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Research article

Existence and uniqueness for Moore–Gibson–Thompson equation with, source terms, viscoelastic memory and integral condition

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Abstract: This manuscript deals with the existence and uniqueness for the fourth order of Moore–Gibson–Thompson equation with, source terms, viscoelastic memory and integral condition by using Galerkin's method.

Keywords: Galerkin's method; Moore–Gibson–Thompson equation; boundary value problem
Mathematics Subject Classification: 35B30, 35B40, 35J60

1. Introduction

Recent research on nonlinear propagation of sound in the case of high amplitude waves has shown that there is a literature on well-grounded partial differential models. (see, e.g., [1, 5, 7, 9–13, 16–18, 20, 21, 23–30, 39, 49]). This highly active field of research is being carried out by a wide range of applications such as the medical and industrial use of high intensity ultrasound in lithotripsy, thermotherapy, ultrasound cleaning and ultrasound chemistry. The classical models of nonlinear acoustics are Kuznetsov's equation, the Westervelt equation, and the KZK (Kokhlov–Zabolotskaya–Kuznetsov) equation. For mathematics. Existence and singularity analysis of several types of initial boundary value problems of this second nonlinear order in evolutionary PDEs, we refer (see [19, 22, 31–38, 40–48, 50, 51]). Focusing on the study of sound wave propagation, it should be noted that the MGT equation is one of the nonlinear sound equations describing the propagation of sound waves in gases and liquids. The behavior of sound waves depends strongly on the average property of scattering, scattering, and nonlinear effects. Arises from high-frequency ultrasound (HFU) modeling see ([16, 25, 41]). The original derivation dates back to [19]. This model is realized through the third order hyperbolic equation

$$\tau u_{ttt} + u_{tt} - c^2 \Delta u - b\Delta u_t = 0,$$

the unknown function $u = u(x, t)$ denotes the scalar acoustic velocity, c denotes the speed of sound and τ denotes the thermal relaxation. Besides, the coefficient $b = \beta c^2$ is related to the diffusively of the sound with $\beta \in (0, \tau]$. In [19], W Chen and A Palmieri studied the blow–up result for the semilinear Moore– Gibson–Thompson equation with nonlinearity of derivative type in the conservative case defined as following

$$\beta u_{ttt} + u_{tt} - \Delta u - \beta\Delta u_t = |u_t|^p, x \in \mathbb{R}^n, t > 0.$$

This paper is related to the following works (see [27, 46]). Now when we talk about the (MGT) equation with memory term, we have I. Lasieka and X.Wang in [29] studied the exponential decay of energy of the temporally third order (Moore-Gibson-Thompson) equation with a memory term as follow

$$\tau u_{ttt} + \alpha u_{tt} - c^2 Au - bAu_t - \int_0^t g(t-s)Aw(s)\,ds = 0,$$

where τ, α, b, c^2 are physical parameters and A is a positive self-adjoint operator on a Hilbert space H. The convolution term $\int_0^t g(t-s)Aw(s)\,ds$ reflects the memory effects of materials due to viscoelasticity. In [13] I. Lasieka and X. Wang studied the general decay of solution of same problem above. Moore–Gibson–Thompson equation with nonlocal condition is a new posed problem. Existence and uniqueness of the generalized solution are established by using Galerkin method. This problems can be encountered in many scientific domains and many engineering models, see previous works ([20, 22, 31–37, 42, 43, 47, 48]). Mesloub and Mesloub in [40] have applied the Galerkin method to a higher dimension mixed nonlocal problem for a Boussinesq equation. While, S. Boulaaras, A. Zaraï and A. Draifia investigated the Moore–Gibson–Thompson equation with integral condition in [17]. In motivate by these outcomes, we improve the existence and uniqueness by Galerkin method of the Fourth-Order Equation of Moore–Gibson–Thompson Type with source term and integral condition, this problem was cited by the work of F. Dell'Oro and V. Pata in [24].

We define the problem as follow

$$\begin{cases} u_{tttt} + \alpha u_{ttt} + \beta u_{tt} - \varrho\Delta u - \delta\Delta u_t - \gamma\Delta u_{tt} + \int_0^t h(t-\sigma)\,\Delta u(\sigma)\,d\sigma = F(x,t), \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),\ u_{tt}(x,0) = u_2(x), u_{ttt}(x,0) = u_3(x) \\ \dfrac{\partial u}{\partial \eta} = \int_0^t \int_\Omega u(\xi,\tau)\,d\xi d\tau, \quad x \in \partial\Omega. \end{cases} \tag{1.1}$$

The convolution term $\int_0^t h(t-s)\Delta u(s)ds$ reflects the memory effect of materials due to vicoelasticity, F is a given function and h is the relaxation function satisfying

(H1) $h \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ is a non-increasing function satisfying

$$h(0) > 0, \ \ \exists h_0 > 0 \ / \ H(\infty) < h_0. \tag{1.2}$$

where $H(\infty) = \int_0^\infty h(s)\,ds > 0$, $H(t) = \int_0^t h(s)\,ds$ and $h'' > 0, h''' < 0$.
(H2) $\exists \zeta > 0$ satisfying

$$h'(t) \le -\zeta h(t), \ \ t \ge 0. \tag{1.3}$$

The impartial of this manuscript is to consider the following nonlocal mixed boundary value problem for the Moore–Gibson–Thompson (MGT) equation for all $(x;t) \in Q_T = (0,T)$, where $\Omega \subset \mathbb{R}^n$ is a bounded domain with sufficiently smooth boundary $\partial\Omega$. solution of the posed problem.

We divide this paper into the following: In the second part, some definitions and appropriate spaces have been given. Then, we use the Galerkin's method to prove the existence, and in the fourth part we demonstrate the uniqueness.

2. Preliminaries

Let $V(Q_T)$ and $W(Q_T)$ be the set spaces defined respectively by

$$V(Q_T) = \left\{ u \in W_2^1(Q_T) : u_t, u_{tt} \in W_2^1(Q_T), u, \nabla u \in L_h^2(Q_T) \right\},$$

and

$$\begin{aligned} W(Q_T) &= \{ u \in V(Q_T) : u(x,T) = 0 \}. \\ L_h^2(Q_T) &= \left\{ u \in V(Q_T) : \int_0^T h \circ u(t)dt < \infty \right\}, \end{aligned}$$

where

$$h \circ u(t) = \int_\Omega \int_0^t h(t-\sigma)(u(t) - u(\sigma))^2 d\sigma dx.$$

Consider the equation

$$\begin{aligned} &(u_{tttt}, v)_{L^2(Q_T)} + \alpha(u_{ttt}, v)_{L^2(Q_T)} + \beta(u_{tt}, v)_{L^2(Q_T)} - \varrho(\Delta u, v)_{L^2(Q_T)} \\ &-\delta(\Delta u_t, v)_{L^2(Q_T)} - \gamma(\Delta u_{tt}, v)_{L^2(Q_T)} + (\Delta w, v)_{L^2(Q_T)} = (F, v)_{L^2(Q_T)}, \end{aligned} \tag{2.1}$$

where

$$w(x,t) = \int_0^t h(t-\sigma)\, u(x,\sigma) d\sigma,$$

and $(.,.)_{L^2(Q_T)}$ defend for the inner product in $L^2(Q_T)$, u is supposed to be a solution of (1.1) and $v \in W(Q_T)$. Upon using (2.1) and (1.1), we find

$$\begin{aligned} &-(u_{ttt}, v_t)_{L^2(Q_T)} - \alpha(u_{tt}, v_t)_{L^2(Q_T)} - \beta(u_t, v_t)_{L^2(Q_T)} + \varrho(\nabla u, \nabla v)_{L^2(Q_T)} \\ &+\delta(\nabla u_t, \nabla v)_{L^2(Q_T)} - \gamma(\nabla u_t, \nabla v_t)_{L^2(Q_T)} - (\nabla w, \nabla v)_{L^2(Q_T)} \\ &= (F, v)_{L^2(Q_T)} + \varrho \int_0^T \int_{\partial\Omega} v \left(\int_0^t \int_\Omega u(\xi,\tau)\, d\xi d\tau \right) ds_x dt + \delta \int_0^T \int_{\partial\Omega} v \int_\Omega u(\xi,t)\, d\xi ds_x dt \\ &-\delta \int_0^T \int_{\partial\Omega} v \int_\Omega u_0(\xi) d\xi ds_x dt - \gamma \int_0^T \int_{\partial\Omega} v_t \left(\int_0^t \int_\Omega u_\tau(\xi,\tau)\, d\xi d\tau \right) ds_x dt \\ &+(u_3(x), v(x,0))_{L^2(\Omega)} + \alpha(u_2(x), v(x,0))_{L^2(\Omega)} + \beta(u_1(x), v(x,0))_{L^2(\Omega)} \\ &-\gamma(\Delta u_1, v(x,0))_{L^2(\Omega)} - \int_0^T \int_{\partial\Omega} v \left(\int_0^t \int_\Omega w(\xi,\tau)\, d\xi d\tau \right) ds_x dt. \end{aligned} \tag{2.2}$$

Now, we give two useful inequalities:

- Gronwall inequality: If for any $t \in I$, we have

$$y(t) \leq h(t) + c \int_0^t y(s)\, ds,$$

where $h(t)$ and $y(t)$ are two nonnegative integrable functions on the interval I with $h(t)$ non decreasing and c is constant, then

$$y(t) \leq h(t) \exp(ct).$$

- Trace inequality: When $w \in W_1^2(\Omega)$, we have

$$\|w\|^2_{L^2(\partial\Omega)} \leq \varepsilon \|\nabla w\|^2_{L^2(\Omega)} + l(\varepsilon) \|w\|^2_{L^2(\Omega)},$$

where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, and $l(\varepsilon)$ is a positive constant.

Definition 1. *If a function $u \in V(Q_T)$ satisfies Eq (2.1) for each $v \in W(Q_T)$ is called a generalized solution of problem (1.1).*

3. Solvability of the problem

Here, by using Galerkin's method, we give the existence of problem (1.1).

Theorem 1. *If $u_0, u_1, u_2 \in W_2^1(\Omega)$, $u_3 \in L^2(\Omega)$ and $F \in L^2(Q_T)$, then there is at least one generalized solution in $V(Q_T)$ to problem (1.1).*

Proof. Let $\{Z_k(x)\}_{k\geq 1}$ be a fundamental system in $W_2^1(\Omega)$, such that

$$(Z_k, Z_l)_{L^2(\Omega)} = \delta_{k,l}.$$

First, we will find an approximate solution of the problem (1.1) in the form

$$u^N(x,t) = \sum_{k=1}^{N} C_k(t) Z_k(x), \tag{3.1}$$

where the constants $C_k(t)$ are defined by the conditions

$$C_k(t) = (u^N(x,t), Z_k(x))_{L^2(\Omega)}, \quad k = 1, ..., N, \tag{3.2}$$

and can be determined from the relations

$$\begin{aligned}
&(u^N_{tttt}, Z_l(x))_{L^2(\Omega)} + \alpha(u^N_{ttt}, Z_l(x))_{L^2(\Omega)} + \beta(u^N_{tt}, Z_l(x))_{L^2(\Omega)} \\
&+\varrho(\nabla u^N, \nabla Z_l(x))_{L^2(\Omega)} + \delta(\nabla u^N_t, \nabla Z_l(x))_{L^2(\Omega)} \\
&+\gamma\left(\nabla u^N_{tt}, \nabla Z_l(x)\right)_{L^2(\Omega)} - \left(\nabla w^N, \nabla Z_l(x)\right)_{L^2(\Omega)} \\
&= (F(x,t), Z_l(x))_{L^2(\Omega)} + \varrho \int_{\partial\Omega} Z_l(x) \left(\int_0^t \int_\Omega u^N(\xi,\tau)\, d\xi d\tau\right) ds_x \\
&+\delta \int_{\partial\Omega} Z_l(x) \left(\int_0^t \int_\Omega u^N_\tau(\xi,\tau)\, d\xi d\tau\right) ds_x \\
&+\gamma \int_{\partial\Omega} Z_l(x) \left(\int_0^t \int_\Omega u^N_{\tau\tau}(\xi,\tau)\, d\xi d\tau\right) ds_x \\
&- \int_{\partial\Omega} Z_l(x) \left(\int_0^t \int_\Omega w^N(\xi,\tau)\, d\xi d\tau\right) ds_x,
\end{aligned} \tag{3.3}$$

Invoking to (3.1) in (3.3) gives for $l = 1, ..., N$.

$$\begin{aligned}
&\int_\Omega \sum_{k=1}^{N} \Big\{ C_k''''(t) Z_k(x) Z_l(x) + \alpha C_k'''(t) Z_k(x) Z_l(x) \\
&+\beta C_k''(t) Z_k(x) Z_l(x) + \varrho C_k(t) \nabla Z_k(x) . \nabla Z_l(x) \\
&+\delta C_k'(t) \nabla Z_k(x) . \nabla Z_l(x) + \gamma C_k''(t) \nabla Z_k . \nabla Z_l \\
&-\Big(\int_0^t h(t-\sigma) C_k(\sigma) d\sigma \Big) \nabla Z_k(x) . \nabla Z_l(x) \Big\} dx \\
= \;& (F(x,t), Z_l(x))_{L^2(\Omega)} + \varrho \sum_{k=1}^{N} \int_0^t C_k(\tau) \Big(\int_{\partial\Omega} Z_l(x) \int_\Omega Z_k(\xi) d\xi ds_x \Big) d\tau \\
&+\delta \sum_{k=1}^{N} \int_0^t C_k'(\tau) \Big(\int_{\partial\Omega} Z_l(x) \int_\Omega Z_k(\xi) d\xi ds_x \Big) d\tau \\
&+\gamma \sum_{k=1}^{N} \int_0^t C_k''(\tau) \Big(\int_{\partial\Omega} Z_l(x) \int_\Omega Z_k(\xi) d\xi ds_x \Big) d\tau \\
&- \sum_{k=1}^{N} \int_0^t \int_0^\tau h(\tau-\sigma) C_k(\sigma) \Big(\int_{\partial\Omega} Z_l(x) \int_\Omega Z_k(\xi) d\xi ds_x \Big) d\sigma d\tau. \qquad (3.4)
\end{aligned}$$

From (3.4) it follows that

$$\begin{aligned}
&\sum_{k=1}^{N} C_k''''(t) (Z_k(x), Z_l(x))_{L^2(\Omega)} + \alpha C_k'''(t) (Z_k(x), Z_l(x))_{L^2(\Omega)} \\
&+\beta C_k''(t) (Z_k(x), Z_l(x))_{L^2(\Omega)} + \varrho C_k(t) (\nabla Z_k, \nabla Z_l)_{L^2(\Omega)} \\
&+\delta C_k'(t) (\nabla Z_k(x), \nabla Z_l(x))_{L^2(\Omega)} + \gamma C_k''(t) (\nabla Z_k(x), \nabla Z_l(x))_{L^2(\Omega)} \\
&-\Big(\int_0^t h(t-\sigma) C_k(\sigma) d\sigma \Big) (\nabla Z_k, \nabla Z_l)_{L^2(\Omega)} \Big\} dx \\
= \;& (F(x,t), Z_l(x))_{L^2(\Omega)} + \varrho \sum_{k=1}^{N} \int_0^t C_k(\tau) \left(\int_{\partial\Omega} Z_l(x) \int_\Omega Z_k(\xi) d\xi ds_x \right) d\tau \\
&+\delta \sum_{k=1}^{N} \int_0^t C_k'(\tau) \left(\int_{\partial\Omega} Z_l(x) \int_\Omega Z_k(\xi) d\xi ds_x \right) d\tau \\
&+\gamma \sum_{k=1}^{N} \int_0^t \left(C_k''(\tau) \int_{\partial\Omega} Z_l(x) \int_\Omega Z_k(\xi) d\xi ds \right) d\tau \qquad (3.5) \\
&- \sum_{k=1}^{N} \int_0^t \int_0^\tau h(\tau-\sigma) C_k(\sigma) \Big(\int_{\partial\Omega} Z_l(x) \int_\Omega Z_k(\xi) d\xi ds_x \Big) d\sigma d\tau, \quad l = 1, ..., N.
\end{aligned}$$

Let

$$(Z_k, Z_l)_{L^2(\Omega)} = \delta_{kl} = \begin{cases} 1, & k = l \\ 0, & k \neq l \end{cases}$$

$$(\nabla Z_k, \nabla Z_l)_{L^2(\Omega)} = \gamma_{kl},$$

$$\int_{\partial\Omega} Z_l(x) \int_{\Omega} Z_k(\xi)d\xi ds = \chi_{kl}.$$
$$(F(x,t), Z_l(x))_{L^2(\Omega)} = F_l(t).$$

Then (3.5) can be written as

$$\begin{aligned}
&\sum_{k=1}^{N} C_k''''(t)\delta_{kl} + \alpha C_k'''(t)\delta_{kl} + C_k''(t)(\beta\delta_{kl} + \gamma\gamma_{kl}) + \delta C_k'(t)\gamma_{kl} + \varrho C_k(t)\gamma_{kl} \\
&- \int_0^t \Big(\varrho C_k(\tau)\chi_{kl} + \delta C_k'(\tau)\chi_{kl} + \gamma C_k''(\tau)\chi_{kl} - h(t-\tau)C_k(\tau)\gamma_{kl}\Big) \\
&- \int_0^t \int_0^\tau h(\tau-\sigma)C_k(\sigma)d\sigma\chi_{kl}d\sigma d\tau = F_l(t).
\end{aligned} \tag{3.6}$$

A differentiation with respect to t (two times), yields

$$\begin{aligned}
&\sum_{k=1}^{N} C_k''''''(t)\delta_{kl} + \alpha C_k'''''(t)\delta_{kl} + C_k''''(t)(\beta\delta_{kl} + \gamma\gamma_{kl}) + C_k'''(t)(\delta\gamma_{kl} - \gamma\chi_{kl}) \\
&+ C_k''(t)(\varrho\gamma_{kl} - \delta\chi_{kl}) - (\varrho + h(0))C_k'(t)\chi_{kl} + h(0)C_k(t)\chi_{kl} = F_l''(t),
\end{aligned} \tag{3.7}$$

$$\begin{cases}
\sum_{k=1}^{N} \Big[C_k''''(0)\delta_{kl} + \alpha C_k'''(0)\delta_{kl} + C_k''(0)(\beta\delta_{kl} + \gamma\gamma_{kl}) \\
\qquad + \delta C_k'(0)\gamma_{kl} + \varrho C_k(0)\gamma_{kl}\Big] = F_l(0) \\
\\
C_k(0) = (Z_k, u_0)_{L^2(\Omega)},\ C_k'(0) = (Z_k, u_1(x))_{L^2(\Omega)}, \\
C_k''(0) = (Z_k, u_2(x))_{L^2(\Omega)}, C_k'''(0) = (Z_k, u_3(x))_{L^2(\Omega)}.
\end{cases} \tag{3.8}$$

Thus for every n there exists a function $u^N(x)$ satisfying (3.3).

Now, we will demonstrate that the sequence u^N is bounded. To do this, we multiply each equation of (3.3) by the appropriate $C_k'(t)$ summing over k from 1 to N then integrating the resultant equality with respect to t from 0 to τ, with $\tau \le T$, yields

$$\begin{aligned}
&\left(u_{tttt}^N, u_t^N\right)_{L^2(Q_\tau)} + \alpha\left(u_{ttt}^N, u_t^N\right)_{L^2(Q_\tau)} + \beta\left(u_{tt}^N, u_t^N\right)_{L^2(Q_\tau)} \\
&+ \varrho\left(\nabla u^N, \nabla u_t^N\right)_{L^2(Q_\tau)} + \delta\left(\nabla u_t^N, \nabla u_t^N\right)_{L^2(Q_\tau)} \\
&+ \gamma\left(\nabla u_{tt}^N, \nabla u_t^N\right)_{L^2(Q_\tau)} - \left(\nabla w^N, \nabla u_t^N\right)_{L^2(Q_\tau)} \\
= \;& \left(F, u_t^N\right)_{L^2(Q_\tau)} + \varrho\int_0^\tau \int_{\partial\Omega} u_t^N(x,t)\left(\int_0^t \int_\Omega u^N(\xi,\eta)d\xi d\eta\right) ds_x dt \\
&+ \delta\int_0^\tau \int_{\partial\Omega} u_t^N(x,t)\left(\int_0^t \int_\Omega u_t^N(\xi,\eta)d\xi d\eta\right) ds_x dt \\
&+ \gamma\int_0^\tau \int_{\partial\Omega} u_t^N(x,t)\left(\int_0^t \int_\Omega u_{tt}^N(\xi,\eta)d\xi d\eta\right) ds_x dt \\
&- \int_0^\tau \int_{\partial\Omega} u_t^N(x,t)\left(\int_0^t \int_\Omega w^N(\xi,\eta)d\xi d\eta\right) ds_x dt,
\end{aligned} \tag{3.9}$$

after a simplification of the LHS of (3.9), we get

$$
\begin{aligned}
\left(u_{tttt}^{N},u_{t}^{N}\right)_{L^{2}(Q_{\tau})} &= -\int_{0}^{\tau}\left(u_{ttt}^{N},u_{tt}^{N}\right)_{L^{2}(\Omega)}dt+\left(u_{\tau\tau\tau}^{N}\left(x,\tau\right),u_{\tau}^{N}\left(x,\tau\right)\right)_{L^{2}(\Omega)},\\
&\quad -\left(u_{ttt}^{N}\left(x,0\right),u_{t}^{N}\left(x,0\right)\right)_{L^{2}(\Omega)},\\
\alpha\left(u_{ttt}^{N},u_{t}^{N}\right)_{L^{2}(Q_{\tau})} &= \alpha\left(u_{\tau\tau}^{N}\left(x,\tau\right),u_{\tau}^{N}\left(x,\tau\right)\right)_{L^{2}(\Omega)}\\
&\quad -\left(u_{tt}^{N}\left(x,0\right),u_{t}^{N}\left(x,0\right)\right)_{L^{2}(\Omega)}-\alpha\int_{0}^{\tau}\left\|u_{tt}\left(x,t\right)\right\|_{L^{2}(\Omega)}^{2}dt,\\
\beta\left(u_{tt}^{N},u_{t}^{N}\right)_{L^{2}(Q_{\tau})} &= \frac{\beta}{2}\left\|u_{\tau}^{N}(x,\tau)\right\|_{L^{2}(\Omega)}^{2}-\frac{\beta}{2}\left\|u_{t}^{N}(x,0)\right\|_{L^{2}(\Omega)}^{2},\\
\varrho\left(\nabla u^{N},\nabla u_{t}^{N}\right)_{L^{2}(Q_{\tau})} &= \frac{\varrho}{2}\left\|\nabla u^{N}(x,\tau)\right\|_{L^{2}(\Omega)}^{2}-\frac{\varrho}{2}\left\|\nabla u^{N}(x,0)\right\|_{L^{2}(\Omega)}^{2},\\
\delta\left(\nabla u_{t}^{N},\nabla u_{t}^{N}\right)_{L^{2}(Q_{\tau})} &= \delta\int_{0}^{\tau}\left\|\nabla u_{t}^{N}\left(x,t\right)\right\|_{L^{2}(\Omega)}^{2}dt,\\
\gamma\left(\nabla u_{tt}^{N},\nabla u_{t}^{N}\right)_{L^{2}(Q_{\tau})} &= \frac{\gamma}{2}\left\|\nabla u_{\tau}^{N}(x,\tau)\right\|_{L^{2}(\Omega)}^{2}-\frac{\gamma}{2}\left\|\nabla u_{t}^{N}(x,0)\right\|_{L^{2}(\Omega)}^{2},\\
-\left(\nabla w^{N},\nabla u_{t}^{N}\right)_{L^{2}(Q_{\tau})} &= \frac{1}{2}h\circ\nabla u^{N}(\tau)-\frac{1}{2}H(\tau)\left\|\nabla u^{N}(x,\tau)\right\|_{L^{2}(\Omega)}^{2}\\
&\quad -\frac{1}{2}\int_{0}^{\tau}h'\circ\nabla u^{N}(t)dt+\frac{1}{2}h(t)\left\|\nabla u^{N}(x,t)\right\|_{L^{2}(\Omega)}^{2}dt,
\end{aligned}
\tag{3.10}
$$

