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Research article

Unit groups of finite group algebras of Abelian groups of order 17 to 20

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Abstract: Let F be a finite field of characteristic p having $q = p^n$ elements and G be an abelian group. In this paper, we determine the structure of the group of units of the group algebra FG, where G is an abelian group of order $17 \leq |G| \leq 20$.

Keywords: Unit group; group algebra; abelian group; finite field
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1. Introduction

Let FG be the group algebra of a finite group G over a finite field F and let $U(FG)$ be the group of units of FG. Determining the structure of $U(FG)$ is a classical problem that has already generated considerable interest in the study of group algebra [1, 10, 12, 14]. In recent years, units of a group algebra were also used as a tool to tackle many research problems in some other areas including coding theory [5–8] and combinatorial number theory [4].

Many researchers have investigated the structure of $U(FG)$ under different conditions. Sandling [17] completely determined $U(FG)$ when G is a finite p-group and F is a field of characteristic p. Creedon [2] and Tang et al. [19] studied the unit groups of group algebras of some small groups. Tang and Gao [18] described the unit group of FG with $|G| = 12$. Maheshwari [11] determined the unit group of group algebras $FSL(2, Z_3)$. Monaghan [13] investigated the units of group algebras of non-abelian groups of order 24 over any finite field of characteristic 3. Sahai and Ansari [15] discussed the unit groups of group algebras of some dihedral groups. In a recent paper, Sahai et al. [16] characterized the unit group of FG when G is an abelian group of order at most 16. In this paper we focus our investigation on the group of units of FG of an abelian group G and determine the structure of $U(FG)$ when G is an abelian group of order between 17 and 20.

This paper is organized as follows. In section 2, we provide some preliminary results. Section 3 deals with the unit group of FG when G is a group of prime order (17 or 19). In the last two sections, we determine the structure of $U(FG)$ when $|G| = 18$ and $|G| = 20$, respectively.

2. Preliminaries

Let F be a finite filed of characteristic p having $q = p^n$ elements and G be a finite abelian group. Denote by C_n the cyclic group of order n and by G^k the direct sum of k copies of an abelian group G. Let F^n be the direct sum of n copies of F and let F_n be the extension of F of degree n. Let $V(FG)$, $\omega(FG)$, and $J(FG)$ be the group of the normalized unit group, the augmentation ideal and the Jacobson radical of FG, respectively. For a subgroup H of G, we denote by $\omega(G, H)$ the left ideal of FG generated by the set $\{h - 1 \mid h \in H\}$.

The number of simple components of $FG/J(FG)$ has been given by Ferraz in [3]. An element $g \in G$ is called p-regular, if $p \nmid o(g)$. In this article we use the same symbols m, η and T as in [3] to represent the least common multiple of the orders of p-regular elements of G, a primitive mth root of unity over the field F, and the set

$$T = \{t : \eta \to \eta^t \text{ is an automorphism of } F(\eta) \text{ over } F\}.$$

Let γ_g be the sum of all conjugates of $g \in G$. If g is a p-regular element, then the cyclotomic F-class of γ_g is

$$S_F(\gamma_g) = \{\gamma_{g^t} : t \in T\}.$$

Lemma 2.1. *[3, Proposition 1.2] The number of simple components of $FG/J(FG)$ is equal to the number of cyclotomic F-classes in G.*

Lemma 2.2. *[3, Theorem 1.3] Suppose that $Gal(F(\eta)/F)$ is cyclic. Let t be the number of cyclotomic F-classes in G. If $K_1, K_2, ..., K_t$ are the simple components of $Z(FG/J(FG))$ and $S_1, S_2, ..., S_t$ are the cyclotomic F classes of G, then with a suitable re-ordering of indices,*

$$|S_i| = [K_i : F],$$

for $i = 1, 2, \ldots, t$.

Remark 2.3. By Lemmas 2.1 and 2.2, we conclude that if G is a finite abelian group and $p \nmid |G|$, then $FG \cong \oplus_i^t K_i$, where K_i's are defined in Lemma 2.2.

We also need the following results.

