$; by Lemma 2.3, we have $m(r, f_c/f) = S(r, f)$; and by Lemma 2.5, we have $m(r, P) = O(r)$. Note that $n \geq 4$ and H, fH are entire, by applying Lemma 2.6 to (4.10), it follows that

$$T(r, H) = m(r, H) = S(r, f) + O(r),$$

and

$$T(r, fH) = m(r, fH) = S(r, f) + O(r).$$

Thus, by $H \not\equiv 0$ we deduce that

$$T(r, f) \leq T(r, fH) + T\left(r, \frac{1}{H}\right) = S(r, f) + O(r),$$

which contradicts with $\sigma(f) > 1$.

Subcase 2.1.2. $H \equiv 0$. Then from (4.10), we have

$$PL - P'(qf_c) = P\left[(qf_c)' + Q'(qf_c)\right] - P'(qf_c) \equiv 0.$$

Noting $qf_cP \not\equiv 0$, it follows that

$$\frac{(qf_c)'}{qf_c} + Q' - \frac{P'}{P} \equiv 0.$$

By integration, we have

$$qf_c = c_6 P e^{-Q}, \ c_6 \in \mathbb{C} \setminus \{0\}. \tag{4.11}$$

Substituting (4.11) into (4.8), we get that

$$f^n + \omega f^{n-1} f' = (1 - c_6)P. \tag{4.12}$$

If $c_6 = 1$, then from (4.12) we have $f + \omega f' = 0$. By integration, we get $f = c_7 e^{-\frac{1}{\omega}z}$, $c_7 \in \mathbb{C} \setminus \{0\}$, which contradicts with $\sigma(f) > 1$. Thus, $c_6 \neq 1$. It follows from (4.11) that

$$f = c_6 \frac{P_{-c}}{q_{-c}} e^{-Q_{-c}}. \tag{4.13}$$

Then $\deg Q = \deg Q_{-c} = \sigma(f) > 1$ since $\sigma(P_{-c}) = 1$ by Lemma 2.5.

By Substituting (4.13) into (4.12), we have

$$\frac{c_6^n}{1 - c_6}\left(\left(\frac{P_{-c}}{q_{-c}}\right)^n + \omega\left(\frac{P_{-c}}{q_{-c}}\right)^{n-1}\left(\left(\frac{P_{-c}}{q_{-c}}\right)' + \frac{P_{-c}}{q_{-c}}(-Q_{-c})'\right)\right)e^{-nQ_{-c}} = P.$$

Thus, from $\deg Q > 1$, $\sigma(P_{-c}) = \sigma(P) = 1$ and Lemma 2.1, we get $P(z) \equiv 0$, which is impossible.

Subcase 2.2. $\omega = 0$ and $n \geq 3$. Eliminating e^Q from (4.8) and (4.9), we obtain

$$f^{n-1}\left(Lf - nqf_c f'\right) = PL - P'(qf_c).$$

Subcase 2.2.1. $Lf - nqf_c f' \not\equiv 0$. Since $n \geq 3$ and $\omega = 0$, similarly as in Subcase 2.1.1, we have

$$T(r, Lf - nqf_c f') = m(r, Lf - nqf_c f') = S(r, f) + O(r),$$

and

$$T(r, f\,(Lf - nqf_c f')) = m(r, f\,(Lf - nqf_c f')) = S(r, f) + O(r).$$

By $Lf - nqf_c f' \not\equiv 0$, we deduce that

$$T(r, f) \leq T(r, f\,(Lf - nqf_c f')) + T\left(r, \frac{1}{Lf - nqf_c f'}\right) = S(r, f) + O(r),$$

which contradicts with $\sigma(f) > 1$.

Subcase 2.2.2. $Lf - nqf_c f' \equiv 0$. Then

$$\frac{(qf_c)'}{qf_c} + Q' - n\frac{f'}{f} \equiv 0.$$

By integration, we obtain

$$qf_c e^Q = c_8 f^n, \ c_8 \in \mathbb{C} \setminus \{0\}. \tag{4.14}$$

Substituting (4.14) into (4.8) yields

$$(1 + c_8)f^n = P.$$

If $c_8 \neq -1$, then we have $nT(r, f) + S(r, f) = T(r, (1 + c_8)f^n) = T(r, P) = O(r)$, which contradicts with $\sigma(f) > 1$. Thus, $c_8 = -1$, and then $p_1 e^{\lambda_1 z} + p_2 e^{\lambda_2 z} = P = (1 + c_8) f^n \equiv 0$, which is impossible.