$$
\begin{aligned}
&\varrho\int_{0}^{\tau}\int_{\partial\Omega}u_{t}^{N}\left(\int_{0}^{t}\int_{\Omega}u^{N}(\xi,\eta)d\xi d\eta\right)ds_{x}dt\\
&=\varrho\int_{\partial\Omega}u^{N}(x,\tau)\int_{0}^{\tau}\int_{\Omega}u^{N}\left(\xi,t\right)d\xi dtds_{x}\\
&-\varrho\int_{\partial\Omega}\int_{0}^{\tau}u^{N}\left(x,t\right)\int_{\Omega}u^{N}\left(\xi,t\right)d\xi dtds_{x},
\end{aligned}
\tag{3.11}
$$

$$
\begin{aligned}
&\delta\int_{0}^{\tau}\int_{\partial\Omega}u_{t}^{N}\left(\int_{0}^{t}\int_{\Omega}u_{t}^{N}(\xi,\eta)d\xi d\eta\right)ds_{x}dt\\
&=\delta\int_{\partial\Omega}\int_{0}^{\tau}u_{t}^{N}\left(x,t\right)\int_{\Omega}u^{N}\left(\xi,t\right)d\xi dtds_{x}\\
&-\delta\int_{\partial\Omega}\int_{0}^{\tau}u_{t}^{N}\left(x,t\right)\int_{\Omega}u^{N}(\xi,0)d\xi dtds_{x},
\end{aligned}
\tag{3.12}
$$

$$
\begin{aligned}
&\gamma\int_{0}^{\tau}\int_{\partial\Omega}u_{t}^{N}\left(x,t\right)\left(\int_{0}^{t}\int_{\Omega}u_{tt}^{N}\left(\xi,\eta\right)d\xi d\eta\right)ds_{x}dt\\
=\ &\gamma\int_{0}^{\tau}\int_{\partial\Omega}u_{t}^{N}\left(x,t\right)\left(\int_{\Omega}u_{t}^{N}\left(\xi,t\right)d\xi\right)ds_{x}dt\\
&-\gamma\int_{0}^{\tau}\int_{\partial\Omega}u_{t}^{N}\left(x,t\right)\left(\int_{\Omega}u_{t}^{N}(\xi,0)d\xi\right)ds_{x}dt.
\end{aligned}
\tag{3.13}
$$

$$
\begin{aligned}
&-\int_{0}^{\tau}\int_{\partial\Omega}u_{t}^{N}\left(\int_{0}^{t}\int_{\Omega}w^{N}(\xi,\eta)d\xi d\eta\right)ds_{x}dt\\
=\ &-\int_{0}^{\tau}\int_{\partial\Omega}u_{t}^{N}\left(\int_{0}^{t}\int_{\Omega}H(\eta)u^{N}(\xi,\eta)d\xi d\eta\right)ds_{x}dt\\
&+\int_{0}^{\tau}\int_{\partial\Omega}u_{t}^{N}\left(\int_{0}^{t}\int_{\Omega}\left[\int_{0}^{\eta}h(\eta-\sigma)(u^{N}(\xi,\eta)-u^{N}(\xi,\sigma))d\sigma\right]d\xi d\eta\right)ds_{x}dt
\end{aligned}
$$

$$
\begin{aligned}
= & -\int_{\partial\Omega} u^N(x,\tau)\int_0^\tau\int_\Omega H(t)u^N(\xi,t)d\xi dt ds_x \\
& +\int_0^\tau\int_{\partial\Omega} u^N(x,t)\int_\Omega H(t)u^N(\xi,t)d\xi ds_x dt \\
& +\int_{\partial\Omega} u^N(x,\tau)\left(\int_0^\tau\int_\Omega\Big[\int_0^t h(t-\sigma)(u^N(\xi,t)-u^N(\xi,\sigma))d\sigma\Big]d\xi\right)ds_x dt \\
& -\int_0^\tau\int_{\partial\Omega} u^N(x,t)\int_\Omega\Big[\int_0^t h(t-\sigma)(u^N(\xi,t)-u^N(\xi,\sigma))d\sigma\Big]d\xi ds_x dt.
\end{aligned} \tag{3.14}
$$

Taking into account the equalities (3.10)–(3.14) in (3.9), we obtain

$$
\begin{aligned}
& \left(u^N_{\tau\tau\tau}(x,\tau),u^N_\tau(x,\tau)\right)_{L^2(\Omega)}+\alpha\left(u^N_{\tau\tau}(x,\tau),u^N_\tau(x,\tau)\right)_{L^2(\Omega)} \\
& +\frac{\beta}{2}\left\|u^N_\tau(x,\tau)\right\|^2_{L^2(\Omega)}+\frac{\varrho}{2}\left\|\nabla u^N(x,\tau)\right\|^2_{L^2(\Omega)}+\frac{\gamma}{2}\left\|\nabla u^N_\tau(x,\tau)\right\|^2_{L^2(\Omega)} \\
& +\frac{1}{2}h\circ\nabla u^N(\tau)-\frac{1}{2}H(\tau)\left\|\nabla u^N(x,\tau)\right\|^2_{L^2(\Omega)} \\
= & \left(u^N_{ttt}(x,0),u^N_t(x,0)\right)_{L^2(\Omega)}+\alpha\left(u^N_{tt}(x,0),u^N_t(x,0)\right)_{L^2(\Omega)} \\
& +\frac{\varrho}{2}\left\|\nabla u^N(x,0)\right\|^2_{L^2(\Omega)}+\frac{\gamma}{2}\left\|\nabla u^N_t(x,0)\right\|^2_{L^2(\Omega)}+\int_0^\tau\left(u^N_{ttt},u^N_{tt}\right)_{L^2(\Omega)}dt \\
& +\alpha\int_0^\tau\|u_{tt}(x,t)\|^2_{L^2(\Omega)}dt-\delta\int_0^\tau\left\|\nabla u^N_t(x,t)\right\|^2_{L^2(\Omega)}dt+\frac{\beta}{2}\left\|u^N_t(x,0)\right\|^2_{L^2(\Omega)} \\
& +\varrho\int_{\partial\Omega}u^N(x,\tau)\int_0^\tau\int_\Omega u^N(\xi,t)\,d\xi dt ds_x+\left(F,u^N_t\right)_{L^2(Q_\tau)} \\
& -\varrho\int_{\partial\Omega}\int_0^\tau u^N(x,t)\int_\Omega u^N(\xi,t)\,d\xi dt ds_x \\
& +\delta\int_{\partial\Omega}\int_0^\tau u^N_t(x,t)\int_\Omega u^N(\xi,t)\,d\xi dt ds_x \\
& -\delta\int_{\partial\Omega}\int_0^\tau u^N_t(x,t)\int_\Omega u^N(\xi,0)d\xi dt ds_x \\
& +\gamma\int_0^\tau\int_{\partial\Omega}u^N_t(x,t)\left(\int_\Omega u^N_t(\xi,t)\,d\xi\right)ds_x dt \\
& -\gamma\int_0^\tau\int_{\partial\Omega}u^N_t(x,t)\left(\int_\Omega u^N_t(\xi,0)d\xi\right)ds_x dt \\
& -\frac{1}{2}\int_0^\tau h'\circ\nabla u^N(t)dt+\frac{1}{2}h(t)\left\|\nabla u^N(x,t)\right\|^2_{L^2(\Omega)}dt \\
& -\int_{\partial\Omega}u^N(x,\tau)\int_0^\tau\int_\Omega H(t)u^N(\xi,t)d\xi dt ds_x \\
& +\int_0^\tau\int_{\partial\Omega}u^N(x,t)\int_\Omega H(t)u^N(\xi,t)d\xi ds_x dt \\
& +\int_{\partial\Omega}u^N(x,\tau)\left(\int_0^\tau\int_\Omega\Big[\int_0^t h(t-\sigma)(u^N(\xi,t)-u^N(\xi,\sigma))d\sigma\Big]d\xi\right)ds_x dt
\end{aligned}
$$

$$-\int_0^\tau\int_{\partial\Omega}u^N(x,t)\int_\Omega\Big[\int_0^t h(t-\sigma)(u^N(\xi,t)-u^N(\xi,\sigma))d\sigma\Big]d\xi ds_x dt. \tag{3.15}$$

Now, multiplying each equation of (3.3) by the appropriate $C_k''(t)$, add them up from 1 to N and them integrate with respect to t from 0 to τ, with $\tau \le T$, we obtain

$$\begin{aligned}
&\left(u_{tttt}^N, u_{tt}^N\right)_{L^2(Q_\tau)} + \alpha\left(u_{ttt}^N, u_{tt}^N\right)_{L^2(Q_\tau)} + \beta\left(u_{tt}^N, u_{tt}^N\right)_{L^2(Q_\tau)} \\
&+\varrho\left(\nabla u^N, \nabla u_{tt}^N\right)_{L^2(Q_\tau)} + \delta\left(\nabla u_t^N, \nabla u_{tt}^N\right)_{L^2(Q_\tau)} \\
&+\gamma\left(\nabla u_{tt}^N, \nabla u_{tt}^N\right)_{L^2(Q_\tau)} - \left(\nabla w^N, \nabla u_{tt}^N\right)_{L^2(Q_\tau)} \\
=\ & \left(F, u_{tt}^N\right)_{L^2(Q_\tau)} + \varrho\int_0^\tau\int_{\partial\Omega}u_{tt}^N(x,t)\left(\int_0^t\int_\Omega u^N(\xi,\eta)d\xi d\eta\right)ds_x dt \\
&+\delta\int_0^\tau\int_{\partial\Omega}u_{tt}^N(x,t)\left(\int_0^t\int_\Omega u_t^N(\xi,\eta)d\xi d\eta\right)ds_x dt \\
&+\gamma\int_0^\tau\int_{\partial\Omega}u_{tt}^N(x,t)\left(\int_0^t\int_\Omega u_{tt}^N(\xi,\eta)d\xi d\eta\right)ds_x dt \\
&-\int_0^\tau\int_{\partial\Omega}u_{tt}^N(x,t)\left(\int_0^t\int_\Omega w^N(\xi,\eta)d\xi d\eta\right)ds_x dt.
\end{aligned} \tag{3.16}$$

With the same reasoning in (3.9), we find

$$\begin{aligned}
\left(u_{tttt}^N, u_{tt}^N\right)_{L^2(Q_\tau)} &= -\int_0^\tau\left\|u_{ttt}^N(x,t)\right\|_{L^2(\Omega)}^2 dt + \left(u_{\tau\tau\tau}^N(x,\tau), u_{\tau\tau}^N(x,\tau)\right)_{L^2(\Omega)} \\
&\quad -\left(u_{ttt}^N(x,0), u_{tt}^N(x,0)\right)_{L^2(\Omega)}, \\
\alpha\left(u_{ttt}^N, u_{tt}^N\right)_{L^2(Q_\tau)} &= \frac{\alpha}{2}\left\|u_{\tau\tau}^N(x,\tau)\right\|_{L^2(\Omega)}^2 - \frac{\alpha}{2}\left\|u_{tt}^N(x,0)\right\|_{L^2(\Omega)}^2, \\
\beta\left(u_{tt}^N, u_{tt}^N\right)_{L^2(Q_\tau)} &= \beta\int_0^\tau\left\|u_{tt}^N(x,t)\right\|_{L^2(\Omega)}^2 dt, \\
\varrho\left(\nabla u^N, \nabla u_{tt}^N\right)_{L^2(Q_\tau)} &= \varrho\left(\nabla u^N(x,\tau), \nabla u_\tau^N(x,\tau)\right)_{L^2(Q_\tau)} \\
&\quad -\varrho\left(\nabla u^N(x,0), \nabla u_t^N(x,0)\right)_{L^2(\Omega)} \\
&\quad -\varrho\int_0^\tau\left\|\nabla u_t^N(x,t)\right\|_{L^2(\Omega)}^2 dt, \\
\delta\left(\nabla u_t^N, \nabla u_{tt}^N\right)_{L^2(Q_\tau)} &= \frac{\delta}{2}\left\|\nabla u_\tau^N(x,\tau)\right\|_{L^2(\Omega)}^2 - \frac{\delta}{2}\left\|\nabla u_t^N(x,0)\right\|_{L^2(\Omega)}^2, \\
\gamma\left(\nabla u_{tt}^N, \nabla u_{tt}^N\right)_{L^2(Q_\tau)} &= \gamma\int_0^\tau\left\|\nabla u_{tt}^N(x,t)\right\|_{L^2(\Omega)}^2 dt \\
-\left(\nabla w^N, \nabla u_{tt}^N\right)_{L^2(Q_\tau)} &= -\frac{1}{2}\Big\{h'\circ\nabla u^N(\tau) + h(\tau)\left\|\nabla u^N(x,\tau)\right\|_{L^2(\Omega)}^2 \\
&\quad -2(\nabla w^N(\tau), \nabla u_\tau^N)_{L^2(\Omega)}\Big\} + \frac{1}{2}\int_0^\tau h''\circ\nabla u^N(t)dt \\
&\quad -\frac{1}{2}\int_0^\tau h'(t)\left\|\nabla u^N(x,t)\right\|_{L^2(\Omega)}^2 dt,
\end{aligned} \tag{3.17}$$

$$\begin{aligned}
&\varrho \int_0^\tau \int_{\partial\Omega} u_{tt}^N \left(\int_0^t \int_\Omega u^N(\xi,\eta) d\xi d\eta \right) ds_x dt \\
&= \varrho \int_{\partial\Omega} u_\tau^N(x,\tau) \int_0^\tau \int_\Omega u^N(\xi,t)\, d\xi dt ds_x \\
&-\varrho \int_{\partial\Omega} \int_0^\tau u_t^N(x,t) \int_\Omega u^N(\xi,t)\, d\xi dt ds_x,
\end{aligned} \tag{3.18}$$

$$\begin{aligned}
&\delta \int_0^\tau \int_{\partial\Omega} u_{tt}^N(x,t) \left(\int_0^t \int_\Omega u_t^N(\xi,\eta) d\xi d\eta \right) ds_x dt \\
&= \delta \int_{\partial\Omega} u_\tau^N(x,\tau) \int_\Omega u^N(\xi,\tau)\, d\xi ds_x - \delta \int_{\partial\Omega} u_\tau^N(x,\tau) \int_\Omega u^N(\xi,0) d\xi ds_x \\
&-\delta \int_{\partial\Omega} \int_0^\tau u_t^N(x,t) \int_\Omega u_t^N(\xi,t)\, d\xi dt ds,
\end{aligned} \tag{3.19}$$

$$\begin{aligned}
&\gamma \int_0^\tau \int_{\partial\Omega} u_{tt}^N(x,t) \left(\int_0^t \int_\Omega u_{tt}^N(\xi,\eta) d\xi d\eta \right) ds_x dt \\
&= \gamma \int_{\partial\Omega} u_\tau^N(x,\tau) \int_\Omega u_\tau^N(\xi,\tau)\, d\xi ds_x - \gamma \int_{\partial\Omega} u_\tau^N(x,\tau) \int_\Omega u_t^N(\xi,0) d\xi ds_x \\
&-\gamma \int_{\partial\Omega} \int_0^\tau u_t^N(x,t) \int_\Omega u_{tt}^N(\xi,t)\, d\xi dt ds,
\end{aligned} \tag{3.20}$$

$$\begin{aligned}
& - \int_0^\tau \int_{\partial\Omega} u_{tt}^N \left(\int_0^t \int_\Omega w^N(\xi,\eta) d\xi d\eta \right) ds_x dt \\
= & - \int_0^\tau \int_{\partial\Omega} u_{tt}^N \left(\int_0^t \int_\Omega H(\eta) u^N(\xi,\eta) d\xi d\eta \right) ds_x dt \\
& + \int_0^\tau \int_{\partial\Omega} u_{tt}^N \left(\int_0^t \int_\Omega \left[\int_0^\eta h(\eta-\sigma)(u^N(\xi,\eta) - u^N(\xi,\sigma)) d\sigma \right] d\xi d\eta \right) ds_x dt \\
= & - \int_{\partial\Omega} u_\tau^N(x,\tau) \int_0^\tau \int_\Omega H(t) u^N(\xi,t) d\xi dt ds_x \\
& + \int_0^\tau \int_{\partial\Omega} u_t^N(x,t) \int_\Omega H(t) u^N(\xi,t) d\xi ds_x dt \\
& + \int_{\partial\Omega} u_\tau^N(x,\tau) \left(\int_0^\tau \int_\Omega \left[\int_0^t h(t-\sigma)(u^N(\xi,t) - u^N(\xi,\sigma)) d\sigma \right] d\xi \right) ds_x dt \\
& - \int_0^\tau \int_{\partial\Omega} u_t^N(x,t) \int_\Omega \left[\int_0^t h(t-\sigma)(u^N(\xi,t) - u^N(\xi,\sigma)) d\sigma \right] d\xi ds_x dt.
\end{aligned} \tag{3.21}$$

Upon using (3.17)–(3.21) into (3.16), we have

$$\begin{aligned}
& \left(u_{\tau\tau\tau}^N(x,\tau), u_{\tau\tau}^N(x,\tau) \right)_{L^2(\Omega)} + \frac{\alpha}{2} \left\| u_{\tau\tau}^N(x,\tau) \right\|_{L^2(\Omega)}^2 + \frac{\delta}{2} \left\| \nabla u_\tau^N(x,\tau) \right\|_{L^2(\Omega)}^2 \\
& + \varrho (\nabla u^N(x,\tau), \nabla u_\tau^N(x,\tau))_{L^2(\Omega)} + \frac{1}{2} h(\tau) \left\| \nabla u^N(x,\tau) \right\|_{L^2(\Omega)}^2 \\
& - \frac{1}{2} h' \circ \nabla u^N(\tau) + (\nabla w^N(\tau), \nabla u_\tau^N)_{L^2(\Omega)} \\
= & \int_0^\tau \left\| u_{ttt}^N(x,t) \right\|_{L^2(\Omega)}^2 dt + \left(u_{ttt}^N(x,0), u_{tt}^N(x,0) \right)_{L^2(\Omega)} + \frac{\alpha}{2} \left\| u_{tt}^N(x,0) \right\|_{L^2(\Omega)}^2 \\
& - \beta \int_0^\tau \left\| u_{tt}^N(x,t) \right\|_{L^2(\Omega)}^2 dt + \varrho (\nabla u^N(x,0), \nabla u_t^N(x,0))_{L^2(\Omega)} \\
& + \varrho \int_0^\tau \| \nabla u_t(x,t) \|_{L^2(\Omega)}^2 dt + \frac{\delta}{2} \left\| \nabla u_t^N(x,0) \right\|_{L^2(\Omega)}^2 + \left(F, u_{tt}^N \right)_{L^2(Q_\tau)} \\
& - \gamma \int_0^\tau \left\| \nabla u_{tt}^N(x,t) \right\|_{L^2(\Omega)}^2 dt + \varrho \int_{\partial\Omega} u_\tau^N(x,\tau) \int_0^\tau \int_\Omega u^N(\xi,t)\, d\xi dt ds_x
\end{aligned}$$

$$
\begin{aligned}
&-\varrho\int_{\partial\Omega}\int_0^\tau u_t^N(x,t)\int_\Omega u^N(\xi,t)\,d\xi dt ds_x+\delta\int_{\partial\Omega}u_\tau^N(x,\tau)\int_\Omega u^N(\xi,\tau)\,d\xi ds_x\\
&-\delta\int_{\partial\Omega}u_\tau^N(x,\tau)\int_\Omega u^N(\xi,0)d\xi ds_x-\delta\int_{\partial\Omega}\int_0^\tau u_t^N(x,t)\int_\Omega u_t^N(\xi,t)\,d\xi dt ds_x\\
&+\gamma\int_{\partial\Omega}u_\tau^N(x,\tau)\int_\Omega u_\tau^N(\xi,\tau)\,d\xi ds_x-\gamma\int_{\partial\Omega}u_\tau^N(x,\tau)\int_\Omega u_t^N(\xi,0)d\xi ds_x\\
&-\gamma\int_{\partial\Omega}\int_0^\tau u_t^N(x,t)\int_\Omega u_{tt}^N(\xi,t)\,d\xi dt ds_x\\
&-\int_{\partial\Omega}u_\tau^N(x,\tau)\int_0^\tau\int_\Omega H(t)u^N(\xi,t)d\xi dt ds_x\\
&+\int_0^\tau\int_{\partial\Omega}u_t^N(x,t)\int_\Omega H(t)u^N(\xi,t)d\xi ds_x dt\\
&+\int_{\partial\Omega}u_\tau^N(x,\tau)\left(\int_0^\tau\int_\Omega\left[\int_0^t h(t-\sigma)(u^N(\xi,t)-u^N(\xi,\sigma))d\sigma\right]d\xi\right)ds_x dt\\
&-\int_0^\tau\int_{\partial\Omega}u_t^N(x,t)\int_\Omega\left[\int_0^t h(t-\sigma)(u^N(\xi,t)-u^N(\xi,\sigma))d\sigma\right]d\xi ds_x dt. \qquad (3.22)
\end{aligned}
$$

Now, multiplying each equation of (3.3) by the appropriate $C_k'''(t)$, add them up from 1 to N and them integrate with respect to t from 0 to τ, with $\tau\le T$, we obtain

$$
\begin{aligned}
&\left(u_{tttt}^N,u_{ttt}^N\right)_{L^2(Q_\tau)}+\alpha\left(u_{ttt}^N,u_{ttt}^N\right)_{L^2(Q_\tau)}+\beta\left(u_{tt}^N,u_{ttt}^N\right)_{L^2(Q_\tau)}\\
&+\varrho\left(\nabla u^N,\nabla u_{ttt}^N\right)_{L^2(Q_\tau)}+\delta\left(\nabla u_t^N,\nabla u_{ttt}^N\right)_{L^2(Q_\tau)}\\
&+\gamma\left(\nabla u_{tt}^N,\nabla u_{ttt}^N\right)_{L^2(Q_\tau)}-\left(\nabla w^N,\nabla u_{ttt}^N\right)_{L^2(Q_\tau)}\\
=\;&\left(F,u_{ttt}^N\right)_{L^2(Q_\tau)}+\varrho\int_0^\tau\int_{\partial\Omega}u_{ttt}^N(x,t)\left(\int_0^t\int_\Omega u^N(\xi,\eta)d\xi d\eta\right)ds_x dt\\
&+\delta\int_0^\tau\int_{\partial\Omega}u_{ttt}^N(x,t)\left(\int_0^t\int_\Omega u_t^N(\xi,\eta)d\xi d\eta\right)ds_x dt\\
&+\gamma\int_0^\tau\int_{\partial\Omega}u_{ttt}^N(x,t)\left(\int_0^t\int_\Omega u_{tt}^N(\xi,\eta)d\xi d\eta\right)ds_x dt\\
&-\int_0^\tau\int_{\partial\Omega}u_{ttt}^N(x,t)\left(\int_0^t\int_\Omega w^N(\xi,\eta)d\xi d\eta\right)ds_x dt. \qquad (3.23)
\end{aligned}
$$