Lemma 2.4. *[16, Lemma 4.1] Let F be a finite field of characteristic p with $|F| = q = p^n$ and let $G = C_{pk^i}$, where k, p are distinct primes and i is a positive integer. Let $V = 1 + J(FG)$. Then*

$$U(FG) \cong V \times U(FC_{k^i}),$$

and

$$V \cong C_p^{n(p-1)k^i}.$$

Lemma 2.5. *[9, Lemma 1.17] Let G be a locally finite p-group, and let F be a field of characteristic p. Then*

$$J(FG) = \omega(FG).$$

Lemma 2.6. *[14, Theorem 7.2.7] Let F be a finite field and let H be a normal subgroup of G with* $[G : H] = n < \infty$*. Then*

$$(J(FG))^n \subseteq J(FH)FG \subseteq J(FG).$$

If in addition $n \neq 0$ *in F, then*

$$J(FG) = J(FH)FG.$$

3. Groups of order 17 and 19

In this section, we describe the structure of $U(FG)$ when the order of the abelian group G is 17 or 19. We need the following two lemmas.

Lemma 3.1. *[2, Lemma 4.1] Let F be a finite field of characteristic p with* $|F| = q = p^n$*, where p is a prime number. Then* $U(FC_p^k) = C_p^{np^k-n} \times C_{p^n-1}$.

Lemma 3.2. *[16, Lemma 2.2] Let F be a finite field of characteristic p with* $|F| = q = p^n$*. If* $p \nmid k$*,then*

$$FC_k \cong \begin{cases} F^k, & \text{if } q \equiv 1 \mod k; \\ F \oplus F_2^{\frac{k-1}{2}}, & \text{if } q \equiv -1 \mod k \text{ and } k \text{ is odd}; \\ F^2 \oplus F_2^{\frac{k-2}{2}}, & \text{if } q \equiv -1 \mod k \text{ and } k \text{ is even}. \end{cases}$$

Now we can state our first result.

Theorem 3.3. *Let F be a finite field of characteristic p with* $|F| = q = p^n$*. Then*

$$U(FC_{17}) \cong \begin{cases} C_{17}^{16n} \times C_{17^n-1}, & \text{if } p = 17; \\ C_{p^n-1}^{17}, & \text{if } q \equiv 1 \mod 17; \\ C_{p^n-1} \times C_{p^{2n}-1}^{8}, & \text{if } q \equiv -1 \mod 17; \\ C_{p^n-1} \times C_{p^{8n}-1}^{2}, & \text{if } q \equiv \pm 2, \pm 8 \mod 17; \\ C_{p^n-1} \times C_{p^{4n}-1}^{4}, & \text{if } q \equiv \pm 4 \mod 17; \\ C_{p^n-1} \times C_{p^{16n}-1}, & \text{if } q \equiv \pm 3, \pm 5, \pm 6, \pm 7 \mod 17. \end{cases}$$

Proof. If $p = 17$, applying Lemma 3.1 with $k = 1$, we get

$$U(FC_{17}) = C_{17}^{16n} \times C_{17^n-1}.$$

Next we assume that $p \neq 17$. Let $C_{17} = \langle x \rangle$. Obviously, $m = 17$.

We divide the rest of the proof into several cases according to the value of q module 17.

Case 1. $q \equiv \pm 1 \mod 17$. By Lemma 3.2, we obtain that

$$U(FC_{17}) \cong \begin{cases} C_{p^n-1}^{17}, & \text{if } q \equiv 1 \mod 17; \\ C_{p^n-1} \times C_{p^{2n}-1}^{8}, & \text{if } q \equiv -1 \mod 17. \end{cases}$$

Case 2. $q \equiv \pm 2, \pm 8 \mod 17$. It is easy to verify that

$$T = \{1, 2, 4, 8, 9, 13, 15, 16\} \mod 17.$$

By an easy calculation we obtain that

$$\begin{aligned} S_F(\gamma_1) &= \{\gamma_1\}, \\ S_F(\gamma_x) &= \{\gamma_x, \gamma_{x^2}, \gamma_{x^4}, \gamma_{x^8}, \gamma_{x^9}, \gamma_{x^{13}}, \gamma_{x^{15}}, \gamma_{x^{16}}\}, \\ S_F(\gamma_{x^3}) &= \{\gamma_{x^3}, \gamma_{x^5}, \gamma_{x^6}, \gamma_{x^7}, \gamma_{x^{10}}, \gamma_{x^{11}}, \gamma_{x^{12}}, \gamma_{x^{14}}\}. \end{aligned}$$

