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Research article

A new product of weighted differentiation and superposition operators between Hardy and Zygmund Spaces

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Abstract: Our goal of this article is to introduce a new product operator that will be called $D_u^n S_\phi$ the product of weighted differentiation and superposition operators from H^∞ to Zygmund spaces. Moreover, we characterize a necessary and sufficient conditions for $D_u^n S_\phi$ operators from H^∞ to Zygmund spaces to be bounded and compact.

Keywords: Superposition operator; weighted differentiation operator; compactness; $D_u^n S_\phi$ operators
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1. Introduction

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk in the complex plane $\mathbb{C}$, $H(\mathbb{D})$ denote the class of all analytic functions in $\mathbb{D}$. Let ϕ be a complex-valued function in the plane $\mathbb{C}$. The superposition operator S_ϕ defined as follow (see [2]):

Definition 1. *Let X and Y be two metric spaces of $H(\mathbb{D})$ and ϕ denote a complex-valued function in the plane $\mathbb{C}$ such that $\phi \circ f \in Y$ whenever $f \in X$, we say that ϕ acts by superposition from X into Y and the superposition operator S_ϕ on X is defined by*

$$S_\phi(f) = \phi \circ f, \qquad f \in X. \tag{1.1}$$

Observe that if, X contains the linear functions and S_ϕ maps X into Y, then S_ϕ must be an entire function.

The problem of boundedness and compactness of S_ϕ has been studied in many Banach spaces of analytic functions and the study of such operators has recently attracted the most attention (see [1, 6–8, 15, 16] and others).

Now, we will introduce a class of nonlinear operators as follows:

Definition 2. *Let n be a nonnegative integer, $u \in H(\mathbb{D})$ and ϕ is a non constant analytic self-map of $\mathbb{D}$. The weighted differentiation superposition operators $(D_u^n S_\phi f)(z)$ defined as*

$$(D_u^n S_\phi f)(z) = u(z)\phi^{(n)}(f(z)), \quad z \in \mathbb{D}, \quad f \in H(\mathbb{D}). \tag{1.2}$$

The Bloch type space defined as follows (see [13, 16]).

Definition 3. *An analytic function f is said to belong to the Bloch space $\mathcal{B}$ if*

$$\sup_{z\in\mathbb{D}}(1 - |z|^2)|f'(z)| < \infty, \tag{1.3}$$

while the little Bloch spaces $\mathcal{B}_0 \subset \mathcal{B}$ consisting of all functions analytic in $\mathbb{D}$ for which

$$\lim_{|z|\to 1^-}(1 - |z|^2)|f'(z)| = 0.$$

Now we present the needed spaces and some facts. The Hardly space can be defined as follows (see [17]).

Definition 4. *The space H^∞ denotes the space of all bounded analytic functions f on the unit disk $\mathbb{D}$ such that*

$$\|f\|_\infty = \sup_{z\in\mathbb{D}} |f(z)| < \infty.$$

Definition 5. *Let $\mathcal{Z}$ denote the space of all $f \in H(\mathbb{D}) \cap C(\bar{\mathbb{D}})$ such that*

$$\|f\|_{\mathcal{Z}} = \sup_{z\in\mathbb{D}} \frac{|f(e^{i\theta+h}) + f(e^{i\theta-h} - 2f(e^{i\theta})|}{h} < \infty,$$

where the supremum is taken over all $e^{i\theta} \in \partial\mathbb{D}$ which denote the boundary of $\mathbb{D}$ and $h > 0$. By the Zygmund theorem and the closed-graph theorem (see [5], Theorem 5.3), we see that $f \in \mathcal{Z}$ if and only if

$$\sup_{z\in\mathbb{D}}(1 - |z|^2)|f''(z)| < \infty.$$

Moreover, the following asymptotic relation holds:

$$\|f\|_{\mathcal{Z}} \asymp \sup_{z\in\mathbb{D}}(1 - |z|^2)|f''(z)| < \infty, \tag{1.4}$$

Therefore, $\mathcal{Z}$ is called the Zygmund class. Since the quantities in (1.4) are semi norms, it is natural to add them the quantity $|f(0)| + |f'(0)|$ to obtain two equivalent norms on the Zygmund class.The Zygmund class with such defined norm will be called the Zygmund space. Some information on Zygmund type spaces on the unit disk and some operators on them can be found in (see [3, 10–12]). This norm will be again denoted by $\| \, . \, \|_{\mathcal{Z}}$. The little Zygmund space $\mathcal{Z}_0$ was introduced by Li and Stević (see [9]) in the following natural way:

$$f \in \mathcal{Z}_0 \Leftrightarrow \lim_{|z|\to 1} |(1-|z|^2)|f''(z)| = 0.$$

It is easy to see that $\mathcal{Z}_0$ is a closed subspace of $\mathcal{Z}$ and the set of all polynomials is dense in $\mathcal{Z}_0$.

Now, we will introduce the definition of boundedness and compactness of the operator $D_u^n S_\phi$: $H^\infty \to \mathcal{Z}$.

Definition 6. *The operators $D_u^n S_\phi : H^\infty \to \mathcal{Z}$ is said to be bounded, if there is a positive constant C such that $\|D_u^n S_\phi f\|_{\mathcal{Z}} \leq C\|f\|_\infty$ for all $f \in H^\infty$.*

Definition 7. *The operators $D_u^n S_\phi : H^\infty \to \mathcal{Z}$ is said to be compact, if it maps any function in unit disk in H^∞ onto a pre-compact set in $\mathcal{Z}$.*

The notation a $\preceq b$ means that there is a positive constant C such that a $\leq Cb$. Also, the notation a $\asymp$ b means that a $\preceq$ b and b $\preceq$ a hold.

In this paper, we study a concerned class of weighted differentiation superposition operators $D_u^n S_\phi$. Furthermore, It has made the discussions on the boundedness and compactness property of the new class of operators from H^∞ to Zygmund spaces. Finally, it has also provided the conditions which grant the product operators $D_u^n S_\phi$ be bounded and compact.

2. The Boundedness of $D_u^n S_\phi : H^\infty \to \mathcal{Z}$

Now we characterize the boundedness of the operators $D_u^n S_\phi : H^\infty \to \mathcal{Z}$.
First we enumerate several useful lemmas. The first one below is well-known.

Lemma 1. *(see([13])) Assume that $f \in H^\infty$. Then for each $n \in \mathbb{N}$, there is a positive constant C independent of f such that*

$$\sup_{z\in\mathbb{D}} (1- | z |)^n | f^{(n)}(z) | \leq \| f \|_\infty .$$

The following lemma is introduced in (see [18]).