With the same reasoning in (3.9), we find

$$
\begin{aligned}
\left(u_{tttt}^N,u_{ttt}^N\right)_{L^2(Q_\tau)} &= \frac{1}{2}\left\|u_{\tau\tau\tau}^N(x,\tau)\right\|_{L^2(\Omega)}^2-\frac{1}{2}\left\|u_{ttt}^N(x,0)\right\|_{L^2(\Omega)}^2\\
\alpha\left(u_{ttt}^N,u_{ttt}^N\right)_{L^2(Q_\tau)} &= \alpha\int_0^\tau\left\|u_{ttt}^N(x,t)\right\|_{L^2(\Omega)}^2,\\
\beta\left(u_{tt}^N,u_{ttt}^N\right)_{L^2(Q_\tau)} &= \frac{\beta}{2}\left\|u_{\tau\tau}^N(x,\tau)\right\|_{L^2(\Omega)}^2-\frac{\beta}{2}\left\|u_{tt}^N(x,0)\right\|_{L^2(\Omega)}^2,\\
\varrho\left(\nabla u^N,\nabla u_{ttt}^N\right)_{L^2(Q_\tau)} &= \varrho\left(\nabla u^N(x,\tau),\nabla u_{\tau\tau}^N(x,\tau)\right)_{L^2(\Omega)}\\
&\quad-\varrho\left(\nabla u^N(x,0),\nabla u_{tt}^N(x,0)\right)_{L^2(\Omega)}
\end{aligned}
$$

$$
\begin{aligned}
& -\varrho \int_0^\tau \left(\nabla u_t^N, \nabla u_{tt}^N\right)_{L^2(\Omega)} dt, \\
\delta\left(\nabla u_t^N, \nabla u_{ttt}^N\right)_{L^2(Q_\tau)} &= -\delta \int_0^\tau \left\|\nabla u_{tt}^N(x,t)\right\|_{L^2(\Omega)}^2 dt \\
& +\delta\left(\nabla u_\tau^N(x,\tau), \nabla u_{\tau\tau}^N(x,\tau)\right)_{L^2(\Omega)} \\
& -\delta\left(\nabla u_t^N(x,0), \nabla u_{tt}^N(x,0)\right)_{L^2(\Omega)}, \\
\gamma\left(\nabla u_{tt}^N, \nabla u_{ttt}^N\right)_{L^2(Q_\tau)} &= \frac{\gamma}{2}\left\|\nabla u_{\tau\tau}^N(x,\tau)\right\|_{L^2(\Omega)}^2 - \frac{\gamma}{2}\left\|\nabla u_{tt}^N(x,0)\right\|_{L^2(\Omega)}^2 \\
-\left(\nabla w^N, \nabla u_{ttt}^N\right)_{L^2(Q_\tau)} &= -H(\tau)(\nabla u_{\tau\tau}^N(x,\tau), \nabla u^N(x,\tau))_{L^2(\Omega)}^2 \\
& +h(\tau)(\nabla u_\tau^N(x,\tau), \nabla u^N(x,\tau))_{L^2(\Omega)}^2 \\
& -\frac{1}{2}\left\|\nabla u^N(x,\tau)\right\|_{L^2(\Omega)}^2 \\
& +\int_\Omega \nabla u_{\tau\tau}^N \int_0^\tau h(\tau-\sigma)(\nabla u^N(\tau) - \nabla u^N(\sigma))d\sigma dx \\
& +\int_\Omega \nabla u_\tau^N \int_0^\tau h'(\tau-\sigma)(\nabla u^N(\tau) - \nabla u^N(\sigma))d\sigma dx \\
& +\frac{1}{2}h'' \circ \nabla u^N(\tau) + \frac{1}{2}\int_0^\tau (h'' - h''') \circ \nabla u^N(t)dt \\
& -h(0)\int_0^\tau \left\|\nabla u_t^N(x,t)\right\|_{L^2(\Omega)}^2 dt, \qquad (3.24)
\end{aligned}
$$

$$
\begin{aligned}
& \varrho \int_0^\tau \int_{\partial\Omega} u_{ttt}^N \left(\int_0^t \int_\Omega u^N(\xi,\eta)d\xi d\eta\right) ds_x dt \qquad (3.25) \\
= \; & \varrho \int_{\partial\Omega} u_{\tau\tau}^N(x,\tau) \int_0^\tau \int_\Omega u^N(\xi,t)\, d\xi dt ds_x \\
& -\varrho \int_{\partial\Omega} \int_0^\tau u_{tt}^N(x,t) \int_\Omega u^N(\xi,t)\, d\xi dt ds_x, \qquad (3.26)
\end{aligned}
$$

$$
\begin{aligned}
& \delta \int_0^\tau \int_{\partial\Omega} u_{ttt}^N(x,t) \left(\int_0^t \int_\Omega u_t^N(\xi,\eta)d\xi d\eta\right) ds_x dt \\
= \; & \delta \int_{\partial\Omega} u_{\tau\tau}^N(x,\tau) \int_\Omega u^N(\xi,\tau)\, d\xi ds_x - \delta \int_{\partial\Omega} u_{\tau\tau}^N(x,\tau) \int_\Omega u^N(\xi,0) d\xi ds_x \\
& -\delta \int_{\partial\Omega} \int_0^\tau u_{tt}^N(x,t) \int_\Omega u_t^N(\xi,t)\, d\xi dt ds, \qquad (3.27)
\end{aligned}
$$

$$
\begin{aligned}
& \gamma \int_0^\tau \int_{\partial\Omega} u_{ttt}^N(x,t) \left(\int_0^t \int_\Omega u_{tt}^N(\xi,\eta)d\xi d\eta\right) ds_x dt \\
= \; & \gamma \int_{\partial\Omega} u_{\tau\tau}^N(x,\tau) \int_\Omega u_\tau^N(\xi,\tau)\, d\xi ds_x - \gamma \int_{\partial\Omega} u_{\tau\tau}^N(x,\tau) \int_\Omega u_t^N(\xi,0) d\xi ds_x
\end{aligned}
$$

$$-\gamma\int_{\partial\Omega}\int_0^\tau u_{tt}^N(x,t)\int_\Omega u_{tt}^N(\xi,t)\,d\xi dtds, \tag{3.28}$$

$$\begin{aligned}
&-\int_0^\tau\int_{\partial\Omega}u_{ttt}^N\left(\int_0^t\int_\Omega w^N(\xi,\eta)d\xi d\eta\right)ds_xdt\\
=&-\int_0^\tau\int_{\partial\Omega}u_{ttt}^N\left(\int_0^t\int_\Omega H(\eta)u^N(\xi,\eta)d\xi d\eta\right)ds_xdt\\
&+\int_0^\tau\int_{\partial\Omega}u_{ttt}^N\left(\int_0^t\int_\Omega\left[\int_0^\eta h(\eta-\sigma)(u^N(\xi,\eta)-u^N(\xi,\sigma))d\sigma\right]d\xi d\eta\right)ds_xdt\\
=&-\int_{\partial\Omega}u_{\tau\tau}^N(x,\tau)\int_0^\tau\int_\Omega H(t)u^N(\xi,t)d\xi dtds_x\\
&+\int_0^\tau\int_{\partial\Omega}u_{tt}^N(x,t)\int_\Omega H(t)u^N(\xi,t)d\xi ds_xdt\\
&+\int_{\partial\Omega}u_{\tau\tau}^N(x,\tau)\left(\int_0^\tau\int_\Omega\left[\int_0^t h(t-\sigma)(u^N(\xi,t)-u^N(\xi,\sigma))d\sigma\right]d\xi\right)ds_xdt\\
&-\int_0^\tau\int_{\partial\Omega}u_{tt}^N(x,t)\int_\Omega\left[\int_0^t h(t-\sigma)(u^N(\xi,t)-u^N(\xi,\sigma))d\sigma\right]d\xi ds_xdt.
\end{aligned} \tag{3.29}$$

A substitution of equalities (3.24)–(3.29) in (3.23), gives

$$\begin{aligned}
&\frac{1}{2}\left\|u_{\tau\tau\tau}^N(x,\tau)\right\|_{L^2(\Omega)}^2+\frac{\beta}{2}\left\|u_{\tau\tau}^N(x,\tau)\right\|_{L^2(\Omega)}^2+\varrho\left(\nabla u^N(x,\tau),\nabla u_{\tau\tau}^N(x,\tau)\right)_{L^2(\Omega)}\\
&+\delta\left(\nabla u_\tau^N(x,\tau),\nabla u_{\tau\tau}^N(x,\tau)\right)_{L^2(\Omega)}+\frac{\gamma}{2}\left\|\nabla u_{\tau\tau}^N(x,\tau)\right\|_{L^2(\Omega)}^2\\
&-H(\tau)(\nabla u_{\tau\tau}^N(x,\tau),\nabla u^N(x,\tau))_{L^2(\Omega)}^2\\
&+h(\tau)(\nabla u_\tau^N(x,\tau),\nabla u^N(x,\tau))_{L^2(\Omega)}^2-\frac{1}{2}\left\|\nabla u^N(x,\tau)\right\|_{L^2(\Omega)}^2\\
&+\int_\Omega\nabla u_{\tau\tau}^N\int_0^\tau h(\tau-\sigma)(\nabla u^N(\tau)-\nabla u^N(\sigma))d\sigma dx\\
&+\int_\Omega\nabla u_\tau^N\int_0^\tau h'(\tau-\sigma)(\nabla u^N(\tau)-\nabla u^N(\sigma))d\sigma dx+\frac{1}{2}h''\circ\nabla u^N(\tau)\\
=&\left(F,u_{ttt}^N\right)_{L^2(Q_\tau)}+\frac{1}{2}\left\|u_{ttt}^N(x,0)\right\|_{L^2(\Omega)}^2-\alpha\int_0^\tau\left\|u_{ttt}^N(x,t)\right\|_{L^2(\Omega)}^2\\
&+\varrho\left(\nabla u^N(x,0),\nabla u_{tt}^N(x,0)\right)_{L^2(\Omega)}+\varrho\int_0^\tau\left(\nabla u_t^N,\nabla u_{tt}^N\right)_{L^2(\Omega)}dt\\
&+\delta\int_0^\tau\left\|\nabla u_{tt}^N(x,t)\right\|_{L^2(\Omega)}^2dt+\delta\left(\nabla u_t^N(x,0),\nabla u_{tt}^N(x,0)\right)_{L^2(\Omega)}\\
&+\varrho\int_{\partial\Omega}u_{\tau\tau}^N(x,\tau)\int_0^\tau\int_\Omega u^N(\xi,t)\,d\xi dtds_x-\frac{\gamma}{2}\left\|\nabla u_{tt}^N(x,0)\right\|_{L^2(\Omega)}^2\\
&-\varrho\int_{\partial\Omega}\int_0^\tau u_{tt}^N(x,t)\int_\Omega u^N(\xi,t)\,d\xi dtds_x-\frac{\beta}{2}\left\|u_{tt}^N(x,0)\right\|_{L^2(\Omega)}^2
\end{aligned}$$

$$
\begin{aligned}
&+\delta\int_{\partial\Omega}u_{\tau\tau}^{N}(x,\tau)\int_{\Omega}u^{N}(\xi,\tau)\,d\xi ds_{x}-\delta\int_{\partial\Omega}u_{\tau\tau}^{N}(x,\tau)\int_{\Omega}u^{N}(\xi,0)d\xi ds_{x}\\
&-\delta\int_{\partial\Omega}\int_{0}^{\tau}u_{tt}^{N}(x,t)\int_{\Omega}u_{t}^{N}(\xi,t)\,d\xi dtds+\delta\int_{\partial\Omega}u_{\tau\tau}^{N}(x,\tau)\int_{\Omega}u_{\tau}^{N}(\xi,\tau)\,d\xi ds_{x}\\
&-\gamma\int_{\partial\Omega}u_{\tau\tau}^{N}(x,\tau)\int_{\Omega}u_{t}^{N}(\xi,0)d\xi ds_{x}-\gamma\int_{\partial\Omega}\int_{0}^{\tau}u_{tt}^{N}(x,t)\int_{\Omega}u_{tt}^{N}(\xi,t)\,d\xi dtds\\
&-\int_{\partial\Omega}u_{\tau\tau}^{N}(x,\tau)\int_{0}^{\tau}\int_{\Omega}H(t)u^{N}(\xi,t)d\xi dtds_{x}\\
&+\frac{1}{2}\int_{0}^{\tau}(h''-h''')\circ\nabla u^{N}(t)dt-h(0)\int_{0}^{\tau}\left\|\nabla u_{t}^{N}(x,t)\right\|_{L^{2}(\Omega)}^{2}dt\\
&+\int_{0}^{\tau}\int_{\partial\Omega}u_{tt}^{N}(x,t)\int_{\Omega}H(t)u^{N}(\xi,t)d\xi ds_{x}dt\\
&+\int_{\partial\Omega}u_{\tau\tau}^{N}(x,\tau)\left(\int_{0}^{\tau}\int_{\Omega}\left[\int_{0}^{t}h(t-\sigma)(u^{N}(\xi,t)-u^{N}(\xi,\sigma))d\sigma\right]d\xi\right)ds_{x}dt\\
&-\int_{0}^{\tau}\int_{\partial\Omega}u_{tt}^{N}(x,t)\int_{\Omega}\left[\int_{0}^{t}h(t-\sigma)(u^{N}(\xi,t)-u^{N}(\xi,\sigma))d\sigma\right]d\xi ds_{x}dt. \qquad (3.30)
\end{aligned}
$$

Multiplying (3.15) by λ_1, (3.22) by λ_2, and (3.30) by λ_3 such as $(\lambda_1+\lambda_2<\lambda_3)$, we get

$$
\begin{aligned}
&\lambda_{1}\left(u_{\tau\tau\tau}^{N}(x,\tau),u_{\tau}^{N}(x,\tau)\right)_{L^{2}(\Omega)}+\lambda_{1}\alpha\left(u_{\tau\tau}^{N}(x,\tau),u_{\tau}^{N}(x,\tau)\right)_{L^{2}(\Omega)}\\
&+\frac{\lambda_{1}\beta}{2}\left\|u_{\tau}^{N}(x,\tau)\right\|_{L^{2}(\Omega)}^{2}+\frac{\lambda_{1}\varrho}{2}\left\|\nabla u^{N}(x,\tau)\right\|_{L^{2}(\Omega)}^{2}\\
&+(\frac{\lambda_{1}\gamma}{2}+\frac{\lambda_{2}\delta}{2})\left\|\nabla u_{\tau}^{N}(x,\tau)\right\|_{L^{2}(\Omega)}^{2}+\lambda_{2}\left(u_{\tau\tau\tau}^{N}(x,\tau),u_{\tau\tau}^{N}(x,\tau)\right)_{L^{2}(\Omega)}\\
&+(\frac{\lambda_{2}\alpha}{2}+\frac{\lambda_{3}\beta}{2})\left\|u_{\tau\tau}^{N}(x,\tau)\right\|_{L^{2}(\Omega)}^{2}+\lambda_{2}\varrho(\nabla u^{N}(x,\tau),\nabla u_{\tau}^{N}(x,\tau))_{L^{2}(\Omega)}\\
&+\frac{\lambda_{3}}{2}\left\|u_{\tau\tau\tau}^{N}(x,\tau)\right\|_{L^{2}(\Omega)}^{2}+\lambda_{3}\varrho\left(\nabla u^{N}(x,\tau),\nabla u_{\tau\tau}^{N}(x,\tau)\right)_{L^{2}(\Omega)}\\
&+\lambda_{3}\delta\left(\nabla u_{\tau}^{N}(x,\tau),\nabla u_{\tau\tau}^{N}(x,\tau)\right)_{L^{2}(\Omega)}+\frac{\lambda_{3}\gamma}{2}\left\|\nabla u_{\tau\tau}^{N}(x,\tau)\right\|_{L^{2}(\Omega)}^{2}\\
&+\frac{\lambda_{1}}{2}h\circ\nabla u^{N}(\tau)-\frac{\lambda_{1}}{2}H(\tau)\left\|\nabla u^{N}(x,\tau)\right\|_{L^{2}(\Omega)}^{2}\\
&-\frac{\lambda_{2}}{2}h'\circ\nabla u^{N}(\tau)+\lambda_{2}(\nabla w^{N}(\tau),\nabla u_{\tau}^{N})_{L^{2}(\Omega)}\\
&-\lambda_{3}H(\tau)(\nabla u_{\tau\tau}^{N}(x,\tau),\nabla u^{N}(x,\tau))_{L^{2}(\Omega)}^{2}\\
&+\lambda_{3}h(\tau)(\nabla u_{\tau}^{N}(x,\tau),\nabla u^{N}(x,\tau))_{L^{2}(\Omega)}^{2}-\frac{\lambda_{3}}{2}\left\|\nabla u^{N}(x,\tau)\right\|_{L^{2}(\Omega)}^{2}\\
&+\lambda_{3}\int_{\Omega}\nabla u_{\tau\tau}^{N}\int_{0}^{\tau}h(\tau-\sigma)(\nabla u^{N}(\tau)-\nabla u^{N}(\sigma))d\sigma dx\\
&+\lambda_{3}\int_{\Omega}\nabla u_{\tau}^{N}\int_{0}^{\tau}h'(\tau-\sigma)(\nabla u^{N}(\tau)-\nabla u^{N}(\sigma))d\sigma dx+\frac{\lambda_{3}}{2}h''\circ\nabla u^{N}(\tau)\\
=\;&\lambda_{1}\left(u_{ttt}^{N}(x,0),u_{t}^{N}(x,0)\right)_{L^{2}(\Omega)}+\lambda_{1}\alpha\left(u_{tt}^{N}(x,0),u_{t}^{N}(x,0)\right)_{L^{2}(\Omega)}\\
&+\frac{\lambda_{1}\varrho}{2}\left\|\nabla u^{N}(x,0)\right\|_{L^{2}(\Omega)}^{2}+\frac{\lambda_{1}\beta}{2}\left\|u_{t}^{N}(x,0)\right\|_{L^{2}(\Omega)}^{2}
\end{aligned}
$$

$$
\begin{aligned}
&+(\frac{\lambda_1\gamma}{2}+\frac{\lambda_2\delta}{2})\left\|\nabla u_t^N(x,0)\right\|^2_{L^2(\Omega)}+\lambda_1\int_0^\tau\left(u_{ttt}^N,u_{tt}^N\right)_{L^2(\Omega)}dt\\
&+(\lambda_1\alpha-\lambda_2\beta)\int_0^\tau\|u_{tt}(x,t)\|^2_{L^2(\Omega)}dt+(\lambda_2\varrho-\lambda_1\delta)\int_0^\tau\left\|\nabla u_t^N(x,t)\right\|^2_{L^2(\Omega)}dt\\
&+(\lambda_2-\lambda_3\alpha)\int_0^\tau\left\|u_{ttt}^N(x,t)\right\|^2_{L^2(\Omega)}dt+\lambda_2\left(u_{ttt}^N(x,0),u_{tt}^N(x,0)\right)_{L^2(\Omega)}\\
&+(\frac{\lambda_2\alpha}{2}-\frac{\lambda_3\beta}{2})\left\|u_{tt}^N(x,0)\right\|^2_{L^2(\Omega)}\frac{\lambda_3}{2}\left\|u_{ttt}^N(x,0)\right\|^2_{L^2(\Omega)}\\
&+\lambda_2\varrho(\nabla u^N(x,0),\nabla u_t^N(x,0))_{L^2(\Omega)}+(\lambda_3\delta-\lambda_2\gamma)\int_0^\tau\left\|\nabla u_{tt}^N(x,t)\right\|^2_{L^2(\Omega)}dt\\
&+\lambda_3\varrho\left(\nabla u^N(x,0),\nabla u_{tt}^N(x,0)\right)_{L^2(\Omega)}+\lambda_3\varrho\int_0^\tau\left(\nabla u_t^N,\nabla u_{tt}^N\right)_{L^2(\Omega)}dt\\
&+\lambda_3\delta\left(\nabla u_t^N(x,0),\nabla u_{tt}^N(x,0)\right)_{L^2(\Omega)}-\frac{\lambda_3\gamma}{2}\left\|\nabla u_{tt}^N(x,0)\right\|^2_{L^2(\Omega)}\\
&+\lambda_1\varrho\int_{\partial\Omega}u^N(x,\tau)\int_0^\tau\int_\Omega u^N(\xi,t)\,d\xi dtds_x\\
&-\lambda_1\varrho\int_{\partial\Omega}\int_0^\tau u^N(x,t)\int_\Omega u^N(\xi,t)\,d\xi dtds_x\\
&+(\lambda_1\delta-\lambda_2\varrho)\int_{\partial\Omega}\int_0^\tau u_t^N(x,t)\int_\Omega u^N(\xi,t)\,d\xi dtds_x\\
&-\lambda_1\delta\int_{\partial\Omega}\int_0^\tau u_t^N(x,t)\int_\Omega u^N(\xi,0)d\xi dtds_x\\
&+(\lambda_1\gamma-\lambda_2\delta)\int_0^\tau\int_{\partial\Omega}u_t^N(x,t)\left(\int_\Omega u_t^N(\xi,t)\,d\xi\right)ds_xdt\\
&-\lambda_1\gamma\int_0^\tau\int_{\partial\Omega}u_t^N(x,t)\left(\int_\Omega u_t^N(\xi,0)d\xi\right)ds_xdt\\
&+\lambda_2\varrho\int_{\partial\Omega}u_\tau^N(x,\tau)\int_0^\tau\int_\Omega u^N(\xi,t)\,d\xi dtds_x\\
&+\lambda_2\delta\int_{\partial\Omega}u_\tau^N(x,\tau)\int_\Omega u^N(\xi,\tau)\,d\xi ds_x-\lambda_2\delta\int_{\partial\Omega}u_\tau^N(x,\tau)\int_\Omega u^N(\xi,0)d\xi ds_x\\
&+\lambda_2\gamma\int_{\partial\Omega}u_\tau^N(x,\tau)\int_\Omega u_\tau^N(\xi,\tau)\,d\xi ds_x-\lambda_2\gamma\int_{\partial\Omega}u_\tau^N(x,\tau)\int_\Omega u_t^N(\xi,0)d\xi ds_x\\
&-\lambda_2\gamma\int_{\partial\Omega}\int_0^\tau u_t^N(x,t)\int_\Omega u_{tt}^N(\xi,t)\,d\xi dtds_x\\
&+\lambda_3\varrho\int_{\partial\Omega}u_{\tau\tau}^N(x,\tau)\int_0^\tau\int_\Omega u^N(\xi,t)\,d\xi dtds_x\\
&-\lambda_3\varrho\int_{\partial\Omega}\int_0^\tau u_{tt}^N(x,t)\int_\Omega u^N(\xi,t)\,d\xi dtds_x\\
&+\lambda_3\delta\int_{\partial\Omega}u_{\tau\tau}^N(x,\tau)\int_\Omega u^N(\xi,\tau)\,d\xi ds_x-\lambda_3\delta\int_{\partial\Omega}u_{\tau\tau}^N(x,\tau)\int_\Omega u^N(\xi,0)d\xi ds_x\\
&-\lambda_3\delta\int_{\partial\Omega}\int_0^\tau u_{tt}^N(x,t)\int_\Omega u_t^N(\xi,t)\,d\xi dtds
\end{aligned}
$$

$$
\begin{aligned}
&+\lambda_3\gamma\int_{\partial\Omega}u_{\tau\tau}^N(x,\tau)\int_{\Omega}u_{\tau}^N(\xi,\tau)\,d\xi ds_x-\lambda_3\gamma\int_{\partial\Omega}u_{\tau\tau}^N(x,\tau)\int_{\Omega}u_t^N(\xi,0)d\xi ds_x\\
&-\lambda_3\gamma\int_{\partial\Omega}\int_0^{\tau}u_{tt}^N(x,t)\int_{\Omega}u_{tt}^N(\xi,t)\,d\xi dt ds\\
&+\lambda_1\left(F,u_t^N\right)_{L^2(Q_\tau)}+\lambda_2\left(F,u_{tt}^N\right)_{L^2(Q_\tau)}+\lambda_3\left(F,u_{ttt}^N\right)_{L^2(Q_\tau)}\\
&+\frac{\lambda_1}{2}\int_0^{\tau}h'\circ\nabla u^N(t)dt-\frac{\lambda_1}{2}h(t)\left\|\nabla u^N(x,t)\right\|_{L^2(\Omega)}^2dt\\
&-\lambda_1\int_{\partial\Omega}u^N(x,\tau)\int_0^{\tau}\int_{\Omega}H(t)u^N(\xi,t)d\xi dt ds_x\\
&+\lambda_1\int_0^{\tau}\int_{\partial\Omega}u^N(x,t)\int_{\Omega}H(t)u^N(\xi,t)d\xi ds_x dt\\
&+\lambda_1\int_{\partial\Omega}u^N(x,\tau)\left(\int_0^{\tau}\int_{\Omega}\left[\int_0^t h(t-\sigma)(u^N(\xi,t)-u^N(\xi,\sigma))d\sigma\right]d\xi\right)ds_x dt\\
&-\lambda_1\int_0^{\tau}\int_{\partial\Omega}u^N(x,t)\int_{\Omega}\left[\int_0^t h(t-\sigma)(u^N(\xi,t)-u^N(\xi,\sigma))d\sigma\right]d\xi ds_x dt\\
&-\lambda_3\int_{\partial\Omega}u_{\tau\tau}^N(x,\tau)\int_0^{\tau}\int_{\Omega}H(t)u^N(\xi,t)d\xi dt ds_x\\
&-\frac{\lambda_3}{2}\int_0^{\tau}(h''-h''')\circ\nabla u^N(t)dt+\lambda_3h(0)\int_0^{\tau}\left\|\nabla u_t^N(x,t)\right\|_{L^2(\Omega)}^2dt\\
&+\lambda_3\int_0^{\tau}\int_{\partial\Omega}u_{tt}^N(x,t)\int_{\Omega}H(t)u^N(\xi,t)d\xi ds_x dt\\
&+\lambda_3\int_{\partial\Omega}u_{\tau\tau}^N(x,\tau)\left(\int_0^{\tau}\int_{\Omega}\left[\int_0^t h(t-\sigma)(u^N(\xi,t)-u^N(\xi,\sigma))d\sigma\right]d\xi\right)ds_x dt\\
&-\lambda_3\int_0^{\tau}\int_{\partial\Omega}u_{tt}^N(x,t)\int_{\Omega}\left[\int_0^t h(t-\sigma)(u^N(\xi,t)-u^N(\xi,\sigma))d\sigma\right]d\xi ds_x dt\\
&-\lambda_2\int_{\partial\Omega}u_{\tau}^N(x,\tau)\int_0^{\tau}\int_{\Omega}H(t)u^N(\xi,t)d\xi dt ds_x\\
&+\lambda_2\int_0^{\tau}\int_{\partial\Omega}u_t^N(x,t)\int_{\Omega}H(t)u^N(\xi,t)d\xi ds_x dt\\
&+\lambda_2\int_{\partial\Omega}u_{\tau}^N(x,\tau)\left(\int_0^{\tau}\int_{\Omega}\left[\int_0^t h(t-\sigma)(u^N(\xi,t)-u^N(\xi,\sigma))d\sigma\right]d\xi\right)ds_x dt\\
&-\lambda_2\int_0^{\tau}\int_{\partial\Omega}u_t^N(x,t)\int_{\Omega}\left[\int_0^t h(t-\sigma)(u^N(\xi,t)-u^N(\xi,\sigma))d\sigma\right]d\xi ds_x dt\\
&-\frac{\lambda_2}{2}\int_0^{\tau}h''\circ\nabla u^N(t)dt+\frac{\lambda_2}{2}\int_0^{\tau}h'(t)\left\|\nabla u^N(x,t)\right\|_{L^2(\Omega)}^2dt.
\end{aligned}
\tag{3.31}
$$

