AIMS Mathematics, 6(7): 7798–7832.
DOI: 10.3934/math.2021454
Received: 16 February 2021
Accepted: 11 May 2021
Published: 17 May 2021

http://www.aimspress.com/journal/Math

Research article

Some average aggregation operators based on spherical fuzzy soft sets and their applications in multi-criteria decision making

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Abstract: Spherical fuzzy soft sets $(SFS_{ft}Ss)$ have its importance in a situation where human opinion is not only restricted to yes or no but some kind of abstinence or refusal aspects are also involved. Moreover, the notion of $SFS_{ft}S$ is free from all that complexities which suffers the contemporary theories because parameterization toll is a more important character of $SFS_{ft}S$. Also, note that aggregation operators are very effective apparatus to convert the overall information into a single value which further helps in decision-making problems. Due to these reasons, based on a spherical fuzzy soft set $(SFS_{ft}S)$, we have first introduced basic operational laws and then based on these introduced operational laws, some new notions like spherical fuzzy soft weighted average $(SFS_{ft}WA)$ aggregation operator, spherical fuzzy soft ordered weighted average $(SFS_{ft}OWA)$ aggregation operator and spherical fuzzy soft hybrid average $(SFS_{ft}HA)$ aggregation operators are introduced. Furthermore, the properties of these aggregation operators are discussed in detail. An algorithm is established in the environment of $SFS_{ft}S$ and a numerical example are given to show the authenticity of the introduced work. Moreover, a comparative study is established with other existing methods to show the validity and superiority of the established work.

Keywords: spherical fuzzy set; spherical fuzzy soft set; aggregation operators; multicriteria decision making
Mathematics Subject Classification: 03E72, 62C86

1. Introduction

Fuzzy set (FS) proposed by Zadeh [1] established the foundation in the field of fuzzy set theory and provided a track to address the difficulties of acquiring accurate data for multi-attribute decision-making problems. Fuzzy set only considers membership grade (MG) "$\mathfrak{a}$" which is bounded to $[0, 1]$ while in many real-life problems we have to deal with not only MG but also non-membership grade (NMG) "$\mathfrak{b}$", and due to this reason, the concept of FS has been extended to intuitionistic fuzzy set (IFS) established by Atanassov [2] that also compensate the drawback of FS. IFS has gained the attention of many researchers and they have used it for their desired results in practical examples for decision-making problems. IFS also enlarges the information space for decision-makers (DMs) because NMG is also involved with the condition that $0 \leq \mathfrak{a} + \mathfrak{b} \leq 1.$ Zhao et al. [3] established the generalized intuitionistic fuzzy aggregation operators. Moreover, some intuitionistic fuzzy weighted average, intuitionistic fuzzy ordered weighted average, and intuitionistic fuzzy hybrid average aggregation operators are introduced in [4]. Moreover, IF interaction aggregation operators and IF hybrid arithmetic and geometric aggregation operators are established in [5]. Later on, the MG and NMG are denoted by interval values and a new notion has been introduced called interval-valued IFS (IVIFS) [6] being a generalization of FS and IFS. The notions of IFS and IVIFS have been applied to many areas like group decision-making [7], similarity measure [8], and multicriteria decision-making problems [9]. Zhang et al. [10] introduce some information measures for interval-valued intuitionistic fuzzy sets. In many decision-making problems when decision-makers prove 0.6 as an MG and "0.5" as NMG then IFS fail to handle this kind of information, so to overcome this issue, the idea of IFS is further extended to the Pythagorean fuzzy set (P_yFS) established by Yager [11] with condition that $0 \leq \mathfrak{a}^2 + \mathfrak{b}^2 \leq 1.$ Therefore, P_yFS can express more information and IFS can be viewed as a particular case of P_yFS. Since aggregation operators are very helpful to change the overall data into single value which help us in decision-making problems for the selection of the best alternative among the given ones, so Khan et al. [12] introduced Pythagorean fuzzy Dombi aggregation operators and their application in decision support system. Moreover, P_yF interaction aggregation operators and their application to multiple-attribute decision-making have been proposed in [13]. In many circumstances, we have information that cannot be tackled by P_yFS like $sum(0.8^2, 0.9^2) \notin [0, 1]$, to compensate for this hurdle the idea of .q-rung orthopair fuzzy set (q-ROFS) is established by Yager [14] with condition that $0 \leq \mathfrak{a}^q + \mathfrak{b}^q \leq 1$ for $q \geq 1.$ It is clear that IFS and P_yFS are special cases of q-ROFS and it is more strong apparatus to deal with the fuzzy information more accurately. Wei et al. [15] introduced some q-rung orthopair fuzzy Heronian mean operators in multiple attribute decision making. Moreover, Liu and Wang [16] established the multiple-attribute decision-making based on Archimedean Bonferroni operators. Since all ideas given above can only consider only MG and NMG, while in many decision-making problems we need to consider the abstinence grade (AG) "$\mathfrak{c}$", hence to overcome this drawback, the idea of picture fuzzy set (PFS) has been introduced by Cuong [17]. Cuong et al. [18] introduce the primary fuzzy logic operators, conjunction, disjunction, negation, and implication based on PFS. Furthermore, some concepts and operational laws are proposed by Wang et al. [19], and also some picture fuzzy geometric aggregation operators and their properties have been discussed by them. Some PF aggregation operators are also discussed in [20,21]. Zeng et al. [22] introduced the extended version of the linguistic picture fuzzy Topsis method and its application in enterprise resource planning systems. In the picture fuzzy set, we have a condition that $0 \leq \mathfrak{a} + \mathfrak{b} + \mathfrak{c} \leq 1,$ but in many decision-making problems, the information given by experts cannot be handled by PFS and PFS fail to hold. For example, when experts provide "0.6" as MG, "0.5" as

NMG, and "0.3" as AG then we can see that $sum(0.6, 0.5, 0.3) \notin [0, 1]$. To overcome this difficulty, the idea of a spherical fuzzy set has been proposed by Mahmood et al. [23] with condition that $0 \leq \mathfrak{a}^2 + \mathfrak{b}^2 + \mathfrak{c}^2 \leq 1$. So, SFS is a more general form and provides more space to decision-makers for making their decision in many multi-attribute group decision-making problems. Jin et al. [24] discover the spherical fuzzy logarithmic aggregation operators based on entropy and their application in decision support systems. Furthermore, some weighted average, weighted geometric, and Harmonic mean aggregation operators based on the SF environment with their application in group decision-making problems have been discussed in [25,26]. Also, some spherical fuzzy Dombi aggregation operators are defined in [27]. Ashraf et al. [28] introduced the GRA method based on a spherical linguistic fuzzy Choquet integral environment. Ali et al. [29] introduced the TOPSIS method based on a complex spherical fuzzy set with Bonferroni mean operator.

Note that all the above existing literature has the characteristic that they can only deal with fuzzy information and cannot consider the parameterization structure. Due to this reason, Molodtsov [30] introduced the idea of a soft set (SS) which is more general than that of FS due to parameterization structure. Moreover, Maji et al. [31,32] use the SS in multi-criteria decision-making problems. The notions of a fuzzy soft set $(FS_{ft}S)$ has been introduced by Maji et al. [33] which is the combination of fuzzy set and soft set. Also, the applications of $FS_{ft}S$ theory to BCK/BC-algebra, in medical diagnosis, and decision-making problems have been established in [34–36]. Similarly as FS set is generalized into IFS, $FS_{ft}S$ is generalized into an intuitionistic fuzzy soft set $(IFS_{ft}S)$ [37] that is more strong apparatus to deal with fuzzy soft theory. Furthermore, Garg and Arora [38] introduced Bonferroni mean aggregation operators under an intuitionistic fuzzy soft set environment and proposed their applications to decision-making problems. Moreover, the concept of intuitionistic fuzzy parameterized soft set theory and its application in decision-making have been established in [39]. Since $IFS_{ft}S$ is a limited notion, so the idea of Pythagorean fuzzy soft set $P_yFS_{ft}S$ has been established by Peng et al. [40]. Furthermore q-rung orthopair fuzzy soft set generalizes the intuitionistic fuzzy soft set, as well as the Pythagorean fuzzy soft set and some q-rung orthopair fuzzy soft aggregation operators, are defined by Husain et al. [41]. Since $FS_{ft}S, IFS_{ft}S, P_yFS_{ft}S$ and $q-ROFS_{ft}S$ can only explore the MG and NMG but they cannot consider the AG, so to overcome this drawback, PFS and SS are combined to introduce a more general notion called picture fuzzy soft set $PFS_{ft}S$ given in [42]. Also, Jan et al. [43] introduced the multi-valued picture fuzzy soft sets and discussed their applications in group decision-making problems. Furthermore, SFS and soft set are combined to introduce a new notion called spherical fuzzy soft set $\left(SFS_{ft}S\right)$ discussed in [44] being the generalization of picture fuzzy soft set. Furthermore, the concepts of interval-valued neutrosophic fuzzy soft set and bipolar fuzzy neutrosophic fuzzy soft set along with their application in decision-making problems have been introduced in [45,46].

The motivation of the study is to use $SFS_{ft}S$ because (1) Most of the existing structure like $FS_{ft}S, IFS_{ft}S, P_yFS_{ft}S, q-ROFS_{ft}S$ and $PFS_{ft}S$ are the special cases of $SFS_{ft}S$. (2) Also, note that $SFS_{ft}S$ can cope with the information involving the human opinion based on MG, AG, NMG, and RG. Consider the example of voting where one can vote in favor of someone or vote against someone or abstain to vote or refuse to vote. $SFS_{ft}S$ can easily handle this situation, while the existing structures like $FS_{ft}S, IFS_{ft}S, P_yFS_{ft}S$ and $q-ROFS_{ft}S$ can note cope this situation due to lack of AG or RG. (3) The aim of using $SFS_{ft}S$ is to enhance the space of $PFS_{ft}S$ because $PFS_{ft}S$ has its limitation in assigning MG, AG, and NMG to the element of a set. (4) Also note that FS, IFS, P_yFS and q-ROFS are non-parameterized structure, while $SFS_{ft}S$ is a parameterized structure, so $SFS_{ft}S$ has more advantages over all these concepts. So in this article, based on $SFS_{ft}S$, we have introduced the idea of $SFS_{ft}WA, SFS_{ft}OWA$ and $SFS_{ft}HA$ aggregation operators. Moreover, their properties are discussed in detail.

Further, we organize our article as follows: Section 2 deal with basic notions of FS, $S_{ft}S$, PFS, $PFS_{ft}S$, SFS and $SFS_{ft}S$ and their operational laws. In section 3, we have introduced the basic operational laws for $SFS_{ft}Ns$. Section 4 deal with some new average aggregation operators called $SFS_{ft}WA, IVT-SFS_{ft}OWA$ and $SFS_{ft}HA$ operators. In section 5, we have established an algorithm and an illustrative example is given to show the validity of the established work. Finally, we have provided the comparative analysis of the proposed work to support the proposed work and show the superiority of established work by comparing it with existing literature.

2. Preliminaries

Definition 1. [1] Fuzzy set (FS) on a nonempty set $\mathbb{U}$ is given by

$$F = \{\langle \varkappa, \mathfrak{a}(\varkappa)\rangle : \varkappa \in \mathbb{U}\}$$

where $\mathfrak{a}: \mathbb{U} \to [0,1]$ denote the MG.

Definition 2. [30] For a fix universal set $\mathbb{U}, E$ a set of parameters and $T \subseteq E$, the pair (Q,T) is said to be soft set $(S_{ft}S)$ over $\mathbb{U}$, where Q is the map given by $Q: T \to P(\mathbb{U})$, where $P(\mathbb{U})$ is the power set of $\mathbb{U}$.

Definition 3. [33] Let $\mathbb{U}$ be a universal set, E be the set of parameters and $H \subseteq$ E. A pair (P,H) is said to be fuzzy soft set $(FS_{ft}S)$ over $\mathbb{U}$, where "P" is the map given by $P: H \to FS^{\mathbb{U}}$, which is defined by

$$P_{\rho_j}(\varkappa_i) = \{\langle \varkappa_i, \mathfrak{a}_j(\varkappa_i)\rangle : \varkappa_i \in \mathbb{U}\}$$

where $FS^{\mathbb{U}}$ is the family of all FSs on $\mathbb{U}$. Here $\mathfrak{a}_j(\varkappa_i)$ represents the MG satisfying the condition that $0 \leq \mathfrak{a}_j(\varkappa_i) \leq 1$.

Definition 4. [17] Let $\mathbb{U}$ be a universal set then a picture fuzzy set (PFS) over $\mathbb{U}$ is given by

$$P = \{\langle \varkappa, \mathfrak{a}(\varkappa), \mathfrak{b}(\varkappa), \mathfrak{c}(\varkappa)\rangle : \varkappa \in \mathbb{U}\}$$

where $\mathfrak{a}: \mathbb{U} \to [0,1]$ is the MG, $\mathfrak{b}: \mathbb{U} \to [0,1]$ is the AG and $\mathfrak{c}: \mathbb{U} \to [0,1]$ is NMG with the condition that $0 \leq \mathfrak{a}(\varkappa) + \mathfrak{b}(\varkappa) + \mathfrak{c}(x) \leq 1$.

Definition 5. [19–21] Let $P_1 = (\mathfrak{a}_1, \mathfrak{b}_1, \mathfrak{c}_1)$, $P_2 = (\mathfrak{a}_2, \mathfrak{b}_2, \mathfrak{c}_2)$ be two $PFNs$ and $\lambda > 0$. Then basic operations on $PFNs$ are defined by

1. $P_1 \cup P_2 = \{(max(\mathfrak{a}_1(\varkappa), \mathfrak{a}_2(\varkappa))), (min(\mathfrak{b}_1(\varkappa), \mathfrak{b}_2(\varkappa))), min(\mathfrak{c}_1(\varkappa), \mathfrak{c}_2(\varkappa))\}$.
2. $P_1 \cap P_2 = \left\{\left(min\left(\mathfrak{a}_1(\varkappa), \mathfrak{a}_2(\varkappa)\right)\right), \left(min\left(\mathfrak{b}_1(\varkappa), \mathfrak{b}_2(\varkappa)\right), \left(max\left(\mathfrak{c}_1(\varkappa), \mathfrak{c}_2(\varkappa)\right)\right)\right)\right\}$.
3. $P_1{}^c = (\mathfrak{c}_1(\varkappa), \mathfrak{b}_1(\varkappa), \mathfrak{a}_1(\varkappa))$.
4. $P_1 \oplus P_2 = \left(\left(\mathfrak{a}_1(\varkappa) + \mathfrak{a}_2(\varkappa) - (\mathfrak{a}_1(\varkappa))(\mathfrak{a}_2(\varkappa))\right), (\mathfrak{b}_1(\varkappa)\mathfrak{b}_2(\varkappa)), (\mathfrak{c}_1(\varkappa)\mathfrak{c}_2(\varkappa))\right)$.

5. $P_1 \otimes P_2 = \begin{pmatrix} (\mathfrak{a}_1(\varkappa)\mathfrak{a}_2(\varkappa)), \left(\mathfrak{b}_1(\varkappa) + \mathfrak{b}_2(\varkappa) - (\mathfrak{b}_1(\varkappa))(\mathfrak{b}_2(\varkappa))\right), \\ (\mathfrak{c}_1(\varkappa) + \mathfrak{c}_2(\varkappa) - (\mathfrak{c}_1(\varkappa))(\mathfrak{c}_2(\varkappa))) \end{pmatrix}$.

6. $P_1{}^{\lambda} = \left((\mathfrak{a}_1(\varkappa))^{\lambda}, (1 - (1 - \mathfrak{b}_1(\varkappa))^{\lambda}), (1 - (1 - \mathfrak{c}_1(\varkappa))^{\lambda})\right)$.

7. $\lambda P_1 = \left((1 - (1 - \mathfrak{a}_1(\varkappa))^{\lambda}), (\mathfrak{b}_1(\varkappa))^{\lambda}, ((\mathfrak{c}_1(\varkappa))^{\lambda})\right)$.

Definition 6. [42] Let $\mathbb{U}$ be a universal set, E be the set of parameters and $H \subseteq E$. A pair (P, H) is said to be picture fuzzy soft set $(PFS_{ft}S)$ over $\mathbb{U}$, where "P" is the map given by $P: H \to PFS^{\mathbb{U}}$, which is defined by

$$P_{\rho_j}(\varkappa_i) = \{\langle \varkappa_i, \mathfrak{a}_j(\varkappa_i), \mathfrak{b}_j(\varkappa_i), \mathfrak{c}_j(\varkappa_i)\rangle : \varkappa_i \in \mathbb{U}\}$$

where $PFS^{\mathbb{U}}$ is the family of all PFSs over $\mathbb{U}$. Here $\mathfrak{a}_j(\varkappa_i), \mathfrak{b}_j(\varkappa_i)$, and $\mathfrak{c}_j(\varkappa_i)$ represent the MG, AG, and NMG respectively satisfying the condition that $0 \leq \mathfrak{a}_j(\varkappa_i) + \mathfrak{b}_j(\varkappa_i) + \mathfrak{c}_j(\varkappa_i) \leq 1$.

Definition 7. [23] Let $\mathbb{U}$ be a universal set then a spherical fuzzy set (SFS) over $\mathbb{U}$ is given by

$$S = \{\langle \varkappa, \mathfrak{a}(\varkappa), \mathfrak{b}(\varkappa), \mathfrak{c}(\varkappa)\rangle : \varkappa \in \mathbb{U}\}$$

where $\mathfrak{a}: \mathbb{U} \to [0, 1]$ is the MG, $\mathfrak{b}: \mathbb{U} \to [0, 1]$ is the AG and $\mathfrak{c}: \mathbb{U} \to [0, 1]$ is NMG with the condition that $0 \leq \left(\mathfrak{a}(\varkappa)\right)^2 + \left(\mathfrak{b}(\varkappa)\right)^2 + (\mathfrak{c}(x))^2 \leq 1$.

Definition 8. [25] Let $S_1 = (\mathfrak{a}_1, \mathfrak{b}_1, \mathfrak{c}_1)$, $S_2 = (\mathfrak{a}_2, \mathfrak{b}_2, \mathfrak{c}_2)$ be two $SFNs$ and $\lambda > 0$. Then basic operations on $SFNs$ are defined by

1. $S_1 \cup S_2 = \left\{\left(max(\mathfrak{a}_1(\varkappa), \mathfrak{a}_2(\varkappa))\right), \left(min(\mathfrak{b}_1(\varkappa), \mathfrak{b}_2(\varkappa))\right), min(\mathfrak{c}_1(\varkappa), \mathfrak{c}_2(\varkappa))\right\}$.

2. $S_1 \cap S_2 = \left\{\left(min(\mathfrak{a}_1(\varkappa), \mathfrak{a}_2(\varkappa))\right), \left(min(\mathfrak{b}_1(\varkappa), \mathfrak{b}_2(\varkappa)), \left(max(\mathfrak{c}_1(\varkappa), \mathfrak{c}_2(\varkappa))\right)\right)\right\}$.

3. $S_1{}^c = (\mathfrak{c}_1(\varkappa), \mathfrak{b}_1(\varkappa), \mathfrak{a}_1(\varkappa))$.

4. $S_1 \oplus S_2 = \left(\left(\sqrt{(\mathfrak{a}_1(\varkappa))^2 + (\mathfrak{a}_2(\varkappa))^2 - (\mathfrak{a}_1(\varkappa))^2(\mathfrak{a}_2(\varkappa))^2}\right), (\mathfrak{b}_1(\varkappa)\mathfrak{b}_2(\varkappa)), (\mathfrak{c}_1(\varkappa)\mathfrak{c}_2(\varkappa))\right)$.

5. $S_1 \otimes S_2 = \left((\mathfrak{a}_1(\varkappa)\mathfrak{a}_2(\varkappa)), (\mathfrak{b}_1(\varkappa)\mathfrak{b}_2(\varkappa)), \left(\sqrt{(\mathfrak{c}_1(\varkappa))^2 + (\mathfrak{c}_2(\varkappa))^2 - (\mathfrak{c}_1(\varkappa))^2(\mathfrak{c}_2(\varkappa))^2}\right)\right)$.