Lemma 2. *Assume that $f \in \mathcal{B}$. Then for each $n \in \mathbb{N}$.*

$$\| f \|_{\mathcal{B}} \asymp \sum_{j=0}^{n-1} | f^{(j)}(0)| + \sup_{z\in\mathbb{D}} (1- | z |^2)^n | f^{(n)}(z) |.$$

Theorem 1. *Suppose ϕ be an entire function and $u \in H(\mathbb{D})$. Then $D_u^n S_\phi : H^\infty \to \mathcal{Z}$ bounded if and only if the following conditions are satisfied,*

$$\sup_{z\in\mathbb{D}} \frac{(1- | z |^2) | u''(z) |}{(1- | z |^{2n})^n} < \infty, \tag{2.1}$$

$$\sup_{z\in\mathbb{D}} \frac{(1- | z |^2) | 2nz^{n-1}u'(z) + n(n-1)z^{n-2}u(z) |}{(1- | z |^{2n})^{n+1}} < \infty, \tag{2.2}$$

and

$$\sup_{z\in\mathbb{D}} \frac{(1-\mid z\mid^2)\mid n^2 z^{2n-2}u(z)\mid}{(1-\mid z\mid^{2n})^{n+2}} < \infty. \tag{2.3}$$

Proof. First direction, we assume that conditions (2.1) – (2.3) hold. So, for every $z \in \mathbb{D}$ and $f \in H^\infty$, by using Lemma 1, we have

$$\begin{aligned}
& \sup_{z\in\mathbb{D}} \mid (1-\mid z\mid^2)(D_u^n S_\phi f)^{''}(z)\mid \\
= & \sup_{z\in\mathbb{D}} (1-\mid z\mid^2)\mid (u(z)\phi^{(n)}(f(z)))^{''}\mid \\
+ & u(z)f^{''}(z)\phi^{(n+1)}(f(z)) + u(z)(f^{'}(z))^2\phi^{(n+2)}\mid \\
\leq & \sup_{z\in\mathbb{D}} (1-\mid z\mid^2)\mid u^{''}(z)\mid \phi^{(n)}(f(z)) + (1-\mid z\mid^2)\mid 2u^{'}(z)f^{'}(z) \\
+ & u(z)f^{''}(z)\mid \phi^{(n+1)}(f(z)) + (1-\mid z\mid^2)\mid u(z)(f^{'}(z))^2\mid \phi^{(n+2)}(f(z)) \\
\leq & \sup_{z\in\mathbb{D}} C(1-\mid z\mid^2)\Big[\frac{\mid u^{''}(z)\mid}{(1-\mid f(z)\mid^2)^n} + \frac{\mid 2u^{'}(z)f^{'}(z) + u(z)f^{''}(z)\mid}{(1-\mid f(z)\mid^2)^{n+1}} \\
+ & \frac{\mid u(z)(f^{'}(z))^2\mid}{(1-\mid f(z)\mid^2)^{n+2}}\Big] \parallel \phi \parallel_\infty.
\end{aligned} \tag{2.4}$$

Since, if we take $f(z) = z^n$, we have

$$\begin{aligned}
& \sup_{z\in\mathbb{D}} \mid (1-\mid z\mid^2)(D_u^n S_\phi f)^{''}(z)\mid \leq C(1-\mid z\mid^2)\Big[\frac{\mid u^{''}(z)\mid}{(1-\mid z\mid^{2n})^n} \\
+ & \frac{\mid 2nz^{n-1}u^{'}(z) + n(n-1)z^{n-2}u(z)\mid}{(1-\mid z\mid^{2n})^{n+1}} + \frac{\mid n^2 z^{2n-2}u(z)\mid}{(1-\mid z\mid^{2n})^{n+2}}\Big] \parallel \phi \parallel_\infty.
\end{aligned}$$

On the other hand, we obtain

$$\begin{aligned}
\mid (D_u^n S_\phi f)(0)\mid &= \mid u(0)D_u^n S_\phi(f(0))\mid \\
&\leq C\frac{\mid u(0)\mid}{(1-\mid f(0)\mid^2)^n} \parallel \phi \parallel_\infty,
\end{aligned}$$

and

$$\begin{aligned}
\mid (D_u^n S_\phi f)^{'}(0)\mid &= \mid u^{'}(0)\phi^{(n)}(f(0)) + u(0)f^{'}(0)\phi^{(n+1)}(f(0))\mid \\
&\leq C\Big(\frac{\mid u^{'}(0)\mid}{(1-\mid f(0)\mid^2)^n} + \frac{\mid u(0)f^{'}(0)\mid}{(1-\mid f(0)\mid^2)^{n+1}}\Big) \parallel \phi \parallel_\infty .
\end{aligned}$$

From the fact $\mid f(0)\mid < 1$ and by applying the conditions (2.1) – (2.3), it follows that the operators $D_u^n S_\phi : H^\infty$ *or* $\mathcal{B} \to \mathcal{Z}$ is bounded.
Now, we will prove the second direction, assume that $D_u^n S_\phi : H^\infty \to \mathcal{Z}$ is bounded, this means that there exists a constant C such that

$$\parallel D_u^n S_\phi f \parallel_{\mathcal{Z}} \leq C \parallel f \parallel_\infty .$$

For all $f \in H^\infty$. From the above inequality and by taking the function $\phi(z) = z^n$ we have

$$\sup_{z\in\mathbb{D}}(1-|z|^2)|u''(z)| \leq C. \tag{2.5}$$

By taking the function $\phi(z) = z^{n+1}$. From the fact that $\|\phi\|_\infty \leq 1$ and using (2.5), it follows that

$$\begin{aligned}
&\sup_{z\in\mathbb{D}}(1-|z|^2)|2u'(z)f'(z)+u(z)f''(z)| \\
\leq\ &\sup_{z\in\mathbb{D}}(1-|z|^2)|u''(z)f(z)| + \sup_{z\in\mathbb{D}}(1-|z|^2)|2u'(z)f'(z)+u(z)f''(z)| \\
\leq\ &C + C\sup_{z\in\mathbb{D}}(1-|z|^2)|2u'(z)f'(z)+u(z)f''(z)| \\
\leq\ &C.
\end{aligned} \tag{2.6}$$

Similarly, by taking the function $\phi(z) = z^{n+2}$. From the fact that $\|\phi\|_\infty \leq 1$ and by using (2.5) , (2.6), it follows that

$$\begin{aligned}
&\sup_{z\in\mathbb{D}}(1-|z|^2)|u(z)(f'(z))^2| \\
\leq\ &\sup_{z\in\mathbb{D}}(1-|z|^2)|u''(z)f(z)| + \sup_{z\in\mathbb{D}}(1-|z|^2)|2u'(z)f'(z)+u(z)f''(z)| \\
+\ &\sup_{z\in\mathbb{D}}(1-|z|^2)|u(z)(f'(z))^2| \\
\leq\ &C + \sup_{z\in\mathbb{D}}(1-|z|^2)|u(z)(f'(z))^2| \\
\leq\ &C.
\end{aligned} \tag{2.7}$$