We can estimate all the terms in the RHS of (3.31) as follows

$$
\begin{aligned}
&\lambda_1\varrho\int_{\partial\Omega}u^N(x,\tau)\int_0^{\tau}\int_{\Omega}u^N(\xi,t)\,d\xi dt ds_x\\
&\leq\tfrac{\lambda_1\varrho}{2\varepsilon_1}\left(\varepsilon\left\|\nabla u^N(x,\tau)\right\|_{L^2(\Omega)}^2+l(\varepsilon)\left\|u^N(x,\tau)\right\|_{L^2(\Omega)}^2\right)\\
&+\tfrac{\lambda_1\varrho}{2}\varepsilon_1T\left|\Omega\right|\left|\partial\Omega\right|\int_0^{\tau}\left\|u^N(x,t)\right\|_{L^2(\Omega)}^2dt,
\end{aligned}
\tag{3.32}
$$

$$
\begin{aligned}
&-\lambda_1\varrho \int_{\partial\Omega}\int_0^\tau u^N(x,t)\int_\Omega u^N(\xi,t)\,d\xi dt ds_x \\
&\le \tfrac{\lambda_1\varrho}{2}\varepsilon\int_0^\tau \left\|\nabla u^N(x,t)\right\|^2_{L^2(\Omega)} dt \\
&+\tfrac{\lambda_1\varrho}{2}\left(l(\varepsilon)+|\Omega|\,|\partial\Omega|\right)\int_0^\tau \left\|u^N(x,t)\right\|^2_{L^2(\Omega)} dt,
\end{aligned}
\tag{3.33}
$$

$$
\begin{aligned}
&(\lambda_1\delta-\lambda_2\varrho)\int_{\partial\Omega}\int_0^\tau u_t^N(x,t)\int_\Omega u^N(\xi,t)\,d\xi dt ds_x \\
&\le \tfrac{(\lambda_1\delta+\lambda_2\varrho)}{2}\left(\varepsilon\int_0^\tau \left\|\nabla u_t^N(x,t)\right\|^2_{L^2(\Omega)} dt + l(\varepsilon)\int_0^\tau \left\|u_t^N(x,t)\right\|^2_{L^2(\Omega)} dt\right) \\
&+\tfrac{(\lambda_1\delta+\lambda_2\varrho)}{2}|\Omega|\,|\partial\Omega|\int_0^\tau \left\|u^N(x,t)\right\|^2_{L^2(\Omega)} dt,
\end{aligned}
\tag{3.34}
$$

$$
\begin{aligned}
&-\lambda_1\delta\int_{\partial\Omega}\int_0^\tau u_t^N(x,t)\int_\Omega u^N(\xi,0)d\xi dt ds_x \\
&\le \tfrac{\lambda_1\delta}{2}\left(\varepsilon\int_0^\tau \left\|\nabla u_t^N(x,t)\right\|^2_{L^2(\Omega)} dt + l(\varepsilon)\int_0^\tau \left\|u_t^N(x,t)\right\|^2_{L^2(\Omega)} dt\right) \\
&+\tfrac{\lambda_1\delta}{2}|\Omega|\,|\partial\Omega|\,T\left\|u^N(x,0)\right\|^2_{L^2(\Omega)},
\end{aligned}
\tag{3.35}
$$

$$
\begin{aligned}
&\lambda_2\varrho\int_{\partial\Omega}u_\tau^N(x,\tau)\int_0^\tau\int_\Omega u^N(\xi,t)\,d\xi dt ds_x \\
&\le \tfrac{\lambda_2\varrho}{2}\left(\tfrac{\varepsilon}{\varepsilon_2}\left\|\nabla u_\tau^N(x,\tau)\right\|^2_{L^2(\Omega)} + \tfrac{l(\varepsilon)}{\varepsilon_2}\left\|u_\tau^N(x,\tau)\right\|^2_{L^2(\Omega)}\right) \\
&+\tfrac{\lambda_2\varrho}{2}\varepsilon_2|\Omega|\,|\partial\Omega|\,T\int_0^\tau\left\|u^N(x,t)\right\|^2_{L^2(\Omega)} dt,
\end{aligned}
\tag{3.36}
$$

$$
\begin{aligned}
&\lambda_2\delta\int_{\partial\Omega}u_\tau^N(x,\tau)\int_\Omega u^N(\xi,\tau)\,d\xi ds_x \\
&\le \tfrac{\lambda_2\delta}{2\varepsilon_3}\left(\varepsilon\left\|\nabla u_\tau^N(x,\tau)\right\|^2_{L^2(\Omega)} + l(\varepsilon)\left\|u_\tau^N(x,\tau)\right\|^2_{L^2(\Omega)}\right) \\
&+\tfrac{\lambda_2\delta}{2}\varepsilon_3|\Omega|\,|\partial\Omega|\left\|u^N(x,\tau)\right\|^2_{L^2(\Omega)},
\end{aligned}
\tag{3.37}
$$

$$
\begin{aligned}
&-\lambda_2\delta\int_{\partial\Omega}u_\tau^N(x,\tau)\int_\Omega u^N(\xi,0)d\xi ds_x \\
&\le \tfrac{\lambda_2\delta}{2\varepsilon_4}\left(\varepsilon\left\|\nabla u_\tau^N(x,\tau)\right\|^2_{L^2(\Omega)} + l(\varepsilon)\left\|u_\tau^N(x,\tau)\right\|^2_{L^2(\Omega)}\right) \\
&+\tfrac{\lambda_2\delta}{2}\varepsilon_4|\Omega|\,|\partial\Omega|\left\|u^N(x,0)\right\|^2_{L^2(\Omega)},
\end{aligned}
\tag{3.38}
$$

$$
\begin{aligned}
&(\lambda_1\gamma-\lambda_2\delta)\int_{\partial\Omega}\int_0^\tau u_t^N(x,t)\int_\Omega u_t^N(\xi,t)\,d\xi dt ds_x \\
&\le \tfrac{(\lambda_1\gamma+\lambda_2\delta)}{2}\varepsilon\int_0^\tau\left\|\nabla u_t^N(x,t)\right\|^2_{L^2(\Omega)} dt \\
&+\tfrac{(\lambda_1\gamma+\lambda_2\delta)}{2}\left(l(\varepsilon)+|\Omega|\,|\partial\Omega|\right)\int_0^\tau\left\|u_t^N(x,t)\right\|^2_{L^2(\Omega)} dt,
\end{aligned}
\tag{3.39}
$$

$$
\begin{aligned}
&-\lambda_1\gamma\int_0^\tau\int_{\partial\Omega}u_t^N(x,t)\left(\int_\Omega u_t^N(\xi,0)d\xi\right)ds_x dt \\
&\le \tfrac{\lambda_1\gamma}{2}\left(\varepsilon\int_0^\tau\left\|\nabla u_t^N(x,t)\right\|^2_{L^2(\Omega)} dt + l(\varepsilon)\int_0^\tau\left\|u_t^N(x,t)\right\|^2_{L^2(\Omega)} dt\right) \\
&+\tfrac{\lambda_1\gamma}{2}|\Omega|\,|\partial\Omega|\,T\left\|u_t^N(x,0)\right\|^2_{L^2(\Omega)},
\end{aligned}
\tag{3.40}
$$

$$
\begin{aligned}
&\lambda_2\gamma\int_{\partial\Omega}u_\tau^N(x,\tau)\int_\Omega u_\tau^N(\xi,\tau)\,d\xi ds_x \\
&\le \tfrac{\lambda_2\gamma}{2\varepsilon_5}\left(\varepsilon\left\|\nabla u_\tau^N(x,\tau)\right\|^2_{L^2(\Omega)} + l(\varepsilon)\left\|u_\tau^N(x,\tau)\right\|^2_{L^2(\Omega)}\right) \\
&+\tfrac{\lambda_2\gamma}{2}\varepsilon_5|\Omega|\,|\partial\Omega|\left\|u_\tau^N(x,\tau)\right\|^2_{L^2(\Omega)},
\end{aligned}
\tag{3.41}
$$

$$
\begin{aligned}
&-\lambda_2\gamma\int_{\partial\Omega}u_\tau^N(x,\tau)\int_\Omega u_t^N(\xi,0)d\xi ds_x \\
&\le \tfrac{\lambda_2\gamma}{2\varepsilon_6}\left(\varepsilon\left\|\nabla u_\tau^N(x,\tau)\right\|^2_{L^2(\Omega)} + l(\varepsilon)\left\|u_\tau^N(x,\tau)\right\|^2_{L^2(\Omega)}\right) \\
&+\tfrac{\lambda_2\gamma}{2}\varepsilon_6|\Omega|\,|\partial\Omega|\left\|u_t^N(x,0)\right\|^2_{L^2(\Omega)},
\end{aligned}
\tag{3.42}
$$

$$
\begin{aligned}
& -\lambda_2\gamma \int_{\partial\Omega}\int_0^\tau u_t^N(x,t)\int_\Omega u_{tt}^N(\xi,t)\,d\xi dt ds_x \\
\leq\ & \frac{\lambda_2\gamma}{2}\varepsilon\int_0^\tau \left\|\nabla u_t^N(x,t)\right\|_{L^2(\Omega)}^2 dt \\
& +\frac{\lambda_2\gamma}{2}l(\varepsilon)\int_0^\tau \left\|u_t^N(x,t)\right\|_{L^2(\Omega)}^2 dt \\
& +\frac{\lambda_2\gamma}{2}|\Omega|\,|\partial\Omega|\int_0^\tau \left\|u_{tt}^N(x,t)\right\|_{L^2(\Omega)}^2 dt, \qquad (3.43)
\end{aligned}
$$

$$
\begin{aligned}
& \lambda_3\varrho\int_{\partial\Omega} u_{\tau\tau}^N(x,\tau)\int_0^\tau\int_\Omega u^N(\xi,t)\,d\xi dt ds_x \\
\leq\ & \frac{\lambda_3\varrho}{2}\left(\frac{\varepsilon}{\varepsilon_7}\left\|\nabla u_{\tau\tau}^N(x,\tau)\right\|_{L^2(\Omega)}^2+\frac{l(\varepsilon)}{\varepsilon_7}\left\|u_{\tau\tau}^N(x,\tau)\right\|_{L^2(\Omega)}^2\right) \\
& +\frac{\lambda_3\varrho}{2}\varepsilon_7|\Omega|\,|\partial\Omega|\,T\int_0^\tau \left\|u^N(x,t)\right\|_{L^2(\Omega)}^2 dt, \qquad (3.44)
\end{aligned}
$$

and

$$
\begin{aligned}
& -\lambda_3\varrho\int_{\partial\Omega}\int_0^\tau u_{tt}^N(x,t)\int_\Omega u^N(\xi,t)\,d\xi dt ds_x \\
\leq\ & \frac{\lambda_3\varrho}{2}\left(\varepsilon\int_0^\tau \left\|\nabla u_{tt}^N(x,t)\right\|_{L^2(\Omega)}^2 dt+l(\varepsilon)\int_0^\tau \left\|u_{tt}^N(x,t)\right\|_{L^2(\Omega)}^2 dt\right) \\
& +\frac{\lambda_3\varrho}{2}|\Omega|\,|\partial\Omega|\int_0^\tau \left\|u^N(x,t)\right\|_{L^2(\Omega)}^2 dt, \qquad (3.45)
\end{aligned}
$$

$$
\begin{aligned}
& \lambda_3\delta\int_{\partial\Omega} u_{\tau\tau}^N(x,\tau)\int_\Omega u^N(\xi,\tau)\,d\xi ds_x \\
\leq\ & \frac{\lambda_3\delta}{2\varepsilon_8}\left(\varepsilon\left\|\nabla u_{\tau\tau}^N(x,\tau)\right\|_{L^2(\Omega)}^2+l(\varepsilon)\left\|u_{\tau\tau}^N(x,\tau)\right\|_{L^2(\Omega)}^2\right) \\
& +\frac{\lambda_3\delta}{2}\varepsilon_8|\Omega|\,|\partial\Omega|\left\|u^N(x,\tau)\right\|_{L^2(\Omega)}^2, \qquad (3.46)
\end{aligned}
$$

$$
\begin{aligned}
& -\lambda_3\delta\int_{\partial\Omega} u_{\tau\tau}^N(x,\tau)\int_\Omega u^N(\xi,0)d\xi ds_x \\
\leq\ & \frac{\lambda_3\delta}{2\varepsilon_9}\left(\varepsilon\left\|\nabla u_{\tau\tau}^N(x,\tau)\right\|_{L^2(\Omega)}^2+l(\varepsilon)\left\|u_{\tau\tau}^N(x,\tau)\right\|_{L^2(\Omega)}^2\right) \\
& +\frac{\lambda_3\delta}{2}\varepsilon_9|\Omega|\,|\partial\Omega|\left\|u^N(x,0)\right\|_{L^2(\Omega)}^2, \qquad (3.47)
\end{aligned}
$$

$$
-\lambda_3\delta\int_{\partial\Omega}\int_0^\tau u_{tt}^N(x,t)\int_\Omega u_t^N(\xi,t)\,d\xi dt ds_x
$$

$$\leq \frac{\lambda_3\delta}{2}\left(\varepsilon\int_0^\tau \left\|\nabla u_{tt}^N(x,t)\right\|_{L^2(\Omega)}^2 dt + l(\varepsilon)\int_0^\tau \left\|u_{tt}^N(x,t)\right\|_{L^2(\Omega)}^2 dt\right)$$
$$+\frac{\lambda_3\delta}{2}|\Omega||\partial\Omega|\int_0^\tau \left\|u_t^N(x,t)\right\|_{L^2(\Omega)}^2 dt, \tag{3.48}$$

$$\lambda_3\gamma\int_{\partial\Omega} u_{\tau\tau}^N(x,\tau)\int_\Omega u_\tau^N(\xi,\tau)\,d\xi ds_x$$
$$\leq \frac{\lambda_3\gamma}{2\varepsilon_{10}}\left(\varepsilon\left\|\nabla u_{\tau\tau}^N(x,\tau)\right\|_{L^2(\Omega)}^2 + l(\varepsilon)\left\|u_{\tau\tau}^N(x,\tau)\right\|_{L^2(\Omega)}^2\right)$$
$$+\frac{\lambda_3\gamma}{2}\varepsilon_{10}|\Omega||\partial\Omega|\left\|u_\tau^N(x,\tau)\right\|_{L^2(\Omega)}^2, \tag{3.49}$$

$$-\lambda_3\gamma\int_{\partial\Omega} u_{\tau\tau}^N(x,\tau)\int_\Omega u_t^N(\xi,0)d\xi ds_x$$
$$\leq \frac{\lambda_3\gamma}{2\varepsilon_{11}}\left(\varepsilon\left\|\nabla u_{\tau\tau}^N(x,\tau)\right\|_{L^2(\Omega)}^2 + l(\varepsilon)\left\|u_{\tau\tau}^N(x,\tau)\right\|_{L^2(\Omega)}^2\right)$$
$$+\frac{\lambda_3\gamma}{2}\varepsilon_{11}|\Omega||\partial\Omega|\left\|u_t^N(x,0)\right\|_{L^2(\Omega)}^2, \tag{3.50}$$

$$-\lambda_3\gamma\int_{\partial\Omega}\int_0^\tau u_{tt}^N(x,t)\int_\Omega u_{tt}^N(\xi,t)\,d\xi dt ds_x$$
$$\leq \frac{\lambda_3\gamma}{2}\varepsilon\int_0^\tau \left\|\nabla u_{tt}^N(x,t)\right\|_{L^2(\Omega)}^2 dt$$
$$+\frac{\lambda_3\gamma}{2}(l(\varepsilon)+|\Omega||\partial\Omega|)\int_0^\tau \left\|u_{tt}^N(x,t)\right\|_{L^2(\Omega)}^2 dt, \tag{3.51}$$

$$-\frac{\lambda_1}{2}\left\|u_{\tau\tau\tau}^N(x,\tau)\right\|_{L^2(\Omega)}^2 - \frac{\lambda_1}{2}\left\|u_\tau^N(x,\tau)\right\|_{L^2(\Omega)}^2$$
$$\leq \lambda_1\left(u_{\tau\tau\tau}^N(x,\tau), u_\tau^N(x,\tau)\right)_{L^2(\Omega)}, \tag{3.52}$$

$$-\frac{\lambda_2}{2}\left\|u_{\tau\tau\tau}^N(x,\tau)\right\|_{L^2(\Omega)}^2 - \frac{\lambda_2}{2}\left\|u_{\tau\tau}^N(x,\tau)\right\|_{L^2(\Omega)}^2$$
$$\leq \lambda_2\left(u_{\tau\tau\tau}^N(x,\tau), u_{\tau\tau}^N(x,\tau)\right)_{L^2(\Omega)}, \tag{3.53}$$

$$-\frac{\lambda_1\alpha}{2}\left\|u_{\tau\tau}^N(x,\tau)\right\|_{L^2(\Omega)}^2 - \frac{\lambda_1\alpha}{2}\left\|u_\tau^N(x,\tau)\right\|_{L^2(\Omega)}^2$$
$$\leq \lambda_1\alpha\left(u_{\tau\tau}^N(x,\tau), u_\tau^N(x,\tau)\right)_{L^2(\Omega)}, \tag{3.54}$$

$$-\frac{\lambda_2\varrho\varepsilon_{12}}{2}\left\|\nabla u^N(x,\tau)\right\|_{L^2(\Omega)}^2 - \frac{\lambda_2\varrho}{2\varepsilon_{12}}\left\|\nabla u_\tau^N(x,\tau)\right\|_{L^2(\Omega)}^2$$

$$\leq \quad \lambda_2\varrho\left(\nabla u^N(x,\tau), \nabla u_\tau^N(x,\tau)\right)_{L^2(\Omega)}, \tag{3.55}$$

$$-\frac{\lambda_2\varrho\varepsilon_{13}}{2}\left\|\nabla u^N(x,\tau)\right\|_{L^2(\Omega)}^2 - \frac{\lambda_2\varrho}{2\varepsilon_{13}}\left\|\nabla u_{\tau\tau}^N(x,\tau)\right\|_{L^2(\Omega)}^2$$
$$\leq \quad \lambda_3\varrho\left(\nabla u^N(x,\tau), \nabla u_{\tau\tau}^N(x,\tau)\right)_{L^2(\Omega)}, \tag{3.56}$$

$$-\frac{\lambda_3\delta\varepsilon_{14}}{2}\left\|\nabla u_\tau^N(x,\tau)\right\|_{L^2(\Omega)}^2 - \frac{\lambda_3\delta}{2\varepsilon_{14}}\left\|\nabla u_{\tau\tau}^N(x,\tau)\right\|_{L^2(\Omega)}^2$$
$$\leq \quad \lambda_3\delta\left(\nabla u_\tau^N(x,\tau), \nabla u_{\tau\tau}^N(x,\tau)\right)_{L^2(\Omega)}, \tag{3.57}$$

$$\lambda_1\left(u_{ttt}^N(x,0), u_t^N(x,0)\right)_{L^2(\Omega)}$$
$$\leq \quad \frac{\lambda_1}{2}\left\|u_{ttt}^N(x,0)\right\|_{L^2(\Omega)}^2 + \frac{\lambda_1}{2}\left\|u_t^N(x,0)\right\|_{L^2(\Omega)}^2 \tag{3.58}$$

$$\lambda_1\alpha\left(u_{tt}^N(x,0), u_t^N(x,0)\right)_{L^2(\Omega)}$$
$$\leq \quad \frac{\lambda_1\alpha}{2}\left\|u_{tt}^N(x,0)\right\|_{L^2(\Omega)}^2 + \frac{\lambda_1\alpha}{2}\left\|u_t^N(x,0)\right\|_{L^2(\Omega)}^2, \tag{3.59}$$

$$\lambda_2\left(u_{ttt}^N(x,0), u_{tt}^N(x,0)\right)_{L^2(\Omega)}$$
$$\leq \quad \frac{\lambda_2}{2}\left\|u_{ttt}^N(x,0)\right\|_{L^2(\Omega)}^2 + \frac{\lambda_2}{2}\left\|u_{tt}^N(x,0)\right\|_{L^2(\Omega)}^2, \tag{3.60}$$

$$\lambda_2\varrho\left(\nabla u^N(x,0), \nabla u_t^N(x,0)\right)_{L^2(\Omega)}$$
$$\leq \quad \frac{\lambda_2}{2}\varrho\left\|\nabla u^N(x,0)\right\|_{L^2(\Omega)}^2 + \frac{\lambda_2}{2}\varrho\left\|\nabla u_t^N(x,0)\right\|_{L^2(\Omega)}^2, \tag{3.61}$$

$$\lambda_3\varrho\left(\nabla u^N(x,0), \nabla u_{tt}^N(x,0)\right)_{L^2(\Omega)}$$
$$\leq \quad \frac{\lambda_3}{2}\varrho\left\|\nabla u^N(x,0)\right\|_{L^2(\Omega)}^2 + \frac{\lambda_3}{2}\varrho\left\|\nabla u_{tt}^N(x,0)\right\|_{L^2(\Omega)}^2, \tag{3.62}$$

$$\lambda_3\delta\left(\nabla u_t^N(x,0), \nabla u_{tt}^N(x,0)\right)_{L^2(\Omega)}$$
$$\leq \quad \frac{\lambda_3}{2}\delta\left\|\nabla u_t^N(x,0)\right\|_{L^2(\Omega)}^2 + \frac{\lambda_3}{2}\delta\left\|\nabla u_{tt}^N(x,0)\right\|_{L^2(\Omega)}^2, \tag{3.63}$$

$$\lambda_1\int_0^\tau\left(u_{ttt}^N, u_{tt}^N\right)_{L^2(\Omega)}dt \leq \frac{\lambda_1}{2}\int_0^\tau\left\|u_{ttt}^N(x,t)\right\|_{L^2(\Omega)}^2 dt + \frac{\lambda_1}{2}\int_0^\tau\left\|u_{tt}^N(x,t)\right\|_{L^2(\Omega)}^2 dt, \tag{3.64}$$

$$\lambda_3\varrho\int_0^\tau\left(\nabla u_t^N,\nabla u_{tt}^N\right)_{L^2(\Omega)}dt\leq\frac{\lambda_3\varrho}{2}\int_0^\tau\left\|\nabla u_t^N(x,t)\right\|_{L^2(\Omega)}^2dt+\frac{\lambda_3\varrho}{2}\int_0^\tau\left\|u_{tt}^N(x,t)\right\|_{L^2(\Omega)}^2dt,\tag{3.65}$$

$$\begin{aligned}&\lambda_1\int_{\partial\Omega}u^N(x,\tau)\int_0^\tau\int_\Omega H(t)u^N(\xi,t)\,d\xi dtds_x\\&\leq\tfrac{\lambda_1h_0}{2}\left(\varepsilon\left\|\nabla u^N(x,\tau)\right\|_{L^2(\Omega)}^2+l(\varepsilon)\left\|u^N(x,\tau)\right\|_{L^2(\Omega)}^2\right)\\&+\tfrac{\lambda_1h_0}{2}T\,|\Omega|\,|\partial\Omega|\int_0^\tau\left\|u^N(x,t)\right\|_{L^2(\Omega)}^2dt,\end{aligned}\tag{3.66}$$

