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Research article

Valuation of bid and ask prices for European options under mixed fractional Brownian motion

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Abstract: In this paper, we study the valuation of the bid and ask prices for European options under the mixed fractional Brownian motion with Hurst index $H > 3/4$, which is able to capture the long range dependence of the underlying asset returns in real markets. As we know, the classical option pricing theories are usually built on the law of one price, while ignoring the impact of market liquidity on bid-ask spreads. The theory of conic finance replaces the law of one price by the law of two prices, allowing for market participants sell to the market at the bid price and buy from the market at the higher ask price. Within the framework of conic finance, we then derive the explicit formulas for the bid and ask prices of European call and put options by using WANG-transform as a distortion function. Moreover, numerical experiment is performed to illustrate the effects of the Hurst index and market liquidity level on bid and ask prices.

Keywords: conic finance; option pricing; mixed fractional Brownian motion; bid-ask spreads; WANG-transform
Mathematics Subject Classification: 91G20, 91-10, 62P05

1. Introduction

The issue of market liquidity has drawn much attention among academic researchers, institutional professionals and financial regulators in various financial markets. However, there are few literatures on how to incorporate liquidity costs into option pricing. From the framework of acceptability indices proposed by [8], Madan and Cherny [19] developed the theory of conic finance which replaces the law of one price by the law of two prices, allowing for market participants sell to the market at the bid price and buy from the market at the higher ask price. The difference between the bid and ask prices is usually called the bid-ask spread which is an indication of the market liquidity.

The bid price of a cash flow X is defined by its discounted distorted expectation and the ask price by minus the discounted distorted expectation of the cash flow $-X$.* Madan and Cherny [19] proposed to model the liquidity of the market by a single market stress level γ and then presented the closed-form expressions for bid and ask prices for European options. This conic option pricing model was further extended and taken to the real market data (S&P 500 option), see [1, 9, 11, 20]. All of their empirical results indicated that the market-implied liquidity was far from being constant. That is to say, there is an implied liquidity risk premium. It is noteworthy that these papers are all studied on the premise of allowing explicit forms for the distribution function of the underlying asset price, see for example Black-Scholes and Variance Gamma models, such that the bid and ask prices can be further numerically calculated. Recently, Guillaume et al. [12] and Sonono and Mashele [26] derived the explicit formulas for the bid and ask prices of OTC interest rate options and European vanilla options, respectively, by using WANG-transform, which is a distortion function induced by a distribution function, in a Black-Scholes setting. Junike et al. [13] studied the convergence of bid and ask prices for various European and American possible path dependent options in a binomial model, where bid and ask prices are defined recursively using nonlinear expectations, which is closely related to discrete time conic finance models, see [15, 17] for details. Based on the combination of Fourier cosine approximations and numerical integration, Li et al. [16] exhibited an efficient and fast numerical method to calculate the bid and ask prices for the European options.

However, the long-range dependence of the underlying asset returns was not considered in the above mentioned researches. It is well known that the existence of long-range dependence in asset returns has been an interesting subject for both academics and market professionals for a long time. In addition, empirical evidence so far suggests that long-range dependence may be a characteristic of both exchange rates and stock markets, see [3, 14, 25, 27, 28]. In this case, it is possible to choose a fractional Brownian motion (hereafter fBm) to describe the dynamic of the financial asset price. Rogers [23] demonstrated that while a fBm could capture the long-range dependence between returns on different days, it also allowed arbitrage opportunities. To overcome this problem and to take into account the long memory property, El-Nouty [10] and Mishura [18] had shown that it was reasonable to use a mixed fractional Brownian motion(hereafter mfBm) to capture the fluctuations of the financial assets from time to time. Cheridito [6] had demonstrated that the mfBm was equivalent to a Brownian motion which means that no arbitrage was allowed. Whereafter, Xiao et al. [29] studied the problem of equity warrants pricing under a mfBm environment and employed a hybrid intelligent algorithm to solve the valuation of equity warrants. Sun [24] investigated pricing currency options when the driving force is a mixed fractional Brownian motion. Ballestra et al. [4] presented an integral representation for the pricing of the barrier options on an underlying asset driven by a mfBm. Both Prakasa Rao [22] and Zhang et al. [31] assumed that the price of the underlying stock followed a mfBm and derived the analytical pricing formula for the geometric Asian option. Recently, Zhang et al. [32] proposed a fuzzy mixed fractional Brownian motion model with jumps, which was to capture the features of both long memory and jump behaviour in financial assets under non-random uncertainty environment.