Case 3. $\sigma(f) = 1$. By (4.1), Lemma 2.3, and the logarithmic derivative lemma, we obtain

$$\begin{aligned} T\left(r, e^Q\right) &= m\left(r, e^Q\right) = m\left(r, \frac{p_1 e^{\lambda_1 z} + p_2 e^{\lambda_2 z} - f^n - \omega f^{n-1} f'}{q f_c}\right) \\ &\leq m\left(r, \frac{1}{q f_c}\right) + m\left(r, p_1 e^{\lambda_1 z} + p_2 e^{\lambda_2 z}\right) + m\left(r, f^n + \omega f^{n-1} f'\right) + O(1) \\ &\leq m\left(r, \frac{f}{f_c}\right) + m\left(r, \frac{1}{f}\right) + m\left(r, \frac{f^n + \omega f^{n-1} f'}{f^n}\right) + m(r, f^n) \\ &\quad + T\left(r, p_1 e^{\lambda_1 z} + p_2 e^{\lambda_2 z}\right) + O(\log r) \\ &\leq (n+1) T(r, f) + T\left(r, p_1 e^{\lambda_1 z} + p_2 e^{\lambda_2 z}\right) + S(r, f). \end{aligned}$$

Note that $\deg Q \geq 1$, then by combining Lemma 2.5, we get

$$1 \leq \deg Q = \sigma\left(e^Q\right) \leq \max\{\sigma(f), \sigma\left(p_1 e^{\lambda_1 z} + p_2 e^{\lambda_2 z}\right)\} = 1,$$

that is $\sigma(f) = \deg Q = 1$. Thus, the Conclusion (1) is proved.

Next, we will prove the Conclusion (2). Suppose that f is a finite-order transcendental entire solution to Eq (1.8) and satisfies $\lambda(f) < \sigma(f)$. Then, similarly as in the proof of Theorem 1.1, by Lemma 2.7 and Remark 2, we have

$$f(z) = d(z) e^{h(z)}, \tag{4.15}$$

where d is the canonical product formed by zeros of f such that $\sigma(d) = \lambda(f) < \sigma(f)$, and h is a polynomial with $\deg h = \sigma(f) \geq 1$.

By substituting (4.15) into (1.8), we get

$$d^{n-1}(d + \omega(d' + dh')) e^{nh} + q d_c e^{Q + h_c} = p_1 e^{\lambda_1 z} + p_2 e^{\lambda_2 z}. \tag{4.16}$$

Dividing both sides of (4.16) by $p_2 e^{\lambda_2 z}$, we obtain

$$f_1 + f_2 + f_3 = 1, \tag{4.17}$$

where

$$f_1 = -\frac{p_1}{p_2} e^{(\lambda_1 - \lambda_2) z}, f_2 = \frac{d^{n-1}(d + \omega(d' + dh'))}{p_2} e^{nh - \lambda_2 z}, f_3 = \frac{q d_c}{p_2} e^{Q + h_c - \lambda_2 z}.$$

Obviously, f_1 is not a constant by the fact that $\lambda_1 \neq \lambda_2$. Let $T(r) = \max\{T(r, f_1), T(r, f_2), T(r, f_3)\}$. Next we distinguish two cases below.

Case 1. $\sigma(f) > 1$. It follows from $\max\{\sigma(d') = \sigma(d), 1\} < \deg h$ that $d^{n-1}(d + \omega(d' + dh'))/p_2 e^{-\lambda_2 z}$ is a small function of e^h. Then $T(r) \geq T(r, f_2) = nT(r, e^h) + S(r, e^h)$. Thus, by Milloux's theorem and Lemma 2.4, we get

$$\frac{N\left(r, \frac{1}{f_2}\right)}{T(r)} = \frac{N\left(r, \frac{1}{d^{n-1}(d + \omega(d' + dh'))}\right)}{T(r)} \leq \frac{T\left(r, d^{n-1}(d + \omega(d' + dh'))\right)}{T(r)}$$

$$= \frac{O(T(r,d)) + O(\log r)}{T(r,e^h)} \cdot \frac{T(r,e^h)}{T(r)} \to 0,$$

and

$$\frac{N\left(r,\frac{1}{f_3}\right)}{T(r)} = \frac{N\left(r,\frac{1}{qd_c}\right)}{T(r)} \le \frac{T(r,qd_c)}{T(r)} \le \frac{T(r,d)+S(r,d)+O(\log r)}{T(r,e^h)} \cdot \frac{T(r,e^h)}{T(r)} \to 0$$

as $r \to \infty$.

