Research article

On a new modeling strategy: The logarithmically-exponential class of distributions

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Abstract: In this paper, a promising modeling strategy for data fitting is derived from a new general class of univariate continuous distributions. This class is governed by an original logarithmically-exponential one-parameter transformation which has the ability to enhance some modeling capabilities of any parental distribution. In relation to the current literature, it appears to be a "limit case" of the well-established truncated generalized Fréchet generated class. In addition, it offers a natural alternative to the famous odd inverse exponential generated class. Some special distributions are presented, with particular interest in a novel heavy-tailed three-parameter distribution based on the Lomax distribution. Functional equivalences, modes analysis, stochastic ordering, functional expansions, moment measures, information measures and reliability measures are derived. From generic or real data, our modeling strategy is based on the new class combined with the maximum likelihood approach. We apply this strategy to the introduced modified Lomax model. The efficiency of the three parameter estimates is validated by a simulation study. Subsequently, two referenced real data sets are adjusted according to the rules of the art; the first one containing environmental data and the second one, financial data. In particular, we show that the proposed model is preferable to four concurrents also derived from the Lomax model, including the odd inverse exponential Lomax model.

1. Introduction

In many applied fields, such as biology, medicine, insurance, engineering, economics, physics and finance, the modeling of the observed phenomenon from experimental data is crucial. The more appropriate and precise the model, the more we are able to understand the phenomenon in great detail. One area of study is the development of flexible probability distributions, which are frequently based on these models. For this aim, the most standard method is based on a parametric transformation of existing distributions. Such a transformation aims to increase the adaptability of a reference distribution for broader perspectives in terms of data analysis. Modern computer technology has contributed to the applicability of these distributions, allowing the processing of complex functions, accessibility of efficient optimization techniques and achievement of a high degree of numerical precision. The most famous and useful classes of distributions are described in detail in the surveys of [19, 23, 38]. Post-2020 classes include the transmuted Muth generated class by [5], Box-Cox gamma generated class by [6], Topp-Leone odd Fréchet generated class by [7], exponentiated power generalized Weibull power series generated class by [8], truncated Cauchy power generated class by [9], exponentiated truncated inverse Weibull generated class by [10], transmuted odd Fréchet generated class by [13], Marshall-Olkin exponentiated generalized generated class by [41], exponentiated M generated class by [16], type II power Topp-Leone generated class by [17], modified T-X generated class by [12], modified odd Weibull generated class by [22], type II general inverse exponential generated class by [27], odd generalized gamma generated class by [32], and truncated generalized Fréchet generated (TGF-G) class by [42].

However, modern classes of distributions based on simple, quite original and motivated transformation of the reference distribution remain rare. In this article, such a candidate is proposed; we introduce the logarithmically-exponential generated (LE-G) class defined by the following cumulative distribution function (cdf):

$$F(x; \alpha, \omega) = \frac{1}{\ln(1 - e^{-\alpha})} \ln\left(1 - e^{-\alpha G(x;\omega)^{-1}}\right), \quad x \in \mathbb{R}, \tag{1.1}$$

where $G(x; \omega)$ denotes the cdf of a reference continuous distribution with ω as parameter vector. One can notice that $F(x; \alpha, \omega) = T_\alpha[G(x; \omega)]$, where $T_\alpha(y)$ denotes the logarithmically-exponential transformation: $T_\alpha(y) = \ln(1 - e^{-\alpha y^{-1}}) / \ln(1 - e^{-\alpha})$, with $y \in (0, 1)$. Three major facts behind the LE-G class are formulated below.

1. We insist on the simplicity and originality of the logarithmically-exponential definition of $F(x; \alpha, \omega)$, with only one transformation-parameter, that has never been considered before to the best of our knowledge.
2. The LE-G class can be viewed as a natural "limit class" of the TGF-G class introduced by [42]; the cdf of TGF-G class tends to Equation (1.1) when β tends to 0.
3. When α tends to $+\infty$, the following equivalence result holds:

$$F(x; \alpha, \omega) \sim e^{-\alpha\left(G(x;\omega)^{-1} - 1\right)} = e^{-\alpha(1 - G(x;\omega))/G(x;\omega)},$$

which corresponds to the cdf defining the odd inverse exponential generated (OIE-G) class formerly introduced in [28] with $\alpha = 1$. See, also, [26] and [33], for further proceedings in this direction. A deeper relation between the LE-G and OIE-G classes will be revealed in this study.

The respective successes of the TGF-G and OIE-G classes in statistics motivate further investigations into the LE-G class, which is the aim of this study. The first part is devoted to the important functions of the class, and especially, the probability density function (pdf), hazard rate function (hrf) and quantile function (qf). Then, as concrete examples, special members of the LE-G class are exposed. In particular, we intend to show the importance of the new class by focusing on a new heavy right-tailed distribution with three parameters derived to the reference Lomax (Lom) distribution. This choice is motivated by the curvature flexibility of the corresponding pdf and hrf, demonstrating rarely observed shapes for a three-parameter heavy-tailed distribution. Especially, we see decreasing and unimodal shapes for the pdf, and increasing, decreasing and unimodal shapes for the hrf. The general properties of the LE-G class are examined, describing equivalences of the cdf, pdf and hrf at the boundaries, modes analysis, some stochastic ordering structure, functional expansions of the cdf and the exponentiated pdf, diverse moment measures, information measures related to the entropy concept and a crucial reliability measure. The properties of the modified Lom distribution are also specified, and supported by numerical and graphical illustrations. Then the practice of the LE-G class is developed for the purpose of data fitting. The parameter estimation of a special model is executed with the famous maximum likelihood method. In particular, by considering the proposed modified Lom model, a short Monte Carlo simulation is performed to assess the efficiency of the obtained estimates. Next, our modeling strategy is applied to two real heavy-tailed data sets. Well-referenced statistical measures are in favor of the proposed modified Lom model, surpassing four competitors also derived from the Lom model.

Section 2 refines the presentation of the LE-G class. Its theoretical properties are developed in Section 3. Our statistical approach is described in Section 4. Applications are given in Section 5. We end the paper with concluding remarks in Section 6.

2. The LE-G class

Here, we refine the presentation of the LE-G class with functions of interest and examples of specific distributions.

2.1. Important functions

Based on Eq (1.1), the pdf of the LE-G class is obtained upon differentiation (in the almost everywhere sense). It is defined by

$$f(x; \alpha, \omega) = -\frac{\alpha}{\ln(1 - e^{-\alpha})} \frac{g(x; \omega)}{G(x; \omega)^2} \frac{e^{-\alpha G(x;\omega)^{-1}}}{1 - e^{-\alpha G(x;\omega)^{-1}}}, \quad x \in \mathbb{R}, \tag{2.1}$$

where $g(x; \omega)$ denotes the pdf associated to $G(x; \omega)$. As usual, if we consider a random variable X with such a pdf, for any a and b with $a < b$, the distribution of X is characterized by $P(a \leq X \leq b) = \int_a^b f(t; \alpha, \omega)dt$. The cdf of X at the point x is rediscovered by taking $a = -\infty$ and $b = x$.

As a crucial reliability function, the corresponding hrf is derived as

$$h(x; \alpha, \omega) = \frac{f(x; \alpha, \omega)}{1 - F(x; \alpha, \omega)}$$

$$= -\frac{\alpha}{\ln(1 - e^{-\alpha}) - \ln\left(1 - e^{-\alpha G(x;\omega)^{-1}}\right)} \frac{g(x; \omega)}{G(x; \omega)^2} \frac{e^{-\alpha G(x;\omega)^{-1}}}{1 - e^{-\alpha G(x;\omega)^{-1}}}, \quad x \in \mathbb{R}. \tag{2.2}$$

The hrf can be increasing or decreasing, non-monotonic, or discontinuous in the broadest sense. In survival analysis, it is informative about the underlying mechanism of failure of a lifetime system. See, for instance, [1].

The qf of the LE-G class is defined as the inverse function of Eq (1.1). After elementary operations, we arrive at

$$Q(u; \alpha, \omega) = Q_G \left[\alpha \left\{ -\ln \left[1 - (1 - e^{-\alpha})^u \right] \right\}^{-1}, \omega \right], \quad u \in (0, 1), \tag{2.3}$$

where $Q_G(u; \omega)$ denotes the qf associated to $G(x; \omega)$, that is, $Q_G(x; \omega) = G^{-1}(u; \omega)$ with $u \in (0, 1)$. This function is fundamental in defining various parameters, measures and statistical models of the LE-G class. About this, all the information is available in [25].

The above three functions are fundamental for further statistical processing of the LE-G class, and their relative simplicity motivates more in this regard.

2.2. List of special distributions

In terms of heterogeneous and motley distributions, the LE-G class is rich. Based on useful referenced distributions, Table 1 lists some special distributions of the class which have various domains and numbers of parameters.

Table 1. Special distributions of the LE-G class based on some reference distributions.