$$\begin{aligned}&-\lambda_1\int_{\partial\Omega}\int_0^\tau u^N(x,t)\int_\Omega H(t)u^N(\xi,t)\,d\xi dtds_x\\&\leq\tfrac{\lambda_1h_0}{2}\varepsilon\int_0^\tau\left\|\nabla u^N(x,t)\right\|_{L^2(\Omega)}^2dt\\&+\tfrac{\lambda_1h_0}{2}\left(l(\varepsilon)+|\Omega|\,|\partial\Omega|\right)\int_0^\tau\left\|u^N(x,t)\right\|_{L^2(\Omega)}^2dt,\end{aligned}\tag{3.67}$$

$$\begin{aligned}&\lambda_1\int_{\partial\Omega}u^N(x,\tau)\int_0^\tau\int_\Omega\int_0^t h(t-\sigma)(u^N(\xi,t)-u^N(\xi,\sigma))d\sigma d\xi dtds_x\\&\leq\tfrac{\lambda_1h_0}{2}\left(\varepsilon\left\|\nabla u^N(x,\tau)\right\|_{L^2(\Omega)}^2+l(\varepsilon)\left\|u^N(x,\tau)\right\|_{L^2(\Omega)}^2\right)\\&+\tfrac{\lambda_1}{2}T\,|\Omega|\,|\partial\Omega|\int_0^\tau h\circ u^N(t)dt,\end{aligned}\tag{3.68}$$

$$\begin{aligned}&-\lambda_1\int_{\partial\Omega}\int_0^\tau u^N(x,t)\int_\Omega\int_0^t h(t-\sigma)(u^N(\xi,t)-u^N(\xi,\sigma))d\sigma d\xi dtds_x\\&\leq\tfrac{\lambda_1h_0}{2}\varepsilon\int_0^\tau\left\|\nabla u^N(x,t)\right\|_{L^2(\Omega)}^2dt+\tfrac{\lambda_1h_0}{2}l(\varepsilon)\int_0^\tau\left\|u^N(x,t)\right\|_{L^2(\Omega)}^2dt\\&+\tfrac{\lambda_1}{2}\,|\Omega|\,|\partial\Omega|\int_0^\tau h\circ u^N(t)dt,\end{aligned}\tag{3.69}$$

$$\begin{aligned}&\lambda_2\int_{\partial\Omega}u_\tau^N(x,\tau)\int_0^\tau\int_\Omega H(t)u^N(\xi,t)\,d\xi dtds_x\\&\leq\tfrac{\lambda_2h_0}{2}\left(\varepsilon\left\|\nabla u_\tau^N(x,\tau)\right\|_{L^2(\Omega)}^2+l(\varepsilon)\left\|u_\tau^N(x,\tau)\right\|_{L^2(\Omega)}^2\right)\\&+\tfrac{\lambda_2h_0}{2}T\,|\Omega|\,|\partial\Omega|\int_0^\tau\left\|u^N(x,t)\right\|_{L^2(\Omega)}^2dt,\end{aligned}\tag{3.70}$$

$$\begin{aligned}&-\lambda_2\int_{\partial\Omega}\int_0^\tau u_\tau^N(x,t)\int_\Omega H(t)u^N(\xi,t)\,d\xi dtds_x\\&\leq\tfrac{\lambda_2h_0}{2}\varepsilon\int_0^\tau\left\|\nabla u_t^N(x,t)\right\|_{L^2(\Omega)}^2dt+\tfrac{\lambda_2h_0}{2}l(\varepsilon)\int_0^\tau\left\|u_t^N(x,t)\right\|_{L^2(\Omega)}^2dt\\&+\tfrac{\lambda_2h_0}{2}\,|\Omega|\,|\partial\Omega|\int_0^\tau\left\|u^N(x,t)\right\|_{L^2(\Omega)}^2dt,\end{aligned}\tag{3.71}$$

$$\begin{aligned}&\lambda_2\int_{\partial\Omega}u_\tau^N(x,\tau)\int_0^\tau\int_\Omega\int_0^t h(t-\sigma)(u^N(\xi,t)-u^N(\xi,\sigma))d\sigma d\xi dtds_x\\&\leq\tfrac{\lambda_2h_0}{2}\left(\varepsilon\left\|\nabla u_\tau^N(x,\tau)\right\|_{L^2(\Omega)}^2+l(\varepsilon)\left\|u_\tau^N(x,\tau)\right\|_{L^2(\Omega)}^2\right)\\&+\tfrac{\lambda_2}{2}T\,|\Omega|\,|\partial\Omega|\int_0^\tau h\circ u^N(t)dt,\end{aligned}\tag{3.72}$$

$$\begin{aligned}&-\lambda_2\int_{\partial\Omega}\int_0^\tau u_t^N(x,t)\int_\Omega\int_0^t h(t-\sigma)(u^N(\xi,t)-u^N(\xi,\sigma))d\sigma d\xi dtds_x\\&\leq\tfrac{\lambda_2h_0}{2}\varepsilon\int_0^\tau\left\|\nabla u_t^N(x,t)\right\|_{L^2(\Omega)}^2dt+\tfrac{\lambda_2}{2}h_0l(\varepsilon)\int_0^\tau\left\|u_t^N(x,t)\right\|_{L^2(\Omega)}^2dt\\&+\tfrac{\lambda_2}{2}\,|\Omega|\,|\partial\Omega|\int_0^\tau h\circ u^N(t)dt,\end{aligned}\tag{3.73}$$

$$\begin{aligned}&\lambda_3\int_{\partial\Omega}u_{\tau\tau}^N(x,\tau)\int_0^\tau\int_\Omega H(t)u^N(\xi,t)\,d\xi dtds_x\\&\leq\tfrac{\lambda_3h_0}{2\varepsilon_{18}}\left(\varepsilon\left\|\nabla u_{\tau\tau}^N(x,\tau)\right\|_{L^2(\Omega)}^2+l(\varepsilon)\left\|u_{\tau\tau}^N(x,\tau)\right\|_{L^2(\Omega)}^2\right)\\&+\tfrac{\lambda_3h_0}{2}\varepsilon_{18}T\,|\Omega|\,|\partial\Omega|\int_0^\tau\left\|u^N(x,t)\right\|_{L^2(\Omega)}^2dt,\end{aligned}\tag{3.74}$$

$$\begin{aligned}&-\lambda_3\int_{\partial\Omega}\int_0^\tau u_{\tau\tau}^N(x,t)\int_\Omega H(t)u^N(\xi,t)\,d\xi dtds_x\\&\leq\tfrac{\lambda_3h_0}{2}\varepsilon\int_0^\tau\left\|\nabla u_{tt}^N(x,t)\right\|_{L^2(\Omega)}^2dt+\tfrac{\lambda_3h_0}{2}l(\varepsilon)\int_0^\tau\left\|u_{tt}^N(x,t)\right\|_{L^2(\Omega)}^2dt\\&+\tfrac{\lambda_3h_0}{2}\,|\Omega|\,|\partial\Omega|\int_0^\tau\left\|u^N(x,t)\right\|_{L^2(\Omega)}^2dt,\end{aligned}\tag{3.75}$$

$$\begin{aligned}&\lambda_3 \int_{\partial\Omega} u_{\tau\tau}^N(x,\tau) \int_0^\tau \int_\Omega \int_0^t h(t-\sigma)(u^N(\xi,t)-u^N(\xi,\sigma))d\sigma d\xi dt ds_x \\ &\leq \tfrac{\lambda_3 h_0}{2\varepsilon_{15}}\left(\varepsilon \left\|\nabla u_{\tau\tau}^N(x,\tau)\right\|_{L^2(\Omega)}^2 + l(\varepsilon)\left\|u_{\tau\tau}^N(x,\tau)\right\|_{L^2(\Omega)}^2\right) \\ &+\tfrac{\lambda_3}{2}\varepsilon_{15} T\,|\Omega|\,|\partial\Omega| \int_0^\tau h\circ u^N(t)dt,\end{aligned} \tag{3.76}$$

$$\begin{aligned}&-\lambda_3 \int_{\partial\Omega} \int_0^\tau u_{tt}^N(x,t) \int_\Omega \int_0^t h(t-\sigma)(u^N(\xi,t)-u^N(\xi,\sigma))d\sigma d\xi dt ds_x \\ &\leq \tfrac{\lambda_3 h_0}{2}\varepsilon \int_0^\tau \left\|\nabla u_{tt}^N(x,t)\right\|_{L^2(\Omega)}^2 dt + \tfrac{\lambda_3}{2} h_0 l(\varepsilon) \int_0^\tau \left\|u_{tt}^N(x,t)\right\|_{L^2(\Omega)}^2 dt \\ &+\tfrac{\lambda_3}{2}\,|\Omega|\,|\partial\Omega| \int_0^\tau h\circ u^N(t)dt,\end{aligned} \tag{3.77}$$

$$\begin{aligned}-\lambda_3 H(\tau)(\nabla u_{\tau\tau}^N, \nabla u^N)_{L^2(\Omega)}^2 \quad &\geq \quad -\frac{\lambda_3 h_0}{2\varepsilon_{16}}\left\|\nabla u_{\tau\tau}^N(x,\tau)\right\|_{L^2(\Omega)}^2 \\ &\quad -\frac{\lambda_3 h_0 \varepsilon_{16}}{2}\left\|\nabla u^N(x,\tau)\right\|_{L^2(\Omega)}^2,\end{aligned} \tag{3.78}$$

$$\begin{aligned}-\lambda_3 h(\tau)(\nabla u_{\tau}^N, \nabla u^N)_{L^2(\Omega)}^2 \quad &\geq \quad -\frac{\lambda_3 h(0)}{2}\left\|\nabla u_{\tau}^N(x,\tau)\right\|_{L^2(\Omega)}^2 \\ &\quad -\frac{\lambda_3 h(0)}{2}\left\|\nabla u^N(x,\tau)\right\|_{L^2(\Omega)}^2,\end{aligned} \tag{3.79}$$

$$\begin{aligned}&\lambda_3 \int_\Omega \nabla u_{\tau\tau}^N\Big[\int_0^\tau h(\tau-\sigma)(\nabla u^N(\tau)-\nabla u^N(\sigma)d\sigma\Big]dx \\ \geq \quad &-\frac{\lambda_3 h_0}{2\varepsilon_{17}}\left\|\nabla u_{\tau\tau}^N(x,\tau)\right\|_{L^2(\Omega)}^2 - \frac{\lambda_3\varepsilon_{17}}{2} h\circ\nabla u^N(\tau),\end{aligned} \tag{3.80}$$

$$\begin{aligned}&-\lambda_3 \int_\Omega \nabla u_{\tau}^N\Big[\int_0^\tau h'(\tau-\sigma)(\nabla u^N(\tau)-\nabla u^N(\sigma)d\sigma\Big]dx \\ \geq \quad &-\frac{\lambda_3 h_0}{2}\left\|\nabla u_{\tau}^N(x,\tau)\right\|_{L^2(\Omega)}^2 + \frac{\lambda_3}{2} h'\circ\nabla u^N(\tau),\end{aligned} \tag{3.81}$$

$$\begin{aligned}&\lambda_2 \int_\Omega \nabla u_{\tau}^N\Big[\int_0^\tau h(\tau-\sigma)\nabla u^N(\sigma)d\sigma\Big]dx \geq -\frac{\lambda_2}{2} h\circ\nabla u^N(\tau) \\ &-\frac{\lambda_2(h_0+1)}{2}\left\|\nabla u_{\tau}^N(x,\tau)\right\|_{L^2(\Omega)}^2 - \frac{\lambda_2 h_0}{2}\left\|\nabla u^N(x,\tau)\right\|_{L^2(\Omega)}^2,\end{aligned} \tag{3.82}$$

$$\begin{aligned}\lambda_1(F, u_t^N)_{L^2(Q_\tau)} \quad &\leq \quad \frac{\lambda_1}{2}\int_0^\tau \|F(x,t)\|_{L^2(\Omega)}^2 dt + \frac{\lambda_1}{2}\int_0^\tau \|u_t(x,t)\|_{L^2(\Omega)}^2 dt \\ \lambda_2(F, u_{tt}^N)_{L^2(Q_\tau)} \quad &\leq \quad \frac{\lambda_2}{2}\int_0^\tau \|F(x,t)\|_{L^2(\Omega)}^2 dt + \frac{\lambda_2}{2}\int_0^\tau \|u_{tt}(x,t)\|_{L^2(\Omega)}^2 dt \\ \lambda_3(F, u_{ttt}^N)_{L^2(Q_\tau)} \quad &\leq \quad \frac{\lambda_3}{2}\int_0^\tau \|F(x,t)\|_{L^2(\Omega)}^2 dt + \frac{\lambda_3}{2}\int_0^\tau \|u_{ttt}(x,t)\|_{L^2(\Omega)}^2 dt.\end{aligned} \tag{3.83}$$

Substituting (3.32)–(3.83) into (3.31) and make use of the following inequality

$$\begin{aligned}
m_1 \left\|u^N(x,\tau)\right\|^2_{L^2(\Omega)} &\leq m_1 \left\|u^N(x,t)\right\|^2_{L^2(Q_\tau)} + m_1 \left\|u^N_t(x,t)\right\|^2_{L^2(Q_\tau)} \\
&\quad + m_1 \left\|u^N(x,0)\right\|^2_{L^2(\Omega)} \\
m_2 \left\|u^N_\tau(x,\tau)\right\|^2_{L^2(\Omega)} &\leq m_2 \left\|u^N_t(x,t)\right\|^2_{L^2(Q_\tau)} + m_2 \left\|u^N_{tt}(x,t)\right\|^2_{L^2(Q_\tau)} \\
&\quad + m_2 \left\|u^N_t(x,0)\right\|^2_{L^2(\Omega)} \\
m_3 \left\|u^N_{\tau\tau}(x,\tau)\right\|^2_{L^2(\Omega)} &\leq m_3 \left\|u^N_{tt}(x,t)\right\|^2_{L^2(Q_\tau)} + m_3 \left\|u^N_{ttt}(x,t)\right\|^2_{L^2(Q_\tau)} \\
&\quad + m_3 \left\|u^N_{tt}(x,0)\right\|^2_{L^2(\Omega)} \\
m_4 \left\|\nabla u^N(x,\tau)\right\|^2_{L^2(\Omega)} &\leq m_4 \left\|\nabla u^N(x,t)\right\|^2_{L^2(Q_\tau)} + m_4 \left\|\nabla u^N_t(x,t)\right\|^2_{L^2(Q_\tau)} \\
&\quad + m_4 \left\|\nabla u^N(x,0)\right\|^2_{L^2(\Omega)} \\
m_5 \left\|\nabla u^N_\tau(x,\tau)\right\|^2_{L^2(\Omega)} &\leq m_5 \left\|\nabla u^N_t(x,t)\right\|^2_{L^2(Q_\tau)} + m_5 \left\|\nabla u^N_{tt}(x,t)\right\|^2_{L^2(Q_\tau)} \\
&\quad + m_5 \left\|\nabla u^N_t(x,0)\right\|^2_{L^2(\Omega)} \\
m_6 h \circ u^N(\tau) &\leq m_6 \left\|u^N_t(x,t)\right\|^2_{L^2(Q_\tau)} + m_6 \int_0^\tau h \circ u^N(t)dt \\
m_7 h \circ \nabla u^N(\tau) &\leq m_7 \left\|\nabla u^N_t(x,t)\right\|^2_{L^2(Q_\tau)} + m_7 \int_0^\tau h \circ \nabla u^N(t)dt \\
-m_8 h' \circ \nabla u^N(\tau) &\leq m_8 \left\|\nabla u^N_t(x,t)\right\|^2_{L^2(Q_\tau)} - m_8 \int_0^\tau h' \circ \nabla u^N(t)dt,
\end{aligned}$$

where

$$\begin{aligned}
m_1 &= \frac{\lambda_1 \varrho}{\varepsilon_1} l(\varepsilon) + \frac{\lambda_2 \delta}{2} \varepsilon_3 |\Omega| |\partial\Omega| + \frac{\lambda_3 \delta}{2} \varepsilon_8 |\Omega| |\partial\Omega| + \lambda_1 h_0 l(\varepsilon), \\
m_2 &= \frac{\lambda_2 \varrho}{2} \frac{l(\varepsilon)}{\varepsilon_2} + \frac{\lambda_2 \delta}{2} \frac{l(\varepsilon)}{\varepsilon_3} + \frac{\lambda_2 \delta}{2} \frac{l(\varepsilon)}{\varepsilon_4} + \frac{\lambda_2 \gamma}{2} \left(\frac{l(\varepsilon)}{\varepsilon_5} + \varepsilon_5 |\Omega| |\partial\Omega| \right) \\
&\quad + \frac{\lambda_2 \gamma}{2} \frac{l(\varepsilon)}{\varepsilon_6} + \frac{\lambda_3 \gamma}{2} \varepsilon_{10} |\Omega| |\partial\Omega| + \frac{\lambda_1 (1+\alpha)}{2} + \lambda_2 h_0 l(\varepsilon), \\
m_3 &= \frac{\lambda_3 \varrho}{2} \frac{l(\varepsilon)}{\varepsilon_7} + \frac{\lambda_3 \delta}{2} \frac{l(\varepsilon)}{\varepsilon_8} + \frac{\lambda_3 \delta}{2} \frac{l(\varepsilon)}{\varepsilon_9} + \frac{\lambda_3 \gamma}{2} \frac{l(\varepsilon)}{\varepsilon_{10}} + \frac{\lambda_3 \gamma}{2} \frac{l(\varepsilon)}{\varepsilon_{11}} + \frac{\lambda_2}{2} \\
&\quad + \frac{\lambda_1 \alpha}{2} + \frac{\lambda_3 h_0}{2\varepsilon_{18}} l(\varepsilon) + \frac{\lambda_3}{2\varepsilon_{15}} l(\varepsilon), \\
m_4 &= \frac{\lambda_1 h_0}{2\varepsilon_1} \varepsilon + \frac{\lambda_2 \varrho}{2} \varepsilon_{12} + \frac{\lambda_2 \varrho}{2} \varepsilon_{13} + \lambda_1 h_0 \varepsilon + \frac{\lambda_3}{2} + \frac{\lambda_3 h_0}{2} \varepsilon_{16} \\
&\quad + \frac{\lambda_3 h(0)}{2} + \frac{\lambda_2 h_0}{2} + \frac{\lambda_1 \varrho}{2\varepsilon_1} \varepsilon, \\
m_5 &= \frac{\lambda_2 \varrho}{2} \frac{\varepsilon}{\varepsilon_2} + \frac{\lambda_2 \delta}{2} \frac{\varepsilon}{\varepsilon_3} + \frac{\lambda_2 \delta}{2} \frac{\varepsilon}{\varepsilon_4} + \frac{\lambda_2 \gamma}{2} \frac{\varepsilon}{\varepsilon_5} + \frac{\lambda_2 \gamma}{2} \frac{\varepsilon}{\varepsilon_6} + \frac{\lambda_2 \varrho}{2\varepsilon_{12}} + \frac{\lambda_3 \delta \varepsilon_{14}}{2} \\
&\quad + \lambda_2 h_0 \varepsilon + \frac{\lambda_3 (h_0 + h(0))}{2} + \frac{\lambda_2 (h_0 + 1)}{2},
\end{aligned}$$

$$m_7 = \frac{\lambda_2\varepsilon_{17}}{2} + \frac{\lambda_2}{2}, \quad m_8 = \frac{\lambda_3}{2}, \quad m_6 = 1,$$

we have

$$\begin{aligned}
&\tfrac{\lambda_1\varrho}{2\varepsilon_1} l(\varepsilon) \left\|u^N(x,\tau)\right\|^2_{L^2(\Omega)} + \tfrac{\lambda_1\beta}{2}\left\|u^N_\tau(x,\tau)\right\|^2_{L^2(\Omega)} + \left(\tfrac{\lambda_2\alpha}{2} + \tfrac{\lambda_3\beta}{2}\right)\left\|u^N_{\tau\tau}(x,\tau)\right\|^2_{L^2(\Omega)} \\
&+\left\{\tfrac{\lambda_3}{2} - \tfrac{\lambda_1}{2} - \tfrac{\lambda_2}{2}\right\}\left\|u^N_{\tau\tau\tau}(x,\tau)\right\|^2_{L^2(\Omega)} + \tfrac{\lambda_1\varrho}{2}\left\|\nabla u^N(x,\tau)\right\|^2_{L^2(\Omega)} \\
&+\left\{\tfrac{\lambda_1\gamma}{2} + \tfrac{\lambda_2\delta}{2}\right\}\left\|\nabla u^N_\tau(x,\tau)\right\|^2_{L^2(\Omega)} + h\circ u^N(\tau) + \tfrac{\lambda_1}{2} h\circ\nabla u^N(\tau) - \tfrac{\lambda_2}{2} h'\circ\nabla u^N(\tau) \\
&+\left\{\tfrac{\lambda_3\gamma}{2} - \tfrac{\lambda_3\varrho}{2}\tfrac{\varepsilon}{\varepsilon_7} - \tfrac{\lambda_3\delta}{2}\tfrac{\varepsilon}{\varepsilon_8} - \tfrac{\lambda_3\delta}{2}\tfrac{\varepsilon}{\varepsilon_9} - \tfrac{\lambda_3\gamma}{2}\tfrac{\varepsilon}{\varepsilon_{10}} - \tfrac{\lambda_3\gamma}{2}\tfrac{\varepsilon}{\varepsilon_{11}} - \tfrac{\lambda_2\varrho}{2\varepsilon_{13}} - \tfrac{\lambda_3\delta}{2\varepsilon_{14}}\right. \\
&\left.-\tfrac{\lambda_3 h_0}{2}\tfrac{\varepsilon}{\varepsilon_{16}} - \tfrac{\lambda_3 h_0}{2}\tfrac{\varepsilon}{\varepsilon_{17}} - \tfrac{\lambda_3 h_0}{2\varepsilon_{18}} - \tfrac{\lambda_3 h_0}{2\varepsilon_{15}}\right\}\left\|\nabla u^N_{\tau\tau}(x,\tau)\right\|^2_{L^2(\Omega)} \\
&\le \gamma_7\left\|u^N(x,0)\right\|^2_{L^2(\Omega)} + \left\{\tfrac{\lambda_2}{2} + \tfrac{\lambda_1\alpha}{2} + \left(\tfrac{\lambda_2\alpha}{2} - \tfrac{\lambda_3\beta}{2}\right) + m_3\right\}\left\|u^N_{tt}(x,0)\right\|^2_{L^2(\Omega)} \\
&+\left\{\tfrac{\lambda_1}{2} + \tfrac{\lambda_2}{2} + \tfrac{\lambda_3}{2}\right\}\left\|u^N_{ttt}(x,0)\right\|^2_{L^2(\Omega)} + \left\{\tfrac{\lambda_1\varrho}{2} + \tfrac{\lambda_2\varrho}{2} + \tfrac{\lambda_3\varrho}{2} + m_4\right\}\left\|\nabla u^N(x,0)\right\|^2_{L^2(\Omega)} \\
&+\gamma_8\left\|u^N_t(x,0)\right\|^2_{L^2(\Omega)} + \left\{\tfrac{\lambda_2\varrho}{2} + \tfrac{\lambda_3\delta}{2} + \tfrac{\lambda_1\gamma}{2} + \tfrac{\lambda_2\delta}{2} + m_5\right\}\left\|\nabla u^N_t(x,0)\right\|^2_{L^2(\Omega)} \\
&+\left\{\tfrac{\lambda_3\varrho}{2} + \tfrac{3\lambda_3\delta}{2} - \tfrac{\lambda_3\gamma}{2} - \lambda_2\gamma\right\}\left\|\nabla u^N_{tt}(x,0)\right\|^2_{L^2(\Omega)} + (\gamma_1 + m_1)\int_0^\tau\left\|u^N(x,t)\right\|^2_{L^2(\Omega)}dt \\
&+(\gamma_2 + m_1 + m_2)\int_0^\tau\left\|u^N_t(x,t)\right\|^2_{L^2(\Omega)}dt \\
&+\left\{\tfrac{\lambda_1}{2} + \lambda_2 - \lambda_3\alpha + m_3\right\}\int_0^\tau\left\|u^N_{ttt}(x,t)\right\|^2_{L^2(\Omega)}dt - m_8\int_0^\tau h'\circ\nabla u^N(t)dt \\
&+\{\gamma_6 + m_4\}\int_0^\tau\left\|\nabla u^N(x,t)\right\|^2_{L^2(\Omega)}dt + (\gamma_3 + m_2 + m_3)\int_0^\tau\left\|u^N_{tt}(x,t)\right\|^2_{L^2(\Omega)}dt \\
&+(\gamma_4 + m_4 + m_5 + m_7 + m_8)\int_0^\tau\left\|\nabla u^N_t(x,t)\right\|^2_{L^2(\Omega)}dt \\
&+(\gamma_5 + m_5)\int_0^\tau\left\|\nabla u^N_{tt}(x,t)\right\|^2_{L^2(\Omega)}dt \\
&+\int_0^\tau h\circ u^N(t)dt + m_7\int_0^\tau h\circ\nabla u^N(t)dt + \tfrac{\lambda_1+\lambda_2+\lambda_3}{2}\int_0^\tau\|F(x,t)\|^2_{L^2(\Omega)}dt,
\end{aligned} \tag{3.84}$$