For a fixed $w \in \mathbb{D}$, we consider the following test functions

$$\begin{aligned}
\phi^*_{f(w)}(z) &= \frac{(n+2)(n+3)(1-|f(w)|^2)}{1-\overline{f(w)}z} - \frac{2(n+3)(1-|f(w)|^2)^2}{(1-\overline{f(w)}z)^2} \\
&+ \frac{2(1-|f(w)|^2)^3}{(1-\overline{f(w)}z)^3}.
\end{aligned} \tag{2.8}$$

By the triangle inequality, we can see that

$$\begin{aligned}
|\phi^*_{f(w)}(z)| &\leq \frac{(n+2)(n+3)(1-|f(w)|^2)}{1-|f(w)z|} + \frac{2(n+3)(1-|f(w)|^2)^2}{(1-|f(w)z|)^2} \\
&+ \frac{2(1-|f(w)|^2)^3}{(1-|f(w)z|)^3} \\
&\leq \frac{(n+2)(n+3)(1-|f(w)|^2)}{1-|f(w)|} + \frac{2(n+3)(1-|f(w)|^2)^2}{(1-|f(w)|)^2} \\
&+ \frac{2(1-|f(w)|^2)^3}{(1-|f(w)|)^3} \\
&\leq (2n^2+18n+52).
\end{aligned}$$

Hence

$$\sup_{z\in\mathbb{D}} \| \phi^*_{f(w)} \|_\infty \leq (2n^2 + 18n + 52).$$

$$\begin{aligned}
\phi^{*(n)}_{f(w)}(z) &= \frac{(n+2)(n+3)n!(1-|f(w)|^2)(\overline{f(w)})^n}{(1-\overline{f(w)}z)^{n+1}} \\
&- \frac{2(n+3)(n+1)!(1-|f(w)|^2)^2(\overline{f(w)})^n}{(1-\overline{f(w)}z)^{n+2}} \\
&+ \frac{(n+2)!(1-|f(w)|^2)^3(\overline{f(w)})^n}{(1-\overline{f(w)}z)^{n+3}},
\end{aligned} \tag{2.9}$$

$$\begin{aligned}
\phi^{*(n+1)}_{f(w)}(z) &= \frac{(n+3)!(1-|f(w)|^2)(\overline{f(w)})^{n+1}}{(1-\overline{f(w)}z)^{n+2}} \\
&- \frac{2(n+3)!(1-|f(w)|^2)^2(\overline{f(w)})^{n+1}}{(1-\overline{f(w)}z)^{n+3}} \\
&+ \frac{(n+3)!(1-|f(w)|^2)^3(\overline{f(w)})^{n+1}}{(1-\overline{f(w)}z)^{n+4}},
\end{aligned}$$

$$\begin{aligned}
\phi^{*(n+2)}_{f(w)}(z) &= \frac{(n+2)(n+3)!(1-|f(w)|^2)(\overline{f(w)})^{n+2}}{(1-\overline{f(w)}z)^{n+3}} \\
&- \frac{2(n+3)(n+3)!(1-|f(w)|^2)^2(\overline{f(w)})^{n+2}}{(1-\overline{f(w)}z)^{n+4}} \\
&+ \frac{(n+4)!(1-|f(w)|^2)^3(\overline{f(w)})^{n+2}}{(1-\overline{f(w)}z)^{n+5}},
\end{aligned}$$

and we have

$$\phi^{*(n)}_{f(w)}(f(w)) = \frac{2n!\overline{f(w)}^n}{(1-|f(w)|^2)^n}, \phi^{*(n+1)}_{f(w)}(f(w)) = 0, \phi^{*(n+2)}_{f(w)}(f(w)) = 0. \tag{2.10}$$

Which follows that

$$\begin{aligned}
(2n^2+18n+52)\| D^n_u S_{\phi^*} \| &\geq \| D^n_u S_{\phi^*}\phi^*_{f(w)} \|_{\mathcal{Z}} \\
&\geq (1-|w|^2) \,|\, u''(w)\phi^{*(n)}_{f(w)}(f(w)) \\
&+ \Big(2u'(w)f'(w) + u(w)f''(w)\Big)\phi^{*(n+1)}_{f(w)}(f(w)) \\
&+ u(w)(f'(w))^2\phi^{*(n+2)}_{f(w)}(f(w)) \,|
\end{aligned}$$

$$= \quad (1-|w|^2)\,|\,\frac{2n!u''(w)\overline{f(w)}^{n}}{(1-|f(w)|^2)^n}\,|\,. \tag{2.11}$$

If we take $f(w) = w^n$, we get

$$(2n^2+18n+52)\| D_u^n S_{\phi^*} \| \quad \geq \quad (1-|w|^2)\,|\,\frac{2n!u''(w)\overline{(w^n)}^{n}}{(1-|w^n|^2)^n}\,|\,.$$

For a fixed $\delta \in (0,1)$ and by using (2.1), (2.5), we obtain

$$\begin{aligned}
& \sup_{w\in\mathbb{D}}\left|\frac{2n!(1-|w|^2)u''(w)}{(1-|w^n|^2)^n}\right| \\
\leq\ & \sup_{|w^n|>\delta}\left|\frac{2n!(1-|w|^2)u''(w)}{(1-|w^n|^2)^n}\right| + \sup_{|w^n|\leq\delta}\left|\frac{2n!(1-|w|^2)u''(w)}{(1-|w^n|^2)^n}\right| \\
\leq\ & \frac{1}{\delta^n}\sup_{|w^n|>\delta}\left|\frac{2n!(1-|w|^2)u''(w)\overline{(w^n)}^{n}}{(1-|w^n|^2)^n}\right| + \frac{2n!}{(1-\delta^2)^n}\sup_{|w^n|\leq\delta}(1-|w|^2)\,|\,u''(w)\,| \\
\leq\ & C.
\end{aligned} \tag{2.12}$$

It follows that the condition(2.1) holds as desired.
Next, we prove the condition (2.3). To see this, for a fixed $w \in \mathbb{D}$, put

$$\begin{aligned}
\phi^{**}_{f(w)}(z) \quad &= \quad \frac{(n+2)(n+1)(1-|f(w)|^2)}{1-\overline{f(w)}z} - \frac{2(n+2)(1-|f(w)|^2)^2}{(1-\overline{f(w)}z)^2} \\
&+ \quad \frac{2(1-|f(w)|^2)^3}{(1-\overline{f(w)}z)^3},
\end{aligned} \tag{2.13}$$