LE-G	Reference	Domain	$G(x; \omega)$	(α, ω)	$F(x; \alpha, \omega)$
LES	Sine	$(0, 1)$	$\sin\left(\dfrac{\pi}{2} x\right)$	(α)	$\dfrac{1}{\ln(1 - e^{-\alpha})} \ln\left(1 - e^{-\alpha\{\sin[(\pi/2)x]\}^{-1}}\right)$
LEU	Uniform	$(0, b)$	$\dfrac{x}{b}$	(α, b)	$\dfrac{1}{\ln(1 - e^{-\alpha})} \ln\left(1 - e^{-\alpha b x^{-1}}\right)$
LETP	Topp-Leone	$(0, 1)$	$x^b(2 - x)^b$	(α, b)	$\dfrac{1}{\ln(1 - e^{-\alpha})} \ln\left(1 - e^{-\alpha x^{-b}(2-x)^{-b}}\right)$
LEK	Kumaraswamy	$(0, 1)$	$1 - (1 - x^a)^b$	(α, a, b)	$\dfrac{1}{\ln(1 - e^{-\alpha})} \ln\left(1 - e^{-\alpha[1-(1-x^a)^b]^{-1}}\right)$
LEE	Exponential	$(0, +\infty)$	$1 - e^{-\theta x}$	(α, θ)	$\dfrac{1}{\ln(1 - e^{-\alpha})} \ln\left(1 - e^{-\alpha(1-e^{-\theta x})^{-1}}\right)$
LEIE	Inverse exponential	$(0, +\infty)$	$e^{-\theta/x}$	(α, θ)	$\dfrac{1}{\ln(1 - e^{-\alpha})} \ln\left(1 - e^{-\alpha e^{\theta/x}}\right)$
LEL	Lindley	$(0, +\infty)$	$1 - \dfrac{\theta + 1 + \theta x}{\theta + 1} e^{-\theta x}$	(α, θ)	$\dfrac{1}{\ln(1 - e^{-\alpha})} \ln\left(1 - e^{-\alpha[1-(\theta+1+\theta x)e^{-\theta x}/(\theta+1)]^{-1}}\right)$
LEW	Weibull	$(0, +\infty)$	$1 - e^{-\theta x^\lambda}$	$(\alpha, \lambda, \theta)$	$\dfrac{1}{\ln(1 - e^{-\alpha})} \ln\left(1 - e^{-\alpha(1-e^{-\theta x^\lambda})^{-1}}\right)$
LELom	Lomax	$(0, +\infty)$	$1 - \left(1 + \dfrac{x}{\rho}\right)^{-\theta}$	(α, ρ, θ)	$\dfrac{1}{\ln(1 - e^{-\alpha})} \ln\left(1 - e^{-\alpha[1-(1+x/\rho)^{-\theta}]^{-1}}\right)$
LEGu	Gumbel	$\mathbb{R}$	$\exp(-e^{-bx})$	(α, b)	$\dfrac{1}{\ln(1 - e^{-\alpha})} \ln\left(1 - e^{-\alpha \exp(e^{-bx})}\right)$
LELog	Logistic	$\mathbb{R}$	$(1 + e^{-bx})^{-1}$	(α, b)	$\dfrac{1}{\ln(1 - e^{-\alpha})} \ln\left(1 - e^{-\alpha(1+e^{-bx})}\right)$
LEC	Cauchy	$\mathbb{R}$	$\dfrac{1}{\pi} \arctan(bx) + \dfrac{1}{2}$	(α, b)	$\dfrac{1}{\ln(1 - e^{-\alpha})} \ln\left(1 - e^{-\alpha[\arctan(bx)/\pi+1/2]^{-1}}\right)$

All of these special distributions have their own mathematical and practical characteristics; they can be studied independently for various purposes. The next part is devoted to the LELom distribution, defined with the Lom distribution as distribution of reference.

2.3. On the LELom distribution

In order to motivate the LELom distribution, a retrospective on the Lom distribution is necessary. First, the cdf and pdf of the Lom distribution are specified by

$$G(x; \rho, \theta) = 1 - \left(1 + \frac{x}{\rho}\right)^{-\theta}, \quad x > 0$$

and $G(x; \rho, \theta) = 0$ for $x \le 0$, and

$$g(x; \rho, \theta) = \frac{\theta}{\rho} \left(1 + \frac{x}{\rho}\right)^{-\theta-1}, \quad x > 0$$

and $g(x; \rho, \theta) = 0$ for $x \le 0$, respectively. Thus, the Lom distribution has a decreasing pdf which tends to 0 when x tends to $+\infty$ with a polynomial decay rate. Moreover, it is skewed to the right with a heavy tail. It is known to be one of the simplest distributions of this type in the literature. For this reason, the Lom distribution has served as a model in various contexts of applied sciences dealing with heavy-tailed data. More information on this topic can be found in [2–4, 14, 15, 20].

Despite undeniable qualities, the Lom distribution has some modeling limitations, including a pdf which is "desperately only decreasing", and a hrf which lacks flexibility; it can not be increasing or have N shapes, among others. In order to overpass these limitations, several modifications or extensions of the Lom distribution have been proposed, such as the power lomax (PLom) distribution by [35], odd inverse exponential Lomax (OInLom) distribution by [26] and exponential Lomax (ExLom) distribution by [24], to cite a few. The LELom distribution, briefly presented in Table 1, constitutes a new modification of the Lom distribution. As a preliminary statement, it may be more adapted to some data sets than the existing modifications. First, the Lelom distribution is defined with the cdf given as

$$F(x; \alpha, \rho, \theta) = \frac{1}{\ln(1 - e^{-\alpha})} \ln\left(1 - e^{-\alpha[1 - (1+x/\rho)^{-\theta}]^{-1}}\right), \quad x > 0,$$

and $F(x; \alpha, \rho, \theta) < 0$ for $x \le 0$. From Equation (2.1), standard operations give the following pdf:

$$f(x; \alpha, \rho, \theta) = -\frac{\alpha\theta}{\rho \ln(1 - e^{-\alpha})} \frac{(1 + x/\rho)^{-\theta-1}}{\left[1 - (1 + x/\rho)^{-\theta}\right]^2} \frac{e^{-\alpha[1-(1+x/\rho)^{-\theta}]^{-1}}}{1 - e^{-\alpha[1-(1+x/\rho)^{-\theta}]^{-1}}}, \quad x > 0,$$

and $f(x; \alpha, \rho, \theta) < 0$ for $x \le 0$. From Equation (2.2), the hrf of the LELom distribution is obtained as

$$h(x; \alpha, \rho, \theta) = -\frac{\alpha\theta}{\rho\left[\ln(1 - e^{-\alpha}) - \ln\left(1 - e^{-\alpha[1-(1+x/\rho)^{-\theta}]^{-1}}\right)\right]} \frac{(1 + x/\rho)^{-\theta-1}}{\left[1 - (1 + x/\rho)^{-\theta}\right]^2} \times$$
$$\frac{e^{-\alpha[1-(1+x/\rho)^{-\theta}]^{-1}}}{1 - e^{-\alpha[1-(1+x/\rho)^{-\theta}]^{-1}}}, \quad x > 0,$$

and $h(x; \alpha, \rho, \theta) < 0$ for $x \le 0$. In order to communicate on the flexibility of the LELom distribution, various curvatures of $f(x; \alpha, \rho, \theta)$ and $h(x; \alpha, \rho, \theta)$ are shown in Figure 1.
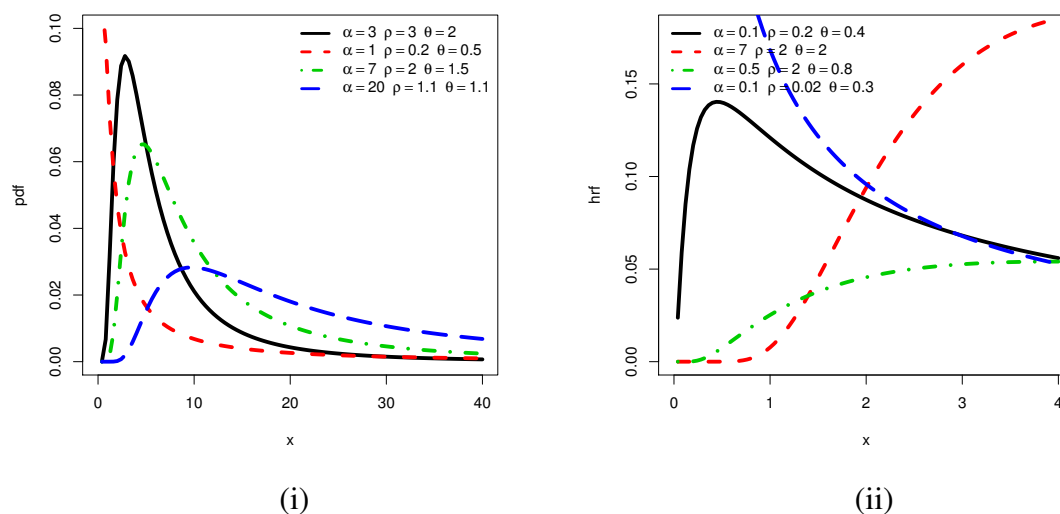
Figure 1. Panel of shapes of the (i) pdf and (ii) hrf of the LELom distribution for several values of the parameters.

In Figure 1 (i), we see that the pdf of the LELom distribution is unimodal and asymmetric on the right with a possibly heavy tail; the mathematical heavy-tailed nature of the LELom distribution will be discussed later. Also, a short left tail is sometimes observed. This detail is more important than it seems at first glance; it can be of interest for the modeling of certain lifetime phenomena presenting low values before a numerical peak. Two concrete examples of this kind will be the subject of all the attention in the applications part of this study (see Section 5). The corresponding hrf is revealed to be very flexible; in Figure 1 (ii), we see that it possesses increasing, decreasing and unimodal (upside-down) shapes, with more or less plateness. The modeling qualities of LELom distributions are thus proven.

We end this part by presenting the qf of the LELom distribution. Based on Equation (2.3), it is given as

$$Q(u; \alpha, \rho, \theta) = \rho \left\{ \left(1 - \alpha \left\{ -\ln\left[1 - (1 - e^{-\alpha})^u\right]\right\}^{-1}\right)^{-1/\theta} - 1 \right\}, \quad u \in (0, 1). \tag{2.4}$$

The analytical expression of this function allows a direct quantile analysis of the LELom distribution, including the expressions of the quantile density function, hazard quantile function, median and the two other quartiles, quantile asymmetry measures and quantile plateness measures. The importance of such a manageable quantile function in survival studies can be found in [25] and [31]. This is also a plus for the LELom distribution.

3. Some theoretical properties

This section is devoted to some theoretical properties of the LE-G class, with applications to the LELom distribution at all stages.

3.1. Functional equivalences

The functional equivalences of the cdf, pdf and hrf of the LE-G class are now discussed, with the cases $G(x; \omega) \to 0$ and $G(x; \omega) \to 1$ distinguished. First, when $G(x; \omega) \to 0$, a direct analysis gives

$$F(x; \alpha, \omega) \sim -\frac{1}{\ln(1 - e^{-\alpha})} e^{-\alpha G(x;\omega)^{-1}}, \quad f(x; \alpha, \omega) \sim -\frac{\alpha}{\ln(1 - e^{-\alpha})} \frac{g(x; \omega)}{G(x; \omega)^2} e^{-\alpha G(x;\omega)^{-1}}.$$

This last equivalence function also holds for $h(x; \alpha, \omega)$. We see that α affects the decay rate of the exponential term for the pdf and hrf; it plays an important role in this regard.