Motivated by the above mentioned insights, the main objective of this paper is to discuss the valuation of the bid and ask prices for European options under the mixed fractional Brownian motion, which is able to capture empirically observed patterns (the long range dependence of the underlying asset returns in real markets). Within the framework of conic finance, we then derive the explicit formulas for bid and ask prices of European call and put options by using WANG-transform as a

*One can refer to Section 2.2 of this paper for more related information on distorted expectation.

distortion function. Finally, we perform numerical experiment for illustrating the effects of the Hurst index and market liquidity level on bid and ask prices.

The remainder of the paper proceeds as follows. In Section 2, we briefly introduce some basic concepts and properties of mixed fractional Brownian motion, distortion function and distorted expectation. Section 3 lays out the mixed fractional Brownian motion model for the underlying asset price. In Section 4, we present the analytical formulas for the bid and ask prices of European options within the framework of conic finance. Numerical experiment is performed in Section 5. Finally, some conclusions are stated in Section 6.

2. Preliminaries

In this section, for better understanding the rest of this paper, we briefly review some basic concepts and the properties of mixed fractional Brownian motion and distorted expectation.

2.1. Mixed fractional Brownian motion

Definition 1. *A mixed fractional Brownian motion $M_t^H(\alpha,\beta)$ is a linear combination of Brownian motion and fractional Brownian motion(fBm), defined on a filtered probability space $(\Omega,\mathcal{F},\{\mathcal{F}_t, t \geq 0\},\mathbb{P})$ by:*

$$M_t^H = \alpha B_t + \beta B_t^H,$$

where α and β are two real constants such that $(\alpha,\beta) \neq (0,0)$, $\mathbb{P}$ is the physical probability measure, B_t is a standard Brownian motion, B_t^H is an independent standard fractional Brownian motion with the Hurst exponent $H \in (0,1)$ and $\{\mathcal{F}_t\}_{t\geq 0}$ denotes the $\mathbb{P}$-augmentation of the filtration generated by (B_τ, B_τ^H) for $\tau \leq t$.

In what follows, some properties of the mfBm are given in the form of proposition described below. For more detailed information about the properties of the mfBm, one can refer to [29,30].

Proposition 1. *The mfBm $M_t^H(\alpha,\beta)$ for $t \in \mathbb{R}^+$ satisfies the following properties:*

(i): *$M_t^H(\alpha,\beta)$ is a centered Gaussian process and not a Markovian one for all $H \in (0,1)\backslash\frac{1}{2}$;*

(ii): *$M_0^H = 0$ $\mathbb{P}$–almost surely;*

(iii): *the covariation function of $M_t^H(\alpha,\beta)$ and $M_s^H(\alpha,\beta)$ for any $t, s \in \mathbb{R}^+$ is given by*

$$cov(M_t^H, M_s^H) = \alpha^2(t \wedge s) + \frac{\beta^2}{2}(t^{2H} + s^{2H} - |t-s|^{2H});$$

(iv): *the increments of M_t^H are stationary and mixed-self-similar for any $h > 0$*

$$M_{ht}^H(\alpha,\beta) \triangleq M_t^H(\alpha h^{\frac{1}{2}},\beta h^H),$$

where $\triangleq$ denotes "to have the same law";

(v): *the increments of M_t^H are positively correlated if $\frac{1}{2} < H < 1$, uncorrelated if $H = \frac{1}{2}$ negatively correlated if $0 < H < \frac{1}{2}$;*

(vi): *the increments of M_t^H are long-range dependent if and only if $H > \frac{1}{2}$.*

2.2. *Distorted expectation*

We fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and denote by $L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ the space of all essentially bounded and $\mathbb{R}$-valued random variables on $(\Omega, \mathcal{F}, \mathbb{P})$.