6. $S_1{}^{\lambda} = \left(((\mathfrak{a}_1(\varkappa))^{\lambda}), (\mathfrak{b}_1(\varkappa))^{\lambda}, \left(\sqrt{1 - (1 - \mathfrak{c}_1(\varkappa)^2)^{\lambda}}\right)\right)$.

7. $\lambda S_1 = \left(\left(\sqrt{1 - (1 - \mathfrak{a}_1(\varkappa)^2)^{\lambda}}\right), (\mathfrak{b}_1(\varkappa))^{\lambda}, ((\mathfrak{c}_1(\varkappa))^{\lambda})\right)$.

3. Operational laws for spherical fuzzy soft numbers

In this section, we will define some basic operational laws for $SFS_{ft}Ns$, score function, accuracy function, and certainty function, which further helps in MCDM problems for the selection of the best alternative.

Definition 9. Let $S_{\rho_{ij}} = (\mathfrak{a}_{ij}, \mathfrak{b}_{ij}, \mathfrak{c}_{ij})$, $S'_{\rho_{ij}} = (\mathfrak{a}'_{ij}, \mathfrak{b}'_{ij}, \mathfrak{c}'_{ij})$ be two $SFS_{ft}Ns$ and $\lambda > 0$. Then basic operational laws for $SFS_{ft}Ns$ are defined by

1. $S_{\rho_{ij}} \subseteq S'_{\rho_{ij}}$ iff $\mathfrak{a}_{ij} \leq \mathfrak{a}'_{ij}$, $\mathfrak{b}_{ij} \leq \mathfrak{b}'_{ij}$ and $\mathfrak{c}_{ij} \geq \mathfrak{c}'_{ij}$.
2. $S_{\rho_{ij}} = S'_{\rho_{ij}}$ iff $S_{\rho_{ij}} \subseteq S'_{\rho_{ij}}$ and $S'_{\rho_{ij}} \subseteq S_{\rho_{ij}}$.
3. $S_{\rho_{ij}} \cup S'_{\rho_{ij}} = \langle max(\mathfrak{a}_{ij}, \mathfrak{a}'_{ij}), min(\mathfrak{b}_{ij}, \mathfrak{b}'_{ij}), min(\mathfrak{c}_{ij}, \mathfrak{c}'_{ij})\rangle$.
4. $S_{\rho_{ij}} \cap S'_{\rho_{ij}} = \langle\, min(\mathfrak{a}_{ij}, \mathfrak{a}'_{ij}), min(\mathfrak{b}_{ij}, \mathfrak{b}'_{ij}), max(\mathfrak{c}_{ij}, \mathfrak{c}'_{ij})\rangle$.
5. ${S_{\rho_{ij}}}^{c} = (\mathfrak{c}_{ij}, \mathfrak{b}_{ij}, \mathfrak{a}_{ij})$.
6. $S_{\rho_{ij}} \oplus S'_{\rho_{ij}} = \left(\sqrt{(\mathfrak{a}_{ij})^2 + (\mathfrak{a}'_{ij})^2 - (\mathfrak{a}_{ij})^2(\mathfrak{a}'_{12})^2}, \mathfrak{b}_{ij}\mathfrak{b}'_{ij}, \mathfrak{c}_{ij}\mathfrak{c}'_{ij}\right)$.
7. $S_{\rho_{ij}} \otimes S'_{\rho_{ij}} = \left(\mathfrak{a}_{ij}\mathfrak{a}'_{ij}, \mathfrak{b}_{ij}\mathfrak{b}'_{ij}, \sqrt{(\mathfrak{c}_{ij})^2 + (\mathfrak{c}'_{ij})^2 - (\mathfrak{c}_{ij})^2(\mathfrak{c}'_{ij})^2}\right)$.
8. $\lambda S_{\rho_{ij}} = (\sqrt{1 - \left(1 - {\mathfrak{a}_{ij}}^2\right)^k}, {\mathfrak{b}_{ij}}^{\lambda}, {\mathfrak{c}_{ij}}^{\lambda})$.
9. ${S_{\rho_{ij}}}^{\lambda} = ({\mathfrak{a}_{ij}}^{\lambda}, {\mathfrak{b}_{ij}}^{\lambda}, \sqrt{1 - \left(1 - {\mathfrak{c}_{ij}}^2\right)^{\lambda}})$.

Example 1. Let $S_{\rho_{11}} = (0.3, 0.5, 0.6)$, $S_{\rho_{12}} = (0.4, 0.7, 0.3)$ and $S = (0.2, 0.6, 0.5)$ be three $SFS_{ft}Ns$ and $\lambda = 3$. Then

1. $S_{\rho_{11}} \cup S_{\rho_{12}} = \langle max(0.3, 0.4), min(0.5, 0.7), min(0.6, 0.3)\rangle = (0.4, 0.5, 0.3)$.
2. $S_{\rho_{11}} \cap S_{\rho_{12}} = \langle\, min(0.3, 0.4), min(0.5, 0.7), max(0.6, 0.3)\rangle = (0.3, 0.5, 0.6)$.
3. $S^c = (0.5, 0.6, 0.2)$.
4. $S_{\rho_{11}} \oplus S_{\rho_{12}} = \left(\sqrt{(0.3)^2 + (0.4)^2 - (0.3)^2(0.4)^2}, (0.5)(0.7), (0.6)(0.3)\right) =$
   $(0.49, 0.35, 0.18)$.
5. $S_{\rho_{11}} \otimes S_{\rho_{12}} = \left((0.3)(0.4), (0.5)(0.7), \sqrt{(0.6)^2 + (0.3)^2 - (0.6)^2(0.3)^2}\right)$
   $= (0.12, 0.35, 0.65)$.
6. $\lambda S = \left(\sqrt{1 - (1 - 0.2^2)^3}, (0.6)^3\,, (0.5)^3\right) = (0.3395, 0.216, 0.125)$.
7. $S^{\lambda} = \left((0.2)^3, (0.6)^3\,, \sqrt{1 - (1 - 0.5^2)^3}\right) = (0.008, 0.216, 0.7603)$.

Definition 10. For a $SFS_{ft}N$, $S_{\rho_{ij}} = (\mathfrak{a}_{ij}, \mathfrak{b}_{ij}, \mathfrak{c}_{ij})$, the score function (SF), accuracy function (AF), and certainty function (CF) are respectively defined by

$$Sc\left(S_{\rho_{ij}}\right) = \frac{(2 + \mathfrak{a}_{ij} - \mathfrak{b}_{ij} - \mathfrak{c}_{ij})}{3}$$

$$Ac\left(S_{\rho_{11}}\right) = \mathfrak{a}_{ij} - \mathfrak{c}_{ij}$$

and

$$CF\left(S_{\rho_{11}}\right) = \mathfrak{a}_{ij}.$$

Note that $SF\left(S_{\rho_{11}}\right) \in [-1, 1]$.

Example 2. For a $SFS_{ft}N$ $S_{\rho_{11}} = (0.5, 0.6, 0.3)$, score values, accuracy values, and certainty values are respectively calculated by

$$Sc(S_{\rho_{11}}) = \frac{(2 + 0.5 - 0.6 - 0.3)}{3} = 0.5333$$

$$Ac(S_{\rho_{11}}) = 0.5 - 0.3 = 0.2$$

and

$$CF(S_{\rho_{11}}) = 0.5.$$

Definition 11. Let $S_{\rho_{11}} = (\mathfrak{a}_{11}, \mathfrak{b}_{11}, \mathfrak{c}_{11})$ and $S_{\rho_{12}} = (\mathfrak{a}_{12}, \mathfrak{b}_{12}, \mathfrak{c}_{12})$ be two $SFS_{ft}Ns$. Then

1. If $Sc(S_{\rho_{11}}) > Sc(S_{\rho_{12}})$, then $S_{\rho_{11}} \geq S_{\rho_{12}}$.
2. If $Sc(S_{\rho_{11}}) < Sc(S_{\rho_{12}})$, then $S_{\rho_{11}} \leq S_{\rho_{12}}$.
3. If $Sc(S_{\rho_{11}}) = Sc(S_{\rho_{12}})$, then
   1) If $r_{S_{\rho_{11}}} > r_{S_{\rho_{12}}}$, then $S_{\rho_{11}} > S_{\rho_{12}}$.
   2) If $r_{S_{\rho_{11}}} = r_{S_{\rho_{12}}}$, then $S_{\rho_{11}} = S_{\rho_{12}}$.

Theorem 1. Let $S_{\rho_{11}} = (\mathfrak{a}_{11}, \mathfrak{b}_{11}, \mathfrak{c}_{11})$ and $S_{\rho_{12}} = (\mathfrak{a}_{12}, \mathfrak{b}_{12}, \mathfrak{c}_{12})$ be two $SFS_{ft}Ns$ and $\lambda > 0$. Then the following properties hold.

1. $S_{\rho_{11}} \oplus S_{\rho_{12}} = S_{\rho_{21}} \oplus S_{\rho_{11}}$.
2. $S_{\rho_{11}} \otimes S_{\rho_{12}} = S_{\rho_{21}} \otimes S_{\rho_{11}}$.
3. $\lambda(S_{\rho_{11}} \oplus S_{\rho_{12}}) = (\lambda S_{\rho_{11}} \oplus \lambda S_{\rho_{12}})$.
4. $(\lambda_1 + \lambda_2)(S_{\rho_{11}}) = \lambda_1(S_{\rho_{11}}) + \lambda_2(S_{\rho_{11}})$.
5. $(S_{\rho_{11}})^{\lambda_1 + \lambda_2} = (S_{\rho_{11}})^{\lambda_1} \otimes (S_{\rho_{11}})^{\lambda_2}$.
6. $(S_{\rho_{11}})^{\lambda} \otimes (S_{\rho_{11}})^{\lambda} = (S_{\rho_{11}} \otimes S_{\rho_{11}})^{\lambda}$.

Proof. Proofs are straightforward and follow immediately from Definition 9.

4. Spherical fuzzy soft average aggregation operators

In this section, basic notions of $SFS_{ft}WA$, $SFS_{ft}OWA$ and $SFS_{ft}HA$ operators are elaborated and further their properties are discussed in detail.

4.1. Spherical fuzzy soft weighted averaging aggregation operators

Here, we present the detailed structure of $SFS_{ft}WA$ aggregation operator and discuss their properties in detail.

Definition 12. For the collection of $SFS_{ft}NsS_{\rho_{ij}} = (\mathfrak{a}_{ij}, \mathfrak{b}_{ij}, \mathfrak{c}_{ij})$, where $i = 1, 2, \dots, n$ and $j = 1, 2, \dots, m$, if $w = \{w_1, w_2, \dots, w_n\}$ denote the weight vector (WV) of e_i experts and $p = \{p_1, p_2, \dots, p_m\}$ denote the WV of parameters ρ_j with condition $w_i, p_j \in [0, 1]$ with $\sum_{i=1}^{n} w_i = 1$ and $\sum_{j=1}^{m} p_j = 1$, then $SFS_{ft}WA$ operator is the mapping defined as $SFS_{ft}WA: \mathcal{R}^n \to \mathcal{R}$, where ($\mathcal{R}$ is the family of all $SFS_{ft}Ns$)

$$SFS_{ft}WA(S_{\rho_{11}}, S_{\rho_{12}}, \dots, S_{\rho_{nm}}) = \oplus_{j=1}^{m} p_j \left(\oplus_{i=1}^{n} w_i S_{\rho_{ij}}\right).$$

Theorem 2. For a collection of $SFS_{ft}Ns$, where $S_{\rho_{ij}} = (\mathfrak{a}_{ij}, \mathfrak{b}_{ij}, \mathfrak{c}_{ij})$, then the aggregated result for $SFS_{ft}WA$ operator is again a $SFS_{ft}N$ and given by

$$SFS_{ft}WA(S_{\rho_{11}}, S_{\rho_{12}}, \dots, S_{\rho_{nm}}) = \oplus_{j=1}^{m} p_j \left(\oplus_{i=1}^{n} w_i S_{\rho_{ij}}\right)$$

$$= \left(\begin{array}{c} \sqrt{1 - \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-(\mathfrak{a}_{ij})^2\right)^{w_i}\right)^{p_j}}, \prod_{j=1}^{m}\left(\prod_{i=1}^{n}(\mathfrak{b}_{ij})^{w_i}\right)^{p_j}, \\ \prod_{j=1}^{m}\left(\prod_{i=1}^{n}(\mathfrak{c}_{ij})^{w_i}\right)^{p_j} \end{array}\right) \tag{1}$$

where $w = \{w_1, w_2, \dots, w_n\}$ denote the WV of "e_i" experts and $p = \{p_1, p_2, \dots, p_m\}$ represent the WV of parameters "ρ_j" with condition $w_i, p_j \in [0, 1]$ with $\sum_{i=1}^{n} w_i = 1$ and $\sum_{i=1}^{n} p_i = 1$.

Proof. Mathematical induction method is to be used to prove this result.

By operational laws, it is clear that

$$S_{\rho_{11}} \oplus S_{\rho_{12}} = \left(\sqrt{(\mathfrak{a}_{11})^2 + (\mathfrak{a}_{12})^2 - (\mathfrak{a}_{11})^2(\mathfrak{a}_{12})^2}, \mathfrak{b}_{11}\mathfrak{b}_{12}, \mathfrak{c}_{11}\mathfrak{c}_{12}\right)$$

and

$$\lambda S = \left(\sqrt{1 - (1-\mathfrak{a}^2)^k}, \mathfrak{b}^{\lambda}, \mathfrak{c}^{\lambda}\right) \quad \text{for } \lambda \geq 1.$$

We will show that Eq (1) is true for $n = 2$ and $m = 2$.

$$SFS_{ft}WA(S_{\rho_{11}}, S_{\rho_{12}}, \dots, S_{\rho_{nm}}) = \oplus_{j=1}^{2} p_j \left(\oplus_{i=1}^{2} w_i S_{\rho_{ij}}\right)$$

$$= p_1 \left(\oplus_{i=1}^{2} w_i S_{\rho_{ij}}\right) \oplus p_2 \left(\oplus_{i=1}^{2} w_i S_{\rho_{ij}}\right)$$

$$= p_1(w_1 S_{\rho_{11}} \oplus w_2 S_{\rho_{21}}) \oplus p_2(w_1 S_{\rho_{12}} \oplus w_2 S_{\rho_{22}})$$

$$= p_1 \left\{\left(\sqrt{1-(1-\mathfrak{a}_{11}{}^2)^{w_1}}, \mathfrak{b}_{11}{}^{w_1}, \mathfrak{c}_{11}{}^{w_1}\right) \oplus \left(\sqrt{1-(1-\mathfrak{a}_{21}{}^2)^{w_2}}, \mathfrak{b}_{21}{}^{w_2}, \mathfrak{c}_{21}{}^{w_2}\right)\right\}$$

$$\oplus\, p_2 \left\{\left(\sqrt{1-(1-\mathfrak{a}_{12}{}^2)^{w_1}}, \mathfrak{b}_{12}{}^{w_1}, \mathfrak{c}_{12}{}^{w_1}\right) \oplus \left(\sqrt{1-(1-\mathfrak{a}_{22}{}^2)^{w_2}}, \mathfrak{b}_{22}{}^{w_2}, \mathfrak{c}_{22}{}^{w_2}\right)\right\}$$

$$= p_1\left(\sqrt{1-\prod_{i=1}^{2}\left(1-\mathfrak{a}_{i1}{}^2\right)^{w_i}},\prod_{i=1}^{2}\mathfrak{b}_{i1}{}^{w_i},\prod_{i=1}^{2}\mathfrak{c}_{i1}{}^{w_i}\right)$$

$$\oplus\, \rho_2\left(\sqrt{1-\prod_{i=1}^{2}\left(1-\mathfrak{a}_{i2}{}^2\right)^{w_i}},\prod_{i=1}^{2}\mathfrak{b}_{i2}{}^{w_i},\prod_{i=1}^{2}\mathfrak{c}_{i2}{}^{w_i}\right)$$

$$=\left(\sqrt{1-\left(\prod_{i=1}^{2}\left(1-\mathfrak{a}_{i1}{}^2\right)^{w_i}\right)^{p_1}},\left(\prod_{i=1}^{2}\mathfrak{b}_{i1}{}^{w_i}\right)^{p_1},\left(\prod_{i=1}^{2}\mathfrak{c}_{i1}{}^{w_i}\right)^{p_1}\right)$$

$$\oplus\left(\sqrt{1-\left(\prod_{i=1}^{2}\left(1-\mathfrak{a}_{i2}{}^2\right)^{w_i}\right)^{p_2}},\left(\prod_{i=1}^{2}\mathfrak{b}_{i2}{}^{w_i}\right)^{p_2},\left(\prod_{i=1}^{2}\mathfrak{c}_{i2}{}^{w_i}\right)^{p_2}\right)$$

$$=\left(\sqrt{1-\prod_{j=1}^{2}\left(\prod_{i=1}^{2}\left(1-\mathfrak{a}_{ij}{}^2\right)^{w_i}\right)^{p_j}},\prod_{j=1}^{2}\left(\prod_{i=1}^{2}\mathfrak{b}_{ij}{}^{w_i}\right)^{p_j},\prod_{j=1}^{2}\left(\prod_{i=1}^{2}\mathfrak{c}_{ij}{}^{w_i}\right)^{p_j}\right).$$

Hence the result is true for $n=2$ and $m=2$.
Next, consider Eq (1) is true for $n=\mathbb{k}_1$ and $m=\mathbb{k}_2$

$$SFS_{ft}WA\left(S_{\rho_{11}},S_{\rho_{12}},\dots,S_{\rho_{\mathbb{k}_1\mathbb{k}_2}}\right)=\oplus_{j=1}^{\mathbb{k}_2}p_j\left(\oplus_{i=1}^{\mathbb{k}_1}w_iS_{\rho_{ij}}\right)$$

$$=\left(\begin{array}{c}\sqrt{1-\prod_{j=1}^{\mathbb{k}_2}\left(\prod_{i=1}^{\mathbb{k}_1}\left(1-\left(\mathfrak{a}_{ij}\right)^2\right)^{w_i}\right)^{p_j}},\prod_{j=1}^{\mathbb{k}_2}\left(\prod_{i=1}^{\mathbb{k}_1}\left(\mathfrak{b}_{ij}\right)^{w_i}\right)^{p_j},\\ \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\mathfrak{c}_{ij}\right)^{w_i}\right)^{p_j}\end{array}\right).$$

Further, suppose that Eq (1) is true for $n=\mathbb{k}_1+1$ and $m=\mathbb{k}_2+1$.

$$SFS_{ft}WA\left(S_{\rho_{11}},S_{\rho_{12}},\dots,S_{\rho_{(\mathbb{k}_1+1)(\mathbb{k}_2+1)}}\right)$$

$$=\left\{\oplus_{j=1}^{\mathbb{k}_2}p_j\left(\oplus_{i=1}^{\mathbb{k}_1}w_iS_{\rho_{ij}}\right)\right\}\oplus p_{(\mathbb{k}_1+1)}\left(w_{(\mathbb{k}_2+1)}S_{\rho_{(\mathbb{k}_1+1)(\mathbb{k}_2+1)}}\right)$$

$$=\left(\begin{array}{c}\sqrt{1-\prod_{j=1}^{\mathbb{k}_2}\left(\prod_{i=1}^{\mathbb{k}_1}\left(1-\left(\mathfrak{a}_{ij}\right)^2\right)^{w_i}\right)^{p_j}},\prod_{j=1}^{\mathbb{k}_2}\left(\prod_{i=1}^{\mathbb{k}_1}\left(\mathfrak{b}_{ij}\right)^{w_i}\right)^{p_j},\\ \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\mathfrak{c}_{ij}\right)^{w_i}\right)^{p_j}\end{array}\right)\oplus p_{\mathbb{k}_1+1}\left(w_{\mathbb{k}_2+1}S_{\rho_{(\mathbb{k}_1+1)(\mathbb{k}_2+1)}}\right)$$

$$=\left(\begin{array}{c}\sqrt{1-\prod_{j=1}^{\mathbb{k}_2+1}\left(\prod_{i=1}^{\mathbb{k}_1+1}\left(1-\left(\mathfrak{a}_{ij}\right)^2\right)^{w_i}\right)^{p_j}},\prod_{j=1}^{\mathbb{k}_2+1}\left(\prod_{i=1}^{\mathbb{k}_1+1}\left(\mathfrak{b}_{ij}\right)^{w_i}\right)^{p_j},\\ \prod_{j=1}^{\mathbb{k}_2+1}\left(\prod_{i=1}^{\mathbb{k}_1+1}\left(\mathfrak{c}_{ij}\right)^{w_i}\right)^{p_j}\end{array}\right).$$

From the above expression, it is clear that aggregated value is also $SFS_{ft}N$. Hence given Eq (1) is true for $n = \Bbbk_1 + 1$ and $m = \Bbbk_2 + 1$. Hence it is true for all $m, n \geq 1$.