For the complementary case $G(x; \omega) \to 1$, by putting $S_G(x; \omega) = 1 - G(x; \omega)$ and using standard equivalence results, we get

$$F(x; \alpha, \omega) = 1 + \frac{\alpha}{(e^\alpha - 1) \ln(1 - e^{-\alpha})} S_G(x; \omega), \quad f(x; \alpha, \omega) \sim -\frac{\alpha}{(e^\alpha - 1) \ln(1 - e^{-\alpha})} g(x; \omega)$$

and $h(x; \alpha, \omega) \sim h(x; \omega)$, where $h(x; \omega) = g(x; \omega)/S_G(x; \omega)$ is the hrf associated to $G(x; \omega)$. Here, the role of α is minor; it is only involved in the proportional constants only. However, these constants can be consequent for a fixed x such that $G(x; \omega) \approx 1$.

As an example of application, if we focus on the LELom distribution, for $x \to 0$, we get

$$F(x; \alpha, \rho, \theta) \sim -\frac{1}{\ln(1 - e^{-\alpha})} e^{-\alpha[1-(1+x/\rho)^{-\theta}]^{-1}} \to 0$$

and

$$f(x; \alpha, \rho, \theta) \sim -\frac{\alpha\rho}{\theta \ln(1 - e^{-\alpha})} x^{-2} e^{-\alpha[1-(1+x/\rho)^{-\theta}]^{-1}} \to 0.$$

Thus, in all circumstances, $f(x; \alpha, \rho, \theta)$ tends to 0 when x tends to 0, and with an exponential decay rate. Note that, in this case, $\alpha[1 - (1 + x/\rho)^{-\theta} \sim \alpha\theta x/\rho$, meaning that all the parameters affect this convergence in the exponential sense. As already stated, the asymptotic behavior of $h(x; \alpha, \rho, \theta)$ corresponds to the one of $f(x; \alpha, \rho, \theta)$. On the other hand, when $x \to +\infty$, by invoking the standard equivalence results, we get

$$F(x; \alpha, \rho, \theta) = 1 + \frac{\alpha\rho^\theta}{(e^\alpha - 1) \ln(1 - e^{-\alpha})} x^{-\theta} \to 1,$$

and

$$f(x; \alpha, \rho, \theta) \sim -\frac{\alpha\theta\rho^\theta}{\rho(e^\alpha - 1) \ln(1 - e^{-\alpha})} x^{-\theta-1} \to 0.$$

Thus, in all cases, $f(x; \alpha, \rho, \theta)$ tends to 0 with a polynomial decay rate when x tends to $+\infty$, more or less rapidly, depending on θ. Also, by this equivalence and the integral Riemann criteria, we have $\int_0^{+\infty} e^{tx} f(x; \alpha, \rho, \theta) dx = +\infty$ for any $t > 0$. Therefore, the LELom distribution is a true heavy-tailed distribution in the mathematical sense. Concerning the hrf, when $x \to +\infty$, we have $h(x; \alpha, \rho, \theta) \sim \theta x^{-1} \to 0$. Thus, only the parameter θ affects the equivalence function as a proportional constant.

3.2. Mode analysis

The mode analysis of a distribution consists in identifying the mode(s) or maximal point(s) of the corresponding pdf. For the LE-G class, a maximal point for $f(x; \alpha, \omega)$ can be determined via the the study of $df(x; \alpha, \omega)/dx$, or more simply, the following derivative:

$$\frac{d}{dx}\ln[f(x;\alpha,\omega)] = \frac{dg(x;\omega)/dx}{g(x;\omega)} - 2\frac{g(x;\omega)}{G(x;\omega)} + \alpha\frac{g(x;\omega)}{G(x;\omega)^2}\frac{e^{\alpha G(x;\omega)^{-1}}}{e^{\alpha G(x;\omega)^{-1}}-1}.$$

Thus, a maximal point of $f(x; \alpha, \omega)$ satisfy the following non-linear equation: $d\ln[f(x;\alpha,\omega)]/dx = 0$, corresponding to

$$\frac{dg(x;\omega)/dx}{g(x;\omega)} + \alpha\frac{g(x;\omega)}{G(x;\omega)^2}\frac{e^{\alpha G(x;\omega)^{-1}}}{e^{\alpha G(x;\omega)^{-1}}-1} = 2\frac{g(x;\omega)}{G(x;\omega)}.$$

If only one maximum is obtained, the LE-G is unimodal, if more, it is multimodal, depending on the definition of the reference distribution. If the above equations are too complicated from the analytical point of view, numerical or graphical approaches are more convenient for the mode analysis.

Concerning the LELom distribution, this aspect has already been commented; it is mainly unimodal and this can be observed in Figure 1, and the mode satisfies the following non-linear equation:

$$-(\theta+1)(1+x/\rho)^{-1} + \alpha\theta\frac{(1+x/\rho)^{-\theta-1}}{\left[1-(1+x/\rho)^{-\theta}\right]^2}\frac{e^{\alpha[1-(1+x/\rho)^{-\theta}]^{-1}}}{e^{\alpha[1-(1+x/\rho)^{-\theta}]^{-1}}-1} = 2\theta\frac{(1+x/\rho)^{-\theta-1}}{1-(1+x/\rho)^{-\theta}}.$$

Obviously, only mathematical software has the ability to evaluate it numerically. Mathematica, Python or R can be used in this regard.

3.3. Stochastic ordering

The following proposition demonstrates an important first order stochastic dominance result between the LE-G and OIE-G classes.

Proposition 1. *The LE-G class first order stochastically dominates the OIE-G class. That is, we have*

$$F(x; \alpha, \omega) \le F^*(x; \alpha, \omega),$$

where $F^*(x; \alpha, \omega) = e^{-\alpha(G(x;\omega)^{-1}-1)}$.

Proof. First, let us set $q(x) = \ln(1-x)/x$ for $x \in (0, 1)$. Then, its derivative with respect to x is $dq(x)/dx = -[\ln(1-x) + x/(1-x)]/x^2$, which is negative since $\ln(1-y) \ge -y/(1-y)$ for $y < 1$, the equality being acheived for $y = 0$ only. Therefore, $q(x)$ is a decreasing function. Thus, for any $(u, v) \in (0, 1)^2$ with $u \le v$, we have $q(v) \le q(u)$, and, since $\ln(1-v) < 0$, it is equivalent to

$$\frac{\ln(1-u)}{\ln(1-v)} \le \frac{u}{v}.$$

By applying this inequality with $u = e^{-\alpha G(x;\omega)^{-1}}$ and $v = e^{-\alpha}$ satisfying $u \le v$ with $(u, v) \in (0, 1)^2$, we get

$$F(x;\alpha,\omega) = \frac{\ln\left(1-e^{-\alpha G(x;\omega)^{-1}}\right)}{\ln(1-e^{-\alpha})} \le \frac{e^{-\alpha G(x;\omega)^{-1}}}{e^{-\alpha}} = F^*(x;\alpha,\omega).$$

This ends the proof of Proposition 1. ☐

Proposition 1 proves that the OIE-G class is not only a "limit case" of the LE-G class; a well-identified stochastic dominance exists. Roughly speaking, the LE-G models can achieve certain objectives that the OIE-G models can not, and vice versa. Also, it motivates the LE-G models as alternatives of the OIE-G models.

In addition, the LE-G class satisfies the following monotone likelihood ratio property.

Proposition 2. *For $\alpha_1 \le \alpha_2$, the following ratio function*

$$\kappa(x; \alpha_1, \alpha_2, \omega) = \frac{f(x; \alpha_1, \omega)}{f(x; \alpha_2, \omega)}, \quad x \in \mathbb{R},$$

is decreasing with respect to x.

Proof. First, after some simplifications, we get

$$\kappa(x; \alpha_1, \alpha_2, \omega) = \frac{\alpha_1 \ln(1 - e^{-\alpha_2})}{\alpha_2 \ln(1 - e^{-\alpha_1})} \frac{e^{\alpha_2 G(x;\omega)^{-1}} - 1}{e^{\alpha_1 G(x;\omega)^{-1}} - 1}.$$

Therefore

$$\frac{d}{dx}\kappa(x; \alpha_1, \alpha_2, \omega) =$$

$$\frac{\alpha_1 \ln(1 - e^{-\alpha_2})}{\alpha_2 \ln(1 - e^{-\alpha_1})} \frac{g(x;\omega)}{G(x;\omega)^2} \frac{\alpha_1 e^{\alpha_1 G(x;\omega)^{-1}} \left[e^{\alpha_2 G(x;\omega)^{-1}} - 1 \right] - \alpha_2 e^{\alpha_2 G(x;\omega)^{-1}} \left[e^{\alpha_1 G(x;\omega)^{-1}} - 1 \right]}{\left(e^{\alpha_1 G(x;\omega)^{-1}} - 1 \right)^2}.$$

Now, let us study the positivity of the numerator term. Let us consider the following intermediary function: $q(\alpha) = \alpha e^{\alpha y^{-1}} / \left[e^{\alpha y^{-1}} - 1 \right]$ with $y = G(x; \omega)$. Then, we have

$$\frac{d}{d\alpha}q(\alpha) = e^{\alpha y^{-1}} \frac{y(e^{\alpha y^{-1}} - 1) - \alpha}{y(e^{\alpha y^{-1}} - 1)^2}.$$

By the general inequality $e^z \ge 1 + z$ for any $z \in \mathbb{R}$, we get $y(e^{\alpha y^{-1}} - 1) - \alpha \ge y\alpha y^{-1} - \alpha = 0$. Therefore, $dq(\alpha)/d\alpha \ge 0$, implying that $q(\alpha)$ is an increasing function with respect to α. Therefore, $\alpha_1 \le \alpha_2$ implies that $q(\alpha_1) \le q(\alpha_2)$, which is equivalent to

$$\alpha_1 e^{\alpha_1 G(x;\omega)^{-1}} \left[e^{\alpha_2 G(x;\omega)^{-1}} - 1 \right] - \alpha_2 e^{\alpha_2 G(x;\omega)^{-1}} \left[e^{\alpha_1 G(x;\omega)^{-1}} - 1 \right] \le 0.$$

It follows that $d\kappa(x; \alpha_1, \alpha_2, \omega)/dx \le 0$, and thus $\kappa(x; \alpha_1, \alpha_2, \omega)$ is a decreasing function with respect to x. This ends the proof of Proposition 2. ☐

The demonstrated stochastic ordering properties involve many other properties described in [37]. We leave the interested reader to this reference for more information in this direction.