Definition 2. *(Distortion function). A function* $\Psi : [0,1] \to [0,1]$ *is a distortion function if and only if it is monotone,* $\Psi(0) = 0$ *and* $\Psi(1) = 1$. *The set function* $\Psi \circ \mathbb{P}$ *defined by*

$$\Psi \circ \mathbb{P}(A) = \Psi(\mathbb{P}(A)), \quad A \in \mathcal{F}, \tag{2.1}$$

is called the distortion of the probability measure $\mathbb{P}$ *with respect to the distortion function* Ψ, *i.e. the distorted probability measure.*

With a probability distortion function Ψ is associated the complementary distortion function $\hat{\Psi}$ given by

$$\hat{\Psi}(x) = 1 - \Psi(1 - x), \quad x \in [0,1]. \tag{2.2}$$

Given the probability distortion function Ψ, the Choquet integral $\mathbb{E}_\Psi[X]$ of $X \in L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ can be defined by

$$\mathbb{E}_\Psi[X] = \int_0^{+\infty} (1 - \Psi(\mathbb{P}(X \le x)))dx - \int_{-\infty}^0 \Psi(\mathbb{P}(X \le x))dx. \tag{2.3}$$

Definition 3. *(Distorted expectation). If we denote the distribution of random* X *by* F_X, *then the Choquet integral* $\mathbb{E}_\Psi[X]$ *can be rewritten as*

$$\mathbb{E}_\Psi[X] = \int_{-\infty}^{+\infty} x d\Psi(F_X(x)). \tag{2.4}$$

Here, $\mathbb{E}_\Psi[X]$ *is generally referred to as the distorted expectation of a random* X *with distribution function* F_X *relative to the distortion function* Ψ. *Note that if* $\Psi(u) = u$, *and thereby* $\mathbb{E}_\Psi[X]$ *is the ordinary expectation.*

In general, the distorted probability $\Psi \circ \mathbb{P}$ is no longer a probability measure. Nevertheless, it is still a finite monotone set function that is submodular, when the distortion function Ψ is concave. Thus, it is possible to define a risk measure based on distorted probability using Choquet integral. Let Ψ be a concave distortion function and a risk X. The function $\varrho^\Psi : L^\infty \to \mathbb{R}$ given by

$$\varrho^\Psi(X) := -\mathbb{E}_\Psi[X], \quad \forall X \in L^\infty(\Omega, \mathcal{F}, \mathbb{P}), \tag{2.5}$$

is called a distortion risk measure induced by Ψ. By the properties of the Choquet integral, we can see that the function ϱ^Ψ is a coherent risk measure.

3. Models settings

Owing to a financial system is a complex system with great flexibility, investors do not make their decisions immediately after receiving the financial information, but rather wait until the information reaches to its threshold limit value. This behavior can lead to the features of "asymmetric leptokurtic" and "long memory". As mentioned above, the mixed fractional Brownian motion may be a useful tool to capture this phenomenon. Thus, in this section, we introduce a mixed fractional Brownian motion model for describing the dynamic of the underlying asset prices, which is a useful tool to capture the long memory of asset prices in real markets.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with information filtration $(\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions, which is defined in Section 2.1. Suppose the underlying asset price S_t takes the following stochastic differential equation:

$$dS_t = \mu S_t dt + \sigma S_t d\bar{B}_t + \sigma S_t d\bar{B}_t^H, \tag{3.1}$$

where the drift parameters μ and the volatility σ denote the expected return rate on the value of the underlying asset, the standard deviation of the return rate on the value of the underlying asset, respectively; $\bar{B}_t$ is a standard Brownian motion and $\bar{B}_t^H$ is an independent standard fractional Brownian motion with Hurst index $H > 3/4$. In fact, the Hurst exponent $H > 3/4$ ensures that the financial market does not allow arbitrage opportunity.

In addition, we also postulate that there are no transaction costs or taxes in purchasing or selling the financial assets, which means the financial market is frictionless. Thus, the market is complete and the risk-neutral martingale measure is unique under the aforementioned conditions, see [5–7] for details.

By using the fractional Girsanov theorem and the change of variables $B_t + B_t^H = \frac{\mu - r}{\sigma} t + \bar{B}_t + \bar{B}_t^H$, thus under the risk-neutral measure $\mathbb{Q}$ we have

$$dS_t = rS_t dt + \sigma S_t dB_t + \sigma S_t dB_t^H, \tag{3.2}$$

where r denotes the risk-free interest rate, B_t is a standard Brownian motion and B_t^H is an independent standard fractional Brownian motion under the risk-neutral measure $\mathbb{Q}$.