It is easy to prove that

$$\sup_{z\in\mathbb{D}} \| \phi^{**}_{f(w)} \|_\infty \leq (2n^2+14n+36). \tag{2.14}$$

$$\begin{aligned}
\phi^{**(n)}_{f(w)}(z) \quad &= \quad \frac{(n+2)!(1-|f(w)|^2)(\overline{f(w)})^n}{(1-\overline{f(w)}z)^{n+1}} \\
&- \quad \frac{2(n+2)!(1-|f(w)|^2)^2(\overline{f(w)})^n}{(1-\overline{f(w)}z)^{n+2}} \\
&+ \quad \frac{(n+2)!(1-|f(w)|^2)^3(\overline{f(w)})^n}{(1-\overline{f(w)}z)^{n+3}},
\end{aligned} \tag{2.15}$$

$$\begin{aligned}
\phi^{**(n+1)}_{f(w)}(z) \quad &= \quad \frac{(n+1)(n+2)!(1-|f(w)|^2)(\overline{f(w)})^{n+1}}{(1-\overline{f(w)}z)^{n+2}} \\
&- \quad \frac{2(n+2)(n+2)!(1-|f(w)|^2)^2(\overline{f(w)})^{n+1}}{(1-\overline{f(w)}z)^{n+3}}
\end{aligned}$$

$$+ \quad \frac{(n+3)!(1-|f(w)|^2)^3(\overline{f(w)})^{n+1}}{(1-\overline{f(w)}z)^{n+4}},$$

$$\begin{aligned}
\phi_{f(w)}^{**(n+2)}(z) &= \frac{(n+1)(n+2)(n+2)!(1-|f(w)|^2)(\overline{f(w)})^{n+2}}{(1-\overline{f(w)}z)^{n+3}} \\
&- \frac{2(n+2)(n+3)!(1-|f(w)|^2)^2(\overline{f(w)})^{n+2}}{(1-\overline{f(w)}z)^{n+4}} \\
&+ \frac{(n+4)!(1-|f(w)|^2)^3(\overline{f(w)})^{n+2}}{(1-\overline{f(w)}z)^{n+5}},
\end{aligned}$$

and we have

$$\phi_{f(w)}^{**(n+2)}(f(w)) = \frac{2(n+2)!\overline{f(w)}^{n+2}}{(1-|f(w)|^2)^{n+2}}, \phi_{f(w)}^{**(n)}(f(w)) = 0, \phi_{f(w)}^{**(n+1)}(f(w)) = 0.$$

Which follows that

$$\begin{aligned}
(2n^2+14n+36)\| D_u^n S_{\phi^{**}} \| &\geq \| D_u^n S_{\phi^{**}} \phi_{f(w)}^{**} \|_{\mathcal{Z}} \\
&\geq (1-|w|^2) \mid u''(w)\phi_{f(w)}^{**(n)}(f(w)) \\
&+ \Big(2u'(w)f'(w) + u(w)f''(w)\Big)\phi_{f(w)}^{**(n+1)}(f(w)) \\
&+ u(w)(f'(w))^2\phi_{f(w)}^{**(n+2)}(f(w)) \mid \\
&= (1-|w|^2)\left|\frac{2(n+2)!u(w)(f'(w))^2\overline{f(w)}^{n+2}}{(1-|f(w)|^2)^{n+2}}\right|. \qquad (2.16)
\end{aligned}$$

If we take $f(w) = w^n$, we get

$$\begin{aligned}
&(2n^2+14n+36)\| D_u^n S_{\phi^{**}} \| \\
\geq\ &(1-|w|^2)\left|\frac{2n^2(n+2)!u(w)w^{2n-2}\overline{(w^n)}^{n+2}}{(1-|w^n|^2)^{n+2}}\right|, \qquad (2.17)
\end{aligned}$$

For a fixed $\delta \in (0,1)$ and by using (2.7) , (2.17), we obtain

$$\begin{aligned}
&\sup_{w\in\mathbb{D}}\left|\frac{(1-|w|^2)2n^2(n+2)!u(w)w^{2n-2}}{(1-|w^n|^2)^{n+2}}\right| \\
\leq\ &\sup_{|w^n|>\delta}\left|\frac{(1-|w|^2)2n^2(n+2)!u(w)w^{2n-2}}{(1-|w^n|^2)^{n+2}}\right| \\
+\ &\sup_{|w^n|\leq\delta}\left|\frac{(1-|w|^2)2n^2(n+2)!u(w)w^{2n-2}}{(1-|w^n|^2)^{n+2}}\right| \\
\leq\ &\frac{1}{\delta^{n+2}}\sup_{|w^n|>\delta}\left|\frac{2n^2(n+2)!u(w)w^{2n-2}\overline{(w^n)}^{n+2}}{(1-|w^n|^2)^{n+2}}\right|
\end{aligned}$$

$$\begin{aligned} &+ \frac{2n^2(n+2)!w^{2n-2}}{(1-\delta^2)^{n+2}} \sup_{|w^n|\leq\delta}(1-|w|^2)|u(w)| \\ &\leq C, \end{aligned} \tag{2.18}$$

It follows that the condition (2.3) holds as desired.
Now, we will prove the condition (2.2), for a fixed $w \in \mathbb{D}$, put

$$\begin{aligned} \phi^{***}_{f(w)}(z) &= \frac{(n+1)(n+3)(1-|f(w)|^2)}{1-\overline{f(w)}z} - \frac{(2n+5)(1-|f(w)|^2)^2}{(1-\overline{f(w)}z)^2} \\ &+ \frac{2(1-|f(w)|^2)^3}{(1-\overline{f(w)}z)^3}, \end{aligned} \tag{2.19}$$

It is easy to see that

$$\sup_{z\in\mathbb{D}} \| \phi^{***}_{f(w)} \|_\infty \leq (2n^2+16n+42). \tag{2.20}$$

$$\begin{aligned} \phi^{***(n)}_{f(w)}(z) &= \frac{(n+3)(n+1)!(1-|f(w)|^2)(\overline{f(w)})^n}{(1-\overline{f(w)}z)^{n+1}} \\ &- \frac{(2n+5)(n+1)!(1-|f(w)|^2)^2(\overline{f(w)})^n}{(1-\overline{f(w)}z)^{n+2}} \\ &+ \frac{(n+2)!(1-|f(w)|^2)^3(\overline{f(w)})^n}{(1-\overline{f(w)}z)^{n+3}}, \end{aligned} \tag{2.21}$$

$$\begin{aligned} \phi^{***(n+1)}_{f(w)}(z) &= \frac{(n+1)(n+3)(n+1)!(1-|f(w)|^2)(\overline{f(w)})^{n+1}}{(1-\overline{f(w)}z)^{n+2}} \\ &- \frac{(2n+5)(n+2)!(1-|f(w)|^2)^2(\overline{f(w)})^{n+1}}{(1-\overline{f(w)}z)^{n+3}} \\ &+ \frac{(n+3)!(1-|f(w)|^2)^3(\overline{f(w)})^{n+1}}{(1-\overline{f(w)}z)^{n+4}}, \end{aligned}$$