It follows from Remark 2.3 that

$$FC_{17} \cong F \oplus F_8^2.$$

So

$$U(FC_{17}) \cong C_{p^n-1} \times C_{p^{8n}-1}^2.$$

Case 3. $q \equiv \pm 4 \mod 17$. Then

$$T = \{1, 4, 13, 16\} \mod 17,$$

and thus,

$$\begin{aligned} S_F(\gamma_1) &= \{\gamma_1\}, \\ S_F(\gamma_x) &= \{\gamma_x, \gamma_{x^4}, \gamma_{x^{13}}, \gamma_{x^{16}}\}, \\ S_F(\gamma_{x^2}) &= \{\gamma_{x^2}, \gamma_{x^8}, \gamma_{x^9}, \gamma_{x^{15}}\}, \\ S_F(\gamma_{x^3}) &= \{\gamma_{x^3}, \gamma_{x^5}, \gamma_{x^{12}}, \gamma_{x^{14}}\}, \\ S_F(\gamma_{x^6}) &= \{\gamma_{x^6}, \gamma_{x^7}, \gamma_{x^{10}}, \gamma_{x^{11}}\}. \end{aligned}$$

It follows from Remark 2.3 that $FC_{17} \cong F \oplus F_4^4$. Therefore,

$$U(FC_{17}) \cong C_{p^n-1} \times C_{p^{4n}-1}^4.$$

Case 4. $q \equiv \pm 3, \pm 5, \pm 6, \pm 7 \mod 17$. Then

$$T = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16\} \mod 17.$$

Thus,

$$\begin{aligned} S_F(\gamma_1) =& \{\gamma_1\}, \\ S_F(\gamma_x) =& \{\gamma_x, \gamma_{x^2}, \gamma_{x^3}, \gamma_{x^4}, \gamma_{x^5}, \gamma_{x^6}, \gamma_{x^7}, \gamma_{x^8}, \\ & \gamma_{x^9}, \gamma_{x^{10}}, \gamma_{x^{11}}, \gamma_{x^{12}}, \gamma_{x^{13}}, \gamma_{x^{14}}, \gamma_{x^{15}}, \gamma_{x^{16}}\}. \end{aligned}$$

As above, we obtain that $FC_{17} \cong F \oplus F_{16}$, and thus

$$U(FC_{17}) \cong C_{p^n-1} \times C_{p^{16n}-1}.$$

This completes the proof. □

Using a similar method as in the proof of Theorem 3.3, we obtain the following result.

Theorem 3.4. *Let F be a finite field of characteristic p with $|F| = q = p^n$. Then*

$$U(FC_{19}) \cong \begin{cases} C_{19}^{18n} \times C_{19^n-1}, & \text{if } p = 19; \\ C_{p^n-1}^{19}, & \text{if } q \equiv 1 \mod 19; \\ C_{p^n-1} \times C_{p^{2n}-1}^{9}, & \text{if } q \equiv -1 \mod 19; \\ C_{p^n-1} \times C_{p^{18n}-1}, & \text{if } q \equiv 2, 3, 10, 13, 14, 15 \mod 19; \\ C_{p^n-1} \times C_{p^{9n}-1}^{2}, & \text{if } q \equiv 4, 5, 6, 9, 16, 17 \mod 19; \\ C_{p^n-1} \times C_{p^{3n}-1}^{6}, & \text{if } q \equiv 7, 11 \mod 19; \\ C_{p^n-1} \times C_{p^{6n}-1}^{3}, & \text{if } q \equiv 8, 12 \mod 19. \end{cases}$$

4. Groups of order 18

In this section, we deal with the unit group of FG, when $|G| = 18$. Note that if G is an abelian group of 18, then $G \cong C_{18}$ or $G \cong C_3 \oplus C_6$. We need a few lemmas.

Lemma 4.1. *[2] Let F be a finite field of characteristic p with $|F| = q = p^n$. Then*

$$U(FC_2) \cong \begin{cases} C_2^n \times C_{2^n-1}, & \text{if } p = 2; \\ C_{p^n-1}^2, & \text{if } p \neq 2. \end{cases}$$