It follows from the mixed fractional pattern of Itô Lemma that the solution of Eq (3.2) can be expressed as

$$S_t = S_0 \exp\left(rt + \sigma\left(B_t + B_t^H\right) - \frac{1}{2}\sigma^2 t - \frac{1}{2}\sigma^2 t^{2H} \right), \tag{3.3}$$

Obviously, the underlying asset price S_t is log-normally distributed with

$$\ln S_t \simeq N\left(\ln S_0 + rt - \frac{1}{2}\sigma^2 t - \frac{1}{2}\sigma^2 t^{2H}, \sigma^2 t + \sigma^2 t^{2H} \right), \tag{3.4}$$

where $N(\tilde{\mu}, \tilde{\sigma}^2)$ represents the Gaussian distribution with mean $\tilde{\mu}$ and variance $\tilde{\sigma}^2$.

4. Bid and ask prices for European options

As mentioned above, the conic finance theory originates from the framework of acceptability indices developed by [8], where risk measures are defined in terms of distorted expectation of zero cash-flows X. Based on the framework of indices of acceptability, we say that a risk X is acceptable or marketed if

$$\mathbb{E}^{\mathrm{Q}}[X] \geq 0, \quad \forall \quad \mathrm{Q} \in \mathcal{M}, \tag{4.1}$$

where $\mathcal{M}$ is a convex set. This convex set $\mathcal{M}$ consists of test measures under which the expected cash-flow needs to have positive expectation in order for X to be acceptable. Under a larger set $\mathcal{M}$, one has a smaller set of acceptable risks since there are more tests to be passed.

On the basis of this framework, cones of acceptability depend solely on the distribution function $F_X(x)$ of X and on the parametric family of distortion functions $\{\Psi_\gamma, \gamma \geq 0\}$, i.e., X is acceptable if its distorted expectation relative to some distortion function Ψ_γ is positive:

$$\int_{-\infty}^{+\infty} xd\Psi_\gamma(F_X(x)) \geq 0, \tag{4.2}$$

where the acceptability index γ quantifies the degree of the distortion. The higher the γ, the higher the degree of distortion, i.e., the above obtained risk-adjusted distribution functions allocate more weight to the downside (losses) than the original distribution function.

For a given distortion function $\Psi_\gamma \in \{\Psi_\gamma, \gamma \geq 0\}$ under which cash-flows are evaluated, and we can also interpret γ as the market liquidity level. The higher the γ (i.e., the more illiquid the market is), the higher the distorted probability measure. A liquidity level of zero (i.e., $\gamma = 0$) implies that there is no distortion at all, which corresponds to the perfect liquidity and hence to the complete market. In this case, the law of one price holds. As we know, the most prominent example of a family of distortion function induced by a distribution function is the WANG-transform, i.e.,

$$\Psi_\gamma(u) = \Phi(\Phi^{-1}(u) + \gamma), \tag{4.3}$$

where $\Phi(\cdot)$ denotes the standard normal cumulate distribution function. In the following, we will focus on the valuation of the bid-ask prices for European options under the WANG-transform.[†]

In conic finance theory, Madan and Cherny [19] assumed that the market is taken as a counterparty willing to accept all stochastic cash-flows X with an acceptability level γ. The ask price of a claim, $a_\gamma(X)$, is defined by the smallest price for which the cash-flow of selling the claim is acceptable at level γ for the market. Similarly, the bid price of a claim, $a_\gamma(X)$, is defined by the highest price for which the cash-flow of buying the claim is acceptable at level γ for the market. Let X be the cash-flow generated by the claim at the future maturity data T. Then, the bid and ask prices of payoff X are, respectively, determined by

$$\begin{aligned} b_\gamma(X) &= \sup\left\{b : \mathbb{E}_{\Psi_\gamma}[e^{-rT}X - b] \geq 0\right\} \\ &= e^{-rT}\mathbb{E}_{\Psi_\gamma}[X] \\ &= e^{-rT}\int_{-\infty}^{+\infty} xd\Psi_\gamma(F_X(x)) \\ &= e^{-rT}\left[-\int_{-\infty}^{0} \Psi_\gamma(F_X(x))dx + \int_{0}^{+\infty} (1 - \Psi_\gamma(F_X(x)))dx\right] \end{aligned} \tag{4.4}$$