$$\begin{aligned} \phi^{***(n+2)}_{f(w)}(z) &= \frac{(n+1)(n+3)!(1-|f(w)|^2)(\overline{f(w)})^{n+2}}{(1-\overline{f(w)}z)^{n+3}} \\ &- \frac{(2n+5)(n+3)!(1-|f(w)|^2)^2(\overline{f(w)})^{n+2}}{(1-\overline{f(w)}z)^{n+4}} \\ &+ \frac{(n+4)!(1-|f(w)|^2)^3(\overline{f(w)})^{n+2}}{(1-\overline{f(w)}z)^{n+5}}, \end{aligned}$$

and we have

$$\phi_{f(w)}^{***(n+1)}(f(w)) = \frac{-(n+2)!\overline{f(w)}^{n+1}}{(1-\mid f(w)\mid^2)^{n+1}}, \phi_{f(w)}^{***(n)}(f(w)) = 0, \phi_{f(w)}^{***(n+2)}(f(w)) = 0.$$

Which follows that

$$\begin{aligned} & (2n^2+16n+42) \| D_u^n S_{\phi^{***}} \| \\ \geq & \; \| D_u^n S_{\phi^{***}} \phi_{f(w)}^{***} \|_{\mathcal{Z}} \\ \geq & \; (1-\mid w\mid^2) \mid u''(w)\phi_{f(w)}^{***(n)}(f(w)) \\ + & \; \Big(2u'(w)f'(w) + u(w)f''(w)\Big)\phi_{f(w)}^{***(n+1)}(f(w)) \\ + & \; u(w)(f'(w))^2\phi_{f(w)}^{***(n+2)}(f(w)) \mid \\ = & \; (1-\mid w\mid^2)\left|\frac{(n+2)!\overline{f(w)}^{n+1}\Big(2u'(w)f'(w)+u(w)f''(w)\Big)}{(1-\mid f(w)\mid^2)^{n+1}}\right|. \end{aligned} \tag{2.22}$$

If we take $f(w) = w^n$, we get

$$\begin{aligned} & (2n^2+16n+42) \| D_u^n S_{\phi^{**}} \| \\ \geq & \; (1-\mid w\mid^2)\left|\frac{(n+2)!\overline{(w^n)}^{n+1}\Big(2nw^{n-1}u'(w)+n(n-1)w^{n-2}u(w)\Big)}{(1-\mid w^n\mid^2)^{n+1}}\right|, \end{aligned} \tag{2.23}$$

from (2.6) and (2.23) simliar to (2.12) we obtain (2.2), finishing the proof of the theorem. □

3. The compactness of $D_u^n S_\phi : H^\infty \to \mathcal{Z}$

Now we characterize the compactness of the operators $D_u^n S_\phi : H^\infty \to \mathcal{Z}$. The next Lemma is often used in dealing the compactness of operators on analytic function spaces. Since the proof standard (see Proposition 3.11 in [4]).

Lemma 3. *Suppose ϕ be an entire function and $u \in H(\mathbb{D})$. Then $D_u^n S_\phi : H^\infty \to \mathcal{Z}$ is compact if and only if $D_u^n S_\phi : H^\infty \to \mathcal{Z}$ is bounded and for any bounded sequence $\{f_k\}$ in H^∞ which converges to zero uniformly on compact subsets of $\mathbb{D}$ as $k \to \infty$, we have $\| D_u^n S_\phi f_n \|_{\mathcal{Z}} \to 0$ as $n \to \infty$.*

The second following lemma was introduced and proved in [9] which is similar to the corresponding lemma in [14].

Lemma 4. *A closed set Kin $\mathcal{Z}_0$ is compact if and only if K is bounded and satisfies*

$$\lim_{|z|\to 1} \sup_{f\in K}(1-\mid z\mid^2)\mid f''(z)\mid = 0.$$

Now, we begin with the sufficient and necessary condition for the compactness of $D_u^n S_\phi : H^\infty \to \mathcal{Z}$

Theorem 2. *Suppose that ϕ be an entire function and $u \in H(\mathbb{D})$. Thus $D_u^n S_\phi : H^\infty \to \mathcal{Z}$ is compact if and only if $D_u^n S_\phi : H^\infty \to \mathcal{Z}$ is bounded and the following conditions are satisfied,*

$$\lim_{|z|\to 1} \frac{(1-|z|^2)\,|u''(z)|}{(1-|z|^{2n})^n} = 0, \tag{3.1}$$

$$\lim_{|z|\to 1} \frac{(1-|z|^2)\,|2nz^{n-1}u'(z) + n(n-1)z^{n-2}u(z)|}{(1-|z|^{2n})^{n+1}} = 0, \tag{3.2}$$

and

$$\lim_{|z|\to 1} \frac{(1-|z|^2)\,|n^2 z^{2n-2}u(z)|}{(1-|z|^{2n})^{n+2}} = 0. \tag{3.3}$$

Proof. Suppose that $D_u^n S_\phi : H^\infty \to \mathcal{Z}$ is bounded and that conditions (3.1) – (3.3) hold. For any bounded sequence $\{f_k\}$ in H^∞ with $f_k \to 0$ uniformly on compact subsets of $\mathbb{D}$. To establish the assertion, it suffices, in view of Lemma 3, to show that

$$\|D_u^n S_\phi\|_{\mathcal{Z}} \to 0 \quad as \quad k \to \infty.$$

We assume that $\|f_k\|_\infty \leq 1$. From (3.1) – (3.3), we have given $\epsilon > 0$, there exists a $\delta \in (0,1)$, when $\delta < |f(z)| < 1$, we have

$$(1-|z|^2)\Big[\frac{|u''(z)|}{(1-|z|^{2n})^n} + \frac{|2nz^{n-1}u'(z) + n(n-1)z^{n-2}u(z)|}{(1-|z|^{2n})^{n+1}} + \frac{|n^2z^{2n-2}u(z)|}{(1-|z|^{2n})^{n+2}}\Big] < \epsilon. \tag{3.4}$$

From the boundedness of $D_u^n S_\phi : H^\infty \to \mathcal{Z}$ by Theorem 1 we see that (2.5) – (2.7) hold. Since $f_k \to 0$ uniformly on compact subsets of $\mathbb{D}$, Cauchy's estimate gives that f_k', f_k'' *and* f_k''' converges to 0 uniformly on compact subsets of $\mathbb{D}$. Hence, there exists a $K_0 \in \mathbb{N}$ such that for for $k > K_0$.