Lemma 4.2. *[16, Theorem 3.6] Let F be a finite field of characteristic p with $|F| = q = p^n$. Then*

$$U(FC_9) \cong \begin{cases} C_3^{4n} \times C_9^{2n} \times C_{3^n-1}, & \text{if } p = 3; \\ C_{p^n-1}^{9}, & \text{if } q \equiv 1 \mod 9; \\ C_{p^n-1} \times C_{p^{2n}-1}^{4}, & \text{if } q \equiv -1 \mod 9; \\ C_{p^n-1} \times C_{p^{2n}-1} \times C_{p^{6n}-1}, & \text{if } q \equiv 2, -4 \mod 9; \\ C_{p^n-1}^{3} \times C_{p^{3n}-1}^{2}, & \text{if } q \equiv -2, 4 \mod 9. \end{cases}$$

Lemma 4.3. *[16, Theorem 3.7] Let F be a finite field of characteristic p with $|F| = q = p^n$. Then*

$$U(FC_3^2) \cong \begin{cases} C_3^{8n} \times C_{3^n-1}, & \text{if } p = 3; \\ C_{p^n-1}^{9}, & \text{if } q \equiv 1 \mod 3; \\ C_{p^n-1} \times C_{p^{2n}-1}^{4}, & \text{if } q \equiv -1 \mod 3. \end{cases}$$

We now state our result on $U(FC_{18})$.

Theorem 4.4. *Let F be a finite field of characteristic p with $|F| = q = p^n$. Then*

(1) *If $p = 2$, then*

$$U(FC_{18}) \cong \begin{cases} C_2^{9n} \times C_{2^n-1}^{9}, & \text{if } q \equiv 1 \mod 9; \\ C_2^{9n} \times C_{2^n-1} \times C_{2^{2n}-1}^{4}, & \text{if } q \equiv -1 \mod 9; \\ C_2^{9n} \times C_{2^n-1} \times C_{2^{2n}-1} \times C_{2^{6n}-1}, & \text{if } q \equiv 2, -4 \mod 9; \\ C_2^{9n} \times C_{2^n-1}^{3} \times C_{2^{3n}-1}^{2}, & \text{if } q \equiv -2, 4 \mod 9. \end{cases}$$

(2) *If* $p = 3$*, then*

$$U(FC_{18}) \cong C_3^{8n} \times C_9^{4n} \times C_{3^n-1}^2.$$

(3) *If* $p \nmid 6$*, then*

$$U(FC_{18}) \cong \begin{cases} C_{p^n-1}^{18}, & \text{if } q \equiv 1 \mod 18; \\ C_{p^n-1}^2 \times C_{p^{2n}-1}^8, & \text{if } q \equiv -1 \mod 18; \\ C_{p^n-1}^2 \times C_{p^{2n}-1}^2 \times C_{p^{6n}-1}^2, & \text{if } q \equiv 5, 11 \mod 18; \\ C_{p^n-1}^6 \times C_{p^{3n}-1}^4, & \text{if } q \equiv 7, 13 \mod 18. \end{cases}$$

Proof. Let $C_{18} = \langle x \rangle$ and $V = 1 + J(FC_{18})$.

(1) If $p = 2$, then applying Lemma 2.4 to $G = C_{18}$, we obtain

$$U(FC_{18}) \cong V \times U(FC_9),$$

and

$$V \cong C_2^{9n}.$$

By Lemma 4.2, we obtain

$$U(FC_{18}) \cong \begin{cases} C_2^{9n} \times C_{2^n-1}^9, & \text{if } q \equiv 1 \mod 9; \\ C_2^{9n} \times C_{2^n-1} \times C_{2^{2n}-1}^4, & \text{if } q \equiv -1 \mod 9; \\ C_2^{9n} \times C_{2^n-1} \times C_{2^{2n}-1} \times C_{2^{6n}-1}, & \text{if } q \equiv 2, -4 \mod 9; \\ C_2^{9n} \times C_{2^n-1}^3 \times C_{2^{3n}-1}^2, & \text{if } q \equiv -2, 4 \mod 9. \end{cases}$$

(2) Suppose $p = 3$. Let $C_2 = \langle x^9 \rangle = \{1, \bar{b}\}$ and $C_9 = \langle x^2 \rangle = \langle \bar{a} \rangle$.