where

$$\begin{aligned}
\gamma_1 &= \frac{\lambda_1\varrho}{2}\varepsilon_1 T|\Omega||\partial\Omega| + \frac{\lambda_1\varrho}{2}(l(\varepsilon) + |\Omega||\partial\Omega|) + \left(\frac{\lambda_1\delta + \lambda_2\varrho}{2}\right)|\Omega||\partial\Omega| \\
&\quad+\frac{\lambda_2\varrho}{2}\varepsilon_2 T|\Omega||\partial\Omega| + \frac{\lambda_3\varrho}{2}\varepsilon_7 T|\Omega||\partial\Omega| + \frac{\lambda_3\varrho}{2}|\Omega||\partial\Omega| + +\frac{\lambda_1 h_0}{2}l(\varepsilon) \\
&\quad+\left[\frac{\lambda_3 h_0}{2}\varepsilon_{18} + \frac{(\lambda_3 + \lambda_2 + \lambda_1)h_0}{2} + \frac{(\lambda_1 + \lambda_2)h_0 T}{2}\right]|\Omega||\partial\Omega| \\
\gamma_2 &= \left(\frac{\lambda_1\delta + \lambda_2\varrho}{2}\right)l(\varepsilon) + \frac{\lambda_1\delta}{2}l(\varepsilon) + \left(\frac{\lambda_1\gamma + \lambda_2\delta}{2}\right)(l(\varepsilon) + |\Omega||\partial\Omega|) \\
&\quad+\frac{\lambda_1\gamma}{2}l(\varepsilon) + \frac{\lambda_2\gamma}{2}l(\varepsilon) + \frac{\lambda_3\delta}{2}|\Omega||\partial\Omega| + \lambda_2 h_0 l(\varepsilon),
\end{aligned}$$

$$\begin{aligned}
\gamma_3 &= \frac{\lambda_2\gamma}{2}|\Omega||\partial\Omega| + \frac{\lambda_3\varrho}{2}l(\varepsilon) + \frac{\lambda_3\delta}{2}l(\varepsilon) + \frac{\lambda_3\gamma}{2}(l(\varepsilon) + |\Omega||\partial\Omega|) + \frac{\lambda_1}{2} \\
&\quad+(\lambda_1\alpha - \lambda_2\beta) + \frac{\lambda_3\varrho}{2} + \lambda_3 h_0 l(\varepsilon), \\
\gamma_4 &= \left(\frac{\lambda_1\delta + \lambda_2\varrho}{2}\right)\varepsilon + \frac{\lambda_1\delta}{2}\varepsilon + \left(\frac{\lambda_1\gamma + \lambda_2\delta}{2}\right)\varepsilon + \frac{\lambda_1\gamma}{2}\varepsilon + \frac{\lambda_2\gamma}{2}\varepsilon + \frac{\lambda_3\varrho}{2} \\
&\quad+(\lambda_2\varrho - \lambda_1\delta) + h(0)\lambda_3 + \lambda_3 h_0\varepsilon, \\
\gamma_5 &= \frac{\lambda_3\delta}{2}\varepsilon + \frac{\lambda_3\gamma}{2}\varepsilon + \frac{\lambda_3\varrho}{2} + (\lambda_3\delta - \lambda_2\gamma) + \lambda_3 h_0\varepsilon,
\end{aligned}$$

$$\begin{aligned}
\gamma_6 &= \frac{\lambda_1\varrho}{2}\varepsilon+\lambda_1 h_0\varepsilon,\\
\gamma_7 &= \frac{\lambda_1\delta}{2}|\Omega||\partial\Omega|T+\frac{\lambda_2\delta}{2}\varepsilon_4|\Omega||\partial\Omega|+\frac{\lambda_3\delta}{2}\varepsilon_9|\Omega||\partial\Omega|+m_1,\\
\gamma_8 &= \frac{\lambda_1\gamma}{2}|\Omega||\partial\Omega|T+\frac{\lambda_2\gamma}{2}\varepsilon_6|\Omega||\partial\Omega|+\frac{\lambda_3\gamma}{2}\varepsilon_{11}|\Omega||\partial\Omega|+\frac{\lambda_1}{2}+\frac{\lambda_1\alpha}{2}+\frac{\lambda_1\beta}{2}+m_2.
\end{aligned}$$

Choosing $\varepsilon_7,\ \varepsilon_8,\ \varepsilon_9,\ \varepsilon_{10},\ \varepsilon_{11},\ \varepsilon_{13}\ ,\varepsilon_{14},\varepsilon_{15},\ \varepsilon_{16},\ \varepsilon_{17}$ and ε_{18} sufficiently large

$$\begin{array}{c}\beta_0 := \frac{\lambda_3\gamma}{2}-\frac{\lambda_3\varrho}{2}\frac{\varepsilon}{\varepsilon_7}-\frac{\lambda_3\delta}{2}\frac{\varepsilon}{\varepsilon_8}-\frac{\lambda_3\delta}{2}\frac{\varepsilon}{\varepsilon_9}-\frac{\lambda_3\gamma}{2}\frac{\varepsilon}{\varepsilon_{10}}-\frac{\lambda_3\gamma}{2}\frac{\varepsilon}{\varepsilon_{11}}-\frac{\lambda_3\delta}{2\varepsilon_{14}}\\ -\frac{\lambda_2\varrho}{2\varepsilon_{13}}-\frac{\lambda_3 h_0}{2}\frac{\varepsilon}{\varepsilon_{16}}-\frac{\lambda_3}{2}\frac{\varepsilon}{\varepsilon_{17}}-\frac{\lambda_3 h_0}{2\varepsilon_{18}}-\frac{\lambda_3}{2\varepsilon_{15}}>0,\end{array} \tag{3.85}$$

the relation (3.84) reduces to

$$\begin{aligned}
&\Big\{\left\|u^N(x,\tau)\right\|^2_{L^2(\Omega)}+\left\|\nabla u^N(x,\tau)\right\|^2_{L^2(\Omega)}+\left\|u^N_\tau(x,\tau)\right\|^2_{L^2(\Omega)}\\
&+\left\|\nabla u^N_\tau(x,\tau)\right\|^2_{L^2(\Omega)}+\left\|u^N_{\tau\tau}(x,\tau)\right\|^2_{L^2(\Omega)}+\left\|\nabla u^N_{\tau\tau}(x,\tau)\right\|^2_{L^2(\Omega)}\\
&+\left\|u^N_{\tau\tau\tau}(x,\tau)\right\|^2_{L^2(\Omega)}+h\circ\nabla u^N(\tau)+h\circ u^N(\tau)-h'\circ\nabla u^N(\tau)\Big\} \qquad (3.86)\\
\leq\ & D\int_0^\tau\Big\{\left\|u^N(x,t)\right\|^2_{L^2(\Omega)}+\left\|\nabla u^N(x,t)\right\|^2_{L^2(\Omega)}+\left\|u^N_t(x,t)\right\|^2_{L^2(\Omega)}\\
&+\left\|\nabla u^N_t(x,t)\right\|^2_{L^2(\Omega)}+\left\|u^N_{tt}(x,t)\right\|^2_{L^2(\Omega)}+\left\|\nabla u^N_{tt}(x,t)\right\|^2_{L^2(\Omega)}\\
&+\left\|u^N_{ttt}(x,t)\right\|^2_{L^2(\Omega)}+h\circ\nabla u^N(t)+h\circ u^N(t)-h'\circ\nabla u^N(t)+\|F\|^2_{L^2(\Omega)}\Big\}dt\\
&+D\Big\{\left\|u^N(x,0)\right\|^2_{W^1_2(\Omega)}+\left\|u^N_t(x,0)\right\|^2_{W^1_2(\Omega)}+\left\|u^N_{tt}(x,0)\right\|^2_{W^1_2(\Omega)}\\
&+\left\|u^N_{ttt}(x,0)\right\|^2_{L^2(\Omega)}+h\circ\nabla u^N(0)+h\circ u^N(0)-h'\circ\nabla u^N(0)\Big\}, \qquad (3.87)
\end{aligned}$$

where

$$D := \frac{\begin{array}{c}\max\Big\{\frac{\lambda_1\delta}{2}|\Omega||\partial\Omega|T+\frac{\lambda_2\delta}{2}\varepsilon_4|\Omega||\partial\Omega|+\frac{\lambda_3\delta}{2}\varepsilon_9|\Omega||\partial\Omega|+m_1\\ ,\ \frac{\lambda_1\gamma}{2}|\Omega||\partial\Omega|T+\frac{\lambda_2\gamma}{2}\varepsilon_6|\Omega||\partial\Omega|\\ +\frac{\lambda_3\gamma}{2}\varepsilon_{11}|\Omega||\partial\Omega|+\frac{\lambda_1}{2}+\frac{\lambda_1\alpha}{2}+\frac{\lambda_1\beta}{2}+m_2,\ \frac{\lambda_2}{2}+\frac{\lambda_1\alpha}{2}+\frac{\lambda_2\alpha}{2}-\frac{\lambda_3\beta}{2}+m_3,\\ \frac{\lambda_1+\lambda_2+\lambda_3}{2},\ \frac{\lambda_1\varrho}{2}+\frac{\lambda_2\varrho}{2}+\frac{\lambda_3\varrho}{2}+m_4,\ \frac{\lambda_2\varrho}{2}+\frac{\lambda_3\delta}{2}+\frac{\lambda_1\gamma}{2}+\frac{\lambda_2\delta}{2}+m_5,\\ \gamma_1+m_1,\gamma_2+m_1+m_2,\gamma_3+m_2+m_3,\ \frac{\lambda_1}{2}+\lambda_2-\lambda_3\alpha+m_3,\\ \frac{\lambda_3\varrho}{2}+\frac{\lambda_3\delta}{2}-\frac{\lambda_3\gamma}{2},\gamma_6+m_4,\gamma_4+m_4+m_5,\gamma_5+m_5,m_7,m_8,1\Big\}\end{array}}{\min\Big\{\frac{\lambda_1\varrho}{2\varepsilon_1}l(\varepsilon),\ \frac{\lambda_1\beta}{2},\ \frac{\lambda_2\alpha}{2}+\frac{\lambda_3\beta}{2},\ \frac{\lambda_3}{2}-\frac{\lambda_1}{2}-\frac{\lambda_2}{2},\ \frac{\lambda_1\varrho}{2},\ \frac{\lambda_1\gamma}{2}+\frac{\lambda_2\delta}{2},1,\ \frac{\lambda_1}{2},\ \frac{\lambda_2}{2},\beta_0\Big\}}. \tag{3.88}$$

Applying the Gronwall inequality to (3.87) and then integrate from 0 to τ appears that

$$\begin{aligned}
&\left\|u^N(x,t)\right\|^2_{W^1_2(Q_\tau)}+\left\|u^N_t(x,t)\right\|^2_{W^1_2(Q_\tau)}+\left\|u^N_{tt}(x,t)\right\|^2_{W^1_2(Q_\tau)}+\|u^N(x,t)\|_h\\
&\leq De^{DT}\Big\{\|u_0(x)\|^2_{W^1_2(\Omega)}+\|u_1(x)\|^2_{W^1_2(\Omega)}+\|u_2(x)\|^2_{L^2(\Omega)}\\
&\qquad+\|u_3(x)\|^2_{L^2(\Omega)}+\|F\|^2_{L^2(\Omega)}\Big\}.
\end{aligned} \tag{3.89}$$

We deduce from (3.89) that

$$\left\|u^{N}(x,t)\right\|^{2}_{W_2^1(Q_\tau)}+\left\|u_t^{N}(x,t)\right\|^{2}_{W_2^1(Q_\tau)}+\left\|u_{tt}^{N}(x,t)\right\|^{2}_{W_2^1(Q_\tau)}+\|u^{N}(x,t)\|_{h}\leq A, \tag{3.90}$$

where

$$\|u^{N}(x,t)\|_{h}:=\int_0^\tau\Big(h\circ\nabla u^{N}(t)+h\circ u^{N}(t)-h'\circ\nabla u^{N}(t)\Big)dt.$$

Therefore the sequence $\left\{u^{N}\right\}_{N\geq 1}$ is bounded in $V(Q_T)$, and we can extract from it a subsequence for which we use the same notation which converges weakly in $V(Q_T)$ to a limit function $u(x,t)$ we have to show that $u(x,t)$ is a generalized solution of (1.1). Since $u^{N}(x,t)\to u(x,t)$ in $L^2(Q_T)$ and $u^N(x,0)\to\zeta(x)$ in $L^2(\Omega)$, then $u(x,0)=\zeta(x)$.

Now to prove that (2.1) holds, we multiply each of the relations (3.5) by a function $p_l(t)\in W_2^1(0,T)$, $p_l(t)=0$, then add up the obtained equalities ranging from $l=1$ to $l=N$, and integrate over t on $(0,T)$. If we let $\eta^N=\sum\limits_{k=1}^{N}p_k(t)Z_k(x)$, then we have

$$\begin{aligned}
&-(u_{ttt}^N,\eta_t^N)_{L^2(Q_T)}-\alpha(u_{tt}^N,\eta_t^N)_{L^2(Q_T)}-\beta(u_t^N,\eta_t^N)_{L^2(Q_T)}+\varrho(\nabla u^N,\nabla\eta^N)_{L^2(Q_T)}\\
&+\delta(\nabla u_t^N,\nabla\eta^N)_{L^2(Q_T)}-\gamma\left(\nabla u_t^N,\nabla\eta_t^N\right)_{L^2(Q_T)}-(\nabla w^N,\nabla\eta^N)_{L^2(Q_T)}\\
&=\varrho\int_{\partial\Omega}\int_0^T\eta^N(x,t)\left(\int_0^t\int_\Omega u^N(\xi,\tau)\,d\xi d\tau\right)dtds_x\\
&+\delta\int_{\partial\Omega}\int_0^T\eta^N(x,t)\int_\Omega u^N(\xi,t)\,d\xi dtds_x\\
&-\delta\int_{\partial\Omega}\int_0^T\eta^N(x,t)\int_\Omega u^N(\xi,0)d\xi dtds_x-\gamma\int_0^T\int_{\partial\Omega}\eta_t^N\left(\int_\Omega u^N(\xi,t)\,d\xi\right)ds_xdt\\
&+\gamma\int_0^T\int_{\partial\Omega}\eta_t^N\left(\int_\Omega u^N(\xi,0)\,d\xi\right)ds_xdt-\gamma\left(\Delta u_t^N(x,0),\eta^N(0)\right)_{L^2(\Omega)}\\
&-\int_{\partial\Omega}\int_0^T\eta^N(x,t)\left(\int_0^t\int_\Omega w^N(\xi,\tau)\,d\xi d\tau\right)dtds_x+\left(F,\eta_t^N\right)_{L^2(Q_T)}\\
&+\left(u_{ttt}^N(x,0),\eta^N(0)\right)_{L^2(\Omega)}+\alpha\left(u_{tt}^N(x,0),\eta^N(0)\right)_{L^2(\Omega)}+\beta\left(u_{tt}^N(x,0),\eta^N(0)\right)_{L^2(\Omega)},
\end{aligned} \tag{3.91}$$

for all η^N of the form $\sum\limits_{k=1}^{N}p_l(t)Z_k(x)$.

Since

$$\int_0^t\int_\Omega(\left(u^N(\xi,\tau)-u(\xi,\tau)\right)d\xi d\tau\leq\sqrt{T|\Omega|}\left\|u^N-u\right\|_{L^2(Q_T)},$$

$$\begin{aligned}
&\int_0^T\eta^N(x,t)\int_\Omega\left(u_t^N(\xi,t)-u_t(\xi,t)\right)d\xi dt\\
\leq\;&\sqrt{|\Omega|}\left(\int_0^T(\eta^N(x,t))^2dt\right)^{1/2}\left\|u_t^N-u_t\right\|_{L^2(Q_T)},
\end{aligned}$$

$$\begin{aligned}
&\int_0^T\eta^N(x,t)\int_\Omega(u\left(^N(\xi,0)-u(\xi,0)\right)d\xi dt\\
\leq\;&\sqrt{|\Omega|}\left(\int_0^T(\eta^N(x,t))^2dt\right)^{1/2}\left\|u^N(x,0)-u(x,0)\right\|_{L^2(Q_T)},
\end{aligned}$$

and

$$\left\|u^N - u\right\|_{L^2(Q_T)} \to 0, \text{ as } N \to \infty,$$

therefore we have

$$\begin{aligned}
&\varrho \int_{\partial\Omega} \int_0^T \eta^N(x,t) \int_0^t \int_\Omega u^N(\xi,\tau)\, d\xi d\tau dt ds_x \\
\to\ &\varrho \int_{\partial\Omega} \int_0^T \eta(x,t) \int_0^t \int_\Omega u(\xi,\tau)\, d\xi d\tau dt ds_x,
\end{aligned}$$

$$\begin{aligned}
&\delta \int_{\partial\Omega} \int_0^T \eta^N(x,t) \int_\Omega u^N(\xi,t)\, d\xi dt ds_x \\
\to\ &\delta \int_{\partial\Omega} \int_0^T \eta(x,t) \int_\Omega u(\xi,t)\, d\xi dt ds_x,
\end{aligned}$$

$$\begin{aligned}
&-\delta \int_{\partial\Omega} \int_0^T \eta^N(x,t) \int_\Omega u^N(\xi,0) d\xi dt ds \\
\to\ &-\delta \int_{\partial\Omega} \int_0^T \eta(x,t) \int_\Omega u(\xi,0) d\xi dt ds,
\end{aligned}$$

$$\begin{aligned}
&-\gamma \int_0^T \int_{\partial\Omega} \eta_t^N \left(\int_\Omega u^N(\xi,t)\, d\xi\right) ds_x dt \\
\to\ &-\gamma \int_0^T \int_{\partial\Omega} \eta_t \left(\int_\Omega u(\xi,t)\, d\xi\right) ds_x dt,
\end{aligned}$$

$$\begin{aligned}
&\gamma \int_0^T \int_{\partial\Omega} \eta_t^N \left(\int_\Omega u^N(\xi,0)\, d\xi\right) ds_x dt \\
\to\ &\gamma \int_0^T \int_{\partial\Omega} \eta_t \left(\int_\Omega u(\xi,0)\, d\xi\right) ds_x dt.
\end{aligned}$$

$$\begin{aligned}
&\int_{\partial\Omega} \int_0^T \eta^N(x,t) \int_0^t \int_\Omega w^N(\xi,\tau)\, d\xi d\tau dt ds_x \\
\to\ &\varrho \int_{\partial\Omega} \int_0^T \eta(x,t) \int_0^t \int_\Omega w(\xi,\tau)\, d\xi d\tau dt ds_x.
\end{aligned}$$

Thus, the limit function u satisfies (2.1) for every $\eta^N = \sum_{k=1}^{N} p_l(t) Z_k(x)$. We denote by $\mathbb{Q}_N$ the totality of all functions of the form $\eta^N = \sum_{k=1}^{N} p_l(t) Z_k(x)$, with $p_l(t) \in W_2^1(0,T)$, $p_l(t) = 0$.

But $\cup_{l=1}^{N} \mathbb{Q}_N$ is dense in $W(Q_T)$, then relation (2.1) holds for all $u \in W(Q_T)$. Thus we have shown that the limit function $u(x,t)$ is a generalized solution of problem (1.1) in $V(Q_T)$. □

4. Uniqueness of solution

Theorem 2. *The problem (1.1) cannot have more than one generalized solution in $V(Q_T)$.*

Proof. Suppose that there exist two different generalized solutions $u_1 \in V(Q_T)$ and $u_2 \in V(Q_T)$ for the problem (1.1). Then, $U = u_1 - u_2$ solves

$$\begin{cases} U_{tttt} + \alpha U_{ttt} + \beta U_{tt} - \varrho \Delta U - \delta \Delta U_t - \gamma \Delta U_{tt} + \int_0^t h(t-\sigma)\Delta u(\sigma)d\sigma = 0, \\ U(x,0) = U_t(x,0) = U_{tt}(x,0) = U_{ttt}(x,0) = 0 \\ \frac{\partial u}{\partial \eta} = \int_0^t \int_\Omega u(\xi,\tau)\, d\xi d\tau, \quad x \in \partial\Omega. \end{cases} \tag{4.1}$$

and (2.1) gives

$$\begin{aligned} & -(U_{ttt}, v_t)_{L^2(Q_T)} - \alpha(U_{tt}, v_t)_{L^2(Q_T)} - \beta(U_t, v_t)_{L^2(Q_T)} + \varrho(\nabla U, \nabla v)_{L^2(Q_T)} \\ & +\delta(\nabla U_t, \nabla v)_{L^2(Q_T)} - \gamma(\nabla U_t, \nabla v_t)_{L^2(Q_T)} - (\nabla W, \nabla v)_{L^2(Q_T)} \\ = \ & \varrho \int_0^T \int_{\partial\Omega} v \left(\int_0^t \int_\Omega u(\xi,\tau)\, d\xi d\tau \right) ds_x dt + \delta \int_0^T \int_{\partial\Omega} v \int_\Omega U(\xi,t)\, d\xi ds_x dt \\ & -\gamma \int_0^T \int_{\partial\Omega} v_t \left(\int_\Omega U_\tau(\xi,t)\, d\xi dt \right) ds_x dt \\ & - \int_0^T \int_{\partial\Omega} v \left(\int_0^t \int_\Omega W(\xi,\tau)\, d\xi d\tau \right) ds_x dt, \end{aligned} \tag{4.2}$$

where

$$W(x,t) := \int_0^t h(t-\sigma)\Delta U(\sigma)d\sigma.$$

Consider the function

$$v(x,t) = \begin{cases} \int_t^\tau U(x,s)ds, & 0 \le t \le \tau, \\ 0, & \tau \le t \le T. \end{cases} \tag{4.3}$$

It is obvious that $v \in W(Q_T)$ and $v_t(x,t) = -U(x,t)$ for all $t \in [0,\tau]$. Integration by parts in the left hand side of (4.2) gives

$$-(U_{ttt}, v_t)_{L^2(Q_T)} = (U_{\tau\tau}(x,\tau), U(x,\tau))_{L^2(\Omega)} - \frac{1}{2} \|U_\tau(x,\tau)\|^2_{L^2(\Omega)}, \tag{4.4}$$

$$-\alpha(U_{tt}, v_t)_{L^2(Q_T)} = \alpha(U_\tau(x,\tau), U(x,\tau))_{L^2(\Omega)} - \alpha \int_0^\tau \|U_t(x,t)\|^2_{L^2(\Omega)} dt, \tag{4.5}$$

$$-\beta(U_t, v_t)_{L^2(Q_T)} = \frac{\beta}{2} \|U(x,\tau)\|^2_{L^2(\Omega)}, \tag{4.6}$$

$$\varrho(\nabla U, \nabla v)_{L^2(Q_T)} = \frac{\varrho}{2} \|\nabla v(x,0)\|^2_{L^2(\Omega)}, \tag{4.7}$$

$$\delta\left(\nabla U_t, \nabla v\right)_{L^2(Q_T)} = \delta \int_0^\tau \|\nabla v_t(x,t)\|^2_{L^2(\Omega)}\,dt, \tag{4.8}$$

$$-\gamma\left(\nabla U_t, \nabla v_t\right)_{L^2(Q_T)} = \frac{\gamma}{2}\|\nabla U(x,\tau)\|^2_{L^2(\Omega)}, \tag{4.9}$$

$$\begin{aligned}-\left(\nabla W, \nabla v\right)_{L^2(Q_T)} &\leq h_0 \int_0^\tau \|\nabla v(x,t)\|^2_{L^2(\Omega)}\,dt + \frac{h_0}{2}\int_0^\tau \|\nabla U(x,t)\|^2_{L^2(\Omega)}\,dt \\ &+\frac{1}{2}\int_0^\tau h \circ \nabla U(t)dt.\end{aligned} \tag{4.10}$$