$$\begin{aligned}
& |u(0)\phi^n(f_k(0))| + |u'(0)\phi^n(f_k(0))| + |u(0)f_k'(0)\phi^{n+1}(f_k(0))| \\
+ \ & \sup_{|z|\leq\delta}(1-|z|^2)|u''(z)\phi^n(f_k(z))| \\
+ \ & \sup_{|z|\leq\delta}(1-|z|^2)\Big|\Big[2u'(z)(f_k'(z)) + u(z)(f_k''(z))\Big]\phi^{n+1}(f_k(z) + u(z)f'^2(z)\phi^{n+2}(f_k(z))\Big| \\
\leq \ & C\epsilon.
\end{aligned} \tag{3.5}$$

From (3.4) and (3.5), we have

$$\begin{aligned}
\| D_u^n S_\phi f_k \|_{\mathcal{Z}} &= \sup_{z\in\mathbb{D}} |D_u^n S_\phi f_k(0)| + |(D_u^n S_\phi f_k)'(0)| + \sup_{z\in\mathbb{D}}(1-|z|^2)|(D_u^n S_\phi f_k)''(z)| \\
&\leq \sup_{z\in\mathbb{D}} |u(0)\phi^{(n)}(f_k(0))| + |u'(0)\phi^{(n)}(f_k(0))| + |u(0)f_k'(0)\phi^{(n+1)}(f_k(0))| \\
&+ \sup_{|f(z)|\leq\delta} (1-|z|^2)|u''(z)\phi^{(n)}(f_k(z))| \\
&+ \sup_{|f(z)|\leq\delta} (1-|z^2|)\Big|(2u'(z)f_k'(z) + u(z)f_k''(z))\phi^{(n+1)}(f_k(z))
\end{aligned}$$

$$
\begin{aligned}
&+ \quad u(z)f_k'^2(z)\phi^{(n+2)}(f_k(z))\Big| \\
&+ \sup_{\delta<|f(z)|<1}(1-|z|^2)|u''(z)\phi^{(n)}(f_k(z))| \\
&+ \sup_{\delta<|f(z)|<1}(1-|z|^2)\Big|(2u'(z)f_k'(z)+u(z)f_k''(z))\phi^{(n+1)}(f_k(z)) \\
&+ \quad u(z)f_k'^2(z)\phi^{(n+2)}(f_k(z))\Big|
\end{aligned}
$$

$$
\begin{aligned}
\le\ & C\epsilon + C\sup_{\delta<|f(z)|<1}(1-|z|^2)\Big[\frac{|u''(z)|}{(1-|f_k(z)|^{2n})^n} \\
+\ & \frac{|2nz^{n-1}u'(z)+n(n-1)z^{n-2}u(z)|}{(1-|f_k(z)|^{2n})^{n+1}} + \frac{|n^2z^{2n-2}u(z)|}{(1-|f_k(z)|^{2n})^{n+2}}\Big].
\end{aligned}
$$

If we take $f(z)=z^n$, we get

$$
\begin{aligned}
\| D_u^nS_\phi f_k\|_{\mathcal{Z}} \le\ & C\epsilon + C\sup_{\delta<|f(z)|<1}(1-|z|^2)\Big[\frac{|u''(z)|}{(1-|z^n|^{2n})^n} \\
+\ & \frac{|2nz^{n-1}u'(z)+n(n-1)z^{n-2}u(z)|}{(1-|z^n|^{2n})^{n+1}} + \frac{|n^2z^{2n-2}u(z)|}{(1-|z^n|^{2n})^{n+2}}\Big] \\
\le\ & 2C\epsilon,
\end{aligned}
$$

when $k>K_0$. It follows that the operators $D_u^nS_\phi : H^\infty \to \mathcal{Z}$ is compact.
Conversely, suppos that $D_u^nS_\phi : H^\infty \to \mathcal{Z}$ is compact. Therefore it is clear that $D_u^nS_\phi : H^\infty \to \mathcal{Z}$ is bounded. Let $\{z_k\}$ be a sequence in $\mathbb{D}$ such that $|f(z_k)|\to 1$ as $k\to\infty$. If such a sequence does not exist, thus (3.1) – (3.3) are automatically holding. Now, we consider the test functions

$$
\begin{aligned}
\phi^*_{f_k(z)}(z_k) =\ & \frac{(n+2)(n+3)(1-|f_k(z_k)|^2)}{1-\overline{f_k(z_k)}z_k} - \frac{2(n+3)(1-|f_k(z_k)|^2)^2}{(1-\overline{f_k(z_k)}z_k)^2} \\
+\ & \frac{2(1-|f_k(z_k)|^2)^3}{(1-\overline{f_k(z_k)}z_k)^3}. \qquad (3.6)
\end{aligned}
$$

From (2.9) and (2.10), we have

$$
\sup_{k\in\mathbb{N}}\|\phi^*_{f_k(z_k)}\|_\infty \le (2n^2+18n+52).
$$

And

$$
\phi^{*(n)}_{f_k(z_k)}(f_k(z_k)) = \frac{2n!\overline{f_k(z_k)}^n}{(1-|f_k(z_k)|^2)^n},
$$
$$
\phi^{*(n+1)}_{f_k(z_k)}(f_k(z_k)) = 0, \phi^{*(n+2)}_{f_k(z_k)}(f_k(z_k)) = 0.
$$

For $|z| = r < 1$, we get

$$\phi^{*}_{f_k(z_k)}(z_k) \leq \frac{2(n+2)(n+3)+8(n+3)+16}{1-r}(1-|(f_k(z_k))|) \to \ 0 \ as \ (k \to \infty),$$

that is, $\phi^{*(n)}_{f_k(z_k)}$ converges to 0 uniformly on compact subsets of $\mathbb{D}$, using (2.11) and the compactness of $D^n_u S_\phi : H^\infty \to \mathcal{Z}$ we get

$$\begin{aligned}
(1-|z_k|^2)\left|\frac{2n!u''(z_k)\overline{f_k(z_k)}^n}{(1-|f_k(z_k)|^2)^n}\right| &= (1-|z_k|^2)\,|\,u''(z_k)\phi^{*(n)}_{f_k(z_k)}(f_k(z_k))| \\
&+ (1-|z_k|^2)\Big|\Big(2u'(z_k)f_k'(z_k) \\
&+ u(z_k)f_k''(z_k)\Big)\phi^{*(n+1)}_{f_k(z_k)}(f_k(z_k)) \\
&+ u(z_k)(f_k'(z_k))^2\phi^{*(n+2)}_{f_k(z_k)}(f_k(z_K))\Big| \\
&\leq \|\, D^n_u S_\phi f_k \,\|_{\mathcal{Z}} \to 0 \ as \ k \to \infty.
\end{aligned}$$

If we take $f_k(z_k) = z_k^n$, we get

$$(1-|z_k|^2)\left|\frac{2n!u''(z_k)\overline{(z_k^n)}^n}{(1-|(z_k)^n|)^{2n}}\right| \leq \|\, D^n_u S_\phi f_k \,\|_{\mathcal{Z}} \to 0 \ as \ k \to \infty.$$