Definition 13. [44] Let $\mathbb{U}$ be a universal set, E be a set of parameters and $H \subseteq E$. A pair (S, H) is said to be spherical fuzzy soft set $(SFS_{ft}S)$ over $\mathbb{U}$, where "S" is the map given by $S: H \to SFS^{\mathbb{U}}$, which is defined to be

$$S_{\rho_j}(x_i) = \{\langle x_i, \mathfrak{a}_j(x_i), \mathfrak{b}_j(x_i), \mathfrak{c}_j(x_i)\rangle : x_i \in \mathbb{U}\}$$

where $SFS^{\mathbb{U}}$ is the family of SFSs over $\mathbb{U}$. Here $\mathfrak{a}_j(x_i), \mathfrak{b}_j(x_i)$, and $\mathfrak{c}_j(x_i)$ represents the MG, AG, and NMG respectively satisfying the condition that $0 \leq \left(\mathfrak{a}_j(x_i)\right)^2 + \left(\mathfrak{b}_j(x_i)\right)^2 + \left(\mathfrak{c}_j(x_i)\right)^2 \leq 1$. For simplicity, the triplet $\{\langle \mathfrak{a}_j(x_i), \mathfrak{b}_j(x_i), \mathfrak{c}_j(x_i)\rangle\}$ is called spherical fuzzy soft number $(SFS_{ft}N)$. Also, refusal grade is defined by

$$r_{S_{\rho_{ij}}} = \sqrt{1 - \left(\left(\mathfrak{a}_j(x_i)\right)^2 + \left(\mathfrak{b}_j(x_i)\right)^2 + \left(\mathfrak{c}_j(x_i)\right)^2\right)} \ .$$

Example 3. From a set of five laptop brands as alternatives $A = \{x_1 = Dell, x_2 = Apple, x_3 = Lenovo, x_4 = Samsung, x_5 = Acer\}$, a person's desire to buy the best brand. Let $\rho = \{\rho_1 = USB\ type - C, \rho_2 = Higher\ resolution\ screen, \rho_3 = Reaonable\ price, \rho_4 = 8\ GB\ of\ RAM\ or\ more\}$ be the set of parameters. Let $w = \{0.25, 0.15, 0.14, 0.3, 0{,}16\}$ denote the WV of "e_i" experts and $p = \{0.27, 0.19, 0.29, 0.25\}$ denote the WV of "ρ_j" parameters. The experts provide their information in the form of $SFS_{ft}Ns$ as given in Table 1.

Table 1. Tabular representation of $SFS_{ft}Ns$.

	ρ_1	ρ_2	ρ_3	ρ_4
x_1	(0.2, 0.1, 0.6)	(0.5, 0.3, 0.1)	(0.4, 0.3, 0.2)	(0.6, 0.1, 0.2)
x_2	(0.1, 0.4, 0.4)	(0.6, 0.3, 0.1)	(0.2, 0.1, 0.7)	(0.5, 0.6, 0.1)
x_3	(0.3, 0.2, 0.2)	(0.6, 0.2, 0.1)	(0.3, 0.3, 0.4)	(0.5, 0.1, 0.3)
x_4	(0.3, 0.1, 0.6)	(0.1, 0.2, 0.6)	(0.2, 0.1, 0.2)	(0.2, 0.3, 0.4)
x_5	(0.7, 0.4, 0.2)	(0.5, 0.3, 0.7)	(0.2, 0.8, 0.3)	(0.7, 0.1, 0.2)

By using Eq (1), we have

$SFS_{ft}WA(S_{\rho_{11}}, S_{\rho_{12}}, \ldots, S_{\rho_{nm}})$

$$= \left(\begin{array}{c} \sqrt{1 - \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1 - (\mathfrak{a}_{ij})^2\right)^{w_i}\right)^{p_j}}, \prod_{j=1}^{m}\left(\prod_{i=1}^{n}(\mathfrak{b}_{ij})^{w_i}\right)^{p_j}, \\ \prod_{j=1}^{m}\left(\prod_{i=1}^{n}(\mathfrak{c}_{ij})^{w_i}\right)^{p_j} \end{array} \right)$$

$$= \left[\begin{array}{c} \left(\sqrt{\begin{array}{c} 1 - \{(1-0.2^2)^{0.25}(1-0.1^2)^{0.15}(1-0.3^2)^{0.14}(1-0.3^2)^{0.3}(1-0.7^2)^{0.16}\}^{0.27} \\ \{(1-0.5^2)^{0.25}(1-0.6^2)^{0.15}(1-0.6^2)^{0.14}(1-0.1^2)^{0.3}(1-0.5^2)^{0.16}\}^{0.19} \\ \{(1-0.4^2)^{0.25}(1-0.2^2)^{0.15}(1-0.3^2)^{0.14}(1-0.2^2)^{0.3}(1-0.2^2)^{0.16}\}^{0.29} \\ \{(1-0.6)^{0.25}(1-0.5^2)^{0.15}(1-0.5^2)^{0.14}(1-0.2^2)^{0.3}(1-0.7^2)^{0.16}\}^{0.25} \end{array}} \right), \\ \left(\begin{array}{c} \{(1-0.1^2)^{0.25}(1-0.4^2)^{0.15}(1-0.2^2)^{0.14}(1-0.1^2)^{0.3}(1-0.4^2)^{0.16}\}^{0.27} \\ \{(1-0.3^2)^{0.25}(1-0.3^2)^{0.15}(1-0.2^2)^{0.14}(1-0.2^2)^{0.3}(1-0.3^2)^{0.16}\}^{0.19} \\ \{(1-0.3^2)^{0.25}(1-0.1^2)^{0.15}(1-0.3^2)^{0.14}(1-0.1^2)^{0.3}(1-0.8^2)^{0.16}\}^{0.29} \\ \{(1-0.1^2)^{0.25}(1-0.6^2)^{0.15}(1-0.1^2)^{0.14}(1-0.3^2)^{0.3}(1-0.1^2)^{0.16}\}^{0.25} \end{array} \right), \\ \left(\begin{array}{c} \{(1-0.6^2)^{0.25}(1-0.4^2)^{0.15}(1-0.2^2)^{0.14}(1-0.6^2)^{0.3}(1-0.2^2)^{0.16}\}^{0.27} \\ \{(1-0.1^2)^{0.25}(1-0.1^2)^{0.15}(1-0.1^2)^{0.14}(1-0.6^2)^{0.3}(1-0.7^2)^{0.16}\}^{0.19} \\ \{(1-0.2^2)^{0.25}(1-0.7^2)^{0.15}(1-0.4^2)^{0.14}(1-0.2^2)^{0.3}(1-0.3^2)^{0.16}\}^{0.29} \\ \{(1-0.2^2)^{0.25}(1-0.1^2)^{0.15}(1-0.3^2)^{0.14}(1-0.4^2)^{0.3}(1-0.2^2)^{0.16}\}^{0.25} \end{array} \right) \end{array} \right]$$

$$= (0.461322, 0.174673, 0.250344).$$

Next, we propose the properties for $SFS_{ft}WA$ aggregation operator, which can be easily proved**.**

Theorem 3. Let $S_{\rho_{ij}} = (\mathfrak{a}_{ij}, \mathfrak{b}_{ij}, \mathfrak{c}_{ij})$ for $i = 1, 2, \dots, n$ and $j = 1, 2, \dots, m$, be the family of $SFS_{ft}Ns$, $w = (w_1, w_2, \dots, w_n)^T$ denote the WV of e_i experts and $p = (p_1, p_2, \dots, p_m)^T$ denote the WV of parameters ρ_j with condition $w_i, p_j \in [0, 1]$ with $\sum_{i=1}^{n} w_i = 1$ and $\sum_{j=1}^{m} p_j = 1$. Then $SFS_{ft}WA$ operator holds the following properties:

1. **(Idempotency):** Let $S_{\rho_{ij}} = (\mathfrak{a}_{ij}, \mathfrak{b}_{ij}, \mathfrak{c}_{ij}) = S'_\rho$ for all $i = 1, 2, \dots, n$ and $j = 1, 2, \dots, m$,

   where $S'_\rho = (\mathfrak{a}', \mathfrak{b}', \mathfrak{c}')$, then $SFS_{ft}WA(S_{\rho_{11}}, S_{\rho_{12}}, \dots, S_{\rho_{nm}}) = S'_\rho$.

Proof. If $S_{\rho_{ij}} = (\mathfrak{a}_{ij}, \mathfrak{b}_{ij}, \mathfrak{c}_{ij}) = S'_\rho$ for all $i = 1, 2, \dots, n$ and $j = 1, 2, \dots, m$, where $S'_\rho = (\mathfrak{a}', \mathfrak{b}', \mathfrak{c}')$, then from Theorem 1, we have

$$SFS_{ft}WA(S_{\rho_{11}}, S_{\rho_{12}}, \dots, S_{\rho_{nm}}) = \left(\begin{array}{c} \sqrt{1 - \prod_{j=1}^{m} \left(\prod_{i=1}^{n} \left(1 - (\mathfrak{a}_{ij})^2\right)^{w_i} \right)^{p_j}}, \prod_{j=1}^{m} \left(\prod_{i=1}^{n} (\mathfrak{b}_{ij})^{w_i} \right)^{p_j}, \\ \prod_{j=1}^{m} \left(\prod_{i=1}^{n} (\mathfrak{c}_{ij})^{w_i} \right)^{p_j} \end{array} \right)$$

$$= \left(\begin{array}{c} \sqrt{1 - \prod_{j=1}^{m} \left(\prod_{i=1}^{n} (1 - (\mathfrak{a}')^2)^{w_i} \right)^{p_j}}, \prod_{j=1}^{m} \left(\prod_{i=1}^{n} (\mathfrak{b}')^{w_i} \right)^{p_j}, \\ \prod_{j=1}^{m} \left(\prod_{i=1}^{n} (\mathfrak{c}')^{w_i} \right)^{p_j} \end{array} \right)$$

$= \left(\sqrt{1-(1-(\mathfrak{a}')^2)}, \mathfrak{b}', \mathfrak{c}'\right) = (\mathfrak{a}', \mathfrak{b}', \mathfrak{c}') = S'_{\rho}$.

Hence $SFS_{ft}WA\left(S_{\rho_{11}}, S_{\rho_{12}}, \dots, S_{\rho_{nm}}\right) = S'_{\rho}$.

2. (Boundedness): If $S^-_{\rho_{ij}} = \left(min_j min_i\{\mathfrak{a}_{ij}\}, max_j max_i\{\mathfrak{b}_{ij}\}, max_j max_i\{\mathfrak{c}_{ij}\}\right)$ and $S^+_{\rho_{ij}} = \left(max_j max_i\{\mathfrak{a}_{ij}\}, min_j min_i\{\mathfrak{b}_{ij}\}, min_j min_i\{\mathfrak{c}_{ij}\}\right)$, then $S^-_{\rho_{ij}} \le SFS_{ft}WA\left(S_{\rho_{11}}, S_{\rho_{12}}, \dots, S_{\rho_{nm}}\right) \le S^+_{\rho_{ij}}$.

Proof. As $S^-_{\rho_{ij}} = \left(min_j min_i\{\mathfrak{a}_{ij}\}, max_j max_i\{\mathfrak{b}_{ij}\}, max_j max_i\{\mathfrak{c}_{ij}\}\right)$ and $S^+_{\rho_{ij}} = \left(max_j max_i\{\mathfrak{a}_{ij}\}, min_j min_i\{\mathfrak{b}_{ij}\}, min_j min_i\{\mathfrak{c}_{ij}\}\right)$, then we have to prove that $S^-_{\rho_{ij}} \le SFS_{ft}WA\left(S_{\rho_{11}}, S_{\rho_{12}}, \dots, S_{\rho_{nm}}\right) \le S^+_{\rho_{ij}}$.

Now for each $i = 1, 2, \dots, n$ and $j = 1, 2, \dots, m$,

$$min_j min_i\{\mathfrak{a}_{ij}\} \le \{\mathfrak{a}_{ij}\} \le max_j max_i\{\mathfrak{a}_{ij}\} \Leftrightarrow 1 - max_j max_i\{\mathfrak{a}_{ij}{}^2\} \le 1 - \mathfrak{a}_{ij}{}^2 \le min_j min_i\{\mathfrak{a}_{ij}{}^2\}$$

$$\Leftrightarrow \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1 - max_j max_i\left(\mathfrak{a}_{ij}\right)^2\right)^{w_i}\right)^{p_j} \le \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1 - \left(\mathfrak{a}_{ij}\right)^2\right)^{w_i}\right)^{p_j}$$

$$\le \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1 - min_j min_i\left(\mathfrak{a}_{ij}\right)^2\right)^{w_i}\right)^{p_j} \Leftrightarrow \left(\left(1 - max_j max_i\left(\mathfrak{a}_{ij}\right)^2\right)^{\sum_{i=1}^{n} w_i}\right)^{\sum_{j=1}^{m} p_j}$$

$$\le \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1 - \left(\mathfrak{a}_{ij}\right)^2\right)^{w_i}\right)^{p_j} \le \left(\left(1 - min_j min_i\left(\mathfrak{a}_{ij}\right)^2\right)^{\sum_{i=1}^{n} w_i}\right)^{\sum_{j=1}^{m} p_j}$$

$$\Leftrightarrow 1 - max_j max_i\left(\mathfrak{a}_{ij}\right)^2 \le \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1 - \left(\mathfrak{a}_{ij}\right)^2\right)^{w_i}\right)^{p_j} \le 1 - min_j min_i\left(\mathfrak{a}_{ij}\right)^2$$

$$\Leftrightarrow 1 - \left(1 - min_j min_i\left(\mathfrak{a}_{ij}\right)^2\right) \le 1 - \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1 - \left(\mathfrak{a}_{ij}\right)^2\right)^{w_i}\right)^{p_j} \le 1 - \left(1 - max_j max_i\left(\mathfrak{a}_{ij}\right)^2\right).$$

Hence

$$min_j min_i\left(\mathfrak{a}_{ij}\right) \le \sqrt{1 - \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1 - \left(\mathfrak{a}_{ij}\right)^2\right)^{w_i}\right)^{p_j}} \le max_j max_i\left(\mathfrak{a}_{ij}\right). \tag{2}$$

Now for each $i = 1, 2, \dots, n$ and $j = 1, 2, \dots, m$, we have

$$min_j min_i\left(\mathfrak{b}_{ij}\right) \le max_j max_i\left(\mathfrak{b}_{ij}\right) \Leftrightarrow \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(min_j min_i\left(\mathfrak{b}_{ij}\right)\right)^{w_i}\right)^{p_j} \le \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\mathfrak{b}_{ij}\right)^{w_i}\right)^{p_j}$$

$$\leq \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(max_j max_i(\mathfrak{b}_{ij})\right)^{w_i}\right)^{p_j} \Leftrightarrow \left(\left(min_j min_i(\mathfrak{b}_{ij})\right)^{\sum_{i=1}^{n} w_i}\right)^{\sum_{j=1}^{m} p_j}$$

$$\leq \prod_{j=1}^{m}\left(\prod_{i=1}^{n}(\mathfrak{b}_{ij})^{w_i}\right)^{p_j} \leq \left(\left(max_j max_i(\mathfrak{b}_{ij})\right)^{\sum_{i=1}^{n} w_i}\right)^{\sum_{j=1}^{m} p_j}$$

$$\Rightarrow min_j min_i(\mathfrak{b}_{ij}) \leq \prod_{j=1}^{m}\left(\prod_{i=1}^{n}(\mathfrak{b}_{ij})^{w_i}\right)^{p_j} \leq max_j max_i(\mathfrak{b}_{ij}). \tag{3}$$

Also for each $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$, we get

$$min_j min_i(\mathfrak{c}_{ij}) \leq max_j max_i(\mathfrak{c}_{ij}) \Leftrightarrow \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(min_j min_i(\mathfrak{c}_{ij})\right)^{w_i}\right)^{p_j} \leq \prod_{j=1}^{m}\left(\prod_{i=1}^{n}(\mathfrak{c}_{ij})^{w_i}\right)^{p_j}$$

$$\leq \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(max_j max_i(\mathfrak{c}_{ij})\right)^{w_i}\right)^{p_j} \Leftrightarrow \left(\left(min_j min_i(\mathfrak{c}_{ij})\right)^{\sum_{i=1}^{n} w_i}\right)^{\sum_{j=1}^{m} p_j}$$

$$\leq \prod_{j=1}^{m}\left(\prod_{i=1}^{n}(\mathfrak{c}_{ij})^{w_i}\right)^{p_j} \leq \left(\left(max_j max_i(\mathfrak{c}_{ij})\right)^{\sum_{i=1}^{n} w_i}\right)^{\sum_{j=1}^{m} p_j}$$

$$\Rightarrow min_j min_i(\mathfrak{c}_{ij}) \leq \prod_{j=1}^{m}\left(\prod_{i=1}^{n}(\mathfrak{c}_{ij})^{w_i}\right)^{p_j} \leq max_j max_i(\mathfrak{c}_{ij}). \tag{4}$$

Therefore from Eqs (2), (3), and (4), it is clear that

$$min_j min_i(\mathfrak{a}_{ij}) \leq \sqrt{1 - \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1 - (\mathfrak{a}_{ij})^2\right)^{w_i}\right)^{p_j}} \leq max_j max_i(\mathfrak{a}_{ij}),$$

$$min_j min_i(\mathfrak{b}_{ij}) \leq \prod_{j=1}^{m}\left(\prod_{i=1}^{n}(\mathfrak{b}_{ij})^{w_i}\right)^{p_j} \leq max_j max_i(\mathfrak{b}_{ij})$$

and

$$min_j min_i(\mathfrak{c}_{ij}) \leq \prod_{j=1}^{m}\left(\prod_{i=1}^{n}(\mathfrak{c}_{ij})^{w_i}\right)^{p_j} \leq max_j max_i(\mathfrak{c}_{ij}).$$

Let $\mathfrak{T} = SFS_{ft}WA\left(S_{\rho_{11}}, S_{\rho_{12}}, \ldots, S_{\rho_{nm}}\right) = (\mathfrak{a}_{\mathfrak{T}}, \mathfrak{b}_{\mathfrak{T}}, \mathfrak{c}_{\mathfrak{T}})$, by Definition 10, we obtain

$$Sc(\mathfrak{T}) = Sc\left(S_{\rho_{11}}\right) = \frac{(2 + \mathfrak{a}_{\mathfrak{T}} - \mathfrak{b}_{\mathfrak{T}} - \mathfrak{c}_{\mathfrak{T}})}{3}$$

$$\leq \frac{\left(2 + max_j max_i(\mathfrak{a}_{ij}) - min_j min_i(\mathfrak{b}_{ij}) - min_j min_i(\mathfrak{c}_{ij})\right)}{3}$$

$$= Sc(S^{+}_{\rho_{ij}})$$

$$\Rightarrow Sc(\mathfrak{T}) \leq Sc\left(S^{+}_{\rho_{ij}}\right)$$

and

$$Sc(\mathfrak{T}) = \frac{(2 + \mathfrak{a}_{\mathfrak{T}} - \mathfrak{b}_{\mathfrak{T}} - \mathfrak{c}_{\mathfrak{T}})}{3}$$

$$\geq \frac{\left(2 + min_{j} min_{i}\left(\mathfrak{a}_{ij}\right) - max_{j} max_{i}\left(\mathfrak{b}_{ij}\right) - max_{j} max_{i}\left(\mathfrak{c}_{ij}\right)\right)}{3}$$

$$= Sc\left(S^{-}_{\rho_{ij}}\right)$$

$$\Rightarrow Sc(\mathfrak{T}) \geq Sc\left(S^{-}_{\rho_{ij}}\right).$$

We have the following cases

Case (1): If $Sc(\mathfrak{T}) < Sc\left(S^{+}_{\rho_{ij}}\right)$ and $Sc(\mathfrak{T}) > Sc\left(S^{-}_{\rho_{ij}}\right)$, then by the Definition 10, we have

$$\left(S^{-}_{\rho_{ij}}\right) \leq SFS_{ft}WA\left(S_{\rho_{11}}, S_{\rho_{12}}, \ldots, S_{\rho_{nm}}\right) \leq \left(S^{+}_{\rho_{ij}}\right)$$

Case (2): If $Sc(\mathfrak{T}) = Sc\left(S^{+}_{\rho_{ij}}\right)$, that is

$\frac{(2+\mathfrak{a}_{\mathfrak{T}}-\mathfrak{b}_{\mathfrak{T}}-\mathfrak{c}_{\mathfrak{T}})}{3} = \frac{\left(2+max_{j} max_{i}\left(\mathfrak{a}_{ij}\right)-min_{j} min_{i}\left(\mathfrak{b}_{ij}\right)-min_{j} min_{i}\left(\mathfrak{c}_{ij}\right)\right)}{3}$, then by using the above inequalities, we get

$$\mathfrak{a}_{\mathfrak{T}} = max_{j} max_{i}\left(\mathfrak{a}_{ij}\right) \text{ and } \mathfrak{b}_{\mathfrak{T}} = min_{j} min_{i}\left(\mathfrak{b}_{ij}\right) \text{ and} \mathfrak{c}_{\mathfrak{T}} = min_{j} min_{i}\left(\mathfrak{c}_{ij}\right).$$

Thus $r_{\mathfrak{T}} = r_{S^{+}_{\rho_{ij}}}$, this implies that $SFS_{ft}WA\left(S_{\rho_{11}}, S_{\rho_{12}}, \ldots, S_{\rho_{nm}}\right) = \left(S^{+}_{\rho_{ij}}\right)$.