Plugging (4.4)–(4.10) into (4.2) we get

$$\begin{aligned}&(U_{\tau\tau}(x,\tau), U(x,\tau))_{L^2(\Omega)} + \alpha\left(U_\tau(x,\tau), U(x,\tau)\right)_{L^2(\Omega)} + \frac{\beta}{2}\|U(x,\tau)\|^2_{L^2(\Omega)} \\ &+\frac{\varrho}{2}\|\nabla v(x,0)\|^2_{L^2(\Omega)} + \frac{\gamma}{2}\|\nabla U(x,\tau)\|^2_{L^2(\Omega)} - \frac{1}{2}\|U_\tau(x,\tau)\|^2_{L^2(\Omega)} \\ \leq\ & \alpha\int_0^\tau \|U_t(x,t)\|^2_{L^2(\Omega)}\,dt - \delta\int_0^\tau \|\nabla v_t(x,t)\|^2_{L^2(\Omega)}\,dt + h_0\int_0^\tau \|\nabla v(x,t)\|^2_{L^2(\Omega)}\,dt \\ &+\frac{h_0}{2}\int_0^\tau \|\nabla U(x,t)\|^2_{L^2(\Omega)}\,dt + \frac{1}{2}\int_0^\tau h\circ\nabla U(t)dt \\ &+\varrho\int_0^T\int_{\partial\Omega} v\left(\int_0^t\int_\Omega U(\xi,\tau)\,d\xi d\tau\right)ds_x dt \\ &+\delta\int_0^T\int_{\partial\Omega} v\int_\Omega U(\xi,t)\,d\xi ds_x dt - \gamma\int_0^T\int_{\partial\Omega} v_t\left(\int_\Omega U(\xi,t)\,d\xi\right)ds dt \\ &-\int_0^T\int_{\partial\Omega} v\left(\int_0^t\int_\Omega W(\xi,\tau)\,d\xi d\tau\right)ds_x dt.\end{aligned} \tag{4.11}$$

Now since

$$v^2(x,t) = \left(\int_t^\tau U(x,s)ds\right)^2 \leq \tau\int_0^\tau U^2(x,s)ds,$$

then

$$\|v\|^2_{L^2(Q_\tau)} \leq \tau^2\|U\|^2_{L^2(Q_\tau)} \leq T^2\|U\|^2_{L^2(Q_\tau)}. \tag{4.12}$$

Using the trace inequality, the RHS of (4.11) can be estimated as follows

$$\begin{aligned}&\varrho\int_0^T\int_{\partial\Omega} v\left(\int_0^t\int_\Omega U(\xi,\tau)\,d\xi d\tau\right)ds_x dt \\ \leq\ & \frac{\varrho}{2}T^2\left\{l(\varepsilon) + |\Omega|\,|\partial\Omega|\right\}\int_0^\tau \|U(x,t)\|^2_{L^2(\Omega)}\,dt + \frac{\varrho}{2}\varepsilon\int_0^\tau \|\nabla v(x,t)\|^2_{L^2(\Omega)}\,dt,\end{aligned} \tag{4.13}$$

and

$$\delta\int_0^T\int_{\partial\Omega} v\int_\Omega U(\xi,t)\,d\xi ds_x dt \tag{4.14}$$

$$\leq \frac{\delta}{2}\left\{T^2 l(\varepsilon)+|\Omega|\,|\partial\Omega|\right\}\int_0^\tau \|U(x,t)\|^2_{L^2(\Omega)}\,dt+\frac{\delta}{2}\varepsilon\int_0^\tau \|\nabla v(x,t)\|^2_{L^2(\Omega)}\,dt,$$

and

$$\begin{aligned}
&-\gamma\int_0^T\int_{\partial\Omega} v_t\left(\int_\Omega U(\xi,t)\,d\xi\right)dsdt\\
=\;&\gamma\int_0^\tau\int_{\partial\Omega} v\left(\int_\Omega U_t(\xi,t)\,d\xi\right)dsdt \qquad (4.15)\\
\leq\;&\frac{\gamma\,|\Omega|\,|\partial\Omega|}{2}\|U_t\|^2_{L^2(Q_\tau)}+\frac{\gamma T^2}{2}\varepsilon\|\nabla v\|^2_{L^2(Q_\tau)}+\frac{\gamma}{2}l(\varepsilon)T^2\|U\|^2_{L^2(Q_\tau)}.
\end{aligned}$$

$$\begin{aligned}
&-\int_0^T\int_{\partial\Omega} v\left(\int_0^t\int_\Omega W(\xi,\tau)\,d\xi d\tau\right)ds_x dt\\
=\;&-\int_0^T\int_{\partial\Omega} v\left(\int_0^t\int_\Omega H(\tau)U(\xi,\tau)\,d\xi d\tau\right)ds_x dt\\
&+\int_0^T\int_{\partial\Omega} v\left(\int_0^t\int_\Omega\left[\int_0^\tau h(\tau-\sigma)(U(\xi,\tau)-U(\xi,\sigma))d\sigma\right]d\xi d\tau\right)ds_x dt\\
\leq\;&\frac{h_0}{2}T^2\{l(\varepsilon)+|\Omega|\,|\partial\Omega|\}\int_0^\tau\|U(x,t)\|^2_{L^2(\Omega)}\,dt+\frac{h_0}{2}\varepsilon\int_0^\tau\|\nabla v(x,t)\|^2_{L^2(\Omega)}\,dt\\
&+\frac{1}{2}l(\varepsilon)\int_0^\tau\|U(x,t)\|^2_{L^2(\Omega)}\,dt+\frac{1}{2}|\Omega|\,|\partial\Omega|\int_0^\tau h\circ U(t)dt\\
&+\frac{1}{2}\varepsilon\int_0^\tau\|\nabla v(x,t)\|^2_{L^2(\Omega)}\,dt. \qquad (4.16)
\end{aligned}$$

Combining the relations (4.13)–(4.16) and (4.11) we get

$$\begin{aligned}
&(U_{\tau\tau}(x,\tau),U(x,\tau))_{L^2(\Omega)}+\alpha\,(U_\tau(x,\tau),U(x,\tau))_{L^2(\Omega)}+\tfrac{\beta}{2}\|U(x,\tau)\|^2_{L^2(\Omega)}\\
&+\tfrac{\varrho}{2}\|\nabla v(x,0)\|^2_{L^2(\Omega)}+\tfrac{\gamma}{2}\|\nabla U(x,\tau)\|^2_{L^2(\Omega)}-\tfrac{1}{2}\|U_\tau(x,\tau)\|^2_{L^2(\Omega)}\\
&\leq\Big\{\tfrac{\varrho}{2}T^2\left(l(\varepsilon)+|\Omega|\,|\partial\Omega|\right)+\tfrac{\delta}{2}\left(T^2 l(\varepsilon)+|\Omega|\,|\partial\Omega|\right)+\tfrac{\gamma}{2}l(\varepsilon)T^2\\
&\qquad+\tfrac{h_0}{2}T^2(l(\varepsilon)+|\Omega|\,|\partial\Omega|)+\tfrac{1}{2}l(\varepsilon)\Big\}\int_0^\tau\|U(x,t)\|^2_{L^2(\Omega)}\,dt \qquad (4.17)\\
&+\left(\alpha+\tfrac{\gamma|\Omega||\partial\Omega|}{2}\right)\int_0^\tau\|U_t(x,t)\|^2_{L^2(\Omega)}\,dt+\tfrac{1}{2}\int_0^\tau\|U_{tt}(x,t)\|^2_{L^2(\Omega)}\,dt\\
&+\left\{\left(\tfrac{\varrho+\delta+\gamma+h_0}{2}\right)\varepsilon+h_0\right\}\int_0^\tau\|\nabla v(x,t)\|^2_{L^2(\Omega)}\,dt+h_0\int_0^\tau\|\nabla v(x,t)\|^2_{L^2(\Omega)}\,dt\\
&+\tfrac{h_0}{2}\int_0^\tau\|\nabla U(x,t)\|^2_{L^2(\Omega)}\,dt+\tfrac{1}{2}\int_0^\tau h\circ\nabla U(t)dt+\tfrac{1}{2}|\Omega||\partial\Omega|\int_0^\tau h\circ U(t)dt.
\end{aligned}$$

Next, multiplying the differential equation in (4.1) by U_{ttt} and integrating over $Q_\tau=\Omega\times(0,\tau)$, we obtain

$$\begin{aligned}
&(U_{tttt},U_{ttt})_{L^2(Q_\tau)}+\alpha(U_{ttt},U_{ttt})_{L^2(Q_\tau)}+\beta(U_{tt},U_{ttt})_{L^2(Q_\tau)}-\varrho(\Delta U,U_{ttt})_{L^2(Q_\tau)}\\
&-\delta(\Delta U_t,U_{ttt})_{L^2(Q_\tau)}-\gamma(\Delta U_t,U_{ttt})_{L^2(Q_\tau)}+(\Delta W,U_{ttt})_{L^2(Q_\tau)}=0. \qquad (4.18)
\end{aligned}$$

An integration by parts in (4.18) yields

$$(U_{tttt},U_{ttt})_{L^2(Q_\tau)}=\frac{1}{2}\|U_{\tau\tau\tau}(x,\tau)\|^2_{L^2(\Omega)}, \qquad (4.19)$$

$$\alpha\left(U_{ttt},U_{ttt}\right)_{L^2(Q_\tau)}=\alpha\int_0^\tau\left\|U_{ttt}\left(x,t\right)\right\|_{L^2(\Omega)}^2dt, \tag{4.20}$$

$$\beta\left(U_{tt},U_{ttt}\right)_{L^2(Q_\tau)}=\frac{\beta}{2}\left\|U_{\tau\tau}(x,\tau)\right\|_{L^2(\Omega)}^2, \tag{4.21}$$

$$\begin{aligned}-\varrho\left(\Delta U,U_{ttt}\right)_{L^2(Q_\tau)} &= \varrho\left(\nabla U(x,\tau),\nabla U_{\tau\tau}(x,\tau)\right)_{L^2(\Omega)}-\frac{\varrho}{2}\left\|\nabla U_\tau\left(x,\tau\right)\right\|_{L^2(\Omega)}^2\\ &\quad -\varrho\int_{\partial\Omega}U_{\tau\tau}(x,\tau)\left(\int_0^\tau\int_\Omega U(\xi,\eta)d\xi d\eta\right)ds_x\\ &\quad +\varrho\int_{\partial\Omega}\int_0^\tau U_{tt}\left(x,t\right)\int_\Omega U\left(\xi,t\right)d\xi dt ds_x,\end{aligned} \tag{4.22}$$

$$\begin{aligned}-\delta\left(\Delta U_t,U_{ttt}\right)_{L^2(Q_\tau)} &= \delta\left(\nabla U_\tau(x,\tau),\nabla U_{\tau\tau}(x,\tau)\right)_{L^2(\Omega)}-\delta\int_0^\tau\left\|\nabla U_{tt}(x,)\right\|_{L^2(\Omega)}^2dt\\ &\quad -\delta\int_{\partial\Omega}U_{\tau\tau}(x,\tau)\int_\Omega U\left(\xi,\tau\right)d\xi ds_x\\ &\quad +\delta\int_0^\tau\int_{\partial\Omega}U_{tt}\left(x,t\right)\int_\Omega U_t\left(\xi,t\right)d\xi ds_x dt,\end{aligned} \tag{4.23}$$

$$\begin{aligned}-\gamma\left(\Delta U_{tt},U_{ttt}\right)_{L^2(Q_\tau)} &= \frac{\gamma}{2}\left\|\nabla U_{\tau\tau}\left(x,\tau\right)\right\|_{L^2(\Omega)}^2-\gamma\int_{\partial\Omega}U_{\tau\tau}\left(x,\tau\right)\int_\Omega U_\tau\left(\xi,\tau\right)d\xi ds_x\\ &\quad +\gamma\int_0^\tau\int_{\partial\Omega}U_{tt}\left(x,t\right)\int_\Omega U_{tt}\left(\xi,t\right)d\xi ds_x dt.\end{aligned} \tag{4.24}$$

$$\begin{aligned}\left(\Delta W,U_{ttt}\right)_{L^2(Q_\tau)} &= -H(\tau)\left(\nabla U(x,\tau),\nabla U_{\tau\tau}(x,\tau)\right)_{L^2(\Omega)}\\ &\quad +\int_\Omega\nabla U_{\tau\tau}\int_0^\tau h(\tau-\sigma)(\nabla U(\tau)-\nabla U(\sigma))d\sigma dx\\ &\quad -\int_0^\tau(\nabla U_{tt},\int_0^t h'(t-\sigma)(\nabla U(t)-\nabla U(\sigma))d\sigma)_{L^2(\Omega)}dt\\ &\quad +\int_0^\tau h(t)(\nabla U_{tt},\nabla U(t))_{L^2(\Omega)}dt\\ &\quad +\int_{\partial\Omega}U_{\tau\tau}(x,\tau)\left(\int_0^\tau\int_\Omega W(\xi,\eta)d\xi d\eta\right)ds_x\\ &\quad -\int_{\partial\Omega}\int_0^\tau U_{tt}\left(x,t\right)\int_\Omega W\left(\xi,t\right)d\xi dt ds_x,\end{aligned} \tag{4.25}$$

Substitution (4.19)–(4.25) into (4.18) we get the equality

$$\begin{aligned}
&\tfrac{1}{2}\|U_{\tau\tau\tau}(x,\tau)\|^2_{L^2(\Omega)} + \tfrac{\beta}{2}\|U_{\tau\tau}(x,\tau)\|^2_{L^2(\Omega)} + \varrho\,(\nabla U(x,\tau),\nabla U_{\tau\tau}(x,\tau))_{L^2(\Omega)}\\
&+\delta\,(\nabla U_\tau(x,\tau),\nabla U_{\tau\tau}(x,\tau))_{L^2(\Omega)} + \tfrac{\gamma}{2}\|\nabla U_{\tau\tau}(x,\tau)\|^2_{L^2(\Omega)} - \tfrac{\varrho}{2}\|\nabla U_\tau(x,\tau)\|^2_{L^2(\Omega)}\\
&-H(\tau)\,(\nabla U(x,\tau),\nabla U_{\tau\tau}(x,\tau))_{L^2(\Omega)}\\
&+\int_\Omega \nabla U_{\tau\tau}\int_0^\tau h(\tau-\sigma)(\nabla U(\tau)-\nabla U(\sigma))d\sigma dx\\
&= -\alpha\int_0^\tau \|U_{ttt}(x,t)\|^2_{L^2(\Omega)}dt + \delta\int_0^\tau \|\nabla U_{tt}(x,)\|^2_{L^2(\Omega)}dt\\
&+\varrho\int_{\partial\Omega} U_{\tau\tau}(x,\tau)\left(\int_0^\tau\int_\Omega U(\xi,\eta)d\xi d\eta\right)ds_x - \varrho\int_{\partial\Omega}\int_0^\tau U_{tt}(x,t)\int_\Omega U(\xi,t)\,d\xi dt ds_x\\
&+\delta\int_{\partial\Omega} U_{\tau\tau}(x,\tau)\int_\Omega U(\xi,\tau)\,d\xi ds_x - \delta\int_0^\tau\int_{\partial\Omega} U_{tt}(x,t)\int_\Omega U_t(\xi,t)\,d\xi ds_x dt\\
&+\gamma\int_{\partial\Omega} U_{\tau\tau}(x,\tau)\int_\Omega U_\tau(\xi,\tau)\,d\xi ds_x - \gamma\int_0^\tau\int_{\partial\Omega} U_{tt}(x,t)\int_\Omega U_{tt}(\xi,t)\,d\xi ds_x dt\\
&-\int_0^\tau(\nabla U_{tt},\int_0^t h'(t-\sigma)(\nabla U(t)-\nabla U(\sigma))d\sigma)_{L^2(\Omega)}dt\\
&+\int_0^\tau h(t)(\nabla U_{tt},\nabla U(t))_{L^2(\Omega)}dt + \int_{\partial\Omega} U_{\tau\tau}(x,\tau)\left(\int_0^\tau\int_\Omega W(\xi,\eta)d\xi d\eta\right)ds_x\\
&-\int_{\partial\Omega}\int_0^\tau U_{tt}(x,t)\int_\Omega W(\xi,t)\,d\xi dt ds_x.
\end{aligned}\tag{4.26}$$

The right hand side of (4.26) can be bounded as follows

$$\begin{aligned}
&\varrho\int_{\partial\Omega} U_{\tau\tau}(x,\tau)\left(\int_0^\tau\int_\Omega U(\xi,\eta)\,d\xi d\eta\right)ds_x\\
\le\;& \frac{\varrho}{2\varepsilon_1'}\left(\varepsilon\|\nabla U_{\tau\tau}(x,\tau)\|^2_{L^2(\Omega)} + l(\varepsilon)\|U_{\tau\tau}(x,\tau)\|^2_{L^2(\Omega)}\right)\\
&+\frac{\varrho}{2}\varepsilon_1' T\,|\partial\Omega|\,|\Omega|\int_0^\tau \|U(x,t)\|^2_{L^2(\Omega)}dt,
\end{aligned}\tag{4.27}$$

$$\begin{aligned}
&-\varrho\int_{\partial\Omega}\int_0^\tau U_{tt}(x,t)\int_\Omega U(\xi,t)\,d\xi dt ds_x\\
\le\;& \frac{\varrho}{2}\int_0^\tau\left\{\varepsilon\|\nabla U_{tt}(x,t)\|^2_{L^2(\Omega)} + l(\varepsilon)\|U_{tt}(x,t)\|^2_{L^2(\Omega)}\right\}dt\\
&+\frac{\varrho}{2}|\Omega|\,|\partial\Omega|\int_0^\tau \|U(x,t)\|^2_{L^2(\Omega)}dt,
\end{aligned}\tag{4.28}$$

$$\begin{aligned}
&\delta\int_{\partial\Omega} U_{\tau\tau}(x,\tau)\int_\Omega U(\xi,\tau)\,d\xi ds_x\\
\le\;& \frac{\delta}{2\varepsilon_2'}\left(\varepsilon\|\nabla U_{\tau\tau}(x,\tau)\|^2_{L^2(\Omega)} + l(\varepsilon)\|U_{\tau\tau}(x,\tau)\|^2_{L^2(\Omega)}\right)\\
&+\frac{\delta}{2}\varepsilon_2' T\,|\Omega|\,|\partial\Omega|\,\|U(x,\tau)\|^2_{L^2(\Omega)},
\end{aligned}\tag{4.29}$$

$$\begin{aligned}
&-\delta\int_0^\tau\int_{\partial\Omega} U_{tt}(x,t)\int_\Omega U_t(\xi,t)\,d\xi ds_x dt\\
\le\;& \frac{\delta}{2}\varepsilon\int_0^\tau \|\nabla U_{tt}(x,t)\|^2_{L^2(\Omega)}dt + \frac{\delta}{2}l(\varepsilon)\int_0^\tau \|U_{tt}(x,t)\|^2_{L^2(\Omega)}dt\\
&+\frac{\delta}{2}T\,|\Omega|\,|\partial\Omega|\int_0^\tau \|U_t(x,t)\|^2_{L^2(\Omega)}dt,
\end{aligned}\tag{4.30}$$

$$\begin{aligned}
&\gamma\int_{\partial\Omega}U_{\tau\tau}(x,\tau)\int_{\Omega}U_{\tau}(\xi,\tau)\,d\xi ds_x\\
\leq\ &\frac{\gamma}{2\varepsilon_3'}\left(\varepsilon\left\|\nabla U_{\tau\tau}(x,\tau)\right\|^2_{L^2(\Omega)}+l(\varepsilon)\left\|U_{\tau\tau}(x,\tau)\right\|^2_{L^2(\Omega)}\right)\\
&+\frac{\gamma}{2}\varepsilon_3' T\,|\Omega|\,|\partial\Omega|\,\|U_{\tau}(x,\tau)\|^2_{L^2(\Omega)}, \qquad (4.31)
\end{aligned}$$

$$\begin{aligned}
&-\gamma\int_0^{\tau}\int_{\partial\Omega}U_{tt}(x,t)\int_{\Omega}U_{tt}(\xi,t)\,d\xi ds_x dt\\
\leq\ &\frac{\gamma}{2}l(\varepsilon)\int_0^{\tau}\|U_{tt}(x,t)\|^2_{L^2(\Omega)}dt+\frac{\gamma}{2}\varepsilon\int_0^{\tau}\|\nabla U_{tt}(x,t)\|^2_{L^2(\Omega)}dt\\
&+\frac{\gamma}{2}T\,|\Omega|\,|\partial\Omega|\int_0^{\tau}\|U_{tt}(x,t)\|^2_{L^2(\Omega)}dt, \qquad (4.32)
\end{aligned}$$

$$\begin{aligned}
&\int_{\partial\Omega}U_{\tau\tau}(x,\tau)\left(\int_0^{\tau}\int_{\Omega}W(\xi,\eta)\,d\xi d\eta\right)ds_x\\
\leq\ &\left(\frac{h_0}{2\varepsilon_6'}+\frac{1}{2\varepsilon_7'}\right)\left(\varepsilon\|\nabla U_{\tau\tau}(x,\tau)\|^2_{L^2(\Omega)}+l(\varepsilon)\|U_{\tau\tau}(x,\tau)\|^2_{L^2(\Omega)}\right) \qquad (4.33)\\
&+\frac{h_0}{2}\varepsilon_6' T\,|\partial\Omega|\,|\Omega|\int_0^{\tau}\|U(x,t)\|^2_{L^2(\Omega)}dt+\frac{1}{2}\varepsilon_7'\,|\partial\Omega|\,|\Omega|\int_0^{\tau}h\circ U(t)dt,
\end{aligned}$$

$$\begin{aligned}
&-\int_{\partial\Omega}\int_0^{\tau}U_{tt}(x,t)\int_{\Omega}W(\xi,t)\,d\xi dt ds_x\\
\leq\ &\frac{h_0+1}{2}\int_0^{\tau}\left\{\varepsilon\|\nabla U_{tt}(x,t)\|^2_{L^2(\Omega)}+l(\varepsilon)\|U_{tt}(x,t)\|^2_{L^2(\Omega)}\right\}dt\\
&+\frac{h_0}{2}\,|\Omega|\,|\partial\Omega|\int_0^{\tau}\|U(x,t)\|^2_{L^2(\Omega)}dt+\frac{1}{2}\,|\partial\Omega|\,|\Omega|\int_0^{\tau}h\circ U(t)dt. \qquad (4.34)
\end{aligned}$$

$$\begin{aligned}
&\int_{\Omega}\nabla U_{\tau\tau}\int_0^{\tau}h(\tau-\sigma)(\nabla U(\tau)-\nabla U(\sigma))d\sigma dx\\
\geq\ &-\frac{1}{2\varepsilon_8'}h_0\|\nabla U_{\tau\tau}(x,\tau)\|^2_{L^2(\Omega)}-\frac{1}{2}\varepsilon_8' h\circ\nabla U(\tau), \qquad (4.35)
\end{aligned}$$

$$\begin{aligned}
&-H(\tau)\left(\nabla U(x,\tau),\nabla U_{\tau\tau}(x,\tau)\right)_{L^2(\Omega)}\\
\geq\ &-\frac{1}{2\varepsilon_9'}h_0\|\nabla U_{\tau\tau}(x,\tau)\|^2_{L^2(\Omega)}-\frac{1}{2}\varepsilon_9' h_0\|\nabla U(x,\tau)\|^2_{L^2(\Omega)}, \qquad (4.36)
\end{aligned}$$

$$\begin{aligned}
&\int_0^{\tau}h(t)\nabla U_{tt}(x,t)\nabla U(x,t)dt\\
\leq\ &\frac{h(0)}{2}\int_0^{\tau}\|\nabla U_{tt}(x,\tau)\|^2_{L^2(\Omega)}dt+\frac{h(0)}{2}\int_0^{\tau}\|\nabla U(x,\tau)\|^2_{L^2(\Omega)}dt, \qquad (4.37)
\end{aligned}$$

$$
\begin{aligned}
&\int_0^\tau \nabla U_{tt} \int_0^t h'(t-\sigma)(\nabla U(t) - \nabla U(\sigma))d\sigma dx \\
\leq\ & -\frac{h(t)-h(0)}{2} \int_0^\tau \|\nabla U_{tt}(x,\tau)\|^2_{L^2(\Omega)} dt - \frac{1}{2}\int_0^\tau h' \circ \nabla U(t)dt.
\end{aligned} \tag{4.38}
$$