3.4. Functional expansions

The following expansion investigates a linear relation between $F(x; \alpha, \omega)$ and basic functions of the reference distribution.

Proposition 3. *The following expansion holds:*

$$F(x; \alpha, \omega) = \sum_{k=1, \ell, m=0}^{+\infty} \beta_{k,\ell,m} S_G(x; \omega)^m,$$

where

$$\beta_{k,\ell,m} = \frac{1}{\ln(1 - e^{-\alpha})} \binom{-\ell}{m} (-1)^{\ell+m+1} \frac{1}{k} \frac{1}{\ell!} \alpha^\ell k^\ell, \tag{3.1}$$

$\binom{-\ell}{m}$ *refers to the general binomial coefficient at m and $-\ell$, and $S_G(x; \omega) = 1 - G(x; \omega)$ defining the survival function associated to $G(x; \omega)$.*

Proof. Based on the logarithmic and exponential series expansions, and the generalized binomial theorem, we have

$$F(x; \alpha, \omega) = -\frac{1}{\ln(1 - e^{-\alpha})} \sum_{k=1}^{+\infty} \frac{1}{k} e^{-\alpha k G(x; \omega)^{-1}} = -\frac{1}{\ln(1 - e^{-\alpha})} \sum_{k=1}^{+\infty} \frac{1}{k} \sum_{\ell=0}^{+\infty} \frac{1}{\ell!} (-\alpha k)^\ell G(x; \omega)^{-\ell}$$

$$= -\frac{1}{\ln(1 - e^{-\alpha})} \sum_{k=1}^{+\infty} \frac{1}{k} \sum_{\ell=0}^{+\infty} \frac{1}{\ell!} (-\alpha k)^\ell \sum_{m=0}^{+\infty} \binom{-\ell}{m} (-1)^m S_G(x; \omega)^m$$

$$= \sum_{k=1, \ell, m=0}^{+\infty} \beta_{k,\ell,m} S_G(x; \omega)^m.$$

Proposition 3 is proved. □

Proposition 3 applied to the LELom distribution gives the following simple expansion:

$$F(x; \alpha, \rho, \theta) = \sum_{k=1, \ell, m=0}^{+\infty} \beta_{k,\ell,m} \left(1 + \frac{x}{\rho}\right)^{-\theta m}, \quad x > 0.$$

The following expansion investigates a linear relation between exponentiated $f(x; \alpha, \omega)$ and functions of the reference distribution.

Proposition 4. *Let $\delta > 0$. Then, the following expansion holds:*

$$f(x; \alpha, \omega)^\delta = \sum_{k, \ell, m=0}^{+\infty} \tau_{k,\ell,m}^{[\delta]} v_m^{[\delta]}(x; \omega),$$

where

$$\tau_{k,\ell,m}^{[\delta]} = \frac{1}{[-\ln(1 - e^{-\alpha})]^\delta} \binom{-\delta}{k} \binom{-\ell - 2\delta}{m} \alpha^{\delta+\ell} \frac{1}{\ell!} (-1)^{k+\ell+m} (k + \delta)^\ell \tag{3.2}$$

and

$$v_m^{[\delta]}(x; \omega) = g(x; \omega)^\delta S_G(x; \omega)^m.$$

Proof. We have

$$f(x; \alpha, \omega)^\delta = \frac{\alpha^\delta}{[-\ln(1 - e^{-\alpha})]^\delta} \frac{g(x; \omega)^\delta}{G(x; \omega)^{2\delta}} \frac{e^{-\alpha\delta G(x;\omega)^{-1}}}{[1 - e^{-\alpha G(x;\omega)^{-1}}]^\delta}, \quad x \in \mathbb{R}.$$

The general binomial theorem is the key of the proof. By applying it three times in a row, it comes

$$
\begin{aligned}
f(x; \alpha, \omega)^\delta &= \frac{\alpha^\delta}{[-\ln(1 - e^{-\alpha})]^\delta} \frac{g(x; \omega)^\delta}{G(x; \omega)^{2\delta}} e^{-\alpha\delta G(x;\omega)^{-1}} \sum_{k=0}^{+\infty} \binom{-\delta}{k} (-1)^k e^{-\alpha k G(x;\omega)^{-1}} \\
&= \frac{\alpha^\delta}{[-\ln(1 - e^{-\alpha})]^\delta} \frac{g(x; \omega)^\delta}{G(x; \omega)^{2\delta}} \sum_{k=0}^{+\infty} \binom{-\delta}{k} (-1)^k e^{-\alpha(k+\delta) G(x;\omega)^{-1}} \\
&= \frac{\alpha^\delta}{[-\ln(1 - e^{-\alpha})]^\delta} g(x; \omega)^\delta \sum_{k=0}^{+\infty} \binom{-\delta}{k} (-1)^k \sum_{\ell=0}^{+\infty} \frac{1}{\ell!} [-\alpha(k+\delta)]^\ell G(x; \omega)^{-\ell-2\delta} \\
&= \frac{\alpha^\delta}{[-\ln(1 - e^{-\alpha})]^\delta} g(x; \omega)^\delta \sum_{k=0}^{+\infty} \binom{-\delta}{k} (-1)^k \sum_{\ell=0}^{+\infty} \frac{1}{\ell!} [-\alpha(k+\delta)]^\ell \sum_{m=0}^{+\infty} \binom{-\ell-2\delta}{m} (-1)^m S_G(x; \omega)^m \\
&= \sum_{k,\ell,m=0}^{+\infty} \tau_{k,\ell,m}^{[\delta]} v_m^{[\delta]}(x; \omega).
\end{aligned}
$$

Proposition 4 is proved. □

Notice that

(i) A direct application of Proposition 4 with $\delta = 1$ gives the following expansion for the pdf of the LE-G class:

$$f(x; \alpha, \omega) = \sum_{k,\ell,m=0}^{+\infty} \tau_{k,\ell,m}^{[1]} v_m^{[1]}(x; \omega),$$

where $v_m^{[1]}(x; \omega) = g(x; \omega) S_G(x; \omega)^m$ corresponds to the pdf of the minimum of $m + 1$ independent random variables with the reference distribution as common distribution when it is multiplied by the constant $m+1$. Thus, it is well identified for most of the standard distributions of the literature. In particular, in the context of the LELom distribution, we have

$$f(x; \alpha, \rho, \theta) = \sum_{k,\ell,m=0}^{+\infty} \tau_{k,\ell,m}^{[1]} v_m^{[1]}(x; \rho, \theta),$$

where

$$v_m^{[1]}(x; \rho, \theta) = \frac{\theta}{\rho} \left(1 + \frac{x}{\rho}\right)^{-\theta(m+1)-1}, \quad x > 0,$$

understanding that $v_m^{[1]}(x; \rho, \theta) = 0$ for $x \le 0$. One can notice that $v_m^{[1]}(x; \rho, \theta) = g(x; \rho, \theta(m + 1))/(m + 1)$, meaning that $f(x; \alpha, \rho, \theta)$ is expressed as an infinite linear combination of pdfs of the Lom distribution.

(ii) Proposition 3 applied to the LELom distribution gives the following simple expansion for $f(x; \alpha, \rho, \theta)^\delta$:

$$f(x; \alpha, \rho, \theta)^\delta = \sum_{k,\ell,m=0}^{+\infty} \tau_{k,\ell,m}^{[\delta]} v_m^{[\delta]}(x; \rho, \theta),$$

where

$$v_m^{[\delta]}(x; \rho, \theta) = \left(\frac{\theta}{\rho}\right)^\delta \left(1 + \frac{x}{\rho}\right)^{-\delta(\theta+1)-\theta m}, \quad x > 0,$$

understanding that $v_m^{[\delta]}(x; \rho, \theta) = 0$ for $x \leq 0$.

These expansion results are useful to obtain manageable analytical expressions of various measures of importance, such as different types of moment, entropy and reliability measures. Some of them will be the subject of further development in the next subsections.

3.5. Moment measures

Now, let us introduce a random variable X whose distribution belongs to the LE-G class and E be the standard expectation. Then, if the rth order moment about the origin of X converges in the mathematical sense, it is given by

$$\mu_r' = E(X^r) = \int_{-\infty}^{+\infty} x^r f(x; \alpha, \omega)dx = -\frac{\alpha}{\ln(1 - e^{-\alpha})} \int_{-\infty}^{+\infty} x^r \frac{g(x; \omega)}{G(x; \omega)^2} \frac{e^{-\alpha G(x;\omega)^{-1}}}{1 - e^{-\alpha G(x;\omega)^{-1}}} dx.$$

The chances of finding a simple expression for this integral are slim. For the evaluation of μ_r', two standard solutions can be studied: (i) a numerical approach by applying approximation integral procedure and (ii) an expansion of μ_r' involving manageable coefficients. This last approach requires the result in Proposition 4 applied with $\delta = 1$; subject to the mathematical validity on the interchanges of the sums and integral signs, we have

$$\mu_r' = \sum_{k,\ell,m=0}^{+\infty} \tau_{k,\ell,m}^{[1]} \xi_m^{[r]}, \tag{3.3}$$

where $\xi_m^{[r]} = \int_{-\infty}^{+\infty} x^r v_m^{[1]}(x; \omega)dx$. Since $v_m^{[1]}(x; \omega)$ is a well identified function for most of the standard distributions, $\xi_m^{[r]}$ is often easily calculable via standard integration techniques or can be extracted in published works. That is, μ_r' can be obtained by summing well-identified coefficients. A correct approximation of μ_r' follows by replacing the infinite bound by any large integer; in most practical situations, the integer 40 gives satisfaction.