Case (3): If $Sc(\mathfrak{T}) = Sc\left(S^{-}_{\rho_{ij}}\right)$, that is

$\frac{(2+\mathfrak{a}_{\mathfrak{T}}-\mathfrak{b}_{\mathfrak{T}}-\mathfrak{c}_{\mathfrak{T}})}{3} = \frac{\left(2+min_{j} min_{i}\left(\mathfrak{a}_{ij}\right)-max_{j} max_{i}\left(\mathfrak{b}_{ij}\right)-max_{j} max_{i}\left(\mathfrak{c}_{ij}\right)\right)}{3}$, then by using the above inequalities, we get

$$\mathfrak{a}_{\mathfrak{T}} = min_{j} min_{i}\left(\mathfrak{a}_{ij}\right) \text{ and } \mathfrak{b}_{\mathfrak{T}} = max_{j} max_{i}\left(\mathfrak{b}_{ij}\right) \text{ and } \mathfrak{c}_{\mathfrak{T}} = max_{j} max_{i}\left(\mathfrak{c}_{ij}\right).$$

Thus $r_{\mathfrak{T}} = r_{S^{-}_{\rho_{ij}}}$, this implies that $SFS_{ft}WA\left(S_{\rho_{11}}, S_{\rho_{12}}, \ldots, S_{\rho_{nm}}\right) = \left(S^{-}_{\rho_{ij}}\right)$.

Hence it is proved that

$$\left(S^{-}_{\rho_{ij}}\right) \leq SFS_{ft}WA\left(S_{\rho_{11}}, S_{\rho_{12}}, \ldots, S_{\rho_{nm}}\right) \leq \left(S^{+}_{\rho_{ij}}\right).$$

3. **(Monotonicity):** Let $S'_{\rho_{ij}} = \left(\mathfrak{a}'_{ij}, \mathfrak{b}'_{ij}, \mathfrak{c}'_{ij}\right)$ be any other collection of $SFS_{ft}Ns$ for all $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$ such that $\mathfrak{a}_{ij} \leq \mathfrak{a}'_{ij}$, $\mathfrak{b}_{ij} \geq \mathfrak{b}'_{ij}$ and $\mathfrak{c}_{ij} \geq \mathfrak{c}'_{ij}$, then

$$SFS_{ft}WA\left(S_{\rho_{11}}, S_{\rho_{12}}, \ldots, S_{\rho_{nm}}\right) \leq SFS_{ft}WA\left(S'_{\rho_{11}}, S'_{\rho_{12}}, \ldots, S'_{\rho_{nm}}\right).$$

Proof. As $\mathfrak{a}_{ij} \leq \mathfrak{a}'_{ij}$, $\mathfrak{b}_{ij} \geq \mathfrak{b}'_{ij}$ and $\mathfrak{c}_{ij} \geq \mathfrak{c}'_{ij}$ for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$, so

$$\mathfrak{a}_{ij} \leq \mathfrak{a}'_{ij} \Rightarrow 1 - \mathfrak{a}'_{ij} \leq 1 - \mathfrak{a}_{ij} \Rightarrow 1 - {\mathfrak{a}'_{ij}}^{2} \leq 1 - {\mathfrak{a}_{ij}}^{2}$$

$$\Rightarrow \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\left(\mathfrak{a}'_{ij}\right)^{2}\right)^{w_i}\right)^{p_j} \leq \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\left(\mathfrak{a}_{ij}\right)^{2}\right)^{w_i}\right)^{p_j}$$

$$\Rightarrow 1-\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\left(\mathfrak{a}_{ij}\right)^{2}\right)^{w_i}\right)^{p_j} \leq 1-\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\left(\mathfrak{a}'_{ij}\right)^{2}\right)^{w_i}\right)^{p_j}$$

$$\Rightarrow \sqrt{1-\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\left(\mathfrak{a}_{ij}\right)^{2}\right)^{w_i}\right)^{p_j}} \leq \sqrt{1-\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\left(\mathfrak{a}'_{ij}\right)^{2}\right)^{w_i}\right)^{p_j}} \quad (5)$$

and

$$\mathfrak{b}_{ij} \geq \mathfrak{b}'_{ij} \Rightarrow \prod_{i=1}^{n}\left(\mathfrak{b}_{ij}\right)^{w_i} \geq \prod_{i=1}^{n}\left(\mathfrak{b}'_{ij}\right)^{w_i} \Rightarrow \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\mathfrak{b}_{ij}\right)^{w_i}\right)^{p_j} \geq \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\mathfrak{b}'_{ij}\right)^{w_i}\right)^{p_j}. \quad (6)$$

Also

$$\mathfrak{c}_{ij} \geq \mathfrak{c}'_{ij} \Rightarrow \prod_{i=1}^{n}\left(\mathfrak{c}_{ij}\right)^{w_i} \geq \prod_{i=1}^{n}\left(\mathfrak{c}'_{ij}\right)^{w_i} \Rightarrow \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\mathfrak{c}_{ij}\right)^{w_i}\right)^{p_j} \geq \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\mathfrak{c}'_{ij}\right)^{w_i}\right)^{p_j}. \quad (7)$$

Let $\mathfrak{T}_S = SFS_{ft}WA\left(S_{\rho_{11}}, S_{\rho_{12}}, \ldots, S_{\rho_{nm}}\right) = \left(\mathfrak{a}_{\mathfrak{T}_S}, \mathfrak{b}_{\mathfrak{T}_S}, \mathfrak{c}_{\mathfrak{T}_S}\right)$ and $\mathfrak{T}_{S'} = SFS_{ft}WA\left(S'_{\rho_{11}}, S'_{\rho_{12}}, \ldots, S'_{\rho_{nm}}\right) = \left(\mathfrak{a}'_{\mathfrak{T}_{S'}}, \mathfrak{b}'_{\mathfrak{T}_{S'}}, \mathfrak{c}'_{\mathfrak{T}_{S'}}\right)$, then from Eqs (5), (6), and (7), we obtain $\mathfrak{a}_{\mathfrak{T}_S} \leq \mathfrak{a}'_{\mathfrak{T}_{S'}}$, $\mathfrak{b}_{\mathfrak{T}_S} \geq \mathfrak{b}'_{\mathfrak{T}_{S'}}$ and $\mathfrak{c}_{\mathfrak{T}_S} \geq \mathfrak{c}'_{\mathfrak{T}_{S'}}$.

Now by using Definition 10, we obtain $Sc(\mathfrak{T}_S) \leq Sc(\mathfrak{T}_{S'})$.
Now we have the following cases

Case (1): If $Sc(\mathfrak{T}_S) < Sc(\mathfrak{T}_{S'})$, then by using Definition 11, we have

$$SFS_{ft}WA\left(S_{\rho_{11}}, S_{\rho_{12}}, \ldots, S_{\rho_{nm}}\right) < SFS_{ft}WA\left(S'_{\rho_{11}}, S'_{\rho_{12}}, \ldots, S'_{\rho_{nm}}\right).$$

Case (2): If $Sc(\mathfrak{T}_S) = Sc(\mathfrak{T}_{S'})$, then

$$Sc(\mathfrak{T}_S) = \frac{\left(2+\mathfrak{a}_{\mathfrak{T}_S}-\mathfrak{b}_{\mathfrak{T}_S}-\mathfrak{c}_{\mathfrak{T}_S}\right)}{3} = \frac{\left(2+\mathfrak{a}_{\mathfrak{T}_{S'}}-\mathfrak{b}_{\mathfrak{T}_{S'}}-\mathfrak{c}_{\mathfrak{T}_{S'}}\right)}{3} = Sc(\mathfrak{T}_{S'}).$$

Hence by using the above inequality, we obtain $\mathfrak{a}_{\mathfrak{T}_S} = \mathfrak{a}'_{\mathfrak{T}_{S'}}, \mathfrak{b}_{\mathfrak{T}_S} = \mathfrak{b}'_{\mathfrak{T}_{S'}}$ and $\mathfrak{c}_{\mathfrak{T}_S} = \mathfrak{c}'_{\mathfrak{T}_{S'}}$.

So we get $r_{\mathfrak{T}_S} = r_{\mathfrak{T}_{S'}} \Rightarrow \left(\mathfrak{a}_{\mathfrak{T}_S}, \mathfrak{b}_{\mathfrak{T}_S}, \mathfrak{c}_{\mathfrak{T}_S}\right) = \left(\mathfrak{a}'_{\mathfrak{T}_{S'}}, \mathfrak{b}'_{\mathfrak{T}_{S'}}, \mathfrak{c}'_{\mathfrak{T}_{S'}}\right)$.

Hence it is proved that

$$SFS_{ft}WA\left(S_{\rho_{11}}, S_{\rho_{12}}, \ldots, S_{\rho_{nm}}\right) < SFS_{ft}WA\left(S'_{\rho_{11}}, S'_{\rho_{12}}, \ldots, S'_{\rho_{nm}}\right).$$

4. **(Shift Invariance):** If $S'_{\rho} = (\mathfrak{a}', \mathfrak{b}', \mathfrak{c}')$ is another family of $SFS_{ft}Ns$, then

$$SFS_{ft}WA\left(S_{\rho_{11}} \oplus S_{\rho}, S_{\rho_{12}} \oplus S_{\rho}, \ldots, S_{\rho_{nm}} \oplus S'_{\rho}\right) = SFS_{ft}WA\left(S_{\rho_{11}}, S_{\rho_{12}}, \ldots, S_{\rho_{nm}}\right) \oplus S'_{\rho}.$$

Proof. Let $S'_{\rho} = (\mathfrak{a}', \mathfrak{b}', \mathfrak{c}')$ and $S_{\rho_{ij}} = \left(\mathfrak{a}_{ij}, \mathfrak{b}_{ij}, \mathfrak{c}_{ij}\right)$ be family of $SFS_{ft}Ns$, then

$$S_{\rho_{ij}} \oplus S'_{\rho} = \left(\sqrt{1-\left(1-\mathfrak{a}_{ij}{}^{2}\right)\left(1-\mathfrak{a}'^{2}\right)}, \mathfrak{b}_{ij}\mathfrak{b}', \mathfrak{c}_{ij}\mathfrak{c}'\right).$$

Therefore,

$$SFS_{ft}WA\left(S_{\rho_{11}}\oplus S_{\rho}, S_{\rho_{12}}\oplus S_{\rho}, \dots, S_{\rho_{nm}}\oplus S'_{\rho}\right) = \oplus_{j=1}^{m} p_j \left(\oplus_{i=1}^{n} w_i \left(S_{\rho_{ij}}\oplus S'_{\rho}\right)\right)$$

$$= \left(\begin{array}{c}\sqrt{1-\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\left(\mathfrak{a}_{ij}\right)^2\right)^{w_i}(1-(\mathfrak{a}')^2)^{w_i}\right)^{p_j}}, \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\mathfrak{b}_{ij}\right)^{w_i}(\mathfrak{b}'^{w_i})\right)^{p_j}, \\ \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\mathfrak{c}_{ij}\right)^{w_i}(\mathfrak{c}'^{w_i})\right)^{p_j}\end{array}\right)$$

$$= \left(\begin{array}{c}\sqrt{1-(1-(\mathfrak{a}')^2)\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\left(\mathfrak{a}_{ij}\right)^2\right)^{w_i}\right)^{p_j}}, \mathfrak{b}'\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\mathfrak{b}_{ij}\right)^{w_i}\right)^{p_j}, \\ \mathfrak{c}'\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\mathfrak{c}_{ij}\right)^{w_i}\right)^{p_j}\end{array}\right)$$

$$= \left(\begin{array}{c}\sqrt{1-\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\left(\mathfrak{a}_{ij}\right)^2\right)^{w_i}\right)^{p_j}}, \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\mathfrak{b}_{ij}\right)^{w_i}\right)^{p_j}, \\ \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\mathfrak{c}_{ij}\right)^{w_i}\right)^{p_j}\end{array}\right)\oplus(\mathfrak{a}', \mathfrak{b}', \mathfrak{c}')$$

$$= \left(\begin{array}{c}\sqrt{1-\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\left(\mathfrak{a}_{ij}\right)^2\right)^{w_i}\right)^{p_j}}, \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\mathfrak{b}_{ij}\right)^{w_i}\right)^{p_j}, \\ \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\mathfrak{c}_{ij}\right)^{w_i}\right)^{p_j}\end{array}\right)\oplus S'_{\rho}$$

$$= SFS_{ft}WA\left(S_{\rho_{11}}, S_{\rho_{12}}, \dots, S_{\rho_{nm}}\right)\oplus S'_{\rho}.$$

Hence the required result is proved.

5. (Homogeneity): For any real number $\lambda \geq 0$,

$$SFS_{ft}WA\left(\lambda S_{\rho_{11}}, \lambda S_{\rho_{12}}, \dots, \lambda S_{\rho_{nm}}\right) \leq \lambda SFS_{ft}WA\left(S_{\rho_{11}}, S_{\rho_{12}}, \dots, S_{\rho_{nm}}\right).$$

Proof. Let $\lambda \geq 0$ be any real number and $S_{\rho_{ij}} = \left(\mathfrak{a}_{ij}, \mathfrak{b}_{ij}, \mathfrak{c}_{ij}\right)$ be a collection of $SFS_{ft}Ns$, then

$$\lambda S_{\rho_{ij}} = \left(\sqrt{\left(1-\left(1-\mathfrak{a}_{ij}{}^2\right)^{\lambda}\right)}, \mathfrak{b}_{ij}{}^{\lambda}, \mathfrak{c}_{ij}{}^{\lambda}\right).$$

Now

$$SFS_{ft}WA\left(\lambda S_{\rho_{11}}, \lambda S_{\rho_{12}}, \dots, \lambda S_{\rho_{nm}}\right)$$

$$= \left(\begin{array}{c} \sqrt{1 - \prod_{j=1}^{m} \left(\prod_{i=1}^{n} \left(1 - \left(\mathfrak{a}_{ij}\right)^2\right)^{\lambda w_i} \right)^{p_j}}, \prod_{j=1}^{m} \left(\prod_{i=1}^{n} \left(\mathfrak{b}_{ij}\right)^{\lambda w_i} \right)^{p_j}, \\ \prod_{j=1}^{m} \left(\prod_{i=1}^{n} \left(\mathfrak{c}_{ij}\right)^{\lambda w_i} \right)^{p_j} \end{array} \right)$$

$$= \left(\begin{array}{c} \sqrt{1 - \left(\prod_{j=1}^{m} \left(\prod_{i=1}^{n} \left(1 - \left(\mathfrak{a}_{ij}\right)^2\right)^{w_i} \right)^{p_j} \right)^{\lambda}}, \left(\prod_{j=1}^{m} \left(\prod_{i=1}^{n} \left(\mathfrak{b}_{ij}\right)^{w_i} \right)^{p_j} \right)^{\lambda}, \\ \left(\prod_{j=1}^{m} \left(\prod_{i=1}^{n} \left(\mathfrak{c}_{ij}\right)^{w_i} \right)^{p_j} \right)^{\lambda} \end{array} \right)$$

$$= \lambda SFS_{ft}WA\left(S_{\rho_{11}}, S_{\rho_{12}}, \dots, S_{\rho_{nm}}\right).$$

Hence the result is proved.

4.2. Spherical fuzzy soft ordered weighted average $\left(SFS_{ft}OWA\right)$ *operator*

From the above analysis, it is clear that $SFS_{ft}WA$ cannot weigh the order position through scoring the SFS_{ft} values, so to overcome this drawback, in this section, we will discuss the notion of $SFS_{ft}OWA$ operator which can weigh the ordered position thorough scoring the $SFS_{ft}Ns$. Also, the properties of established operators are discussed.

Definition 14. Let $S_{\rho_{ij}} = \left(\mathfrak{a}_{ij}, \mathfrak{b}_{ij}, \mathfrak{c}_{ij}\right)$ for $i = 1, 2, \dots, n$ and $j = 1, 2, \dots, m$, be the collection of $SFS_{ft}Ns$, $w = \{w_1, w_2, \dots, w_n\}$ and $p = \{p_1, p_2, \dots, p_m\}$ are the WVs of "e_i" experts and parameters ρ_j respectively with condition $w_i, p_j \in [0, 1]$ and $\sum_{i=1}^{n} w_i = 1$, $\sum_{j=1}^{m} p_j = 1$. Then $SFS_{ft}OWA$ operator is the mapping defined by $SFS_{ft}OWA: \mathcal{R}^n \to \mathcal{R}$, where ($\mathcal{R}$ is the family of all $SFS_{ft}Ns$)

$$SFS_{ft}OWA\left(S_{\rho_{11}}, S_{\rho_{12}}, \dots, S_{\rho_{nm}}\right) = \oplus_{j=1}^{m} p_j \left(\oplus_{i=1}^{n} w_i S_{\mathfrak{d}\rho_{ij}}\right).$$

Theorem 4. Let $S_{\rho_{ij}} = \left(\mathfrak{a}_{ij}, \mathfrak{b}_{ij}, \mathfrak{c}_{ij}\right)$ for $i = 1, 2, \dots, n$ and $j = 1, 2, \dots, m$, be the family of $SFFNs$. Then the aggregated result for $SFS_{ft}OWA$ operator is again a $SFS_{ft}N$ given by

$$SFS_{ft}OWA\left(S_{\rho_{11}}, S_{\rho_{12}}, \dots, S_{\rho_{nm}}\right) = \oplus_{j=1}^{m} p_j \left(\oplus_{i=1}^{n} w_i S_{\mathfrak{d}\rho_{ij}}\right)$$

$$= \left(\begin{array}{c} \sqrt{1 - \prod_{j=1}^{m} \left(\prod_{i=1}^{n} \left(1 - \left(\mathfrak{a}_{\mathfrak{d}ij}\right)^2\right)^{w_i} \right)^{p_j}}, \prod_{j=1}^{m} \left(\prod_{i=1}^{n} \left(\mathfrak{b}_{\mathfrak{d}ij}\right)^{w_i} \right)^{p_j}, \\ \prod_{j=1}^{m} \left(\prod_{i=1}^{n} \left(\mathfrak{c}_{\mathfrak{d}ij}\right)^{w_i} \right)^{p_j} \end{array} \right) \quad (8)$$

where $S_{\mathfrak{d}\rho_{ij}} = (\mathfrak{a}_{\mathfrak{d}ij}, \mathfrak{b}_{\mathfrak{d}ij}, \mathfrak{c}_{\mathfrak{d}ij})$ denote the permutation of ith and jth largest object of the collection of $i \times j$ $SFS_{ft}Ns S_{\rho_{ij}} = (\mathfrak{a}_{ij}, \mathfrak{b}_{ij}, \mathfrak{c}_{ij})$.