Note that

$$[C_{18} : C_9] = 2 \neq 0 \in F.$$

By Lemmas 2.5 and 2.6,

$$J(FC_{18}) = J(FC_9)FC_{18} = \omega(FC_9)FC_{18} = \omega(C_{18}, C_9),$$

and

$$FC_{18}/J(FC_{18}) \cong FC_2.$$

From the ring epimorphism

$$FC_{18} \to FC_2,$$

we deduce a group epimorphism

$$\varphi : U(FC_{18}) \to U(FC_2),$$

and

$$\ker \varphi = V = 1 + J(FC_{18}) = 1 + \omega(FC_9)FC_{18} = 1 + \omega(C_{18}, C_9).$$

The ring monomorphism

$$FC_2 \to FC_{18}$$

given by

$$\alpha_0 + \alpha_1\bar{b} \to \alpha_0 + \alpha_1\bar{b}$$

induces a group monomorphism

$$\sigma : U(FC_2) \to U(FC_{18}).$$

And we can verify that $\varphi\sigma = 1_{U(FC_2)}$. Thus $U(FC_{18})$ is an extension of $U(FC_2)$ by V. So

$$U(FC_{18}) \cong V \times U(FC_2).$$

By Lemma 4.1 we have $U(FC_2) \cong C_{3^n-1}^2$. We next determine V.

Note that

$$\alpha = \sum_{i=0}^{17} a_i x^i \in J(FC_{18}) = \omega(FC_9)FC_{18} = \omega(C_{18}, C_9) \text{ if and only if } \sum_{j=0}^{8} a_{2j+i} = 0, i = 0, 1.$$

If $\alpha \in J(FC_{18})$, a straight forward computation shows that

$$\alpha^3 = \sum_{i=0}^{5}(a_i^3 + a_{6+i}^3 + a_{12+i}^3)x^{3i},$$

and

$$\alpha^9 = \sum_{i=0}^{1}\sum_{j=0}^{8} a_{2j+i}^9 x^{9i} = 0.$$

It follows that $V = 1 + J(FC_{18})$ is an abelian 3-group with exponent dividing 9. Let

$$V \cong C_3^{\ell_1} \times C_9^{\ell_2}.$$

It remains to determine ℓ_1 and ℓ_2.

Since $\dim_F(V) = \dim_F(J(FC_{18})) = \dim_F(FC_{18}/FC_2) = 16$, we have $|V| = 3^{16n}$. So $\ell_1 + 2\ell_2 = 16n$. Let

$$S = \{\alpha \in J(FC_{18}) | \alpha^3 = 0, \text{ and } \exists\, \beta \in \omega(FC_9) \text{ such that } \alpha = \beta^3\}.$$

Then

$$S = \{\Sigma_{i=0}^{1}(a_{3i}x^{3i} + a_{3i+6}x^{3i+6} + (2a_{3i} + 2a_{3i+6})x^{3i+12}) : a_j \in F\}.$$

It follows that $|S| = 3^{4n}$, and thus $\ell_2 = 4n$. So $\ell_1 = 8n$ and hence

$$V \cong C_3^{8n} \times C_9^{4n}.$$

Therefore,

$$U(FC_{18}) \cong C_3^{8n} \times C_9^{4n} \times C_{3^n-1}^2.$$

(3) If $p \nmid 6$, then $m = 18$.

We divide the following proof into several cases according to the value of q module 18.

Case 3.1. $q \equiv \pm 1 \mod 18$. By Lemma 3.2, we can get

$$U(FC_{18}) \cong \begin{cases} C_{p^n-1}^{18}, & \text{if } q \equiv 1 \mod 18; \\ C_{p^n-1}^2 \times C_{p^{2n}-1}^8, & \text{if } q \equiv -1 \mod 18. \end{cases}$$

Case 3.2. $q \equiv 5, 11 \mod 18$. Then $T = \{1, 5, 7, 13, 11, 17\} \mod 18$. It follows from Remark 2.3 that

$$S_F(\gamma_1) = \{\gamma_1\},\ S_F(\gamma_{x^9}) = \{\gamma_{x^9}\},$$

$$\begin{aligned}
S_F(\gamma_x) &= \{\gamma_x, \gamma_{x^5}, \gamma_{x^7}, \gamma_{x^{11}}, \gamma_{x^{13}}, \gamma_{x^{17}}\},\\
S_F(\gamma_{x^2}) &= \{\gamma_{x^2}, \gamma_{x^4}, \gamma_{x^8}, \gamma_{x^{10}}, \gamma_{x^{14}}, \gamma_{x^{16}}\},\\
S_F(\gamma_{x^3}) &= \{\gamma_{x^3}, \gamma_{x^{15}}\},\ S_F(\gamma_{x^6}) = \{\gamma_{x^6}, \gamma_{x^{12}}\}.
\end{aligned}$$