From this, and $|z_k^n| \to 1$, it follows that

$$\lim_{k\to\infty} \frac{(1-|z_k|^2)\,|\,u''(z_k)\,|}{(1-|z_k|^{2n})^n} = 0. \tag{3.7}$$

We get (3.1).
In order to prove (3.2), consider

$$\begin{aligned}
\phi^{**}_{f_k(z)}(z_k) &= \frac{(n+2)(n+1)(1-|f_k(z_k)|^2)}{1-\overline{f_k(z_k)}z_k} - \frac{2(n+2)(1-|f_k(z_k)|^2)^2}{(1-\overline{f_k(z_k)}z_k)^2} \\
&+ \frac{2(1-|f_k(z_k)|^2)^3}{(1-\overline{f_k(z_k)}z_k)^3}.
\end{aligned}$$

It follows from (2.14) and (2.16) that

$$\sup_{k\in\mathbb{N}} \|\, \phi^{**}_{f_k(z_k)}(z_k) \,\|_\infty \leq (2n^2+14n+36),$$

and

$$\begin{aligned}
\phi^{**(n+2)}_{f_k(z_k)}(f_k(z_k)) &= \frac{2(n+2)!\overline{f_k(z_k)}^{n+2}}{(1-|f_k(z_k)|^2)^{n+2}}, \\
\phi^{**(n)}_{f_k(z_k)}(f_k(z_k)) &= 0, \\
\phi^{**(n+1)}_{f_k(z_k)}(f_k(z_k)) &= 0.
\end{aligned}$$

That is, $\phi^{**(n)}_{f_k(z_k)}$ converges to 0 uniformly on compact subsets of $\mathbb{D}$, using (2.11) and the compactness of $D^n_u S_\phi : H^\infty \to \mathcal{Z}$ tends to

$$\lim_{k\to\infty} \|D^n_u S_{\phi^{**}_{f_k(z_k)}}\|_{\mathcal{Z}} = 0.$$

From (2.16), we obtain

$$(1- \mid z_k \mid^2)\left|\frac{u(z_k)(f_k^{'}(z_k))^2\overline{f_k(z_k)}^{n+2}}{(1- \mid f_k(z_k) \mid^2)^{n+2}}\right| \le C\|D^n_u S_{\phi^{**}_{f_k(z_k)}}\|_{\mathcal{Z}} \to 0 \ as \ k \to \infty. \tag{3.8}$$

If we take $f_k(z_k) = z_k^n$ we have

$$(1- \mid z_k \mid^2)\left|\frac{n^2 z_k^{2n-2}u(z_k)\overline{z_k^n}^{n+2}}{(1- \mid z_k \mid^{2n})^{n+2}}\right| \le C\|D^n_u S_{\phi^{**}_{z_k^n}}\|_{\mathcal{Z}} \to 0 \ as \ k \to \infty.$$

Thus,

$$\lim_{k\to\infty} \frac{(1- \mid z_k \mid^2)|n^2 z_k^{2n-2}u(z_k)\overline{z_k^n}^{n+2}|}{(1- \mid z_k \mid^{2n})^{n+2}}.$$

Eq (3.3) satisfied. Next, consider

$$\begin{aligned}\phi^{***}_{f_k(z_k)}(z_k) &= \frac{(n+1)(n+3)(1- \mid f_k(z_k) \mid^2)}{1-\overline{f_k(z_k)}z_k} - \frac{(2n+5)(1- \mid f_k(z_k) \mid^2)^2}{(1-\overline{f_k(z_k)}z_k)^2} \\ &+ \frac{2(1- \mid f_k(z_k) \mid^2)^3}{(1-\overline{f_k(z_k)}z_k)^3}.\end{aligned}$$

From (2.20) and (2.22), we get

$$\sup_{k\in\mathbb{N}} \| \phi^{***}_{f_k(z_k)}(z_k) \|_\infty \le (2n^2 + 16n + 42).$$

and

$$\begin{aligned}\phi^{***(n+1)}_{f_k(z_k)}(f_k(z_k)) = \frac{-(n+2)!\overline{f_k(z_k)}^{n+1}}{(1- \mid f_k(z_k) \mid^2)^{n+1}}, \\ \phi^{**(n+1)}_{f_k(z_k)}(f_k(z_k)) = 0, \\ \phi^{**(n+2)}_{f_k(z_k)}(f_k(z_k)) = 0,\end{aligned}$$

and $\phi^{***}_{f_k(z_k)}(f_k(z_k))$ converges to 0 uniformly on compact subsets of $\mathbb{D}$, the compactness of $D^n_u S_\phi : H^\infty \to \mathcal{Z}$ implies that

$$\lim_{k\to\infty} \|D^n_u S_{\phi^{***}_{f_k(z_k)}}\|_{\mathcal{Z}} = 0.$$

From this and (2.23), we obtain (3.2), the proof of the theorem is complete. □

Theorem 3. *Suppose that ϕ be an entire function and $u \in H(\mathbb{D})$. Thus, $D^n_u S_\phi : H^\infty \to \mathcal{Z}_0$ is compact if and only if the following conditions are holding.*

$$\lim_{|z|\to 1^1} \frac{(1- \mid z \mid^2) \mid u^{''}(z) \mid}{(1- \mid z \mid^{2n})^n} = 0, \tag{3.9}$$

$$\lim_{|z|\to 1^-} \frac{(1-|z|^2)\,|\,2nz^{n-1}u'(z)+n(n-1)z^{n-2}u(z)\,|}{(1-|z|^{2n})^{n+1}}=0, \tag{3.10}$$

and

$$\lim_{|z|\to 1^-} \frac{(1-|z|^2)\,|\,n^2z^{2n-2}u(z)\,|}{(1-|z|^{2n})^{n+2}}=0. \tag{3.11}$$

Proof. Suppose that conditions (3.9)–(3.11) are satisfied. Consider the supremum in inequality (2.4) over all $f\in H^\infty$ such that $\|f\|_\infty \le 1$ and letting $|z|\to 1$ yield

$$\lim_{|z|\to 1^-} \sup_{\|f\|_\infty\le 1} (1-|z|^2)\,|\,(D_u^nS_\phi f)''(z)\,|=0.$$

Therfore , by Lemma 3, we see that the operators $D_u^nS_\phi : H^\infty \to \mathcal{Z}_0$is compact. Now suppose that $D_u^nS_\phi : H^\infty \to \mathcal{Z}_0$ is compact. Then $D_u^nS_\phi : H^\infty \to \mathcal{Z}_0$ is bounded, and by considering the function $\phi(z)=z^n$ we have

$$\lim_{|z|\to 1^-} (1-|z|^2)|\,u''(z)\,|=0. \tag{3.12}$$

By considering the function $\phi(z)=z^{n+1}$. We have

$$\lim_{|z|\to 1^-} (1-|z|^2)\,|\,u''(z)f(z)+2u'(z)f'(z)+u(z)f''(z)\,|=0. \tag{3.13}$$