Proof. The proof is similar to Theorem 2.

Example 4. Consider the collection of $SFS_{ft}Ns\, S_{\rho_{ij}} = (\mathfrak{a}_{ij}, \mathfrak{b}_{ij}, \mathfrak{c}_{ij})$ as given in Table 1, of Example 3, then tabular depiction of $S_{\mathfrak{d}\rho_{ij}} = (\mathfrak{a}_{\mathfrak{d}ij}, \mathfrak{b}_{\mathfrak{d}ij}, \mathfrak{c}_{\mathfrak{d}ij})$ is given in Table 2.

Table 2. Tabular depiction of $S_{\mathfrak{d}\rho_{ij}} = (\mathfrak{a}_{\mathfrak{d}ij}, \mathfrak{b}_{\mathfrak{d}ij}, \mathfrak{c}_{\mathfrak{d}ij})$.

	ρ_1	ρ_2	ρ_3	ρ_4
$\varkappa_1$	(0.7, 0.4, 0.6)	(0.6, 0.3, 0.7)	(0.4, 0.8, 0.7)	(0.7, 0.6, 0.4)
$\varkappa_2$	(0.3, 0.4, 0.6)	(0.6, 0.3, 0.6)	(0.3, 0.3, 0.4)	(0.6, 0.3, 0.3)
$\varkappa_3$	(0.3, 0.2, 0.4)	(0.5, 0.3, 0.1)	(0.2, 0.3, 0.3)	(0.5, 0.1, 0.2)
$\varkappa_4$	(0.2, 0.1, 0.2)	(0.5, 0.2, 0.1)	(0.2, 0.1, 0.2)	(0.5, 0.1, 0.1)
$\varkappa_5$	(0.1, 0.1, 0.2)	(0.1, 0.2, 0.1)	(0.2, 0.1, 0.2)	(0.2, 0.1, 0.1)

Now by using Eq (8) of Theorem 4, we have

$SFS_{ft}OWA(S_{\rho_{11}}, S_{\rho_{12}}, \dots, S_{\rho_{nm}})$

$$= \left(\begin{array}{c} \sqrt{1 - \prod_{j=1}^{m} \left(\prod_{i=1}^{n} \left(1 - (\mathfrak{a}_{\mathfrak{d}ij})^2\right)^{w_i} \right)^{p_j}}, \prod_{j=1}^{m} \left(\prod_{i=1}^{n} (\mathfrak{b}_{\mathfrak{d}ij})^{w_i} \right)^{p_j}, \\ \prod_{j=1}^{m} \left(\prod_{i=1}^{n} (\mathfrak{c}_{\mathfrak{d}ij})^{w_i} \right)^{p_j} \end{array} \right)$$

$$= \left[\begin{array}{c} \left(\sqrt{\begin{array}{c} 1 - \{(1-0.7^2)^{0.25}(1-0.3^2)^{0.15}(1-0.3^2)^{0.14}(1-0.2^2)^{0.3}(1-0.1^2)^{0.16}\}^{0.27} \\ \{(1-0.6^2)^{0.25}(1-0.6^2)^{0.15}(1-0.5^2)^{0.14}(1-0.5^2)^{0.3}(1-0.1^2)^{0.16}\}^{0.19} \\ \{(1-0.4^2)^{0.25}(1-0.3^2)^{0.15}(1-0.2^2)^{0.14}(1-0.2^2)^{0.3}(1-0.2^2)^{0.16}\}^{0.29} \\ \{(1-0.7)^{0.25}(1-0.6^2)^{0.15}(1-0.5^2)^{0.14}(1-0.5^2)^{0.3}(1-0.2^2)^{0.16}\}^{0.25} \end{array}} \right), \\ \left(\begin{array}{c} \{(1-0.4^2)^{0.25}(1-0.4^2)^{0.15}(1-0.2^2)^{0.14}(1-0.1^2)^{0.3}(1-0.1^2)^{0.16}\}^{0.27} \\ \{(1-0.3^2)^{0.25}(1-0.3^2)^{0.15}(1-0.3^2)^{0.14}(1-0.2^2)^{0.3}(1-0.2^2)^{0.16}\}^{0.19} \\ \{(1-0.8^2)^{0.25}(1-0.3^2)^{0.15}(1-0.3^2)^{0.14}(1-0.1^2)^{0.3}(1-0.1^2)^{0.16}\}^{0.29} \\ \{(1-0.6^2)^{0.25}(1-0.3^2)^{0.15}(1-0.1^2)^{0.14}(1-0.1^2)^{0.3}(1-0.1^2)^{0.16}\}^{0.25} \end{array} \right), \\ \left(\begin{array}{c} \{(1-0.6^2)^{0.25}(1-0.6^2)^{0.15}(1-0.4^2)^{0.14}(1-0.2^2)^{0.3}(1-0.2^2)^{0.16}\}^{0.27} \\ \{(1-0.7^2)^{0.25}(1-0.6^2)^{0.15}(1-0.1^2)^{0.14}(1-0.1^2)^{0.3}(1-0.1^2)^{0.16}\}^{0.19} \\ \{(1-0.7^2)^{0.25}(1-0.4^2)^{0.15}(1-0.3^2)^{0.14}(1-0.2^2)^{0.3}(1-0.2^2)^{0.16}\}^{0.29} \\ \{(1-0.4^2)^{0.25}(1-0.3^2)^{0.15}(1-0.2^2)^{0.14}(1-0.1^2)^{0.3}(1-0.1^2)^{0.16}\}^{0.25} \end{array} \right) \end{array} \right]$$

$= (0.456533, 0.16599, 0.21286)$.

Theorem 5. For the collection of $SFS_{ft}Ns$, $S_{\rho_{ij}} = (\mathfrak{a}_{ij}, \mathfrak{b}_{ij}, \mathfrak{c}_{ij})$ for $i = 1, 2, \dots, n$ and $j = 1, 2, \dots, m$, $w = \{w_1, w_2, \dots, w_n\}$ being WV of e_i experts and $p = \{p_1, p_2, \dots, p_m\}$ being

WV of parameters ρ_j with condition $w_i, p_j \in [0,1]$ and $\sum_{i=1}^{n} w_i = 1$, $\sum_{j=1}^{m} p_j = 1$, then $SFS_{ft}OWA$ operator preserves the following properties.

1. **(Idempotency):** Let $S_{\rho_{ij}} = (\mathfrak{a}_{ij}, \mathfrak{b}_{ij}, \mathfrak{c}_{ij}) = S'_{\mathfrak{d}\rho}$ for all $i = 1,2,\dots,n$ and $j = 1,2,\dots,m$, where $S'_{\mathfrak{d}\rho} = (\mathfrak{a}', \mathfrak{b}', \mathfrak{c}')$, then $SFS_{ft}OWA(S_{\rho_{11}}, S_{\rho_{12}}, \dots, S_{\rho_{nm}}) = S'_{\mathfrak{d}\rho}$.
2. **(Boundedness):** If $S^{-}_{\mathfrak{d}\rho_{ij}} = (min_j min_i\{\mathfrak{a}_{\mathfrak{d}ij}\}, max_j max_i\{\mathfrak{b}_{\mathfrak{d}ij}\}, max_j max_i\{\mathfrak{c}_{\mathfrak{d}ij}\})$ and $S^{+}_{\mathfrak{d}\rho_{ij}} = (max_j max_i\{\mathfrak{a}_{\mathfrak{d}ij}\}, min_j min_i\{\mathfrak{b}_{\mathfrak{d}ij}\}, min_j min_i\{\mathfrak{c}_{\mathfrak{d}ij}\})$, then
$$S^{-}_{\mathfrak{d}\rho_{ij}} \leq SFS_{ft}OWA(S_{\rho_{11}}, S_{\rho_{12}}, \dots, S_{\rho_{nm}}) \leq S^{+}_{\mathfrak{d}\rho_{ij}}.$$
3. **(Monotonicity):** Let $S'_{\rho_{ij}} = (\mathfrak{a}'_{ij}, \mathfrak{b}'_{ij}, \mathfrak{c}'_{ij})$ be any other collection of $SFS_{ft}Ns$ for all $i = 1,2,\dots,n$ and $j = 1,2,\dots,m$ such that $\mathfrak{a}_{ij} \leq \mathfrak{a}'_{ij}$, $\mathfrak{b}_{ij} \geq \mathfrak{b}'_{ij}$ and $\mathfrak{c}_{ij} \geq \mathfrak{c}'_{ij}$, then
$$SFS_{ft}OWA(S_{\rho_{11}}, S_{\rho_{12}}, \dots, S_{\rho_{nm}}) \leq SFS_{ft}OWA(S'_{\rho_{11}}, S'_{\rho_{12}}, \dots, S'_{\rho_{nm}}).$$
4. **(Shift Invariance):** If $S'_{\rho} = (\mathfrak{a}', \mathfrak{b}', \mathfrak{c}')$ is another family of $SFS_{ft}Ns$, then
$$SFS_{ft}OWA(S_{\rho_{11}} \oplus S_{\rho}, S_{\rho_{12}} \oplus S_{\rho}, \dots, S_{\rho_{nm}} \oplus S'_{\rho}) = SFS_{ft}OWA(S_{\rho_{11}}, S_{\rho_{12}}, \dots, S_{\rho_{nm}}) \oplus S'_{\rho}.$$
5. **(Homogeneity):** For any real number $\lambda \geq 0$,
$$SFS_{ft}OWA(\lambda S_{\rho_{11}}, \lambda S_{\rho_{12}}, \dots, \lambda S_{\rho_{nm}}) \leq \lambda SFS_{ft}OWA(S_{\rho_{11}}, S_{\rho_{12}}, \dots, S_{\rho_{nm}}).$$

Proof. The proof is simple and follows from Theorem 3.

4.3. Spherical fuzzy soft hybrid aggregation $(SFS_{ft}HA)$ operator

As spherical fuzzy soft hybrid average ($SFS_{ft}HA$) aggregation operator can deal with both situations like measuring the values of $SFS_{ft}Ns$ and also considering the ordered position of SFS_{ft} values, so due to this reason here we elaborate the $SFS_{ft}HA$ and discuss properties related to these operators.

Definition 15. For a collection of $SFS_{ft}NsS_{\rho_{ij}} = (\mathfrak{a}_{ij}, \mathfrak{b}_{ij}, \mathfrak{c}_{ij})$; $i = 1,2,\dots,n$ and $j = 1,2,\dots,m$, and $w = \{w_1, w_2, \dots, w_n\}$ being WV of "e_i" experts and $p = \{p_1, p_2, \dots, p_m\}$ being WV of parameters ρ_j with condition $w_i, p_j \in [0,1]$ and $\sum_{i=1}^{n} w_i = 1$, $\sum_{j=1}^{m} p_j = 1$, the $SFS_{ft}HA$ operator is the mapping defined by $SFS_{ft}HA: \mathcal{R}^n \to \mathcal{R}$, where ($\mathcal{R}$ denote the family of all $SFS_{ft}Ns$)
$$SFS_{ft}HA(S_{\rho_{11}}, S_{\rho_{12}}, \dots, S_{\rho_{nm}}) = \oplus_{j=1}^{m} p_j \left(\oplus_{i=1}^{n} w_i \tilde{S}_{\rho_{ij}}\right).$$

Theorem 6. Let $S_{\rho_{ij}} = (\mathfrak{a}_{ij}, \mathfrak{b}_{ij}, \mathfrak{c}_{ij})$ for $i = 1,2,\dots,n$ and $j = 1,2,\dots,m$, be the family of $SFS_{ft}Ns$ having WVs $\mathfrak{v} = \{\mathfrak{v}_1, \mathfrak{v}_2, \dots, \mathfrak{v}_n\}^T$ and $\mathfrak{u} = \{\mathfrak{u}_1, \mathfrak{u}_2, \dots, \mathfrak{u}_n\}^T$ with condition $\mathfrak{v}_i, \mathfrak{u}_j \in$

$[0,1], \sum_{i=1}^{n} \mathfrak{v}_i = 1, \sum_{j=1}^{m} \mathfrak{u}_j = 1$. Also "$n$" represents the corresponding coefficient for the number of elements in ith row and jth column with WVs $w = (w_1, w_2, \ldots, w_n)^T$ denote the WVs of "e_i" experts and $p = \{p_1, p_2, \ldots, p_m\}^T$ being WVs of parameters "ρ_j" with condition $w_i, p_j \in [0,1], \sum_{i=1}^{n} w_i = 1$ and $\sum_{j=1}^{m} p_j = 1$, then

$$SFS_{ft}HA\left(S_{\rho_{11}}, S_{\rho_{12}}, \ldots, S_{\rho_{nm}}\right) = \oplus_{j=1}^{m} p_j \left(\oplus_{i=1}^{n} w_i \tilde{S}_{\rho_{ij}}\right)$$

$$= \left(\begin{array}{c} \sqrt{1 - \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1 - \left(\tilde{\mathfrak{a}}_{ij}\right)^2\right)^{w_i}\right)^{p_j}}, \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\tilde{\mathfrak{b}}_{ij}\right)^{w_i}\right)^{p_j}, \\ \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\tilde{\mathfrak{c}}_{ij}\right)^{w_i}\right)^{p_j} \end{array} \right) \tag{9}$$

where $\tilde{S}_{\rho_{ij}} = n\mathfrak{v}_i\mathfrak{u}_j S_{\rho_{ij}}$ denote the permutation of ith and jth largest object of the family of $i \times j$ $SFSNs\tilde{S}_{\rho_{ij}} = \left(\tilde{\mathfrak{a}}_{ij}, \tilde{\mathfrak{b}}_{ij}, \tilde{\mathfrak{c}}_{ij}\right)$.

Proof. The proof is similar to Theorem 1.

Example 5. Consider the family of $SFS_{ft}Ns$ $S_{\rho_{ij}} = \left(\mathfrak{a}_{ij}, \mathfrak{b}_{ij}, \mathfrak{c}_{ij}\right)$ as given in Table 1 with WV $\mathfrak{v} = \{0.17, 0.19, .12, 0.16, 0.36\}^T$ and $\mathfrak{u} = \{0.23, 0.2, 0.29, 0.28\}^T$ and having the associated vector as $w = (0.23, 0.18, 0.1, 0.27, 00.22)^T$ and $p = \{0.23, 0.24, 0.18, 0.35\}^T$. Then by using Eq (10) the corresponding $SFS_{ft}Ns\tilde{S}_{\rho_{ij}} = \left(\tilde{\mathfrak{a}}_{ij}, \tilde{\mathfrak{b}}_{ij}, \tilde{\mathfrak{c}}_{ij}\right)$ of the permutation of ith and jth largest object of the family of $i \times j$ $SFS_{ft}Ns\tilde{S}_{\rho_{ij}} = \left(\tilde{\mathfrak{a}}_{ij}, \tilde{\mathfrak{b}}_{ij}, \tilde{\mathfrak{c}}_{ij}\right)$ are given in Table 3. Since

$$\tilde{S}_{\rho_{ij}} = n\mathfrak{v}_i\mathfrak{u}_j S_{\rho_{ij}} = \left(\sqrt{1 - \left(\mathfrak{a}_{ij}{}^2\right)^{n\mathfrak{v}_i\mathfrak{u}_j}}, \left(\mathfrak{b}_{ij}\right)^{n\mathfrak{v}_i\mathfrak{u}_j}, \left(\mathfrak{c}_{ij}\right)^{n\mathfrak{v}_i\mathfrak{u}_j}\right). \tag{10}$$

Table 3. Tabular presentation of $\tilde{S}_{\rho_{ij}} = n\mathfrak{v}_i\mathfrak{u}_j S_{\rho_{ij}}$.

	ρ_1	ρ_2	ρ_3	ρ_4
$\varkappa_1$	(0.03128, 0.01564, 0.09348)	(0.068, 0.0408, 0.0136)	(0.07888, 0.05916, 0.03944,)	(0.11424, 0.01904, 0.03808)
$\varkappa_2$	(0.017480, 0.06992, 0.06992)	(0.0912, 0.0456, 0.0152)	(0.04408, 0.02204, 0.15428,)	(0.1064, 0.1064, 0.02128)
$\varkappa_3$	(0.03312, 0.02208, 0.02208)	(0.0576, 0.01920, 0.096)	(0.04176, 0.04176, 0.05568,)	(0.0672, 0.01344, 0.04032)
$\varkappa_4$	(0.04416, 0.01472, 0.08832)	(0.0128, 0.0256, 0.0768)	(0.01856, 0.01856, 0.03712,)	(0.03548, 0.5376, 0.07168)
$\varkappa_5$	(0.23184, 0.13248, 0.06624)	(0.144, 0.0864, 0.2016)	(0.334008, 0.33408, 0.125280)	(0.28224, 0.04032, 0.08064)

Now by using Eq (9), we get

$SFS_{ft}HA\left(S_{\rho_{11}}, S_{\rho_{12}}, \ldots, S_{\rho_{nm}}\right)$

$$= \left(\sqrt{1 - \prod_{j=1}^{m} \left(\prod_{i=1}^{n} \left(1 - (\tilde{\mathfrak{a}}_{ij})^2 \right)^{w_i} \right)^{p_j}}, \prod_{j=1}^{m} \left(\prod_{i=1}^{n} (\tilde{\mathfrak{b}}_{ij})^{w_i} \right)^{p_j}, \prod_{j=1}^{m} \left(\prod_{i=1}^{n} (\tilde{\mathfrak{c}}_{ij})^{w_i} \right)^{p_j} \right)$$

$= (0.114201, 0.025752, 0.034869)$.