Therefore,

$$FC_{18} \cong F^2 \oplus F_2^2 \oplus F_6^2.$$

So

$$U(FC_{18}) \cong C_{p^n-1}^2 \times C_{p^{2n}-1}^2 \times C_{p^{6n}-1}^2.$$

Case 3.3. $q \equiv 7, 13 \mod 18$. Then $T = \{1, 7, 13\} \mod 18$. Thus,

$$\begin{aligned}
&S_F(\gamma_1) = \{\gamma_1\},\ S_F(\gamma_{x^3}) = \{\gamma_{x^3}\},\\
&S_F(\gamma_{x^6}) = \{\gamma_{x^6}\},\ S_F(\gamma_{x^9}) = \{\gamma_{x^9}\},\\
&S_F(\gamma_{x^{12}}) = \{\gamma_{x^{12}}\},\ S_F(\gamma_{x^{15}}) = \{\gamma_{x^{15}}\},\\
&S_F(\gamma_x) = \{\gamma_x, \gamma_{x^7}, \gamma_{x^{13}}\},\ S_F(\gamma_{x^2}) = \{\gamma_{x^2}, \gamma_{x^8}, \gamma_{x^{14}}\},\\
&S_F(\gamma_{x^4}) = \{\gamma_{x^4}, \gamma_{x^{10}}, \gamma_{x^{16}}\},\ S_F(\gamma_{x^5}) = \{\gamma_{x^2}, \gamma_{x^{11}}, \gamma_{x^{17}}\}.
\end{aligned}$$

Therefore,

$$FC_{18} \cong F^6 \oplus F_3^4.$$

Thus

$$U(FC_{18}) \cong C_{p^n-1}^6 \times C_{p^{3n}-1}^4.$$

This completes the proof. □

Next we determine the structure of $U(F(C_3 \times C_6))$.

Theorem 4.5. *Let F be a finite field of characteristic p with $|F| = q = p^n$ and let $G = C_3 \times C_6$.*

(1) *If $p = 2$, then*

$$U(FG) \cong \begin{cases} C_2^{9n} \times C_{2^n-1}^9, & \text{if } q \equiv 1 \mod 3;\\ C_2^{9n} \times C_{2^n-1} \times C_{2^{2n}-1}^4, & \text{if } q \equiv -1 \mod 3.\end{cases}$$

(2) *If $p = 3$, then*

$$U(FG) \cong C_3^{16n} \times C_{3^n-1}^2.$$

(3) *If $p \nmid 6$, then*

$$U(FG) \cong \begin{cases} C_{p^n-1}^{18}, & \text{if } q \equiv 1 \mod 6;\\ C_{p^n-1}^2 \times C_{p^{2n}-1}^8, & \text{if } q \equiv -1 \mod 6.\end{cases}$$

Proof. Let $G = \langle x, y \mid x^3 = y^6 = 1, xy = yx\rangle$ and $V = 1 + J(FG)$.

(1) If $p = 2$, then let $H = \langle y^3\rangle$. We know that $[G : H] = 9 \neq 0 \in F$. By Lemmas 2.5 and 2.6,

$$J(FG) = J(FH)FG = \omega(FH)FG = \omega(G, H),$$

and

$$FG/J(FG) \cong F(C_3 \times C_3).$$

From the ring epimorphism

$$FG \to F(C_3 \times C_3),$$

we deduce a group epimorphism

$$\varphi : U(FG) \to U(F(C_3 \times C_3)),$$

and

$$\ker\varphi = V = 1 + J(FG) = 1 + \omega(G, H).$$

The ring monomorphism

$$F(C_3 \times C_3) \to FG,$$

induces a group monomorphism

$$\sigma : U(F(C_3 \times C_3)) \to U(FG).$$

It is not hard to show that $\varphi\sigma = 1_{U(F(C_3 \times C_3))}$. Thus $U(FG)$ is an extension of $U(F(C_3 \times C_3))$ by V. So

$$U(FG) \cong V \times U(F(C_3 \times C_3)).$$

By Lemma 4.3 we have

$$U(FC_3^2) \cong \begin{cases} C_{2^n-1}^9, & q \equiv 1 \mod 3; \\ C_{2^n-1} \times C_{2^{2n}-1}^4, & q \equiv -1 \mod 3. \end{cases}$$