Therefore, by applying Lemma 2.2, we can deduce that $f_2 \equiv 1$ or $f_3 \equiv 1$.

If $f_2 \equiv 1$, then $d^{n-1}(d+\omega(d'+dh'))e^{nh-\lambda_2 z} \equiv p_2$. We deduce that $d^{n-1}(d+\omega(d'+dh')) \not\equiv 0$. Otherwise, $p_2 \equiv 0$, which is a contradiction. So, by Lemma 2.5 and Milloux's theorem, we obtain

$$\begin{aligned} S\left(r,e^h\right) + nT\left(r,e^h\right) &= T\left(r,e^{nh}\right) = T\left(r, \frac{p_2 e^{\lambda_2 z}}{d^{n-1}(d+\omega(d'+dh'))}\right) \\ &\le T\left(r, p_2 e^{\lambda_2 z}\right) + T\left(r, d^{n-1}(d+\omega(d'+dh'))\right) + O(1) \\ &= O(r) + O(T(r,d)), \end{aligned}$$

which contradicts with $\deg h = \sigma(f) > \max\{\sigma(d), 1\}$.

If $f_3 \equiv 1$, then by (4.17), we have $f_1 + f_2 \equiv 0$. It follows that

$$d^{n-1}(d+\omega(d'+dh'))e^{nh} = p_1 e^{\lambda_1 z}.$$

By a similar discussion as above, we can also get a contradiction.

Case 2. $\sigma(f) = 1$. Then we have $\sigma(d) < 1 = \deg h = \sigma(e^{(\lambda_1-\lambda_2)z})$ and $T(r) \ge T(r,f_1) = T\left(r, e^{(\lambda_1-\lambda_2)z}\right) + S\left(r, e^{(\lambda_1-\lambda_2)z}\right)$. Thus, by Milloux's theorem and Lemma 2.4, we obtain

$$\frac{N\left(r,\frac{1}{f_2}\right)}{T(r)} = \frac{O(T(r,d)) + O(\log r)}{T\left(r,e^{(\lambda_1-\lambda_2)z}\right)} \cdot \frac{T\left(r,e^{(\lambda_1-\lambda_2)z}\right)}{T(r)} \to 0,$$

and

$$\frac{N\left(r,\frac{1}{f_3}\right)}{T(r)} \le \frac{T(r,d)+S(r,d)+O(\log r)}{T\left(r,e^{(\lambda_1-\lambda_2)z}\right)} \cdot \frac{T\left(r,e^{(\lambda_1-\lambda_2)z}\right)}{T(r)} \to 0,$$

as $r \to \infty$.

Therefore, by applying Lemma 2.2, we can deduce that $f_2 \equiv 1$ or $f_3 \equiv 1$.

If $f_2 \equiv 1$, then

$$d^{n-1}(d+\omega(d'+dh'))e^{nh-\lambda_2 z} = p_2. \tag{4.18}$$

We assert that $h' = \lambda_2/n$. Otherwise, since $\sigma(d') = \sigma(d) < 1 = \deg(nh - \lambda_2 z)$, by applying Lemma 2.1 to (4.18), we get $p_2 \equiv 0$, which is a contradiction. Thus $h' = \lambda_2/n$. We set $h = \lambda_2 z/n + B$, where B is a constant. Substituting this into (4.18), we have

$$d^{n-1}\left(d + \omega\left(d' + \frac{\lambda_2 d}{n}\right)\right) = p_2 e^{-nB}. \tag{4.19}$$

Next, we deduce that d is a constant. Otherwise, if d is a non-constant entire function, then it follows from (4.19) that 0 is a Picard exceptional value of d. Thus by Lemma 2.8, we have $d = e^{\alpha}$, where α is a non-constant polynomial, which contradicts with $\sigma(d) < 1$. So d is a non-zero constant, and (4.19) can be written as

$$d^n e^{nB}\left(1 + \omega\frac{\lambda_2}{n}\right) = p_2.$$

Therefore,

$$f = de^h = de^B e^{\lambda_2 z/n} = \left(\frac{p_2 n}{n + \omega\lambda_2}\right)^{\frac{1}{n}} e^{\frac{\lambda_2 z}{n}}.$$

Moreover, by $f_2 \equiv 1$ and (4.17), we also have $f_1 + f_3 \equiv 0$. That is

$$p_1 e^{\lambda_1 z} = qd_c e^{Q+h_c},$$

which implies that

$$Q = \left(\lambda_1 - \frac{\lambda_2}{n}\right)z + b_1,$$

where b_1 satisfies $p_1 = q\left(\frac{p_2 n}{n+\omega\lambda_2}\right)^{\frac{1}{n}} e^{\frac{\lambda_2 c}{n}+b_1}$.