[†]Moreover, there are many possible distortion functions, such as MINVAR, MAXVAR, MAXMINVAR and MINMAXVAR. For more related information one can refer to [8, 21]. Unfortunately, there are basically no analytical expressionss for bid and ask prices in these distortion functions. Even so, Li et al. [16] presented a numerical method consisting of Fourier cosine approximations and numerical integration to calculate the bid and ask prices for European options as long as the characteristic function was known.

and

$$
\begin{aligned}
a_\gamma(X) &= \inf\left\{a : \mathbb{E}_{\Psi_\gamma}[a - e^{-rT}X] \geq 0\right\} \\
&= -e^{-rT}\mathbb{E}_{\Psi_\gamma}[-X] \\
&= -e^{-rT}\int_{-\infty}^{+\infty} x d\Psi_\gamma(F_{-X}(x)) \\
&= -e^{-rT}\left[\int_{-\infty}^{0}(1 - \Psi_\gamma(1 - F_X(x)))dx + \int_{0}^{+\infty}\Psi_\gamma(1 - F_X(x))dx\right].
\end{aligned}
\tag{4.5}
$$

We next consider a European call option $C = (S_T - K)^+$ and a put option $P = (K - S_T)^+$ with strike price K and maturity T on the underlying S_T. Based on the above general expressions for bid and ask prices and the mixed fractional underlying asset price model (3.2), we can derive the analytical formulas of bid and ask prices for European can and put options by using WANG-transform. The results are given in the form of theorems stated below.

Theorem 1. *If the underlying asset price S_t satisfies the mfBm model (3.2) and the distortion function $\Psi_\gamma(u)$ is the WANG-transform, then the bid and ask prices of the European call option with strike price K and maturity T are, respectively, given by*

$$
b_\gamma(C) = S_0 e^{-\sqrt{\sigma^2 T + \sigma^2 T^{2H}}\gamma}\Phi(d_1 - \gamma) - Ke^{-rT}\Phi(d_2 - \gamma) \tag{4.6}
$$

and

$$
a_\gamma(C) = S_0 e^{\sqrt{\sigma^2 T + \sigma^2 T^{2H}}\gamma}\Phi(d_1 + \gamma) - Ke^{-rT}\Phi(d_2 + \gamma), \tag{4.7}
$$

where

$$
\begin{aligned}
d_1 &= \frac{\ln\frac{S_0}{K} + rT + \frac{1}{2}\sigma^2 T + \frac{1}{2}\sigma^2 T^{2H}}{\sqrt{\sigma^2 T + \sigma^2 T^{2H}}}, \\
d_2 &= d_1 - \sqrt{\sigma^2 T + \sigma^2 T^{2H}},
\end{aligned}
$$

and $\Phi(\cdot)$ denotes the standard normal cumulative distribution function.

Proof. By using the distorted expectation in Eq (4.4), we can derive the bid price of the European call option as follows:

$$
\begin{aligned}
b_\gamma(C) &= e^{-rT}\mathbb{E}^{\Psi_\gamma}\left[(S_T - K)^+\right] \\
&= e^{-rT}\int_{K}^{+\infty}(x - K)d\Psi_\gamma\left(F_{S_T}(x)\right) \\
&= \underbrace{e^{-rT}\int_{K}^{+\infty} x d\Psi_\gamma\left(F_{S_T}(x)\right)}_{A_1} - \underbrace{e^{-rT}\int_{K}^{+\infty} K d\Psi_\gamma\left(F_{S_T}(x)\right)}_{B_1}.
\end{aligned}
\tag{4.8}
$$

It is easy to see that the distribution function of S_T is

$$\begin{aligned} F_{S_T}(x) &= P(S_T \le x) \\ &= P\left(S_0 e^{rT+\sigma\left(B_T+B_T^H\right)-\frac{1}{2}\sigma^2 T-\frac{1}{2}\sigma^2 T^{2H}} \le x\right) \\ &= P\left(B_T + B_T^H \le \frac{\ln x - \ln S_0 - rT + \frac{1}{2}\sigma^2 T + \frac{1}{2}\sigma^2 T^{2H}}{\sigma}\right) \\ &= \Phi\left(\frac{\ln x - \ln S_0 - rT + \frac{1}{2}\sigma^2 T + \frac{1}{2}\sigma^2 T^{2H}}{\sqrt{\sigma^2 T + \sigma^2 T^{2H}}}\right), \quad x > 0. \end{aligned} \tag{4.9}$$