We next determine V. It is clear that

$$\alpha = \sum_{i=0}^{2}\sum_{j=0}^{5} a_{6i+j} x^i y^j \in \omega(G, H)$$

if and only if

$$a_i + a_{3+i} = 0, \quad i = 0, 1, 2, 6, 7, 8, 12, 13, 14.$$

A straight forward calculation gives that $\alpha^2 = 0$. Thus, it is not hard to show that $\dim_F(J(FG)) = 9$, and $V \cong C_2^{9n}$. Therefore

$$U(FG) \cong \begin{cases} C_2^{9n} \times C_{2^n-1}^9, & \text{if } q \equiv 1 \mod 3; \\ C_2^{9n} \times C_{2^n-1} \times C_{2^{2n}-1}^4, & \text{if } q \equiv -1 \mod 3. \end{cases}$$

(2) If $p = 3$, then let $H = \langle x \rangle \times \langle y^2 \rangle$. We know $[G : H] = 2 \neq 0 \in F$. As in the proof of (1) we can show that

$$U(FG) \cong V \times U(FC_2).$$

By Lemma 4.1 we have $U(FC_2) \cong C_{3^n-1}^2$. We next determine V. It is clear that

$$\alpha = \textstyle\sum_{i=0}^{2}\sum_{j=0}^{5} a_{6i+j} x^i y^j \in \omega(FH) \text{ if and only if } \sum_{i=0}^{8} a_{2i+j} = 0,\ j = 0, 1.$$

It is not hard to show that $\alpha^3 = 0$. Thus we obtain that $\dim_F(J(FG)) = 16$, and $V \cong C_3^{16n}$. Therefore,

$$U(FG) \cong C_3^{16n} \times C_{3^n-1}^2.$$

(3) If $p \nmid 6$, then $m = 6$.

If $q \equiv 1 \mod 6$, then $T = \{1\} \mod 6$. Thus,

$$S_F(\gamma_x) = \{\gamma_x\},\ \forall x \in G.$$

So

$$FG \cong F^{18}.$$

Therefore,

$$U(FG) \cong C_{p^n-1}^{18}.$$

If $q \equiv -1 \mod 6$, then $T = \{1, 5\} \mod 6$. Thus,

$$\begin{aligned}
&S_F(\gamma_1) = \{\gamma_1\},\ S_F(\gamma_{y^3}) = \{\gamma_{y^3}\},\\
&S_F(\gamma_y) = \{\gamma_y, \gamma_{y^5}\},\ S_F(\gamma_{y^2}) = \{\gamma_{y^2}, \gamma_{y^4}\},\\
&S_F(\gamma_{xy}) = \{\gamma_{xy}, \gamma_{x^2y^5}\},\ S_F(\gamma_{xy^2}) = \{\gamma_{xy^2}, \gamma_{x^2y^4}\},\\
&S_F(\gamma_{xy^3}) = \{\gamma_{xy^3}, \gamma_{x^2y^3}\},\ S_F(\gamma_{x^2y}) = \{\gamma_{x^2y}, \gamma_{xy^5}\},\\
&S_F(\gamma_{x^2y^2}) = \{\gamma_{x^2y^2}, \gamma_{xy^4}\},\ S_F(\gamma_x) = \{\gamma_x, \gamma_{x^2}\}.
\end{aligned}$$

Therefore,

$$FG \cong F^2 \oplus F_2^8.$$

Thus

$$U(FG) \cong C_{p^n-1}^2 \times C_{p^{2n}-1}^8.$$

This completes the proof. □

5. Groups of order 20

In this section, we investigate the unit group of FG when $|G| = 20$. Since G is an abelian group of 20, $G \cong C_{20}$ or $G \cong C_2 \oplus C_{10}$.

Lemma 5.1. *[16, Theorem 2.3] Let F be a finite field of characteristic p with $|F| = q = p^n$. Then*

$$U(FC_5) \cong \begin{cases} C_p^{4n} \times C_{p^n-1}, & \text{if } p = 5;\\ C_{p^n-1}^5, & \text{if } q \equiv 1 \mod 5;\\ C_{p^n-1} \times C_{p^{4n}-1}, & \text{if } q \equiv \pm 2 \mod 5;\\ C_{p^n-1} \times C_{p^{2n}-1}^2, & \text{if } q \equiv -1 \mod 5.\end{cases}$$