Theorem 7. Let $S_{\rho_{ij}} = (\mathfrak{a}_{ij}, \mathfrak{b}_{ij}, \mathfrak{c}_{ij})$ for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$, be the family of $SFS_{ft}Ns$ with WVs $\mathfrak{v} = \{\mathfrak{v}_1, \mathfrak{v}_2, \ldots, \mathfrak{v}_n\}^T$ and $\mathfrak{u} = \{\mathfrak{u}_1, \mathfrak{u}_2, \ldots, \mathfrak{u}_n\}^T$ having condition $\mathfrak{v}_i, \mathfrak{u}_j \in [0, 1]$ and $\sum_{i=1}^{n} \mathfrak{v}_i = 1$, $\sum_{j=1}^{m} \mathfrak{u}_j = 1$. Also "$n$" represents the corresponding coefficient for the number of elements in ith row and jth column linked with vectors $w = (w_1, w_2, \ldots, w_n)^T$ denote the WV of "e_i" experts and $p = \{p_1, p_2, \ldots, p_m\}^T$ denote the WV of parameters "ρ_j" having condition $w_i, p_j \in [0, 1]$ and $\sum_{i=1}^{n} w_i = 1$, $\sum_{j=1}^{m} p_j = 1$. Then $SFS_{ft}HA$ operator contains the subsequent properties:

1. **(Idempotency):** Let $S_{\rho_{ij}} = \widetilde{S'}_\rho$ for all $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$, $\widetilde{S'}_\rho = n\mathfrak{v}_i\mathfrak{u}_j S'_\rho$ then

$$SFS_{ft}HA(S_{\rho_{11}}, S_{\rho_{12}}, \ldots, S_{\rho_{nm}}) = \widetilde{S'}_\rho.$$

2. **(Boundedness):** If $\tilde{S}^{-}_{\rho_{ij}} = \left(min_j min_i\{\tilde{\mathfrak{a}}_{ij}\}, max_j max_i\{\tilde{\mathfrak{b}}_{ij}\}, max_j max_i\{\tilde{\mathfrak{c}}_{ij}\} \right)$ and $\tilde{S}^{+}_{\rho_{ij}} = \left(max_j max_i\{\tilde{\mathfrak{a}}_{ij}\}, min_j min_i\{\tilde{\mathfrak{b}}_{ij}\}, min_j min_i\{\tilde{\mathfrak{c}}_{ij}\} \right)$, then

$$\tilde{S}^{-}_{\rho_{ij}} \le SFS_{ft}HA(S_{\rho_{11}}, S_{\rho_{12}}, \ldots, S_{\rho_{nm}}) \le \tilde{S}^{+}_{\rho_{ij}}.$$

3. **(Monotonicity):** Let $S'_{\rho_{ij}} = (\mathfrak{a}'_{ij}, \mathfrak{b}'_{ij}, \mathfrak{c}'_{ij})$ be any other family of $SFS_{ft}Ns$ for all $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$ such that $\mathfrak{a}_{ij} \le \mathfrak{a}'_{ij}$, $\mathfrak{b}_{ij} \ge \mathfrak{b}'_{ij}$ and $\mathfrak{c}_{ij} \ge \mathfrak{c}'_{ij}$, then

$$SFS_{ft}HA(S_{\rho_{11}}, S_{\rho_{12}}, \ldots, S_{\rho_{nm}}) \le SFS_{ft}HA(S'_{\rho_{11}}, S'_{\rho_{12}}, \ldots, S'_{\rho_{nm}}).$$

4. **(Shift Invariance):** If $S'_\rho = (\mathfrak{a}', \mathfrak{b}', \mathfrak{c}')$ is another family of $SFS_{ft}Ns$, then

$$SFS_{ft}HA(S_{\rho_{11}} \oplus S_\rho, S_{\rho_{12}} \oplus S_\rho, \ldots, S_{\rho_{nm}} \oplus S'_\rho) = SFS_{ft}HA(S_{\rho_{11}}, S_{\rho_{12}}, \ldots, S_{\rho_{nm}}) \oplus S'_\rho.$$

5. **(Homogeneity):** For any real number $\lambda \ge 0$,

$$SFS_{ft}HA(\lambda S_{\rho_{11}}, \lambda S_{\rho_{12}}, \ldots, \lambda S_{\rho_{nm}}) \le \lambda SFS_{ft}HA(S_{\rho_{11}}, S_{\rho_{12}}, \ldots, S_{\rho_{nm}}).$$

Proof. The proof is simple and follows from Theorem 3.

5. A multicriteria decision making method based on spherical fuzzy soft average aggregation operators

In this section, we will discuss a new MCDM method based on $SFS_{ft}WA, SFS_{ft}OWA$ and

$SFS_{ft}HA$ aggregation operators to solve MCDM problems under the environment of $SFS_{ft}Ns$.

Consider a common MCDM problem. Let $A = \{\varkappa_1, \varkappa_2, \varkappa_3, \dots, \varkappa_r\}$ be the set of "r" alternative, $E = \{E_1, E_2, E_3, \dots, E_n\}$ be the family of "n" senior experts with $\rho = \{\rho_1, \rho_2, \rho_3, \dots, \rho_m\}$ as a family of "m" parameters. The experts evaluate each alternative $\varkappa_l (l = 1, 2, 3, \dots, r)$ according to their respective parameters ρ_j ($j = 1, 2, 3, \dots, m$). Suppose evaluation information given by experts is in the form of $SFS_{ft}Ns$ $S_{\rho_{ij}} = (\mathfrak{a}_{ij}, \mathfrak{b}_{ij}, \mathfrak{c}_{ij})$ for $i = 1, 2, \dots, n$ and $j = 1, 2, \dots, m$. Let $w = \{w_1, w_2, \dots, w_n\}$ denote the WV of "e_i" experts and $p = \{p_1, p_2, \dots, p_m\}$ represent the WV of parameters "ρ_j" with a condition that $w_i, p_j \in [0, 1]$ and $\sum_{i=1}^{n} w_i = 1, \sum_{i=1}^{m} p_j = 1$. The overall information is given in matrix $M = \left[S_{\rho_{ij}}\right]_{n \times m}$. By using the aggregation operator for assessment values, the aggregated $SFS_{ft}N$ "$\mathfrak{B}_l$" for alternative $\varkappa_l (l = 1, 2, 3, \dots, r)$ are given by $\mathfrak{B}_l = (\mathfrak{a}_l, \mathfrak{b}_l, \mathfrak{c}_l)(l = 1, 2, \dots, r)$. Use the Definition 10 to find the score values for $SFS_{ft}Ns$ and rank them.

Step vise algorithm is given by

Step 1: Arrange all assessment information given by experts for each alternative to their corresponding parameters to construct an overall decision matrix $M = \left[S_{\rho_{ij}}\right]_{n \times m}$ given by:

$$M = \begin{bmatrix} (\mathfrak{a}_{11}, \mathfrak{b}_{11}, \mathfrak{c}_{11}) & (\mathfrak{a}_{12}, \mathfrak{b}_{12}, \mathfrak{c}_{12}) & \cdots & (\mathfrak{a}_{1m}, \mathfrak{b}_{1m}, \mathfrak{c}_{1m}) \\ (\mathfrak{a}_{21}, \mathfrak{b}_{21}, \mathfrak{c}_{21}) & (\mathfrak{a}_{22}, \mathfrak{b}_{22}, \mathfrak{c}_{22}) & \cdots & (\mathfrak{a}_{2m}, \mathfrak{b}_{2m}, \mathfrak{c}_{2m}) \\ \vdots & \vdots & & \\ (\mathfrak{a}_{n1}, \mathfrak{b}_{n1}, \mathfrak{c}_{n1}) & (\mathfrak{a}_{n2}, \mathfrak{b}_{n2}, \mathfrak{c}_{n2}) & \cdots & (\mathfrak{a}_{nm}, \mathfrak{b}_{nm}, \mathfrak{c}_{nm}) \end{bmatrix}.$$

Step 2: Normalize the SFS_{ft} decision matrix that is given in step 1, because there are two type of parameters, cost type parameters and benefit type parameters if it is needed according to the following formula

$$\rho_{ij} = \begin{cases} S^c{}_{\rho_{ij}}, & \text{for cost type parameter} \\ S_{\rho_{ij}}, & \text{for a benefit type parameter} \end{cases}$$

where $S^c{}_{\rho_{ij}} = (\mathfrak{c}_{ij}, \mathfrak{b}_{ij}, \mathfrak{a}_{ij})$ denote the complement of $S_{\rho_{ij}} = (\mathfrak{a}_{ij}, \mathfrak{b}_{ij}, \mathfrak{c}_{ij})$.

Step 3: Aggregate $SFS_{ft}Ns$ by using the proposed aggregation operators for each parameter $\rho_l (l = 1, 2, \dots, r)$ to get $\mathfrak{B}_l = (\mathfrak{a}_l, \mathfrak{b}_l, \mathfrak{c}_l)$.

Step 4: Using Definition 10 to calculate the score values for each "$\mathfrak{B}_l$".

Step 5: Rank the results for each alternative $\varkappa_l (l = 1, 2, 3, \dots, r)$ and choose the best result.

5.1. Application steps for the proposed method

In this section, we describe the detailed explanation of the above-given algorithm through an illustrative example to show the effectiveness of the established work.

Example 6. Suppose a person wants to select the best tyre brand from a set of four alternatives $A = \{\varkappa_1 = Bridgestone, \varkappa_2 = Hankook, \varkappa_3 = Dunlop, \varkappa_4 = MRF\ tyres\}$. Let a team of experts consisting of five members $E = \{E_1, E_2, E_3, E_4, E_5\}$ with WVs $w = \{0.15, 0.13, 0.25, 0.23, 0.24\}$ provide their information about alternatives having parameters $\rho = \{\rho_1 = Cornering\ grip, \rho_2 =$

$Durability, \rho_3 = Fuel\ comsumption, \rho_4 = Internal\ noise, \rho_5 = Aquaplaning\}$ with WVs $p = \{0.19, 0.29, 0.18, 0.23, 0.11\}$ in the form of $SFS_{ft}Ns$. Now we use the proposed algorithm for the selection of the best alternative.

By using $SFSWA$ operators.
Step 1: The overall expert information based on $SFS_{ft}Ns$ is given in Table 4–7.
Step 2: There is no need for normalization of SFS_{ft} matrix because of similar kinds of parameters.
Step 3: Using the Eq (1) for each alternative $\varkappa_i$ $(i = 1, 2, 3, 4)$, we have
$\mathfrak{B}_1 = (0.5878, 0.2090, 0.2439)$, $\mathfrak{B}_2 = (0.5695, 0.2081, 0.2602)$.

Table 4. SFS_{ft} matrix for alternative $\varkappa_1$.

	ρ_1	ρ_2	ρ_3	ρ_4	ρ_5
E_1	(0.2, 0.1, 0.2)	(0.5, 0.1, 0.1)	(0.2, 0.1, 0.2)	(0.5, 0.3, 0.1)	(0.3, 0.2, 0.1)
E_2	(0.2, 0.3, 0.3)	(0.7, 0.5, 0.4)	(0.3, 0.2, 0.4)	(0.5, 0.2, 0.1)	(0.2, 0.1, 0.3)
E_3	(0.4, 0.8, 0.3)	(0.2, 0.1, 0.1)	(0.1, 0.1, 0.2)	(0.6, 0.3, 0.6)	(0.4, 0.3, 0.6)
E_4	(0.2, 0.1, 0.2)	(0.5, 0.1, 0.2)	(0.3, 0.4, 0.6)	(0.1, 0.2, 0.1)	(0.6, 0.4, 0.1)
E_5	(0.3, 0.3, 0.4)	(0.6, 0.3, 0.3)	(0.5, 0.4, 0.6)	(0.6, 0.3, 0.7)	(0.6, 0.2, 0.4)

Table 5. SFS_{ft} matrix for alternative $\varkappa_2$.

	ρ_1	ρ_2	ρ_3	ρ_4	ρ_5
E_1	(0.6, 0.1, 0.2)	(0.5, 0.3, 0.1)	(0.5, 0.1, 0.3)	(0.2, 0.3, 0.4)	(0.7, 0.3, 0.2)
E_2	(0.4, 0.4, 0.1)	(0.6, 0.3, 0.1)	(0.5, 0.2, 0.2)	(0.7, 0.1, 0.2)	(0.2, 0.7, 0.3)
E_3	(0.2, 0.2, 0.3)	(0.6, 0.2, 0.1)	(0.4, 0.1, 0.3)	(0.4, 0.3, 0.3)	(0.3, 0.8, 0.1)
E_4	(0.6, 0.1, 0.3)	(0.5, 0.1, 0.2)	(0.1, 0.3, 0.5)	(0.2, 0.3, 0.2)	(0.1, 0.2, 0.7)
E_5	(0.4, 0.3, 0.3)	(0.1, 0.2, 0.6)	(0.1, 0.2, 0.6)	(0.2, 0.2, 0.5)	(0.3, 0.2, 0.5)

Table 6. SFS_{ft} matrix for alternative $\varkappa_3$.

	ρ_1	ρ_2	ρ_3	ρ_4	ρ_5
E_1	(0.7, 0.1, 0.4)	(0.5, 0.1, 0.8)	(0.4, 0.6, 0.5)	(0.8, 0.3, 0.4)	(0.6, 0.4, 0.2)
E_2	(0.4, 0.4, 0.4)	(0.5, 0.3, 0.2)	(0.5, 0.2, 0.2)	(0.3, 0.3, 0.3)	(0.2, 0.3, 0.6)
E_3	(0.2, 0.1, 0.6)	(0.2, 0.1, 0.2)	(0.2, 0.1, 0.3)	(0.4, 0.3, 0.5)	(0.6, 0.3, 0.5)
E_4	(0.8, 0.2, 0.5)	(0.5, 0.5, 0.5)	(0.5, 0.3, 0.4)	(0.2, 0.8, 0.4)	(0.1, 0.4, 0.5)
E_5	(0.8, 0.3, 0.5)	(0.5, 0.4, 0.3)	(0.1, 0.2, 0.6)	(0.2, 0.5, 0.7)	(0.6, 0.3, 0.6)

Table 7. SFS_{ft} matrix for alternative $\varkappa_4$.

	ρ_1	ρ_2	ρ_3	ρ_4	ρ_5
E_1	(0.1, 0.2, 0.6)	(0.4, 0.3, 0.5)	(0.8, 0.3, 0.5)	(0.5, 0.5, 0.5)	(0.7, 0.5, 0.1)
E_2	(0.5, 0.2, 0.2)	(0.2, 0.8, 0.4)	(0.8, 0.2, 0.5)	(0.2, 0.1, 0.2)	(0.5, 0.2, 0.2)
E_3	(0.2, 0.1, 0.3)	(0.3, 0.3, 0.3)	(0.7, 0.1, 0.4)	(0.5, 0.4, 0.3)	(0.4, 0.3, 0.2)
E_4	(0.5, 0.3, 0.4)	(0.2, 0.5, 0.7)	(0.2, 0.1, 0.6)	(0.5, 0.3, 0.2)	(0.1, 0.1, 0.8)
E_5	(0.4, 0.6, 0.5)	(0.8, 0.3, 0.4)	(0.4, 0.4, 0.4)	(0.5, 0.1, 0.8)	(0.4, 0.5, 0.5)

$\mathfrak{B}_3 = (0.6330, 0.2626, 0.4138)$, $\mathfrak{B}_4 = (0.6341, 0.2623, 0.3955)$.

Step 4: To find out the score values, use Definition 10 for each $\mathfrak{B}_i (i = 1, 2, 3, 4, 5)$ given in step 3, i.e.

$Sc(\mathfrak{B}_1) = 0.7116$, $Sc(\mathfrak{B}_2) = 0.7004$,
$Sc(\mathfrak{B}_3) = 0.6522$, $Sc(\mathfrak{B}_4) = 0.6587$.

Step 5: Select the best solution by ranking the score values.

$$Sc(\mathfrak{B}_1) > Sc(\mathfrak{B}_2) > Sc(\mathfrak{B}_4) > Sc(\mathfrak{B}_3).$$

Hence it is clear that "$\varkappa_1$" is the best result.

By using $SFS_{ft}OWA$ operators.
Step 1: Same as above.
Step 2: Same as above.
Step 3: Using the Eq (8) for each alternative $\varkappa_i$ $(i = 1, 2, 3, 4)$, we have

$\mathfrak{B}_1 = \big((0.5670, 0.1843, 0.2113)\big)$, $\mathfrak{B}_2 = \big((0.5601, 0.1936, 0.2211)\big)$,
$\mathfrak{B}_3 = \big((0.6145, 0.2379, 0.3771)\big)$, $\mathfrak{B}_4 = \big((0.6098, 0.2433, 0.3591)\big)$.

Step 4: To find out the score values, use Definition 10 for each $\mathfrak{B}_i (i = 1, 2, 3, 4, 5)$ given in step 3, i.e.

$Sc(\mathfrak{B}_1) = 0.7238$, $Sc(\mathfrak{B}_2) = 0.7151$,
$Sc(\mathfrak{B}_3) = 0.6665$, $Sc(\mathfrak{B}_4) = 0.6691$.

Step 5: Select the best solution by ranking the score values.

$$Sc(\mathfrak{B}_1) > Sc(\mathfrak{B}_2) > Sc(\mathfrak{B}_4) > Sc(\mathfrak{B}_3).$$

Note that the aggregated result for $SFS_{ft}OWA$ operator is same as result obtained for $SFS_{ft}WA$ operator. Hence "$\varkappa_1$" is the best result.

By using $SFS_{ft}HA$ operators.
Step 1: Same as above.
Step 2: Same as above.
Step 3: Using the Eq (9) for each alternative$\varkappa_i$ $(i = 1, 2, 3, 4)$, with $\mathfrak{v} = \{0.12, 0.13, 0.2, 0.4, 0.15\}$ and $\mathfrak{u} = \{0.11, 0.14, 0.2, 0.3, 0.25\}$ being the WVs of $S_{\rho_{ij}} = \big(\mathfrak{a}_{ij}, \mathfrak{b}_{ij}, \mathfrak{c}_{ij}\big)$. Also "$n$" represents the corresponding balancing coefficient for the number of elements in ith row and jth column. Let $w = \{0.15, 0.13, 0.25, 0.23, 0.24\}$ be the WV of "e_i" experts and $p = \{0.19, 0.29, 0.18, 0.23, 0.11\}$ denote the WV of "ρ_j" parameters, so we get

$\mathfrak{B}_1 = \big((0.3823, 0.6357, 0.6422)\big)$, $\mathfrak{B}_2 = \big((0.3801, 0.6379, 0.6411)\big)$,
$\mathfrak{B}_3 = \big((0.3579, 0.6347, 0.6645)\big)$, $\mathfrak{B}_4 = \big((0.3616, 0.6454, 0.6567)\big)$.

Step 4: To find out the score values, use Definition 10 for each $\mathfrak{B}_i (i = 1, 2, 3, 4, 5)$ given in step 3, i.e.

$Sc(\mathfrak{B}_1) = 0.3681$, $Sc(\mathfrak{B}_2) = 0.3670$,
$Sc(\mathfrak{B}_3) = 0.3529$, $Sc(\mathfrak{B}_4) = 0.3531$.

Step 5: Select the best solution by ranking the score values.

$$Sc(\mathfrak{B}_1) > Sc(\mathfrak{B}_2) > Sc(\mathfrak{B}_4) > Sc(\mathfrak{B}_3).$$

Hence it is noted that the aggregated result for $SFS_{ft}HA$ operator is same as result obtained for $SFS_{ft}WA$ and $SFS_{ft}OWA$ operator. Hence "$\varkappa_1$" is the best alternative.

5.2. Comparative analysis

In this section, we are desire to establish the comparative analysis of proposed work with some existing operators to discuss the superiority and validity of established work. The overall analysis is captured in the following examples.

Example 7. Let an American movie production company want to select the best movie of the year from a set of five alternatives $A = \{\varkappa_1 = Bad\ education, \varkappa_2 = The\ invisible\ man, \varkappa_3 = Birds\ of\ prey, \varkappa_4 = Onward, \varkappa_5 = Underwater\}$ under the set of parameters given as $\rho = \{\rho_1 = Casting, \rho_2 = Originality, \rho_3 = Dialogues, \rho_4 = Overall\ story, \rho_5 = Discipline\}$.
Let $w = \{0.12, 0.26, 0.16, 0.22, 0.24\}$ be the WV of "e_i" experts and $p = \{0.15, 0.21, 0.28, 0.13, 0.23\}$ denote the WV of "ρ_j" parameters. Suppose experts provide their evaluation data in the form of picture fuzzy soft numbers as shown in Table 8. We use the Garg method [20], Wei method [21], Jin et al. [24] method, Ashraf et al. [25] method to compare with established work. The overall score values and their ranking results for all these methods are given in Table 9.

Table 8. Picture fuzzy soft information.