So, combining inequalities (4.27)–(4.38) and equality (4.26) we obtain

$$
\begin{aligned}
&\tfrac{1}{2}\|U_{\tau\tau\tau}(x,\tau)\|^2_{L^2(\Omega)} + \left\{\tfrac{\beta}{2} - \tfrac{\varrho}{2\varepsilon_1'}l(\varepsilon) - \tfrac{\delta}{2\varepsilon_2'}l(\varepsilon) - \tfrac{\gamma}{2\varepsilon_3'}l(\varepsilon) - (\tfrac{h_0}{2\varepsilon_6'} + \tfrac{1}{2\varepsilon_7'})l(\varepsilon)\right\}\|U_{\tau\tau}(x,\tau)\|^2_{L^2(\Omega)} \\
&-\tfrac{\gamma}{2}\varepsilon_3' T|\Omega||\partial\Omega|\|U_\tau(x,\tau)\|^2_{L^2(\Omega)} - \tfrac{\delta}{2}\varepsilon_2' T|\Omega||\partial\Omega|\|U(x,\tau)\|^2_{L^2(\Omega)} \\
&+\left\{\tfrac{\gamma}{2} - \tfrac{\varrho}{2\varepsilon_1'}\varepsilon - \tfrac{\delta}{2\varepsilon_2'}\varepsilon - \tfrac{\gamma}{2\varepsilon_3'}\varepsilon + (\tfrac{h_0}{2\varepsilon_6'} + \tfrac{1}{2\varepsilon_7'})\varepsilon - (\tfrac{1}{2\varepsilon_8'} + \tfrac{1}{2\varepsilon_9'})h_0\right\}\|\nabla U_{\tau\tau}(x,\tau)\|^2_{L^2(\Omega)} \\
&-\tfrac{\varepsilon_8'}{2}h\circ\nabla U(\tau) - \tfrac{\varepsilon_9'}{2}h_0\|\nabla U(x,\tau)\|^2_{L^2(\Omega)} - \tfrac{\varrho}{2}\|\nabla U_\tau(x,\tau)\|^2_{L^2(\Omega)} \\
&+\varrho(\nabla U(x,\tau),\nabla U_{\tau\tau}(x,\tau))_{L^2(\Omega)} + \delta(\nabla U_\tau(x,\tau),\nabla U_{\tau\tau}(x,\tau))_{L^2(\Omega)} \\
&\leq -\alpha\int_0^\tau\|U_{ttt}(x,t)\|^2_{L^2(\Omega)}dt + \left\{\tfrac{\varrho}{2}l(\varepsilon) + \tfrac{\delta}{2}l(\varepsilon) + \tfrac{\gamma}{2}l(\varepsilon) + \tfrac{\gamma}{2}T|\Omega||\partial\Omega| + (\tfrac{h_0+1}{0})l(\varepsilon)\right\}\int_0^\tau\|U_{tt}(x,t)\|^2_{L^2(\Omega)}dt \\
&+\left\{\tfrac{\varrho}{2}\varepsilon_1' T|\partial\Omega||\Omega| + (\tfrac{\varrho}{2} + \tfrac{h_0}{2}(1+T\varepsilon_6'))|\Omega||\partial\Omega|\right\}\int_0^\tau\|U(x,t)\|^2_{L^2(\Omega)}dt + \tfrac{\delta}{2}T|\Omega||\partial\Omega|\int_0^\tau\|U_t(x,t)\|^2_{L^2(\Omega)}dt \\
&+\left\{\delta + \tfrac{\varrho}{2}\varepsilon + \tfrac{\delta}{2}\varepsilon + \tfrac{\gamma}{2}\varepsilon + \tfrac{h_0+1}{2}\varepsilon + \tfrac{3h(0)}{2}\right\}\int_0^\tau\|\nabla U_{tt}(x,)\|^2_{L^2(\Omega)}dt \\
&-\tfrac{1}{2}\int_0^\tau h'\circ\nabla U(t)dt - \tfrac{h(0)}{2}\int_0^\tau\|\nabla U(x,\tau)\|^2_{L^2(\Omega)}dt + \tfrac{1+\varepsilon_7'}{2}|\partial\Omega||\Omega|\int_0^\tau h'\circ U(t)dt.
\end{aligned} \tag{4.39}
$$

Adding side to side (4.17) and (4.39), we obtain

$$
\begin{aligned}
&\left\{\tfrac{\beta}{2} - \tfrac{\delta}{2}\varepsilon_2' T|\Omega||\partial\Omega| - \tfrac{1+\alpha}{2}\right\}\|U(x,\tau)\|^2_{L^2(\Omega)} + \tfrac{1}{2}\|U_{\tau\tau\tau}(x,\tau)\|^2_{L^2(\Omega)} \\
&+\left\{-\tfrac{1+\alpha}{2} - \tfrac{\gamma}{2}\varepsilon_3' T|\Omega||\partial\Omega|\right\}\|U_\tau(x,\tau)\|^2_{L^2(\Omega)} \\
&+\left\{\tfrac{\beta}{2} - \tfrac{\varrho}{2\varepsilon_1'}l(\varepsilon) - l(\varepsilon)\tfrac{\delta}{2\varepsilon_2'} - \tfrac{\gamma}{2\varepsilon_3'}l(\varepsilon) - (\tfrac{h_0}{2\varepsilon_6'} + \tfrac{1}{2\varepsilon_7'}) - \tfrac{1}{2}\right\}\|U_{\tau\tau}(x,\tau)\|^2_{L^2(\Omega)} \\
&+(U_{\tau\tau}(x,\tau),U(x,\tau))_{L^2(\Omega)} + \alpha(U_\tau(x,\tau),U(x,\tau))_{L^2(\Omega)} + \tfrac{\varrho}{2}\|\nabla v(x,0)\|^2_{L^2(\Omega)} \\
&+\varrho(\nabla U(x,\tau),\nabla U_{\tau\tau}(x,\tau))_{L^2(\Omega)} + \delta(\nabla U_\tau(x,\tau),\nabla U_{\tau\tau}(x,\tau))_{L^2(\Omega)} \\
&+(\tfrac{\gamma}{2} - \tfrac{\varepsilon_9'}{2}h_0)\|\nabla U(x,\tau)\|^2_{L^2(\Omega)} - \tfrac{\varrho}{2}\|\nabla U_\tau(x,\tau)\|^2_{L^2(\Omega)} - \tfrac{\varepsilon_8'}{2}h\circ\nabla U(\tau) \\
&+\left\{\tfrac{\gamma}{2} - \tfrac{\varrho}{2\varepsilon_1'}\varepsilon - \tfrac{\delta}{2\varepsilon_2'} - \tfrac{\gamma}{2\varepsilon_3'}\varepsilon - (\tfrac{h_0}{2\varepsilon_6'} + \tfrac{1}{2\varepsilon_7'})\varepsilon - (\tfrac{1}{2\varepsilon_8'} + \tfrac{1}{2\varepsilon_9'})h_0\right\}\|\nabla U_{\tau\tau}(x,\tau)\|^2_{L^2(\Omega)} \\
&\leq\left\{\tfrac{\varrho}{2}\varepsilon_1' T|\partial\Omega||\Omega| + \tfrac{\varrho}{2}|\Omega||\partial\Omega| + \tfrac{\varrho}{2}T^2(l(\varepsilon) + |\Omega||\partial\Omega|) + \tfrac{\delta}{2}\left(T^2 l(\varepsilon) + |\Omega||\partial\Omega|\right)\right. \\
&\left.+\tfrac{\gamma}{2}l(\varepsilon)T^2 + \tfrac{h_0}{2}T^2(l(\varepsilon) + |\Omega||\partial\Omega|) + \tfrac{1}{2}l(\varepsilon)\right\}\int_0^\tau\|U(x,t)\|^2_{L^2(\Omega)}dt \\
&+\left(\alpha + \tfrac{\gamma|\Omega||\partial\Omega|}{2} + \tfrac{\delta}{2}T|\Omega||\partial\Omega|\right)\int_0^\tau\|U_t(x,t)\|^2_{L^2(\Omega)}dt \\
&+\left\{\tfrac{1}{2} + l(\varepsilon)\tfrac{\varrho}{2} + \tfrac{\delta}{2}l(\varepsilon) + \tfrac{\gamma}{2}l(\varepsilon) + \tfrac{\gamma}{2}T|\Omega||\partial\Omega|\right\}\int_0^\tau\|U_{tt}(x,t)\|^2_{L^2(\Omega)}dt \\
&+\left\{\tfrac{\delta}{2}\varepsilon + \tfrac{\gamma}{2}\varepsilon + \varepsilon\tfrac{\varrho}{2} + \delta + \tfrac{h_0+1}{2}\varepsilon + \tfrac{3h(0)}{2}\right\}\int_0^\tau\|\nabla U_{tt}(x,t)\|^2_{L^2(\Omega)}dt \\
&-\alpha\int_0^\tau\|U_{ttt}(x,t)\|^2_{L^2(\Omega)}dt + \left(h_0 + \left(\tfrac{\varrho+\delta+\gamma+h_0}{2}\right)\varepsilon\right)\int_0^\tau\|\nabla v(x,t)\|^2_{L^2(\Omega)}dt \\
&+\tfrac{h_0+h(0)}{2}\int_0^\tau\|\nabla U(x,t)\|^2_{L^2(\Omega)}dt + \tfrac{1}{2}\int_0^\tau h\circ\nabla U(t)dt + \tfrac{1}{2}|\Omega||\partial\Omega|\int_0^\tau h\circ U(t)dt \\
&-\tfrac{1}{2}\int_0^\tau h'\circ\nabla U(t)dt + \tfrac{1+\varepsilon_7'}{2}|\partial\Omega||\Omega|\int_0^\tau h'\circ U(t)dt.
\end{aligned} \tag{4.40}
$$

Now to deal with the last term on the right hand side of (4.40), we define the function $\theta(x,t)$ by the relation

$$\theta(x,t) := \int_0^t U(x,s)ds.$$

Hence using (4.12) it follows that

$$v(x,t) = \theta(x,\tau) - \theta(x,t),\ \nabla v(x,0) = \nabla\theta(x,\tau), \tag{4.41}$$

and

$$\begin{aligned} \|\nabla v\|^2_{L^2(Q_\tau)} &= \|\nabla\theta(x,\tau) - \nabla\theta(x,t)\|^2_{L^2(\Omega)} \\ &\le 2\left(\tau\|\nabla\theta(x,\tau)\|^2_{L^2(\Omega)} + \|\nabla\theta(x,t)\|^2_{L^2(Q_\tau)}\right). \end{aligned} \tag{4.42}$$

And make use of the following inequality

$$-\frac{\alpha}{2}\|U_\tau(x,\tau)\|^2_{L^2(\Omega)} - \frac{\alpha}{2}\|U(x,\tau)\|^2_{L^2(\Omega)} \le \alpha(U_\tau(x,\tau), U(x,\tau))_{L^2(\Omega)}, \tag{4.43}$$

$$-\frac{1}{2}\|U_{\tau\tau}(x,\tau)\|^2_{L^2(\Omega)} - \frac{1}{2}\|U(x,\tau)\|^2_{L^2(\Omega)} \le (U_{\tau\tau}(x,\tau), U(x,\tau))_{L^2(\Omega)}, \tag{4.44}$$

$$-\frac{\varrho}{2\varepsilon_4'}\|\nabla U_{\tau\tau}(x,\tau)\|^2_{L^2(\Omega)} - \frac{\varrho}{2}\varepsilon_4'\|\nabla U(x,\tau)\|^2_{L^2(\Omega)} \le \varrho\left(\nabla U(x,\tau), \nabla U_{\tau\tau}(x,\tau)\right)_{L^2(\Omega)}, \tag{4.45}$$

$$-\frac{\delta}{2\varepsilon_5'}\|\nabla U_{\tau\tau}(x,\tau)\|^2_{L^2(\Omega)} - \frac{\delta}{2}\varepsilon_5'\|\nabla U_\tau(x,\tau)\|^2_{L^2(\Omega)} \le \delta\left(\nabla U_\tau(x,\tau), \nabla U_{\tau\tau}(x,\tau)\right)_{L^2(\Omega)}. \tag{4.46}$$

$$\begin{aligned} m_1\|U(x,\tau)\|^2_{L^2(\Omega)} &\le m_1\|U(x,t)\|^2_{L^2(Q_\tau)} + m_1\|U_t(x,t)\|^2_{L^2(Q_\tau)}, \\ m_2\|U_\tau(x,\tau)\|^2_{L^2(\Omega)} &\le m_2\|U_t(x,t)\|^2_{L^2(Q_\tau)} + m_2\|U_{tt}(x,t)\|^2_{L^2(Q_\tau)}, \\ m_3\|U_{\tau\tau}(x,\tau)\|^2_{L^2(\Omega)} &\le m_3\|U_{tt}(x,t)\|^2_{L^2(Q_\tau)} + m_3\|U_{ttt}(x,t)\|^2_{L^2(Q_\tau)}, \\ m_4\|\nabla U(x,\tau)\|^2_{L^2(\Omega)} &\le m_4\|\nabla U(x,t)\|^2_{L^2(Q_\tau)} + m_4\|\nabla U_t(x,t)\|^2_{L^2(Q_\tau)}, \\ m_5\|\nabla U_\tau(x,\tau)\|^2_{L^2(\Omega)} &\le m_5\|\nabla U_t(x,t)\|^2_{L^2(Q_\tau)} + m_5\|\nabla U_{tt}(x,t)\|^2_{L^2(Q_\tau)}, \\ m_6 h\circ\nabla U(\tau) &\le m_6\|\nabla U_t(x,t)\|^2_{L^2(Q_\tau)} + m_6\int_0^\tau h\circ\nabla U(t)dt \\ m_7 h\circ U(\tau) &\le m_7\|U_t(x,t)\|^2_{L^2(Q_\tau)} + m_7\int_0^\tau h\circ U(t)dt \end{aligned} \tag{4.47}$$

$$-m_8 h'\circ\nabla U(\tau) \le m_8\|\nabla U_t(x,t)\|^2_{L^2(Q_\tau)} - m_8\int_0^\tau h'\circ\nabla U(t)dt. \tag{4.48}$$

Let

$$\begin{cases} m_1 := \frac{1+\alpha}{2} + \frac{\delta}{2}\varepsilon_2' T\,|\Omega|\,|\partial\Omega|, \\ m_2 := 1 + \frac{\gamma}{2}\varepsilon_3' T\,|\Omega|\,|\partial\Omega| + \frac{\alpha}{2} \\ m_3 := \left(\frac{\varrho}{2\varepsilon_1'} + \frac{\delta}{2\varepsilon_2'} + \frac{\gamma}{2\varepsilon_3'} + \frac{h_0}{2\varepsilon_6'} + \frac{1}{2\varepsilon_7'}\right) l(\varepsilon) + \frac{1}{2} \\ m_4 := \frac{\varrho}{2}\varepsilon_4' + \frac{h_0}{2} \\ m_5 := 1 + \frac{\varrho}{2} + \frac{\delta}{2\varepsilon_5'} \\ m_6 := \frac{1}{2}\varepsilon_8' + 1, \quad m_7 := 1, \quad m_8 := 1, \end{cases} \tag{4.49}$$

choosing ε_1', ε_2', ε_3', ε_4',ε_5', ε_6', ε_7', ε_8' and ε_9' sufficiently large

$$\alpha_0 := \frac{\gamma}{2} - \frac{\varrho}{2\varepsilon_1'}\varepsilon - \frac{\delta}{2\varepsilon_2'} - \frac{\gamma}{2\varepsilon_3'}\varepsilon - \frac{\varrho}{2\varepsilon_4'} - \frac{\delta}{2\varepsilon_5'}\varepsilon - (\frac{h_0}{2\varepsilon_6'} + \frac{1}{2\varepsilon_7'})\varepsilon - (\frac{1}{2\varepsilon_8'} + \frac{1}{2\varepsilon_9'})h_0 > 0. \tag{4.50}$$

Since τ is arbitrary we get that $\alpha_1 := \frac{\varrho}{2} - 2\tau\Big(h_0 + \varepsilon\frac{(\varrho+\delta+\gamma+h_0)}{2}\Big) > 0$, thus inequality (4.40) takes the form

$$\begin{aligned}
&\tfrac{\beta}{2}\|U(x,\tau)\|^2_{L^2(\Omega)} + \tfrac{1}{2}\|U_\tau(x,\tau)\|^2_{L^2(\Omega)} + \tfrac{\beta}{2}\|U_{\tau\tau}(x,\tau)\|^2_{L^2(\Omega)} + \tfrac{1}{2}\|U_{\tau\tau\tau}(x,\tau)\|^2_{L^2(\Omega)}\\
&+\Big\{\tfrac{\varrho}{2} - 2\tau(h_0 + \varepsilon\tfrac{(\varrho+\delta+\gamma+h_0)}{2})\Big\}\|\nabla\theta(x,\tau)\|^2_{L^2(\Omega)} + \tfrac{\gamma}{2}\|\nabla U(x,\tau)\|^2_{L^2(\Omega)}\\
&+\|\nabla U_\tau(x,\tau)\|^2_{L^2(\Omega)} + \alpha_0\|\nabla U_{\tau\tau}(x,\tau)\|^2_{L^2(\Omega)} + h\circ\nabla U(\tau) + h\circ U(\tau) - h'\circ\nabla U(\tau)\\
&\le \Big\{\gamma_1' + m_1\Big\}\int_0^\tau \|U(x,t)\|^2_{L^2(\Omega)}dt + \Big(\gamma_2' + m_1 + m_2 + m_7\Big)\int_0^\tau\|U_t(x,t)\|^2_{L^2(\Omega)}dt\\
&+\Big\{\gamma_3' + m_2 + m_3\Big\}\int_0^\tau\|U_{tt}(x,t)\|^2_{L^2(\Omega)}dt + (m_3 - \alpha)\int_0^\tau\|U_{ttt}(x,t)\|^2_{L^2(\Omega)}dt\\
&+(2h_0 + \varepsilon(\varrho+\delta+\gamma+h_0))\int_0^\tau\|\nabla\theta(x,t)\|^2_{L^2(\Omega)}dt + (\gamma_4' + m_5)\int_0^\tau\|\nabla U_{tt}(x,t)\|^2_{L^2(\Omega)}dt\\
&+(m_4 + \tfrac{h_0+h(0)}{2})\int_0^\tau\|\nabla U(x,t)\|^2_{L^2(\Omega)}dt\\
&+(m_4 + m_5 + m_6 + m_8)\int_0^\tau\|\nabla U_t(x,t)\|^2_{L^2(\Omega)}dt + (\tfrac{1}{2} + m_6)\int_0^\tau h\circ\nabla U(t)dt\\
&+(\gamma_5' + m_7)\int_0^\tau h\circ U(t)dt - (\tfrac{1}{2} + m_8)\int_0^\tau h'\circ\nabla U(t)dt,
\end{aligned} \tag{4.51}$$

where

$$\begin{cases}
\gamma_1' := \tfrac{\varrho}{2}\varepsilon_1' T|\partial\Omega||\Omega| + \tfrac{\varrho}{2}|\Omega||\partial\Omega| + \tfrac{\varrho}{2}T^2(l(\varepsilon) + |\Omega||\partial\Omega|)\\
\qquad + \tfrac{\delta}{2}\Big(T^2 l(\varepsilon) + |\Omega||\partial\Omega|\Big) + \tfrac{\gamma}{2}l(\varepsilon)T^2\\
\\
\gamma_2' := \alpha + \tfrac{\gamma|\Omega||\partial\Omega|}{2} + \tfrac{\delta}{2}T|\Omega||\partial\Omega|\\
\gamma_3' := \tfrac{1}{2} + l(\varepsilon)\tfrac{\varrho}{2} + \tfrac{\delta}{2}l(\varepsilon) + \tfrac{\gamma}{2}l(\varepsilon) + \tfrac{\gamma}{2}T|\Omega||\partial\Omega|\\
\gamma_4' := \tfrac{\delta}{2}\varepsilon + \tfrac{\gamma}{2}\varepsilon + \varepsilon\tfrac{\varrho}{2} + \delta\\
\gamma_5' := (1 + \tfrac{1}{2}\varepsilon_7')|\Omega||\partial\Omega|
\end{cases} \tag{4.52}$$

We obtain

$$\begin{aligned}
&\|U(x,\tau)\|^2_{L^2(\Omega)} + \|U_\tau(x,\tau)\|^2_{L^2(\Omega)} + \|U_{\tau\tau}(x,\tau)\|^2_{L^2(\Omega)} + \|U_{\tau\tau\tau}(x,\tau)\|^2_{L^2(\Omega)}\\
&+\|\nabla U(x,\tau)\|^2_{L^2(\Omega)} + \|\nabla U_\tau(x,\tau)\|^2_{L^2(\Omega)} + \|\nabla U_{\tau\tau}(x,\tau)\|^2_{L^2(\Omega)}\\
&+\|\nabla\theta(x,\tau)\|^2_{L^2(\Omega)} + h\circ\nabla U(\tau) + h\circ U(\tau) - h'\circ\nabla U(\tau)\\
\le\ & D\int_0^\tau\Big\{\|U(x,t)\|^2_{L^2(\Omega)} + \|U_t(x,t)\|^2_{L^2(\Omega)} + \|U_{tt}(x,t)\|^2_{L^2(\Omega)} + h\circ U(t)\\
&+\|\nabla U(x,t)\|^2_{L^2(\Omega)} + \|\nabla U_t(x,t)\|^2_{L^2(\Omega)} + \|\nabla U_{tt}(x,t)\|^2_{L^2(\Omega)} - h'\circ\nabla U(t)\\
&+\|U_{ttt}(x,t)\|^2_{L^2(\Omega)} + \|\nabla\theta(x,t)\|^2_{L^2(\Omega)} + h\circ\nabla U(t)\Big\}dt,
\end{aligned} \tag{4.53}$$

where

$$D := \frac{\begin{array}{c}\max\Big\{(\gamma_1' + m_1), \Big(\gamma_2' + m_1 + m_2 + m_7\Big), \gamma_3' + m_2 + m_3, m_3 - \alpha,\\ m_4 + m_5 + m_6 + m_8, \gamma_4' + m_5, (2h_0 + \varepsilon(\varrho+\delta+\gamma+h_0)),\\ m_4 + \frac{h_0+h(0)}{2}, \frac{1}{2} + m_6, \gamma_5' + m_7, \frac{1}{2} + m_8\Big\}\end{array}}{\min\Big\{\frac{\beta}{2}, \frac{1}{2},, \frac{\gamma}{2}, \alpha_0, \alpha_1\Big\}}. \tag{4.54}$$

Further, applying Gronwall's lemma to (4.53), we deduce that

$$\begin{aligned} &\|U(x,\tau)\|^2_{L^2(\Omega)} + \|U_\tau(x,\tau)\|^2_{L^2(\Omega)} + \|U_{\tau\tau}(x,\tau)\|^2_{L^2(\Omega)} + \|U_{\tau\tau\tau}(x,\tau)\|^2_{L^2(\Omega)} \\ &+ \|\nabla U(x,\tau)\|^2_{L^2(\Omega)} + \|\nabla U_\tau(x,\tau)\|^2_{L^2(\Omega)} + \|\nabla U_{\tau\tau}(x,\tau)\|^2_{L^2(\Omega)} \\ &+ \|\nabla\theta(x,\tau)\|^2_{L^2(\Omega)} + h\circ\nabla U(\tau) + h\circ U(\tau) - h'\circ\nabla U(\tau) \le 0, \forall\tau\in[0,\alpha_2]\,. \end{aligned} \tag{4.55}$$

where $\alpha_2 := \frac{\varrho}{4h_0+2\varepsilon(\varrho+\delta+\gamma+h_0)}$.

Proceeding in the same way for the intervals $\tau\in[(m-1)\alpha_2, m\alpha_2]$ to cover the whole interval $[0,T]$, and thus proving that $U(x,\tau)=0$, for all τ in $[0,T]$. Thus, the uniqueness is proved. □

5. Conclusions

Study of sound wave propagation, it should be noted that the Moore–Gibson–Thomson equation is one of the nonlinear sound equations that describes the propagation of sound waves in gases and liquids. The behavior of sound waves depends strongly on the average scattering, scattering and nonlinear effects. Arises from high-frequency ultrasound (HFU) modeling (see [16, 25, 41]). In this work, we have studied the solvability of the nonlocal mixed boundary value problem for the fourth order of Moore–Gibson–Thompson equation with source and memory terms. Galerkin's method was the main used tool for proving the solvability of the given non local problem. In the next work, we will try to using the same method with Hall-MHD equations which are nonlinear partial differential equation that arises in hydrodynamics and some physical applications (see for example [2–4, 6]) by using some famous algorithms (see [8, 14, 15]).

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Conflict of interest

This work does not have any conflicts of interest.

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