Lemma 5.2. *[16, Theorem 3.1] Let F be a finite field of characteristic p with $|F| = q = p^n$. Then*

$$U(FC_4) \cong \begin{cases} C_2^n \times C_4^n \times C_{2^n-1}, & \text{if } p = 2;\\ C_{p^n-1}^4, & \text{if } q \equiv 1 \mod 4;\\ C_{p^n-1}^2 \times C_{p^{2n}-1}, & \text{if } q \equiv -1 \mod 4.\end{cases}$$

Lemma 5.3. *[16, Theorem 3.2] Let F be a finite field of characteristic p with* $|F| = q = p^n$. *Then*

$$U(FC_2^2) \cong \begin{cases} C_2^{3n} \times C_{2^n-1}, & \text{if } p = 2; \\ C_{p^n-1}^4, & \text{if } p \neq 2. \end{cases}$$

The next two theorems provide complete characterizations of the structures of $U(FC_{20})$ and $U(F(C_2 \oplus C_{10}))$, respectively. As their proofs are very much similar to those of Theorem 4.4 and Theorem 4.5, we omit the detailed computation and state only the results.

Theorem 5.4. *Let F be a finite field of characteristic p with* $|F| = q = p^n$.

(1) *If* $p = 2$, *then*

$$U(FC_{20}) \cong \begin{cases} C_4^{5n} \times C_2^{5n} \times C_{2^n-1}^5, & \text{if } q \equiv 1 \mod 5; \\ C_4^{5n} \times C_2^{5n} \times C_{2^n-1} \times C_{2^{4n}-1}, & \text{if } q \equiv \pm 2 \mod 5; \\ C_4^{5n} \times C_2^{5n} \times C_{2^n-1} \times C_{2^{2n}-1}^2, & \text{if } q \equiv -1 \mod 5. \end{cases}$$

(2) *If* $p = 5$, *then*

$$U(FC_{20}) \cong \begin{cases} C_5^{16n} \times C_{5^n-1}^4, & \text{if } q \equiv 1 \mod 4; \\ C_5^{16n} \times C_{5^n-1}^2 \times C_{5^{2n}-1}, & \text{if } q \equiv -1 \mod 4. \end{cases}$$

(3) *If* $p \neq 2$ *and* $p \neq 5$, *then*

$$U(FC_{20}) \cong \begin{cases} C_{p^n-1}^{20}, & \text{if } q \equiv 1 \mod 20; \\ C_{p^n-1}^2 \times C_{p^{2n}-1}^9, & \text{if } q \equiv -1 \mod 20; \\ C_{p^n-1}^2 \times C_{p^{2n}-1} \times C_{p^{4n}-1}^4, & \text{if } q \equiv 3, 7 \mod 20; \\ C_{p^n-1}^4 \times C_{p^{4n}-1}^4, & \text{if } q \equiv 13, 17 \mod 20; \\ C_{p^n-1}^4 \times C_{p^{2n}-1}^8, & \text{if } q \equiv 9 \mod 20; \\ C_{p^n-1}^{10} \times C_{p^{2n}-1}^5, & \text{if } q \equiv 11 \mod 20. \end{cases}$$

Theorem 5.5. *Let F be a finite field of characteristic p with* $|F| = q = p^n$ *and let* $G = C_2 \times C_{10}$.

(1) *If* $p = 2$, *then*

$$U(FG) \cong \begin{cases} C_2^{15n} \times C_{2^n-1}^5, & \text{if } q \equiv 1 \mod 5; \\ C_2^{15n} \times C_{2^n-1} \times C_{2^{4n}-1}, & \text{if } q \equiv \pm 2 \mod 5; \\ C_2^{15n} \times C_{2^n-1}^2, & \text{if } q \equiv -1 \mod 5. \end{cases}$$

(2) *If* $p = 5$, *then*

$$U(FG) \cong C_5^{16n} \times C_{5^n-1}^4.$$

(3) *If* $p \neq 2$ *and* $p \neq 5$, *then*

$$U(FG) \cong \begin{cases} C_{p^n-1}^{20}, & \text{if } q \equiv 1 \mod 10; \\ C_{p^n-1}^4 \times C_{p^{2n}-1}^8, & \text{if } q \equiv -1 \mod 10; \\ C_{p^n-1}^4 \times C_{p^{4n}-1}^4, & \text{if } q \equiv 3, 7 \mod 10. \end{cases}$$

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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