	$\varkappa_1$	$\varkappa_2$	$\varkappa_3$	$\varkappa_4$	$\varkappa_5$
$\mathfrak{B}_1$	$((0.6, 0.1, 0.3))$	$((0.2, 0.3, 0.3))$	$((0.3, 0.4, 0.3))$	$((0.2, 0.2, 0.5))$	$((0.3, 0.2, 0.4))$
$\mathfrak{B}_2$	$((0.2, 0.3, 0.4))$	$((0.1, 0.2, 0.6))$	$((0.2, 0.1, 0.7))$	$((0.3, 0.5, 0.1_l))$	$((0.2, 0.6, 0.1))$
$\mathfrak{B}_3$	$((0.5, 0.1, 0.3))$	$((0.4, 0.1, 0.3))$	$((0.4, 0.4, 0.1))$	$((0.1, 0.3, 0.5))$	$((0.2, 0.3, 0.4))$
$\mathfrak{B}_4$	$((0.5, 0.3, 0.1))$	$((0.2, 0.2, 0.5))$	$((0.3, 1, 0.4))$	$((0.4, 0.3, 0.1))$	$((0.3, 0.1, 0.5))$
$\mathfrak{B}_5$	$((0.2, 0.3, 0.2))$	$((0.6, 0.1, 0.2))$	$((0.4, 0.4, 0.1))$	$((0.1, 0.3, 0.5))$	$((0.3, 0.3, 0.4))$

It is clear from the above analysis that "$\varkappa_4$" is the best alternative for all methods given in Table 9 which shows the validity of the proposed work. Also, note that

(1) If we use only one parameter i.e., ρ_1 mean $(m = 1)$, then $SFS_{ft}WA, SFS_{ft}OWA$ and $SFS_{ft}HA$ aggregation operators will reduce to simply spherical fuzzy weighted average (SFWA), spherical fuzzy ordered weighted average (SFOWA), and spherical fuzzy hybrid average (SFHA) aggregation operators that are discussed in Jin et al. method [24] and Ashraf et al. method [25]. It means given work is more general. Also the aggregated results for the Jin et al. method [24] and Ashraf et al. method [25] given in Table 9.

(2) If we replace 2 by 1 in the power of established operators, then $SFS_{ft}WA, SFS_{ft}OWA$ and $SFS_{ft}HA$ aggregation operators will reduce to $PFS_{ft}WA, PFS_{ft}OWA$ and $PFS_{ft}HA$ aggregation operators that show that established operators are more general. Also aggregated results of these reduced operators are given in Table 9.

(3) If we use only one parameter i.e., ρ_1 mean $(m = 1)$ and replace 2 by 1 in the power of established operators, then $SFS_{ft}WA, SFS_{ft}OWA$ and $SFS_{ft}HA$ aggregation operators will reduce to simply picture fuzzy weighted average (PFWA), picture fuzzy ordered weighted average (PFOWA), and picture fuzzy hybrid average (PFHA) aggregation operators given in Garg method [20] and Wei method [21]. So in this case, again the established operators are also more general. Also, the aggregated results for the Garg method [20] and Wei method [21] are given in Table 9.

(4) Also note that the Garg method [20]. Wei method [21], Jin et al. method [24], and Ashraf et al. method [25] are non–parameterize structure while the established work is parameterized structure, so establish work is more general.

Table 9. Overall results for all methods.

Methods	Score values					**Ranking results**
	$\varkappa_1$	$\varkappa_2$	$\varkappa_3$	$\varkappa_4$	$\varkappa_5$	
Garg method [20]	0.3624	0.3543	0.4320	0.4321	0.3781	$\varkappa_4 > \varkappa_3 > \varkappa_5 > \varkappa_1 > \varkappa_2$
Wei method [21]	0.4478	0.3640	0.4467	0.4478	0.3898	$\varkappa_4 > \varkappa_3 > \varkappa_5 > \varkappa_1 > \varkappa_2$
$\boldsymbol{PFS_{ft}WA}$	0.4620	0.4527	0.4733	0.4834	0.4627	$\varkappa_4 > \varkappa_3 > \varkappa_5 > \varkappa_1 > \varkappa_2$
$\boldsymbol{PFS_{ft}OWA}$	0.4587	0.4301	0.4995	0.5083	0.4888	$\varkappa_4 > \varkappa_3 > \varkappa_5 > \varkappa_1 > \varkappa_2$
$\boldsymbol{PFS_{ft}HA}$	0.3071	0.2942	0.3242	0.3305	0.3171	$\varkappa_4 > \varkappa_3 > \varkappa_5 > \varkappa_1 > \varkappa_2$
Jin et al. method [24]	0.4083	0.3543	0.4320	0.4434	0.4312	$\varkappa_4 > \varkappa_3 > \varkappa_5 > \varkappa_1 > \varkappa_2$
Ashraf et al. method [25]	0.3209	0.3364	0.3498	0.3501	0.3142	$\varkappa_3 > \varkappa_2 > \varkappa_5 > \varkappa_1 > \varkappa_4$
$\boldsymbol{SFS_{ft}WA}$ **operator (Proposed work)**	0.4739	0.4737	0.4911	0.4975	0.4855	$\varkappa_4 > \varkappa_3 > \varkappa_5 > \varkappa_1 > \varkappa_2$
$\boldsymbol{SFS_{ft}OWA}$ **operator (Proposed work)**	0.4145	0.4015	0.4648	0.5207	0.4579	$\varkappa_4 > \varkappa_3 > \varkappa_5 > \varkappa_1 > \varkappa_2$
$\boldsymbol{SFS_{ft}HA}$ **operator (Proposed work)**	0.2145	0.2075	0.2367	0.2489	0.2338	$\varkappa_4 > \varkappa_3 > \varkappa_5 > \varkappa_1 > \varkappa_2$

Moreover, Figure 1 shows the graphical representation of the above analysis given in Table 9.

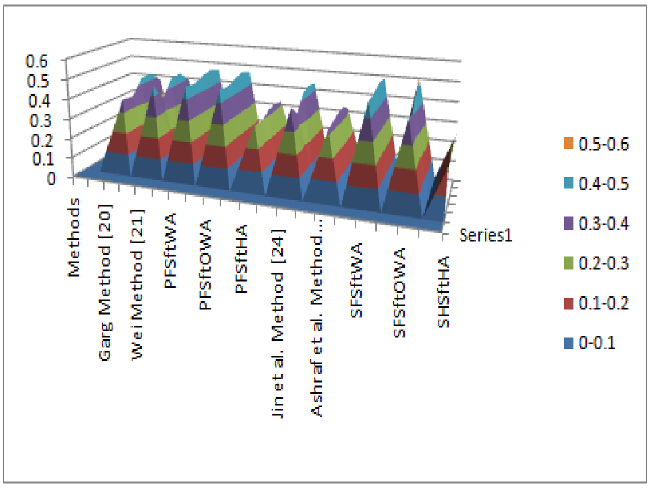

Figure 1. Graphical representations of data given in Table 9.

Example 8. Let an American movie production company want to select the best movie of the year from a set of five alternatives ($\varkappa_1 = Bad\ education, \varkappa_2 = The\ invisible\ man, \varkappa_3 = Birds\ of\ prey, \varkappa_4 = Onward, \varkappa_5 = Underwater$) under the set of parameters given as ($\rho_1 = Casting, \rho_2 = Originality, \rho_3 = Dialogues, \rho_4 = Overall\ story, \rho_5 = Discipline$). Let $w = \{0.12, 0.26, 0.16, 0.22, 0.24\}$ be the WV of "e_i" experts and $p = \{0.15, 0.21, 0.28, 0.13, 0.23\}$ denote the WV of "ρ_j" parameters.

Suppose experts provide their evaluation data in the form of spherical fuzzy soft numbers as shown in Table 10. We use the Garg method [20], Wei method [21], Jin et al. method [24], Ashraf et al. [25] method to compare with established work. The overall score values and their ranking results for all these methods are given in Table 11.

Table 10. Information based on $SFS_{ft}Ns$.

	$\varkappa_1$	$\varkappa_2$	$\varkappa_3$	$\varkappa_4$	$\varkappa_5$
$\mathfrak{B}_1$	$((0.7, 0.1, 0.2))$	$((0.7, 0.1, 0.4))$	$((0.4, 0.4, 0.4))$	$((0.5, 0.6, 0.1))$	$((0.4, 0.4, 0.4))$
$\mathfrak{B}_2$	$((0.4, 0.3, 0.4))$	$((0.5, 0.6, 0.4))$	$((0.5, 0.4, 0.6))$	$((0.4, 0.5, 0.3))$	$((0.2, 0.6, 0.5))$
$\mathfrak{B}_3$	$((0.5, 0.5, 0.3))$	$((0.5, 0.5, 0.7))$	$((0.2, 0.7, 0.3))$	$((0.1, 0.7, 0.5))$	$((0.3, 0.4, 0.5))$
$\mathfrak{B}_4$	$((0.3, 0.3, 0.6))$	$((0.5, 0.5, 0.5))$	$((0.2, 0.8, 0.1))$	$((0.3, 0.6, 0.2))$	$((0.5, 0.6, 0.3))$
$\mathfrak{B}_5$	$((0.6, 0.1, 0.4))$	$((0.4, 0.3, 0.6))$	$((0.9, 0.2, 0.2))$	$((0.5, 0.5, 0.4))$	$((0.4, 0.7, 0.4))$

Table 11. Overall results for all methods.

Methods	**Score values**					**Ranking results**
	$\varkappa_1$	$\varkappa_2$	$\varkappa_3$	$\varkappa_4$	$\varkappa_5$	
Garg method [20]	Failed	Failed	Failed	Failed	Failed	Cannot be calculated
Wei method [21]	Failed	Failed	Failed	Failed	Failed	Cannot be calculated
$PFS_{ft}WA$	Failed	Failed	Failed	Failed	Failed	Cannot be calculated
$PFS_{ft}OWA$	Failed	Failed	Failed	Failed	Failed	Cannot be calculated
$PFS_{ft}HA$	Failed	Failed	Failed	Failed	Failed	Cannot be calculated
Jin et al. method [24]	0.4583	0.3781	0.3798	0.3831	0.4286	$\varkappa_1 > \varkappa_5 > \varkappa_4 > \varkappa_2 > \varkappa_3$
Ashraf et al. method [25]	0.4321	0.4150	0.3925	0.4221	0.4230	$\varkappa_1 > \varkappa_5 > \varkappa_4 > \varkappa_2 > \varkappa_3$
$SFS_{ft}WA$ **operator (Proposed work)**	0.4709	0.4150	0.4125	0.4327	0.4379	$\varkappa_1 > \varkappa_5 > \varkappa_4 > \varkappa_2 > \varkappa_3$
$SFS_{ft}OWA$ **operator (Proposed work)**	0.5198	0.4062	0.3930	0.4604	0.5026	$\varkappa_1 > \varkappa_5 > \varkappa_4 > \varkappa_2 > \varkappa_3$
$SFS_{ft}HA$ **operator (Proposed work)**	0.3302	0.2737	0.2685	0.2977	0.3135	$\varkappa_1 > \varkappa_5 > \varkappa_4 > \varkappa_2 > \varkappa_3$

(1) It is clear that when decision-makers provide their assessment value in the form of $SFS_{ft}Ns$ then the Garg method [20], Wei method [21], $PFS_{ft}WA$ operator, $PFS_{ft}OWA$ operator and $PFS_{ft}HA$ operator fails to tackle such kind of information because when decision-maker provides the data as $(0.5, 0.4, 0.6)$, where 0.5 is MG, 0.4 is an AG and 0.6 is a NMG, then necessary condition i.e., $sum\ (0.5, 0.4, 0.6)$ must belong to [0, 1] fail to hold that is the necessary condition for the Garg method [20], Wei method [21], $PFS_{ft}WA$ operator, $PFS_{ft}OWA$ operator and $PFS_{ft}HA$ operator, while establishing work along with Jin et al. [24] method and Ashraf et al, [25] method can cope with this situation. So introduced work is more efficient.

(2) Also, the Garg method [20], Wei method [21], Jin et al. [24] method, and Ashraf et al. [25] method cannot consider the parameterization structure while the established work can do so. Also proposed work provides more space to decision-makers to deal with MCDM problems. Hence, established work is more superior to existing literature.

Furthermore, Figure 2 shows the graphical representation of the data given in Table 11.

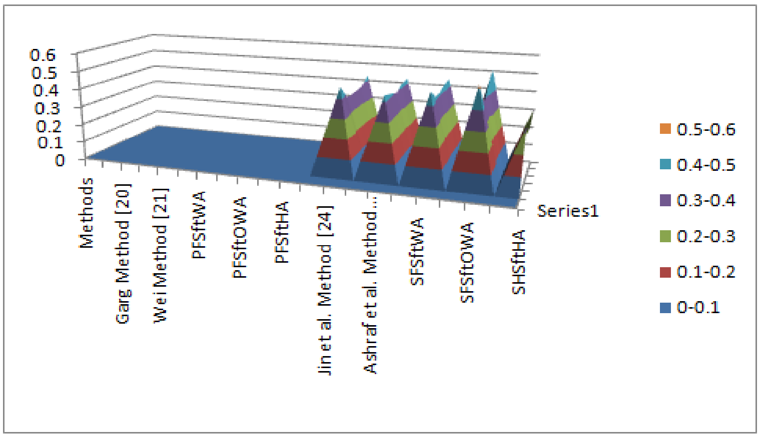

Figure 2. Graphical representations of data given in Table 11.

Example 9. Let an American movie production company want to select the best movie of the year from a set of five alternatives $(\varkappa_1 = Bad\ education, \varkappa_2 = The\ invisible\ man, \varkappa_3 = Birds\ of\ prey, \varkappa_4 = Onward)$ under the set of parameters given as $(\rho_1 = Casting, \rho_2 = Originality, \rho_3 = Dialogues, \rho_4 = Overall\ story, \rho_5 = Discipline)$. Let $w = \{0.12, 0.26, 0.16, 0.22, 0.24\}$ be the WV of "e_i" experts and $p = \{0.15, 0.21, 0.28, 0.13, 0.23\}$ denote the WV of "ρ_j" parameters. These different parameters of $SFS_{ft}Ns$ have been aggregated by using Eq (1) with $w = \{0.12, 0.26, 0.16, 0.22, 0.24\}$ and get overall decision matrix for different alternatives $\varkappa_i (i = 1, 2, 3, 4)$ given in Table 12. We still use the Garg method [20], Wei method [21], Jin et al. method [24], Ashraf et al. [25] method to compare with established work. The overall score values and their ranking results for all these methods are given in Table 13.

Table 12. Overall decision matrix based on $SFS_{ft}Ns$.

	$\varkappa_1$	$\varkappa_2$	$\varkappa_3$	$\varkappa_4$
$\mathfrak{B}_1$	$\left(\begin{pmatrix}0.1762, 0.79611,\\ 0.6786\end{pmatrix}\right)$	$\left(\begin{pmatrix}0.2342, 0.7133,\\ 0.6471\end{pmatrix}\right)$	$\left(\begin{pmatrix}0.3138, 0.6985,\\ 0.6249\end{pmatrix}\right)$	$\left(\begin{pmatrix}0.2867, 0.7224,\\ 0.6047\end{pmatrix}\right)$
$\mathfrak{B}_2$	$\left(\begin{pmatrix}0.1144, 0.7166,\\ 0.6843\end{pmatrix}\right)$	$\left(\begin{pmatrix}0.2573, 0.7094,\\ 0.6105\end{pmatrix}\right)$	$\left(\begin{pmatrix}0.1850, 0.7018,\\ 0.6849\end{pmatrix}\right)$	$\left(\begin{pmatrix}0.2517, 0.7435,\\ 0.6121\end{pmatrix}\right)$
$\mathfrak{B}_3$	$\left(\begin{pmatrix}0.1159, 0.7123,\\ 0.6553\end{pmatrix}\right)$	$\left(\begin{pmatrix}0.1919, 0.7366,\\ 0.6409\end{pmatrix}\right)$	$\left(\begin{pmatrix}0.1588, 0.7036,\\ 0.6578\end{pmatrix}\right)$	$\left(\begin{pmatrix}0.2310, 0.7446,\\ 0.6247\end{pmatrix}\right)$
$\mathfrak{B}_4$	$\left(\begin{pmatrix}0.16783, 0.7020,\\ 0.6618\end{pmatrix}\right)$	$\left(\begin{pmatrix}0.1615, 0.7314,\\ 0.6619\end{pmatrix}\right)$	$\left(\begin{pmatrix}0.1638, 0.7493,\\ 0.6376\end{pmatrix}\right)$	$\left(\begin{pmatrix}0.1633, 0.74936,\\ 0.6398\end{pmatrix}\right)$
$\mathfrak{B}_5$	$\left(\begin{pmatrix}0.2461, 0.6879,\\ 0.6714\end{pmatrix}\right)$	$\left(\begin{pmatrix}0.1018, 0.7445,\\ 0.6367\end{pmatrix}\right)$	$\left(\begin{pmatrix}0.2620, 0.7113,\\ 0.6153\end{pmatrix}\right)$	$\left(\begin{pmatrix}0.2620, 0.7055,\\ 0.6458\end{pmatrix}\right)$

Table 13. Overall results for all methods.

Methods	Score values				**Ranking results**
	$\varkappa_1$	$\varkappa_2$	$\varkappa_3$	$\varkappa_4$	
Garg method [20]	Failed	Failed	Failed	Failed	Cannot be calculated
Wei method [21]	Failed	Failed	Failed	Failed	Cannot be calculated
$\boldsymbol{PFS_{ft}WA}$	Failed	Failed	Failed	Failed	Cannot be calculated
$\boldsymbol{PFS_{ft}OWA}$	Failed	Failed	Failed	Failed	Cannot be calculated
$\boldsymbol{PFS_{ft}HA}$	Failed	Failed	Failed	Failed	Cannot be calculated
Jin et al. method [24]	0.3158	0.3312	0.3498	0.3323	$\varkappa_3 > \varkappa_4 > \varkappa_2 > \varkappa_1$
Ashraf et al. method [25]	0.2999	0.3102	0.3223	0.3192	$\varkappa_3 > \varkappa_4 > \varkappa_2 > \varkappa_1$
$\boldsymbol{SFS_{ft}WA}$ **operator (Proposed work)**	0.3706	0.3733	0.3752	0.3742	$\varkappa_3 > \varkappa_4 > \varkappa_2 > \varkappa_1$
$\boldsymbol{SFS_{ft}OWA}$ **operator (Proposed work)**	0.3731	0.3740	0.3762	0.3748	$\varkappa_3 > \varkappa_4 > \varkappa_2 > \varkappa_1$
$\boldsymbol{SFS_{ft}HA}$ **operator (Proposed work)**	0.2479	0.2491	0.2504	0.2496	$\varkappa_3 > \varkappa_4 > \varkappa_2 > \varkappa_1$

It is clear that the overall information given in Table 12 again consist of $SFS_{ft}Ns$ and this type of information cannot be tackles by the Garg method [20], Wei method [21], $PFS_{ft}WA$ operator, $PFS_{ft}OWA$ operator and $PFS_{ft}HA$ operator, because necessary condition i.e.,

$sum(0.1919, 0.7366, 0.6409)$ fail to hold for all above-given methods for the data $\begin{pmatrix}0.1919, 0.7366,\\ 0.6409\end{pmatrix}$ given in Table 12, while established work can handle this kind of information. So, the proposed work is more general. Also, we can see from Table 13 that the Garg method [20], Wei method [21], Jin et al. method [24], and Ashraf et al method [25] cannot consider parameterization structure, while established work can do so. Hence, the proposed operators are more superior to that of the existing operators. Also, graphical representation of data given in Table 13 is given in Figure 3.