If we apply the WANG-transform to the distribution function $F_{S_T}(x)$, then the distorted distribution function $\Psi_\gamma(F_{S_T}(x))$ has the following representation:

$$\begin{aligned} \Psi_\gamma(F_{S_T}(x)) &= \Phi\left(\Phi^{-1}(F_{S_T}(x)) + \gamma\right) \\ &= \Phi\left(\frac{\ln x - \ln S_0 - rT + \frac{1}{2}\sigma^2 T + \frac{1}{2}\sigma^2 T^{2H}}{\sqrt{\sigma^2 T + \sigma^2 T^{2H}}} + \gamma\right), \quad x > 0. \end{aligned} \tag{4.10}$$

Thus, we can calculate the integrals A_1 and B_1 in Eq (4.8) as follows:[‡]

$$\begin{aligned} A_1 &= e^{-rT}\int_K^{+\infty} x d\Psi_\gamma(F_{S_T}(x)) \\ &= e^{-rT}\frac{1}{\sqrt{2\pi}}\int_K^{+\infty}\exp\left[-\frac{\left(\frac{\ln x - \ln S_0 - rT + \frac{1}{2}\sigma^2 T + \frac{1}{2}\sigma^2 T^{2H}}{\sqrt{\sigma^2 T + \sigma^2 T^{2H}}} + \gamma\right)^2}{2}\right]\frac{1}{\sqrt{\sigma^2 T + \sigma^2 T^{2H}}}dx \\ &= S_0 e^{-\sqrt{\sigma^2 T + \sigma^2 T^{2H}}\gamma}\frac{1}{\sqrt{2\pi}}\int_{\frac{\ln K - \ln S_0 - rT + \frac{1}{2}\sigma^2 T + \frac{1}{2}\sigma^2 T^{2H}}{\sqrt{\sigma^2 T + \sigma^2 T^{2H}}}+\gamma}^{+\infty} e^{-\frac{\left(y-\sqrt{\sigma^2 T + \sigma^2 T^{2H}}\right)^2}{2}}dy \\ &= S_0 e^{-\sqrt{\sigma^2 T + \sigma^2 T^{2H}}\gamma}\frac{1}{\sqrt{2\pi}}\int_{-\frac{\ln\frac{S_0}{K} + rT + \frac{1}{2}\sigma^2 T + \frac{1}{2}\sigma^2 T^{2H}}{\sqrt{\sigma^2 T + \sigma^2 T^{2H}}}+\gamma}^{+\infty} e^{-\frac{z^2}{2}}dz \end{aligned} \tag{4.11}$$

and

$$\begin{aligned} B_1 &= e^{-rT}\int_K^{+\infty} K d\Psi_\gamma(F_{S_T}(x)) \\ &= Ke^{-rT}\Phi\left(\frac{\ln x - \ln S_0 - rT + \frac{1}{2}\sigma^2 T + \frac{1}{2}\sigma^2 T^{2H}}{\sqrt{\sigma^2 T + \sigma^2 T^{2H}}} + \gamma\right)\Bigg|_K^{+\infty} \\ &= Ke^{-rT}\left[1 - \Phi\left(\frac{\ln K - \ln S_0 - rT + \frac{1}{2}\sigma^2 T + \frac{1}{2}\sigma^2 T^{2H}}{\sqrt{\sigma^2 T + \sigma^2 T^{2H}}} + \gamma\right)\right] \\ &= Ke^{-rT}\Phi\left(\frac{\ln\frac{S_0}{K} + rT - \frac{1}{2}\sigma^2 T - \frac{1}{2}\sigma^2 T^{2H}}{\sqrt{\sigma^2 T + \sigma^2 T^{2H}}} - \gamma\right). \end{aligned} \tag{4.12}$$

[‡]In the calculation of integral A_1, we have made the following transformation variables:

$$y = \frac{\ln x - \ln S_0 - rT + \frac{1}{2}\sigma^2 T + \frac{1}{2}\sigma^2 T^{2H}}{\sqrt{\sigma^2 T + \sigma^2 T^{2H}}} + \gamma, \quad z = y - \sqrt{\sigma^2 T + \sigma^2 T^{2H}}.$$

Substituting Eqs (4.11) and (4.12) to (4.8), we can obtain the bid price (4.6).