Let us now study the moments about the origin of a random variable following the LELom distribution, starting by establishing their convergence. First, at the neighborhood of $x = 0$, we recall that the following function equivalence holds: $f(x; \alpha, \rho, \theta) \sim \{-\alpha\rho/(\theta\ln(1 - e^{-\alpha}))\}x^{-2}e^{-\alpha[1-(1+x/\rho)^{-\theta}]^{-1}}$ implying that $x^r f(x; \alpha, \rho, \theta) = o(x^{-1/2})$ for any $r \geq 0$ and, by the Riemann integrability criterion, the integral of $x^r f(x; \alpha, \rho, \theta)$ over $(0, \epsilon)$ with $\epsilon > 0$ converges. At the neighborhood of $x = +\infty$, we recall that $f(x; \alpha, \rho, \theta) \sim \{-\alpha\theta\rho^\theta/[\rho(e^\alpha - 1)\ln(1 - e^{-\alpha})]\}x^{-\theta-1}$ and, by the Riemann integrability criterion, the integral of $x^{r-\theta-1}$ over $(\epsilon, +\infty)$ with $\epsilon > 0$ converges if and only if $r < \theta$. Hence, the condition $r < \theta$ is

necessary to guarantee the convergence of μ_r'; the LELom distribution does not admit moments about the origin of all orders. Thus, under the condition $r < \theta$, numerical approximations of μ_r' have sense. Also, for an evaluation of μ_r', the complementary expansion approach described in Eq (3.3) gives

$$\mu_r' = \sum_{k,\ell,m=0}^{+\infty} \tau_{k,\ell,m}^{[1]} \xi_m^{[r]},$$

where, by introducing the beta function defined by $B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1} dt$ with $a > 0$ and $b > 0$,

$$\xi_m^{[r]} = \int_{-\infty}^{+\infty} x^r v_m^{[1]}(x; \rho, \theta) dx = \frac{\theta}{\rho} \int_0^{+\infty} x^r \left(1 + \frac{x}{\rho}\right)^{-\theta(m+1)-1} dx$$

$$= \theta \rho^r \int_0^{+\infty} \frac{u^{(r+1)-1}}{(1+u)^{(r+1)+\theta(m+1)-r}} du = \theta \rho^r B\left(r+1, \theta(m+1) - r\right).$$

It comes

$$\mu_r' = \sum_{k,\ell,m=0}^{+\infty} \tau_{k,\ell,m}^{[1]} \theta \rho^r B\left(r+1, \theta(m+1) - r\right).$$

As sketched previously, a precise approximation of μ_r' follows by replacing the infinite bound with 40. As an example of numerical work, some values for the mean and variance of X are collected in Table 2.

Table 2. Values of mean and variance of the LELom distribution for diverse values of the parameters.

$\theta\downarrow$	$\alpha = 2, \rho = 3$		$\theta\downarrow$	$\alpha = 0.5, \rho = 1.5$	
	mean	variance		mean	variance
5	1.301	1.36	5	0.368	0.197
5.5	1.152	0.991	5.5	0.328	0.146
6.5	0.938	0.589	6.5	0.269	0.09
7	0.858	0.473	7	0.247	0.073
7.5	0.79	0.388	7.5	0.228	0.061
8	0.733	0.324	8	0.212	0.051
8.5	0.683	0.274	8.5	0.198	0.044

In Table 2, we see some monotonic behavior of the mean and variance of X, being more or less small for the considered values.

In order to complete the study above, one can mention that the moment generating function of X is specified by

$$M(t) = E(e^{tX}) = \int_{-\infty}^{+\infty} e^{tx} f(x; \alpha, \omega) dx$$

$$= -\frac{\alpha}{\ln(1 - e^{-\alpha})} \int_{-\infty}^{+\infty} e^{tx} \frac{g(x; \omega)}{G(x; \omega)^2} \frac{e^{-\alpha G(x;\omega)^{-1}}}{1 - e^{-\alpha G(x;\omega)^{-1}}} dx,$$

with $t \in \mathbb{R}$ such that the integral converges. Subject to the logical validity of the interchanges of the sums and integral signs, the consequence of Proposition 4 with $\delta = 1$ is

$$M(t) = \sum_{k,\ell,m=0}^{+\infty} \tau_{k,\ell,m}^{[1]} \upsilon_m(t),$$

where $\upsilon_m(t) = \int_{-\infty}^{+\infty} e^{tx} v_m^{[1]}(x; \omega) dx$. This last term is already available in the litterature for a wide panel of parental distributions. If we deal with the parental Lom distribution as in the LELom distribution, the moment generating function exists at least for $t \leq 0$. It is however quite complex, but two analytical results involving special functions are provided in [29, Section 5].

Alternatively, if X admits moments of all orders, which is not the case for the LELom distribution, then we can use the following formula:

$$M(t) = \sum_{k=0}^{+\infty} \frac{t^k}{k!} \mu_k'.$$

The characteristic function can be dettermined through the following relationship: $\varphi(t) = M(it)$, where $i^2 = -1$.

In addition, the incomplete version of the rth moment about the origin of X can be treated in a similar way. From it, one can define various measures of deviations, and diverse types of residual and reversed residual lifetime functions. Further developments on this topic can be found in [23].

3.6. Information measures

In order to measure the variation or uncertainty of distributions belonging to the LE-G class, several information measures can be used. The most well-known are described in [11]. Here, we focus our attention on the Rényi and Tsallis entropy measures introduced by [36, 39], respectively. First, if the Rényi entropy measure of the LE-G class converges in the mathematical sense, it can be defined as

$$\mathcal{I}_R^{[p]} = \frac{1}{1-p} \ln \left[\int_{-\infty}^{+\infty} f(x; \alpha, \omega)^p dx \right]$$

$$= \frac{p}{1-p} \ln \alpha - \frac{p}{1-p} \ln \left[-\ln(1 - e^{-\alpha}) \right] + \frac{1}{1-p} \ln \left[\int_{-\infty}^{+\infty} \frac{g(x; \omega)^p}{G(x; \omega)^{2p}} \frac{e^{-\alpha p G(x;\omega)^{-1}}}{\left[1 - e^{-\alpha G(x;\omega)^{-1}} \right]^p} dx \right],$$

with $p > 0$ and $p \neq 1$. By arguing similarly as for moments about the origin, the integral term can be evaluated numerically by any integral approximation procedure. A more direct approach, which is "exact" in the mathematical sense, consists of giving a simple expansion of $\mathcal{I}_R^{[p]}$. Owing to Proposition 4, subject to the mathematical validity on the interchanges of the sums and integral signs, we have

$$\int_{-\infty}^{+\infty} f(x; \alpha, \omega)^p dx = \sum_{k,\ell,m=0}^{+\infty} \tau_{k,\ell,m}^{[p]} \phi_m^{[p]}, \tag{3.4}$$

where $\phi_m^{[p]} = \int_{-\infty}^{+\infty} v_m^{[p]}(x; \omega)dx$, which is quite manageable for most of the standard distributions.

In the setting of the LELom distribution, let us discuss the convergence of $\mathcal{I}_R^{[p]}$ deduced to the convergence of the integral term. First, at the neighborhood of $x = 0$, we have $f(x; \alpha, \rho, \theta)^p \sim \{-\alpha\rho/(\theta \ln(1 - e^{-\alpha})]\}^p x^{-2p} e^{-\alpha p[1-(1+x/\rho)^{-\theta}]^{-1}}$ implying that $f(x; \alpha, \rho, \theta)^p = o(x^{-1/2})$. By the Riemann integrability criterion, the integral of $f(x; \alpha, \rho, \theta)^p$ over $(0, \epsilon)$ with $\epsilon > 0$ converges. At the neighborhood of $x = +\infty$, we have $f(x; \alpha, \rho, \theta)^p \sim \{-\alpha\theta\rho^\theta/[\rho(e^\alpha - 1)\ln(1 - e^{-\alpha})]\}^p x^{-p(\theta+1)}$ and, by the Riemann integrability criterion, the integral of $x^{-p(\theta+1)}$ over $(\epsilon, +\infty)$ with $\epsilon > 0$ converges if and only if $p(\theta + 1) > 1$. Therefore, we must have $p(\theta + 1) > 1$ to ensure the convergence of $\mathcal{I}_R^{[p]}$. Under this condition, numerical approximations of $\mathcal{I}_R^{[p]}$ are possible. Also, owing to Equation (3.4), one can write

$$\int_{-\infty}^{+\infty} f(x; \alpha, \rho, \theta)^p dx = \sum_{k,\ell,m=0}^{+\infty} \tau_{k,\ell,m}^{[p]} \phi_m^{[p]},$$

where

$$\phi_m^{[p]} = \int_{-\infty}^{+\infty} v_m^{[p]}(x; \rho, \theta)dx = \left(\frac{\theta}{\rho}\right)^p \int_0^{+\infty} \left(1 + \frac{x}{\rho}\right)^{-p(\theta+1)-\theta m} dx = \theta^p \rho^{1-p} \frac{1}{p(\theta + 1) + \theta m - 1}.$$

Therefore, the following formula is valid:

$$\mathcal{I}_R^{[p]} = \frac{1}{1-p} \ln\left[\sum_{k,\ell,m=0}^{+\infty} \tau_{k,\ell,m}^{[p]} \theta^p \rho^{1-p} \frac{1}{p(\theta + 1) + \theta m - 1}\right],$$

as well as the following practical approximation:

$$\mathcal{I}_R^{[p]} \approx \frac{1}{1-p} \ln\left[\sum_{k,\ell,m=0}^{40} \tau_{k,\ell,m}^{[p]} \theta^p \rho^{1-p} \frac{1}{p(\theta + 1) + \theta m - 1}\right].$$

Similarly, the Tsallis entropy measure has an integral of the form $\int_{-\infty}^{+\infty} f(x; \alpha, \omega)^q dx$ as main term; it can be defined as

$$\mathcal{I}_T^{[q]} = \frac{1}{q-1}\left[1 - \int_{-\infty}^{+\infty} f(x; \alpha, \omega)^q dx\right],$$

where $q > 0$ and $q \neq 1$. A mathematical treatment similar to the one applied to $\mathcal{I}_R^{[p]}$ can be performed. In particular, for the LELom distribution, $\mathcal{I}_T^{[q]}$ converges if and only if $q(\theta + 1) > 1$. Moreover, the following result holds:

$$\mathcal{I}_T^{[q]} = \frac{1}{q-1}\left[1 - \sum_{k,\ell,m=0}^{+\infty} \tau_{k,\ell,m}^{[q]} \theta^q \rho^{1-q} \frac{1}{q(\theta + 1) + \theta m - 1}\right]$$

$$\approx \frac{1}{q-1}\left[1 - \sum_{k,\ell,m=0}^{40} \tau_{k,\ell,m}^{[q]} \theta^q \rho^{1-q} \frac{1}{q(\theta + 1) + \theta m - 1}\right].$$

As an example of numerical work, some values for the above entropy measures are presented in Table 3.