If $f_3 \equiv 1$, then by (4.17) we have $f_1 + f_2 = 0$. It follows that

$$d^{n-1}(d + \omega(d' + dh'))e^{nh-\lambda_1 z} = p_1.$$

By using a similar method as in the case $f_2 \equiv 1$, we get

$$f(z) = \left(\frac{p_1 n}{n + \omega\lambda_1}\right)^{\frac{1}{n}} e^{\frac{\lambda_1 z}{n}}.$$

Furthermore, it follows from $f_3 \equiv 1$ that $qd_c e^{Q+h_c-\lambda_2 z} \equiv p_2$. Then we can deduce

$$Q = \left(\lambda_2 - \frac{\lambda_1}{n}\right)z + b_2,$$

where b_2 satisfies $p_2 = q\left(\frac{p_1 n}{n+\omega\lambda_1}\right)^{\frac{1}{n}} e^{\frac{\lambda_1 c}{n}+b_2}$. From the above discussion, the proof of the Conclusion (2) is complete.

5. Proof of Theorem 1.5.

Suppose that f is a finite-order transcendental entire solution to Eq (1.9).

If $\sigma(f) < 1$, then by a similar method as in Theorem 1.4 (Case 1), we can get a contradiction.

If $\sigma(f) > 1$, we denote $P = p_1 e^{\lambda_1 z} + p_2 e^{\lambda_2 z}$. Then Eq (1.9) can be written as

$$f^3 + \omega f^2 f' + (qf_c)e^Q = P. \tag{5.1}$$

Differentiating (5.1) yields

$$3f^2f' + \omega 2f(f')^2 + \omega f^2 f'' + Le^Q = P', \tag{5.2}$$

where $L = (qf_c)' + Q'(qf_c)$.

Eliminating e^Q from (5.1) and (5.2), we obtain

$$fH = PL - P'(qf_c), \tag{5.3}$$

where

$$H = Lf^2 + (\omega L - nqf_c)ff' - (n-1)\omega qf_c(f')^2 - \omega qf_c ff''.$$

If $H \equiv 0$, then from (5.3) we have $PL - P'(qf_c) \equiv 0$. By a similar reasoning as in Theorem 1.4 (Subcase 2.1.2), we get a contradiction. Therefore, $H \not\equiv 0$. Noting that H is entire and $PL - P'(qf_c)$, H/f are differential polynomials with meromorphic coefficients, similarly as in Theorem 1.4 (Subcase 2.1.1), by applying Lemma 2.6 to (5.3), we obtain

$$T(r, H) = m(r, H) = S(r, f) + O(r),$$

and

$$m(r, H/f) = S(r, f) + O(r).$$

Obviously, the poles of H/f arise from the poles of $(f')^2/f$. Suppose that z_0 is a zero of f with multiplicity p, then it is a simple pole of $(f')^2/f$ when $p = 1$, and a zero of $(f')^2/f$ with multiplicity $p-2$ when $p \geq 2$. Noting that f is entire, we obtain

$$\begin{aligned} T(r, H/f) &= m(r, H/f) + N(r, H/f) = m(r, H/f) + N_{1)}(r, 1/f) \\ &< (\kappa + o(1))T(r, f) + S(r, f) + O(r). \end{aligned}$$

Therefore, by $H \not\equiv 0$, it follows that

$$\begin{aligned} T(r, f) &= T(r, 1/f) + O(1) \leq T\left(r, \frac{H}{f}\right) + T\left(r, \frac{1}{H}\right) + O(1) \\ &< (\kappa + o(1))T(r, f) + S(r, f) + O(r), \quad 0 \leq \kappa < 1. \end{aligned}$$

Thus we have

$$T(r, f) = S(r, f) + O(r),$$

which contradicts with $\sigma(f) > 1$.

If $\sigma(f) = 1$, then by a similar method as in Theorem 1.4(Case 3), we can get that $\sigma(f) = \deg Q = 1$.

6. Conclusions

By using the Nevanlinna theory and its difference analogues, we study the transcendental entire solutions for two types of nonlinear differential-difference equations, and obtain three main theorems, which are improvements and complements of some previous results. Meanwhile, some examples are given to illustrate the conclusions.

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Conflict of interest

The authors declare no conflict of interest.

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