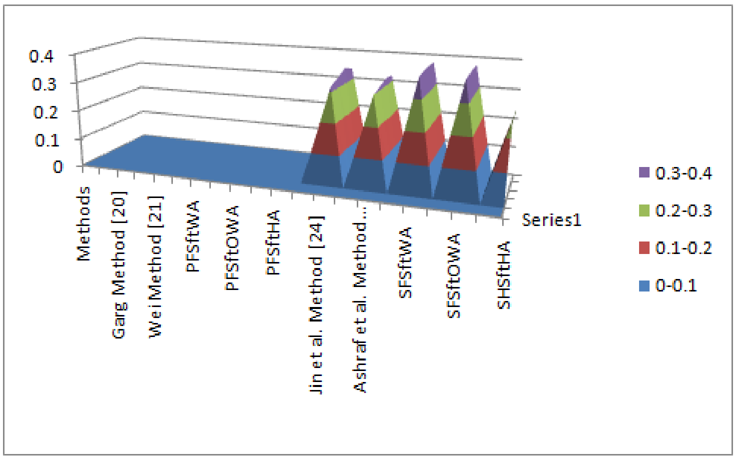

Figure 3. Graphical representations of data given in Table 13.

Table 14. Information based on $SFS_{ft}Ns$.

	$\varkappa_1$	$\varkappa_2$	$\varkappa_3$	$\varkappa_4$	$\varkappa_5$
$\mathfrak{B}_1$	$\left(\begin{pmatrix}0.71, 0.10,\\ 0.21\end{pmatrix}\right)$	$\left(\begin{pmatrix}0.7, 0.19,\\ 0.34\end{pmatrix}\right)$	$\left(\begin{pmatrix}0.70, 0.34,\\ 0.43\end{pmatrix}\right)$	$\left(\begin{pmatrix}0.71, 0.46,\\ 0.45\end{pmatrix}\right)$	$\left(\begin{pmatrix}0.70, 0.34,\\ 0.54\end{pmatrix}\right)$
$\mathfrak{B}_2$	$\left(\begin{pmatrix}0.61, 0.24,\\ 0.16\end{pmatrix}\right)$	$\left(\begin{pmatrix}0.61, 0.36,\\ 0.44\end{pmatrix}\right)$	$\left(\begin{pmatrix}0.62, 0.33,\\ 0.42\end{pmatrix}\right)$	$\left(\begin{pmatrix}0.63, 0.35,\\ 0.33\end{pmatrix}\right)$	$\left(\begin{pmatrix}0.64, 0.40,\\ 0.50\end{pmatrix}\right)$
$\mathfrak{B}_3$	$\left(\begin{pmatrix}0.55, 0.45,\\ 0.47\end{pmatrix}\right)$	$\left(\begin{pmatrix}0.53, 0.52,\\ 0.33\end{pmatrix}\right)$	$\left(\begin{pmatrix}0.52, 0.37,\\ 0.43\end{pmatrix}\right)$	$\left(\begin{pmatrix}0.51, 0.37,\\ 0.54\end{pmatrix}\right)$	$\left(\begin{pmatrix}0.56, 0.41,\\ 0.51\end{pmatrix}\right)$
$\mathfrak{B}_4$	$\left(\begin{pmatrix}0.43, 0.43,\\ 0.36\end{pmatrix}\right)$	$\left(\begin{pmatrix}0.44, 0.35,\\ 0.65\end{pmatrix}\right)$	$\left(\begin{pmatrix}0.46, 0.48,\\ 0.12\end{pmatrix}\right)$	$\left(\begin{pmatrix}0.47, 0.46,\\ 0.24\end{pmatrix}\right)$	$\left(\begin{pmatrix}0.45, 0.62,\\ 0.3\end{pmatrix}\right)$
$\mathfrak{B}_5$	$\left(\begin{pmatrix}0.65, 0.11,\\ 0.41\end{pmatrix}\right)$	$\left(\begin{pmatrix}0.63, 0.32,\\ 0.61\end{pmatrix}\right)$	$\left(\begin{pmatrix}0.71, 0.12,\\ 0.22\end{pmatrix}\right)$	$\left(\begin{pmatrix}0.70, 0.35,\\ 0.44\end{pmatrix}\right)$	$\left(\begin{pmatrix}0.64, 0.60,\\ 0.41\end{pmatrix}\right)$

Example 10. During the pandemic situation of Covid-19, the selection of Covid-19 vaccine is difficult challenge for the countries. Let the a country X want to import the best vaccine for their Covide-19 patients. Let the set $P = \{\varkappa_1 = Pfizer - BioNTech, \varkappa_2 = Sinopharm, \varkappa_3 = Oxford - Astrazeneca, \varkappa_4 = Novavax, \varkappa_5 = Moderna\}$ denote the set of different vaccines as an alternative under the parameter set given as $\rho = \{\rho_1 = Protection\ against\ disease, \rho_2 = Side\ effects, \rho_3 = Delivery\ time, \rho_4 = Effectiveness, \rho_5 = Experimental\ results\}$. Let $w = \{0.22, 0.26, 0.11, 0.28, 0.13\}$ be the WV of "e_i" experts and $p = \{0.25, 0.19, 0.24, 0.10, 0.22\}$ denote the WV of "ρ_j" parameters. We use the Garg method [20],

Wei method [21], Jin et al. method [24], Ashraf et al. [25] method and Deli and Broumi methods [51–52] to compare with established work. Now we use the data given in Table 14 provided by the experts in the form of $SFS_{ft}Ns$ and overall score values and their ranking results for all above given methods are given in Table 15.

Table 15. Overall results for all methods.

Methods	**Score values**					**Ranking results**
	$\varkappa_1$	$\varkappa_2$	$\varkappa_3$	$\varkappa_4$	$\varkappa_5$	
Garg method [20]	Failed	Failed	Failed	Failed	Failed	Cannot be calculated
Wei method [21]	Failed	Failed	Failed	Failed	Failed	Cannot be calculated
$\boldsymbol{PFS_{ft}WA}$	Failed	Failed	Failed	Failed	Failed	Cannot be calculated
$\boldsymbol{PFS_{ft}OWA}$	Failed	Failed	Failed	Failed	Failed	Cannot be calculated
$\boldsymbol{PFS_{ft}HA}$	Failed	Failed	Failed	Failed	Failed	Cannot be calculated
Jin et al. method [24]	0.4634	0.3833	0.3437	0.3926	0.4359	$\varkappa_1 > \varkappa_5 > \varkappa_4 > \varkappa_2 > \varkappa_3$
Ashraf et al. method [25]	0.4521	0.4323	0.4125	0.4401	0.4424	$\varkappa_1 > \varkappa_5 > \varkappa_4 > \varkappa_2 > \varkappa_3$
Deli and Broumi Method [51]	0.69	0.72	0.53	0.90	0.83	$\varkappa_4 > \varkappa_5 > \varkappa_3 > \varkappa_2 > \varkappa_1$
Deli and Broumi Method [52]	0.4516	0.4100	0.2292	0.0270	0.2640	$\varkappa_1 > \varkappa_2 > \varkappa_5 > \varkappa_3 > \varkappa_4$
$\boldsymbol{SFS_{ft}WA}$ **operator (Proposed work)**	0.5123	0.4424	0.4413	0.4621	0.4724	$\varkappa_1 > \varkappa_5 > \varkappa_4 > \varkappa_2 > \varkappa_3$
$\boldsymbol{SFS_{ft}OWA}$ **operator (Proposed work)**	0.5524	0.44314	0.4312	0.4923	0.55414	$\varkappa_1 > \varkappa_5 > \varkappa_4 > \varkappa_2 > \varkappa_3$
$\boldsymbol{SFS_{ft}HA}$ **operator (Proposed work)**	0.3549	0.2912	0.2908	0.3181	0.3379	$\varkappa_1 > \varkappa_5 > \varkappa_4 > \varkappa_2 > \varkappa_3$

(1) It is clear that the best alternative i.e., $\varkappa_1 = Pfizer - BioNTech$ for Jin et al. method [24], Ashraf et al. method [25], $SFS_{ft}WA, SFS_{ft}OWA$ and $SFS_{ft}HA$ aggregation operators are the same that show the validity of introduced work.

(2) Also note that the results for Deli and Broumi [51] and Deli and Broumi [52] are slightly different from the results for the introduced operators. It is because the methods that are given in [51] and [52] are based on neutrosophic soft set ($NS_{ft}S$) and $NS_{ft}S$ do not consider the refusal grade (RG) while computing the scores. Infect there is no concept of RG in the neutrosophic soft set, while the established work can do so. That is the reason that the introduced work and methods that are given in [51] and [52] produce different results.

5.3. Conclusion

In the basic notions of $FS_{ft}S, IFS_{ft}, P_yFS_{ft}S$ and $q-ROFS_{ft}S,$ the yes or no type of aspects have been denoted by MG or NMG. But note that, in real-life problems, human opinion is not restricted to MG and NMD but it has AG or RG as well. So the all above given methods cannot cope with this situation, while $SFS_{ft}S$ has the characteristics to handle this situation. Since the MCDM method is a renowned method for the selection of the best alternative among a given one and aggregation operators are very efficient apparatus to convert the overall information into a single value so based on spherical fuzzy soft set $SFS_{ft}S$, the notions of SFS_{ft} average aggregation operators are introduced like spherical fuzzy soft weighted average aggregation $(SFS_{ft}WA)$ operator, spherical fuzzy soft ordered weighted average aggregation $(SFS_{ft}OWA)$ operator and spherical fuzzy soft hybrid average aggregation $(SFS_{ft}HA)$ operator. Moreover, the properties of these aggregation operators are discussed in detail. An algorithm is established and a numerical example is given to show the authenticity of established work. Furthermore, a comparative study is proposed with other existing methods to show the strength and advantages of established work.

In the future direction, based on the operational laws for $SFS_{ft}S,$ some other aggregation operators and similarities measure for medical diagnosis and pattern recognition can be defined as given in [47–48]. Furthermore, this work can be extended to a T-spherical fuzzy set and real-life problems can be resolved as given in [49–50].

Acknowledgments

This paper was supported by Algebra and Applications Research Unit, Division of Computational Science, Faculty of Science, Prince of Songkla University.

Conflict of interest

The authors declare no conflict of interest.

References

1. G. J. Klir, B. Yuan, *Fuzzy sets, fuzzy logic, and fuzzy systems: selected papers by Lotfi A Zadeh*, Vol. 6, World Scientific, 1996.
2. K. T. Atanassov, New operations defined over the intuitionistic fuzzy sets, *Fuzzy set. Syst.*, **61** (1994), 137–142.
3. H. Zhao, Z. Xu, M. Ni, S. Liu, Generalized aggregation operators for intuitionistic fuzzy sets, *Int. J. Intell. Syst.*, **25** (2010), 1–30.
4. Z. Xu, Intuitionistic fuzzy aggregation operators, *IEEE T. Fuzzy Syst.*, **15** (2007), 1179–1187.

5. Y. He, H. Chen, L. Zhou, J. Liu, Z. Tao, Intuitionistic fuzzy geometric interaction averaging operators and their application to multi-criteria decision making, *Inform. Sciences*, **259** (2014), 142–159.
6. K. T. Atanassov, Interval valued intuitionistic fuzzy sets. In *Intuitionistic Fuzzy Sets* (pp. 139–177), Physica, Heidelberg, 1999.
7. E. Szmidt, J. Kacprzyk, Intuitionistic fuzzy sets in group decision making, *Notes on IFS*, **2** (1996).
8. Z. Liang, P. Shi, Similarity measures on intuitionistic fuzzy sets, *Pattern Recogn. Lett*., **24** (2003), 2687–2693.
9. V. L. G. Nayagam, S. Muralikrishnan, G. Sivaraman, Multi-criteria decision-making method based on interval-valued intuitionistic fuzzy sets, *Expert Syst. Appl*., **38** (2011), 1464–1467.
10. Q. S. Zhang, S. Jiang, B. Jia, S. Luo, Some information measures for interval-valued intuitionistic fuzzy sets, *Inform. sciences*, **180** (2010), 5130–5145.
11. R. R. Yager, Pythagorean fuzzy subsets, *2013 joint IFSA world congress and NAFIPS annual meeting (IFSA/NAFIPS)*, (2013), 57–61.
12. A. A. Khan, S. Ashraf, S. Abdullah, M. Qiyas, J. Luo, S. U. Khan, Pythagorean fuzzy Dombi aggregation operators and their application in decision support system, *Symmetry*, **11** (2019), 383.
13. G. Wei, Pythagorean fuzzy interaction aggregation operators and their application to multiple attribute decision making, *J. Intell. Fuzzy Syst*., **33** (2017), 2119–2132.
14. R. R. Yager, Generalized orthopair fuzzy sets, *IEEE T. Fuzzy Syst*., **25** (2016), 1222–1230.
15. G. Wei, H. Gao, Y. Wei, Some q-rung orthopair fuzzy Heronian mean operators in multiple attribute decision making, *Int. J. Intell. Syst*., **33** (2018), 1426–1458.
16. P. Liu, P. Wang, Multiple-attribute decision-making based on Archimedean Bonferroni Operators of q-rung orthopair fuzzy numbers, *IEEE T. Fuzzy Syst*., **27** (2018), 834–848.
17. B. C. Cuong, Picture fuzzy sets-first results. Part 2, seminar neuro-fuzzy systems with applications, *Institute of Mathematics*, Hanoi, 2013.
18. B. C. Cuong, P. Van Hai, Some fuzzy logic operators for picture fuzzy sets. In *2015 seventh international conference on knowledge and systems engineering (KSE)* (pp. 132–137). IEEE, 2015.
19. C. Wang, X. Zhou, H. Tu, S. Tao, Some geometric aggregation operators based on picture fuzzy sets and their application in multiple attribute decision making, *Ital. J. Pure Appl. Math.*, **37** (2017), 477–492.
20. H. Garg, Some picture fuzzy aggregation operators and their applications to multicriteria decision-making, *Arab. J. Sci. Eng*., **42** (2017), 5275–5290.
21. G. Wei, Picture fuzzy aggregation operators and their application to multiple attribute decision making, *J. Intell. Fuzzy Syst*., **33** (2017), 713–724.
22. S. Zeng, M. Qiyas, M. Arif, T. Mahmood, Extended version of linguistic picture fuzzy TOPSIS method and its applications in enterprise resource planning systems, *Math. Probl. Eng*., 2019.
23. T. Mahmood, K. Ullah, Q. Khan, N. Jan, An approach toward decision-making and medical diagnosis problems using the concept of spherical fuzzy sets, *Neural Computing and Applications*, **31** (2019), 7041–7053.
24. Y. Jin, S. Ashraf, S. Abdullah, Spherical fuzzy logarithmic aggregation operators based on entropy and their application in decision support systems, *Entropy*, **21** (2019), 628.

25. S. Ashraf, S. Abdullah, T. Mahmood, F. Ghani, T. Mahmood, Spherical fuzzy sets and their applications in multi-attribute decision making problems. *J. Intell. Fuzzy Syst*., **36** (2019), 2829–2844.
26. Y. Donyatalab, E. Farrokhizadeh, S. D. S. Garmroodi, S. A. S. Shishavan, Harmonic Mean Aggregation Operators in Spherical Fuzzy Environment and Their Group Decision Making Applications, *J. Multi-Valued Log. S*., **33** (2019).
27. S. Ashraf, S. Abdullah, T. Mahmood, Spherical fuzzy Dombi aggregation operators and their application in group decision making problems, *J. Amb. Intel. Hum. Comp*., (2019), 1–19.
28. S. Ashraf, S. Abdullah, T. Mahmood, GRA method based on spherical linguistic fuzzy Choquet integral environment and its application in multi-attribute decision-making problems, *Math. Sci*., **12** (2018), 263–275.
29. Z. Ali, T. Mahmood, M. S. Yang, TOPSIS Method Based on Complex Spherical Fuzzy Sets with Bonferroni Mean Operators, *Mathematics,* **8** (2020), 1739.
30. D. Molodtsov, Soft set theory—first results, *Comput. Math. Appl*., **37** (1999), 19–31.
31. P. K. Maji, R. Biswas, A. Roy, Soft set theory, *Comput. Math. Appl*., **45** (2003), 555–562.
32. P. K. Maji, A. R. Roy, R. Biswas, An application of soft sets in a decision making problem, *Comput. Math. Appl*., **44** (2002), 1077–1083.
33. P. K. Maji, R. K. Biswas, A. Roy, *Fuzzy soft sets*, 2001
34. Y. B. Jun, K. J. Lee, C. H. Park, Fuzzy soft set theory applied to BCK/BCI-algebras, *Comput. Math. Appl.*, **59** (2010), 3180–3192.
35. Z. Kong, L. Wang, Z. Wu, Application of fuzzy soft set in decision making problems based on grey theory, *J. Comput. Appl. Math*., **236** (2011), 1521–1530.
36. T. J. Neog, D. K. Sut, An application of fuzzy soft sets in medical diagnosis using fuzzy soft complement, *Int. J. Comput. Appl*., **33** (2011), 30–33.
37. P. K. Maji, A. R. Roy, R. Biswas, On intuitionistic fuzzy soft sets, *Journal of fuzzy mathematics*, **12** (2004), 669–684.
38. H. Garg, R. Arora, Bonferroni mean aggregation operators under intuitionistic fuzzy soft set environment and their applications to decision-making, *J. Oper. Res. Soc*., **69** (2018), 1711–1724.
39. I. Deli, N. Çağman, Intuitionistic fuzzy parameterized soft set theory and its decision making, *Appl. Soft Comput*., **28** (2015), 109–113.
40. X. D. Peng, Y. Yang, J. P. Song, Y. Jiang, Pythagorean fuzzy soft set and its application, *Computer Engineering*, **41** (2015), 224–229.
41. A. Hussain, M. I. Ali, T. Mahmood, M. Munir, q-Rung orthopair fuzzy soft average aggregation operators and their application in multicriteria decision-making, *Int. J. Intell. Syst*., **35** (2020), 571–599.
42. Y. Yang, C. Liang, S. Ji, T. Liu, Adjustable soft discernibility matrix based on picture fuzzy soft sets and its applications in decision making, *J. Intell. Fuzzy Syst*., **29** (2015), 1711–1722.
43. N. Jan, T. Mahmood, L. Zedam, Z. Ali, Multi-valued picture fuzzy soft sets and their applications in group decision-making problems, *Soft Comput*., **24** (2020), 18857–18879.
44. F. Perveen PA, J. J. Sunil, K. V. Babitha, H. Garg, Spherical fuzzy soft sets and its applications in decision-making problems, *J. Intell. Fuzzy Syst*., **37** (2019), 8237–8250.
45. I. Deli, Interval-valued neutrosophic soft sets and its decision making, *Int. J. Mach. Learn. Cyb*., **8** (2017), 665–676.

46. M. Ali, L. H. Son, I. Deli, N. D. Tien, Bipolar neutrosophic soft sets and applications in decision making, *J. Intell. Fuzzy Syst.*, **33** (2017), 4077–4087.
47. Y. Donyatalab, E. Farrokhizadeh, S. D. S. Garmroodi, S. A. S. Shishavan, Harmonic Mean Aggregation Operators in Spherical Fuzzy Environment and Their Group Decision Making Applications, *J. Multi-Valued Log. S.*, **33** (2019).
48. T. Mahmood, M. Ilyas, Z. Ali, A. Gumaei, Spherical Fuzzy Sets-Based Cosine Similarity and Information Measures for Pattern Recognition and Medical Diagnosis, *IEEE Access*, **9** (2021), 25835–25842.
49. K. Ullah, T. Mahmood, H. Garg, Evaluation of the performance of search and rescue robots using T-spherical fuzzy hamacher aggregation operators, *Int. J. Fuzzy Syst.*, **22** (2020), 570–582.
50. S. Zeng, M. Munir, T. Mahmood, M. Naeem, Some T-Spherical Fuzzy Einstein Interactive Aggregation Operators and Their Application to Selection of Photovoltaic Cells, *Math. Probl. Eng.*, 2020.
51. I. Deli, S. Broumi, Neutrosophic soft relations and some properties, *Annals of fuzzy mathematics and informatics*, **9** (2015), 169–182.
52. I. Deli, S. Broumi, Neutrosophic soft matrices and NSM-decision making, *J. Intell. Fuzzy Syst.*, **28** (2015), 2233–2241.