Table 3. Values of the Rényi and Tsallis entropy measures of the LELom distribution for diverse values of the parameters.

$\alpha = 0.5, \theta = 2.0, \rho = 1.5$				$\alpha = \rho = p = q = 0.5$		
$p\downarrow$	Rényi entropy	$q\downarrow$	Tsallis entropy	$\theta\downarrow$	Rényi entropy	Tsallis entropy
0.4	3.448	0.4	11.529	2	2.378	4.568
0.5	2.378	0.5	4.568	2.5	1.8	2.918
0.6	1.894	0.6	2.832	3	1.412	2.052
0.7	1.595	0.7	2.046	3.5	1.123	1.507
1.2	0.925	1.2	0.845	4	0.894	1.128
1.5	0.743	1.5	0.62	4.5	0.705	0.846

From Table 3, we see that the amount of uncertainty of the LELOm distribution is versatile, reaching small or large values. This aspect also reveals the statistical richness and complexity of the LELom distribution.

3.7. A crucial reliability measure

The stress-strength model concerns the service life of a component exposed to a certain stress and having strength (or resistance). The stress and the strength are modeled by two random variables, say Y and X, respectively. It is assumed that if the stress exceeds the strength, the component can not live. The reliability of such a random system is expressed by the following measure: $\mathcal{R} = P(Y < X)$. Here, we investigate the expression of $\mathcal{R}$ in the context of the LE-G family. More precisely, we suppose that X and Y are independent variables, and come from LE-G family in the following scenario; X is with the cdf $F(x; \alpha_1, \omega)$ and pdf $f(x; \alpha_1, \omega)$, and Y is with the cdf $F(x; \alpha_2, \omega)$. Thus, the parameter α can differ from one distribution to the other, but the reference distribution is the same. In this setting, we have

$$\mathcal{R} = \int_{-\infty}^{+\infty} F(x; \alpha_2, \omega) f(x; \alpha_1, \omega) dx$$

$$= -\frac{\alpha_1}{\ln(1 - e^{-\alpha_2}) \ln(1 - e^{-\alpha_1})} \int_{-\infty}^{+\infty} \ln\left(1 - e^{-\alpha_2 G(x; \omega)^{-1}}\right) \frac{g(x; \omega)}{G(x; \omega)^2} \frac{e^{-\alpha_1 G(x; \omega)^{-1}}}{1 - e^{-\alpha_1 G(x; \omega)^{-1}}} dx$$

$$= -\frac{\alpha_1}{\ln(1 - e^{-\alpha_2}) \ln(1 - e^{-\alpha_1})} \int_1^{+\infty} \ln\left(1 - e^{-\alpha_2 u}\right) \frac{e^{-\alpha_1 u}}{1 - e^{-\alpha_1 u}} du.$$

The obtained integral remains manageable; its numerical computation is quite possible. Alternatively, by virtue of Propositions 3 and 4, subject to the mathematical validity on the interchanges of the sums and integral signs, we obtain

$$\mathcal{R} = \sum_{k=1, \ell, m=0}^{+\infty} \sum_{k', \ell', m'=0}^{+\infty} \beta_{k, \ell, m}^{(\alpha_2)} \tau_{k', \ell', m'}^{[1]; (\alpha_1)} \int_{-\infty}^{+\infty} S_G(x; \omega)^m v_{m'}^{[1]}(x; \omega) dx,$$

where $\beta_{k,\ell,m}^{(\alpha_2)}$ is defined by Eq (3.1) with $\alpha = \alpha_2$ and $\tau_{k',\ell',m'}^{[1];(\alpha_1)}$ is defined by Eq (3.2) with $\delta = 1$ and $\alpha = \alpha_1$ and

$$\psi_{m,m'} = \int_{-\infty}^{+\infty} S_G(x;\omega)^m v_{m'}^{[1]}(x;\omega)dx = \int_{-\infty}^{+\infty} g(x;\omega)S_G(x;\omega)^{m+m'}dx$$

$$= \int_0^1 (1-u)^{m+m'}du = \frac{1}{m+m'+1}.$$

Therefore, we have

$$\mathcal{R} = \sum_{k=1,\ell,m=0}^{+\infty} \sum_{k',\ell',m'=0}^{+\infty} \beta_{k,\ell,m}^{(\alpha_2)} \tau_{k',\ell',m'}^{[1];(\alpha_1)} \frac{1}{m+m'+1} \approx \sum_{k=1,\ell,m=0}^{40} \sum_{k',\ell',m'=0}^{40} \beta_{k,\ell,m}^{(\alpha_2)} \tau_{k',\ell',m'}^{[1];(\alpha_1)} \frac{1}{m+m'+1}.$$

Thus, it can be approximated via standard computation techniques. Discussions and applications on the role of $\mathcal{R}$ in various settings can be found in [18].

4. Statistical approach

An important objective of the LE-G class is to produce flexible parametric models for concrete problems dealing with data. Here, we present a common strategy to construct accurate LE-G models for data fitting. Let $x_1, \ldots, x_n$ be n values supposed to be generated from a given LE-G distribution. Then, we can estimate the unknown parameters of the corresponding LE-G model by the maximum likelihood approach. Specifically, by denoting (α, ω) the vector of unknown parameters, its maximum likelihood estimate (MLE) is given as

$$(\hat{\alpha}, \hat{\omega}) = \text{argmax}_{(\alpha,\omega)} \mathcal{L}(x_1, \ldots, x_n; \alpha, \omega),$$

where $\mathcal{L}(x_1, \ldots, x_n; \alpha, \omega)$ denotes the likelihood function defined by

$$\mathcal{L}(x_1, \ldots, x_n; \alpha, \omega) = \prod_{i=1}^{n} f(x_i; \alpha, \omega) = \prod_{i=1}^{n} \left[-\frac{\alpha}{\ln(1-e^{-\alpha})} \frac{g(x_i;\omega)}{G(x_i;\omega)^2} \frac{e^{-\alpha G(x_i;\omega)^{-1}}}{1-e^{-\alpha G(x_i;\omega)^{-1}}} \right].$$

The components of $(\hat{\alpha}, \hat{\omega})$ are called MLEs of those of (α, ω), respectively. Subject to mathematical validity, one can find the MLEs through the maximization of the log-likelihood function defined by

$$\ell(x_1, \ldots, x_n; \alpha, \omega) = n \ln \alpha - n \ln[-\ln(1-e^{-\alpha})] + \sum_{i=1}^{n} \ln[g(x_i;\omega)] - 2 \sum_{i=1}^{n} \ln[G(x_i;\omega)]$$

$$- \sum_{i=1}^{n} \ln\left(e^{\alpha G(x_i;\omega)^{-1}} - 1\right).$$

The desired MLEs are solutions of the following system of equations:

$$\frac{\partial}{\partial \alpha} \ell(x_1, \ldots, x_n; \alpha, \omega) = \frac{n}{\alpha} - \frac{n}{(e^\alpha - 1)\ln(1-e^{-\alpha})} - \sum_{i=1}^{n} \frac{e^{\alpha G(x_i;\omega)^{-1}}}{G(x_i;\omega)\left[e^{\alpha G(x_i;\omega)^{-1}} - 1\right]} = 0$$

and, by denoting $\omega = (\omega_1, \ldots, \omega_p)$, for $j = 1, \ldots, p$,

$$\frac{\partial}{\omega_j}\ell(x_1, \ldots, x_n; \alpha, \omega) = \sum_{i=1}^{n} \frac{\partial g(x_i; \omega)/\partial\omega_j}{g(x_i; \omega)} - 2\sum_{i=1}^{n} \frac{\partial G(x_i; \omega)/\partial\omega_j}{G(x_i; \omega)}$$

$$+ \alpha \sum_{i=1}^{n} \frac{[\partial G(x_i; \omega)/\partial\omega_j]e^{\alpha G(x_i;\omega)^{-1}}}{G(x_i; \omega)^2 \left[e^{\alpha G(x_i;\omega)^{-1}} - 1\right]} = 0.$$

For the great majority of the referenced distributions, nice expressions of these solutions are impossible. Therefore, the use of a numerical procedure is necessary to obtain an acceptable approximation of $(\hat{\alpha}, \hat{\omega})$. For the sake of clarity, the existence and uniqueness of MLEs are not proven here, but it is of course an essential aspect of the method. The established theory of the maximum likelihood approach is in effect; the constructions of confidence intervals or likelihood ratio tests for the parameters of LE-G models are entirely possible. We may refer to [21] for the generalities.