From (3.12), (3.13) and the fact that $\|f\|_\infty \le 1$, we get

$$\lim_{|z|\to 1^-} (1-|z|^2)\,|\,2u'(z)f'(z)+u(z)f''(z)\,|=0. \tag{3.14}$$

By taking the function $\phi(z)=z^{n+2}$. From (3.12), (3.14) and the fact that $\|f\|_\infty \le 1$, we get

$$\lim_{|z|\to 1^-} (1-|z|^2)\,|\,u(z)(f'(z))^2\,|=0. \tag{3.15}$$

By (2.12), (2.18), (2.23), and observing that $D_u^nS_{\phi^*}\phi^*_{f(w)}$, $D_u^nS_{\phi^{**}}\phi^{**}_{f(w)}$ and $D_u^nS_{\phi^{***}}\phi^{***}_{f(w)}$ we know that

$$\lim_{|z|\to 1^-} \frac{(1-|z|^2)\,|\,u''(z)\,|}{(1-|z|^{2n})^{n}}=0, \tag{3.16}$$

$$\lim_{|z|\to 1^-} \frac{(1-|z|^2)\,|\,2nz^{n-1}u'(z)+n(n-1)z^{n-2}u(z)\,|}{(1-|z|^{2n})^{n+1}}=0, \tag{3.17}$$

and

$$\lim_{|z|\to 1^-} \frac{(1-|z|^2)\,|\,n^2z^{2n-2}u(z)\,|}{(1-|z|^{2n})^{n+2}}=0. \tag{3.18}$$

We prove that (3.12) and(3.16) imply (3.9). The proof of (3.10) and (3.11) by the same way. Then, it will be held.
From (3.16), it follows that for every $\epsilon > 0$, there exists $\delta \in (0, 1)$ such that

$$\frac{(1- \mid z \mid^2) \mid u^{''}(z) \mid}{(1- \mid z \mid^{2n})^n} < \epsilon, \tag{3.19}$$

when $\delta < |z| < 1$. Using (3.12), we see that there exists $\tau \in (0, 1)$ such that

$$(1- \mid z \mid^2) \mid u^{''}(z) \mid < \epsilon(1 - \delta^{2n})^n, \tag{3.20}$$

when $\tau <\mid z \mid< 1$.
Thus, when $\tau <\mid z \mid< 1$ and $\delta < |z| < 1$, by (3.19) we have

$$\frac{(1- \mid z \mid^2) \mid u^{''}(z) \mid}{(1- \mid z \mid^{2n})^n} < \epsilon. \tag{3.21}$$

On the other hand, when $\tau <\mid z \mid< 1$ and $|z| \leq \delta$, by (3.20) we obtain

$$\frac{(1- \mid z \mid^2) \mid u^{''}(z) \mid}{(1- \mid z \mid^{2n})^n} \leq \frac{(1- \mid z \mid^2) \mid u^{'''}(z) \mid}{(1 - \delta^{2n})^n} < \epsilon. \tag{3.22}$$

From (3.21) and (3.22), we obtain (3.9) as desired. This is the end of the proof. □

4. Conclusions

The present study dealt with a radical study of a concerned class of weighted differentiation superposition operators $D^n_u S_\phi$. Furthermore, It has made the discussions on the boundedness and compactness property of the new class of operators from H^∞ to Zygmund spaces. Finally, it has also provided the conditions which grant the product operators $D^n_u S_\phi$ be bounded and compact.

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Conflict of interest

There is no any conflict.

References

1. V. Álvarez, M. A. Márquez, D. Vukotić, Superposition operators between the Bloch space and Bergman spaces, *Ark. mat.*, **42** (2004), 205–216.

2. S. Buckley, J. Fernández, D. Vukotić, Superposition operators on Dirichlet type spaces, *Report Uni. Jyvaskyla*, **83** (2001), 41–61.
3. B. R. Choe, H. W. Koo, W. Smith, Composition operators on small spaces, *Integr. Equ. Oper. Theory*, **56** (2006), 357–380.
4. C. C. Cowen Jr, *Composition operators on spaces of analytic functions*, Routledge, 2019.
5. P. L. Duren, *Theory of H^p spaces*, New York: Academic press, **38** (1970), 1–261.
6. A. E. S. Ahmed, A. Kamal, T. I. Yassen, Natural metrics and boundedness of the superposition operator acting between $\mathcal{B}^*_\alpha$ and $f^*(p, q, s)$, *Electronic J. Math. Anal. Appl.*, **3** (2015), 195–203.
7. A. K. Mohamed, On generalized superposition operator acting of analytic function spaces, *J. Egyp. Math. Soc.*, **23** (2015), 134–138.
8. A. Kamal, Properties of superposition operators acting between b^*_μ and q^*_K, *J. Egyp. Math. Soc.*, **23** (2015), 507–512.
9. S. X. Li, S. Stević, Volterra-type operators on zygmund spaces, *J. Inequal. Appl.*, **2007** (2007), 32124.
10. S. X. Li, S. Stević, Generalized composition operators on Zygmund spaces and Bloch type spaces, *J. Math. Anal. Appl.*, **338** (2008), 1282–1295.
11. S. X. Li, S. Stević, Products of Volterra type operator and composition operator from H^∞ and Bloch spaces to Zygmund spaces, *J. Math. Anal. Appl.*, **345** (2008), 40–52.
12. S. X. Li, S. Stević, Weighted composition operators from Zygmund spaces into Bloch spaces, *Appl. Math. Comput.*, **206** (2008), 825–831.
13. Y. M. Liu, Y. Y. Yu, Composition followed by differentiation between H^∞ and zygmund spaces, *Complex Anal. Oper. Theory*, **6** (2012), 121–137.
14. K. Madigan, A. Matheson, Compact composition operators on the bloch space, *T. Am. Math. Soc.*, **347** (1995), 2679–2687.
15. C. J. Xiong, Superposition operators between Q_p spaces and Bloch-type spaces, *Complex Var. Elliptic*, **50** (2005), 935–938.
16. W. Xu, Superposition operators on Bloch-type spaces, *Comput. Methods Funct. Theory*, **7** (2007), 501–507.
17. Y. Y. Yu, Y. M. Liu, On stević type operator from H^∞ space to the logarithmic bloch spaces, *Complex Anal. Oper. Theory*, **9** (2015), 1759–1780.
18. K. H. Zhu, *Spaces of holomorphic functions in the unit ball*, Springer Science & Business Media, **26** (2015).

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