The above theory holds for the LELom model, under the following configuration: $\omega = (\rho, \theta)$, for $x > 0$,

$$G(x; \rho, \theta) = 1 - \left(1 + \frac{x}{\rho}\right)^{-\theta}, \quad g(x; \rho, \theta) = \frac{\theta}{\rho}\left(1 + \frac{x}{\rho}\right)^{-\theta-1},$$

$$\frac{\partial}{\partial\rho}G(x; \rho, \theta) = -\frac{\theta}{\rho^2}x\left(1 + \frac{x}{\rho}\right)^{-\theta-1}, \quad \frac{\partial}{\partial\rho}g(x; \rho, \theta) = \frac{\theta}{\rho^3}\left(1 + \frac{x}{\rho}\right)^{-\theta-2}(\theta x - \rho),$$

and

$$\frac{\partial}{\partial\theta}G(x; \rho, \theta) = \left(1 + \frac{x}{\rho}\right)^{-\theta}\ln\left(1 + \frac{x}{\rho}\right), \quad \frac{\partial}{\partial\theta}g(x; \rho, \theta) = \left(1 + \frac{x}{\rho}\right)^{-\theta}\frac{1}{\rho + x}\left[1 - \theta\ln\left(1 + \frac{x}{\rho}\right)\right].$$

Then, precise numerical approximations of the MLEs of (α, ρ, θ) can be obtained. We illustrate this claim by conducting a Monte Carlo simulation study with respect to the sample size n. The R software is used (see [34]) and we proceed as follows:

1. We draw $N = 5000$ samples of size $n = 100, 200, 300$ and 1000 from the LELom distribution under the following sets of parameters: Set1 = ($\alpha = 0.5, \rho = 0.5, \theta = 0.5$), Set2 = ($\alpha = 1.5, \rho = 0.5, \theta = 0.8$), Set3 = ($\alpha = 1.2, \rho = 0.5, \theta = 0.5$) and Set4 = ($\alpha = 1.2, \rho = 0.5, \theta = 0.8$).
2. We calculate the MLEs of α, ρ and θ for each of the N samples, say $\hat{\alpha}_j$, $\hat{\rho}_j$ and $\hat{\theta}_j$, respectively, for $j = 1, \ldots, N$.
3. We compute the average MLEs (AMLEs) and mean-squared errors (MSEs) for $c = \alpha$, ρ and θ through the following formulas:

$$\text{AMLE} = \frac{1}{N}\sum_{j=1}^{N}\hat{c}_j, \quad \text{MSE} = \frac{1}{N}\sum_{j=1}^{N}(\hat{c}_j - c)^2.$$

4. The obtained numerical results are indicated in Table 4 for Sets 1 and 2, and Table 5 for Sets 3 and 4.

Table 4. AMLEs and MSEs of the LELom model parameters for Set1 and Set2.

n	Set1		Set2	
	AMLE	MSE	AMLE	MSE
	0.7670	0.2798	1.8867	1.0433
100	0.4495	0.1524	0.8046	0.7718
	0.5190	0.0059	0.8857	0.0148
	0.5958	0.1583	1.8090	0.8734
200	0.5419	0.0930	0.7320	0.4211
	0.5104	0.0045	0.8686	0.0083
	0.5489	0.1035	1.6867	0.4326
300	0.4724	0.0760	0.6046	0.1718
	0.4989	0.0019	0.8357	0.0048
	0.5317	0.0912	1.5399	0.3495
1000	0.5007	0.0547	0.5926	0.0806
	0.5127	0.0009	0.8236	0.0032

Table 5. AMLEs and MSEs of the LELom model parameters for Set3 and Set4.

n	Set3		Set4	
	AMLE	MSE	AMLE	MSE
	0.7140	0.2910	1.7910	1.5110
100	0.7964	0.2267	0.7286	0.2649
	0.4564	0.0054	0.8505	0.0344
	0.9254	0.2573	1.2968	0.5961
200	0.7204	0.1465	0.6183	0.1950
	0.4654	0.0021	0.8301	0.0102
	0.9573	0.2451	1.2183	0.5758
300	0.6250	0.1093	0.5931	0.1057
	0.4704	0.0017	0.7965	0.0038
	1.1425	0.1476	1.2010	0.1748
1000	0.5142	0.0310	0.5788	0.0893
	0.5197	0.0011	0.8091	0.0027

From Tables 4 and 5, the following comments can be formulated. As the value of n increases, the AMLEs tend to the corresponding true values of the parameters. The MSEs decrease to 0 as the

value of *n* increases. These two observations are consistent with the theoretical first-order properties of the MLEs. Therefore, the LELom model combined with the maximum likelihood estimation method seems to be a good solution for fitting data with a heavy tail, and a possible short left tail, a claim that will be confirmed in the next section.

5. Applications

Here, the accuracy of the fit of the LELom model is compared to that of some competing models such as the three-parameter power lomax (PLom) model by [35], two-parameter odd inverse exponential Lomax (OInLom) model by [26], three-parameter exponential Lomax (ExLom) model by [24], and standard two-parameter Lomax (Lom) model.

Based on a given data set, the following fit measures are taken into account: Cramér-von Mises (W), Anderson-Darling (A) and Kolmogorov-Smirnov (KS) statistics with the p-value of the related KS statistical test. In addition, the following criteria based on the maximum likelihood approach are considered: Akaike information criterion (AIC) and Bayesian information criterion (BIC), both defined from the estimated log-likelihood function denoted as $\hat{\ell}$. For such a data set, the model with the smallest values for W, A, KS, AIC, and BIC, and the largest p-value is considered the best model.

The results are obtained using the R software.

The first data set was originally reported by [40]. It represents the Maximum Annual Flood Discharges (MAFD) of the North Saskachevan River over a period of 47 years. It contains 47 observations, one per year, and the unit is 1000 cubic feet per second. The data set is given as {19.885, 20.940, 21.820, 23.700, 24.888, 25.460, 25.760, 26.720, 27.500, 28.100, 28.600, 30.200, 30.380, 31.500, 32.600, 32.680, 34.400, 35.347, 35.700, 38.100, 39.020, 39.200, 40.000, 40.400, 40.400, 42.250, 44.020, 44.730, 44.900, 46.300, 50.330, 51.442, 57.220, 58.700, 58.800, 61.200, 61.740, 65.440, 65.597, 66.000, 74.100, 75.800, 84.100, 106.600, 109.700, 121.970, 121.970, 185.560}.

The second data set includes the actual monthly tax revenue in Egypt from January 2006 to November 2010. It was originally provided by [30]. The data set is presented as {5.9, 20.4, 14.9, 16.2, 17.2, 7.8, 6.1, 9.2, 10.2, 9.6, 13.3, 8.5, 21.6, 18.5, 5.1, 6.7, 17.0, 8.6, 9.7, 39.2, 35.7, 15.7, 9.7, 10.0, 4.1, 36.0, 8.5, 8.0, 9.2, 26.2, 21.9, 16.7, 21.3, 35.4, 14.3, 8.5, 10.6, 19.1, 20.5, 7.1, 7.7, 18.1, 16.5, 11.9, 7.0, 8.6, 12.5, 10.3, 11.2, 6.1, 8.4, 11.0, 11.6, 11.9, 5.2, 6.8, 8.9, 7.1, 10.8}.

Let us begin with the analysis of the first data set. The MLEs related to the considered models are indicated in Table 6, along with their related standard errors (SEs).

Table 6. Values of the MLEs related to the models with the related SEs (in parentheses) for the first data set.

Models	MLEs and SEs (in parentheses)		
LELom	30.1240	24.4863	3.8502
(α, ρ, θ)	(0.0078)	(0.7501)	(0.9975)
PLom	1.5941	1.3786	248.2217
(α, β, λ)	(0.5966)	(0.1462)	(5.0557)
ExLom	1.7866	2.4702	0.0032
(α, β, λ)	(0.2061)	(1.1485)	(0.0009)
OInLom	7.8195	127.1995	8.4659
(α, ρ, θ)	(2.1153)	(5.1333)	(0.2254)
Lom	112.9830	2.7932	-
(ρ, θ)	(7.7611)	(0.8682)	-

In order to measure the accuracy of the fits of the models, Table 7 gives the values of $-\hat{\ell}$, AIC, BIC, W, A and KS (statistic and p-value) for the considered models.

Table 7. Values of $-\hat{\ell}$, AIC, BIC, W, A, KS and the KS p-value of the models for the first data set.

Models	$-\hat{\ell}$	AIC	BIC	W	A	KS	KS p-value
LELom	216.5707	439.1414	444.7550	0.0280	0.1990	0.0793	0.9234
PLom	235.2859	476.5718	482.1854	0.0960	0.6430	0.2979	0.0004
ExLom	226.5981	459.1963	464.8099	0.28960	1.80388	0.1552	0.1976
OInLom	216.6007	439.2014	444.8150	0.0537	0.3709	0.0882	0.8484
Lom	243.1155	490.2310	493.9734	0.1149	0.7643	0.3641	5.907×10^{-06}

Based on the results in Table 7, we see that the LELom model is the best with AIC = 439.1414, BIC = 444.7550, W = 0.0280, A = 0.1990, KS = 0.0793 and the p-value of the KS test given as p-value = 0.9234. We can notice the proximity of this last value to the optimal unit. The second best model is the OInLom model. We illustrate that in Figure 2 by plotting the estimated pdfs and cdfs of the models over the histogram and empirical cdf, respectively.
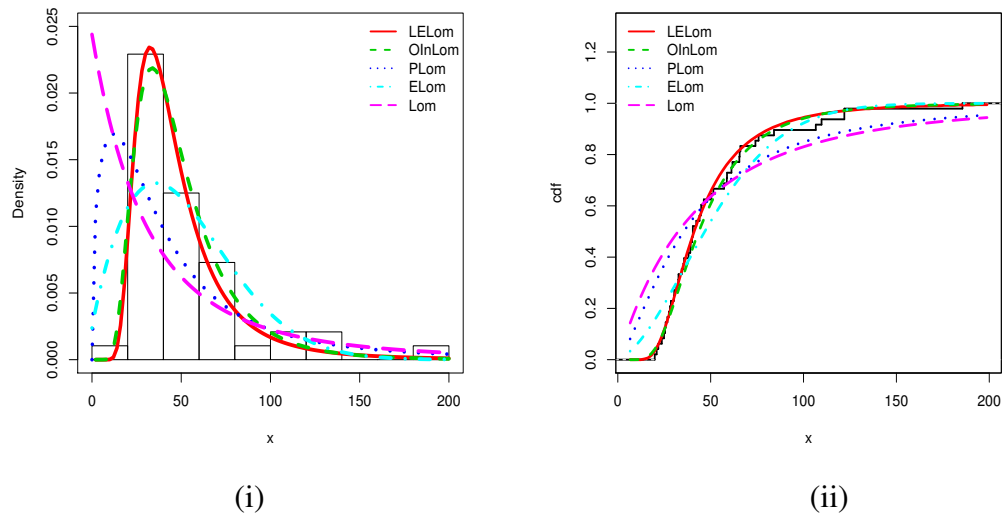
Figure 2. Plots of the (i) estimated pdfs of the models over the histogram and (ii) estimated cdfs of the models over the empirical cdf for the first data set.

Thanks to its small left tail and peakness properties, the estimated functions of the LELom model have well captured the characteristics of the data, contrary to the other models.

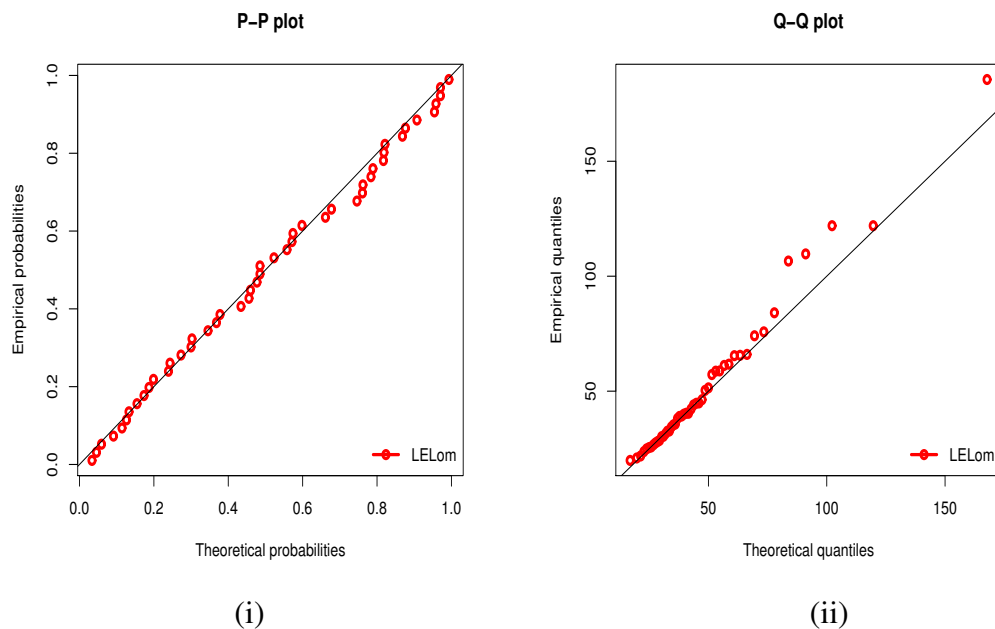Figure 3 completes the previous study by providing the probability-probability (P-P) and quantile-quantile (Q-Q) plots of the LELom model.

Figure 3. Graphics of the (i) P-P plot and (ii) Q-Q plot of the LELom model for the first data set.

This figure shows that the black lines in the P-P and Q-Q plots well adjust the corresponding scatter plots, demonstrating the good fit of the LELom model to the data.

We now use our methodology for the second data set. The MLEs related to the considered models are indicated in Table 8, along with their related SEs.

Table 8. Values of the MLEs related to the models with the related SEs (in parentheses) for the second data set.

Models	MLEs and SEs (in parentheses)		
LELom	27.6902	5.4759	3.4196
(α, ρ, θ)	(0.0243)	(1.8775)	(0.7343)
PLom	1.1901	2.2445	253.0403
(α, β, λ)	(0.4637)	(0.2773)	(7.6449)
ExLom	1.8875	1.4853	0.0103
(α, β, λ)	(0.2739)	(1.3272)	(0.0108)
OInLom	4.2666	142.7981	25.0416
(α, ρ, θ)	(0.6823)	(3.8372)	(1.8329)
Lom	143.6553	11.4152	-
(ρ, θ)	(9.6621)	(6.2237)	-

Table 9 provides the values of AIC, BIC, W, A and KS (statistic and p-value) for the considered models.

Table 9. Values of $-\hat{\ell}$, AIC, BIC, W, A, KS and the KS p-value of the models for the second data set.

Models	$-\hat{\ell}$	AIC	BIC	W	A	KS	KS p-value
LELom	192.3681	390.7361	396.9687	0.0433	0.2617	0.0823	0.8189
PLom	195.8858	397.6563	403.8889	0.1145	0.6521	0.1546	0.1192
ExLom	199.5225	405.0450	411.2776	0.3220	2.0531	0.1418	0.1858
OInLom	192.5344	391.0688	397.3014	0.0908	0.5380	0.0955	0.6538
Lom	214.3048	432.5045	436.6596	0.1764	1.0830	0.3175	0.0013

From Table 9, it is noted that the best model is the LELom model with AIC = 390.7361, BIC = 396.9687, W = 0.0433, A = 0.2617, KS = 0.0823 and p-value = 0.8189. The second best model is the OInLom model. However, it remains far from the performance of the LELom model on the basis of the benchmarks considered.

The main estimated functions of the models over the corresponding empirical objects are presented in Figure 4.
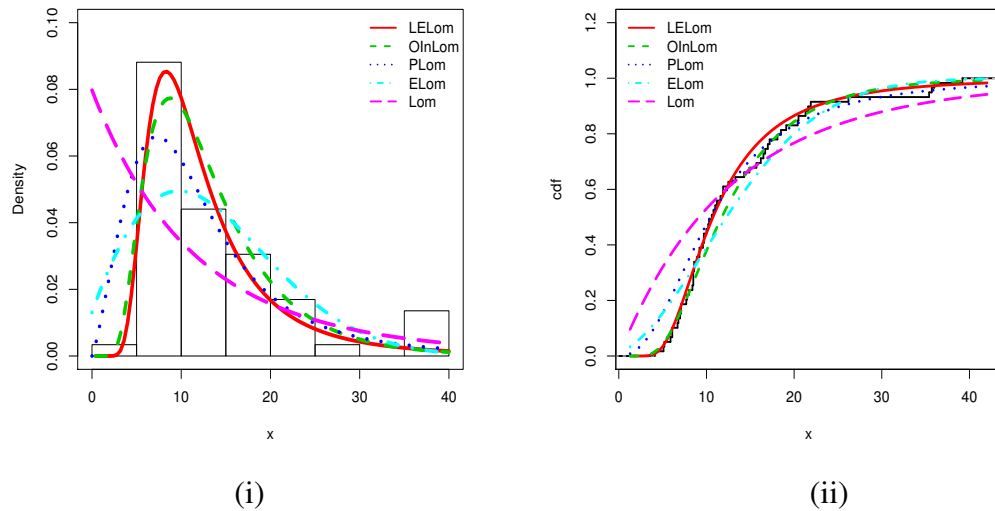

Figure 4. Plots of the (i) estimated pdfs of the models over the histogram and (ii) estimated cdfs of the models over the empirical cdf for the second data set.

We see that the small left tail and peak properties of the LELom model made the difference with the fits of the other models.
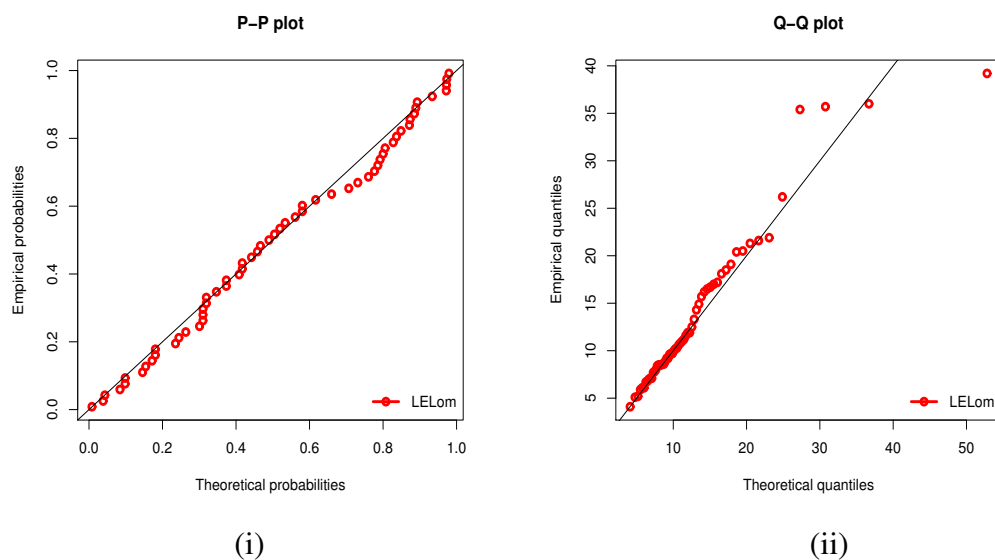
Figure 5 concludes the previous study by showing the P-P and Q-Q plots of the LELom model.

Figure 5. Graphics of the (i) P-P plot and (ii) Q-Q plot of the LELom model for the second data set.

The black lines in the P-P and Q-Q plots adjust the scatter plots nicely, demonstrating the good fit of the LELom model to the data.

In view of the above results, we can recommend the LELom model for the analysis of data similar to those considered.

6. Conclusions

In this paper, we have motivated a modeling strategy based on a class of distributions defined from an original and simple logarithmically-exponential transformation. A new flexible modification of the Lomax model with three parameters is derived, allowing the fit of data having a heavy right tail and an eventual short left tail. The proposed class is the subject of in-depth work on its probabilistic and statistical properties, exploring the functional equivalences, modes analysis, stochastic ordering, functional expansions, moment measures, information measures and reliability measures. The new Lomax model is then applied to two real-life data sets, one related to the environment and the other to finance. It is shown to provide a near perfect fit, outperforming some existing models also based on the reference Lomax model, and with the same number of parameters. It is understood that only a special distribution of the new class has been applied; others distributions with different features can be used quite efficiently for applications in medicine, economics, insurance, reliability, and life testing, among others.

Conflict of interest

The authors declare that the publication of this paper does not involve any conflicts of interest.

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