Applying similar arguments, we have the following expression for the ask price of the European call option:

$$\begin{aligned}a_\gamma(C) &= -e^{-rT}\mathbb{E}^{\Psi_\gamma}\left[-(S_T-K)^+\right]\\&= -e^{-rT}\int_{-\infty}^{0} x d\Psi_\gamma\left(1-F_{S_T}(K-x)\right)\\&= -e^{-rT}\int_{0}^{+\infty} x d\Psi_\gamma\left(1-F_{S_T}(K+x)\right)\\&= -e^{-rT}\int_{K}^{+\infty} (x-K) d\Psi_\gamma\left(1-F_{S_T}(x)\right)\\&= \underbrace{-e^{-rT}\int_{K}^{+\infty} x d\Psi_\gamma\left(1-F_{S_T}(x)\right)}_{A_2} + \underbrace{e^{-rT}\int_{K}^{+\infty} K d\Psi_\gamma\left(1-F_{S_T}(x)\right)}_{B_2}.\end{aligned} \tag{4.13}$$

Note from Eq (4.10) and WANG-transform that the distorted distribution function $\Psi_\gamma\left(1-F_{S_T}(x)\right)$ has the following representation:

$$\begin{aligned}\Psi_\gamma\left(1-F_{S_T}(x)\right) &= \Phi\left(\Phi^{-1}\left(1-F_{S_T}(x)\right)+\gamma\right)\\&= \Phi\left(\frac{-\ln x+\ln S_0+rT-\frac{1}{2}\sigma^2T-\frac{1}{2}\sigma^2T^{2H}}{\sqrt{\sigma^2T+\sigma^2T^{2H}}}+\gamma\right), \quad x>0.\end{aligned} \tag{4.14}$$

Therefore, we also can calculate the integrals A_2 and B_2 in Eq (4.13) by

$$\begin{aligned}A_2 &= -e^{-rT}\int_{K}^{+\infty} x d\Psi_\gamma\left(1-F_{S_T}(x)\right)\\&= e^{-rT}\frac{1}{\sqrt{2\pi}}\int_{K}^{+\infty}\exp\left[-\frac{\left(\frac{-\ln x+\ln S_0+rT-\frac{1}{2}\sigma^2T-\frac{1}{2}\sigma^2T^{2H}}{\sqrt{\sigma^2T+\sigma^2T^{2H}}}+\gamma\right)^2}{2}\right]\frac{1}{\sqrt{\sigma^2T+\sigma^2T^{2H}}}dx\\&= S_0e^{\sqrt{\sigma^2T+\sigma^2T^{2H}}\gamma}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\frac{\ln\frac{S_0}{K}+rT+\frac{1}{2}\sigma^2T+\frac{1}{2}\sigma^2T^{2H}}{\sqrt{\sigma^2T+\sigma^2T^{2H}}}+\gamma} e^{-\frac{y^2}{2}}dy,\end{aligned} \tag{4.15}$$

and

$$\begin{aligned}B_2 &= e^{-rT}\int_{K}^{+\infty} K d\Psi_\gamma\left(1-F_{S_T}(x)\right)\\&= Ke^{-rT}\Phi\left(\frac{-\ln x+\ln S_0+rT-\frac{1}{2}\sigma^2T-\frac{1}{2}\sigma^2T^{2H}}{\sqrt{\sigma^2T+\sigma^2T^{2H}}}+\gamma\right)\Bigg|_K^{+\infty}\\&= -Ke^{-rT}\Phi\left(\frac{\ln\frac{S_0}{K}+rT-\frac{1}{2}\sigma^2T-\frac{1}{2}\sigma^2T^{2H}}{\sqrt{\sigma^2T+\sigma^2T^{2H}}}+\gamma\right),\end{aligned} \tag{4.16}$$

Finally, by combining Eqs (4.13), (4.15) and (4.16), we can get the ask price (4.7). □

Theorem 2. *If the underlying asset price S_t satisfies the mfBm model (3.2) and the distortion function $\Psi_\gamma(u)$ is the WANG-transform, then the bid and ask prices of the European put option with strike price K and maturity T are, respectively, given by*

$$b_\gamma(P) = Ke^{-rT}\Phi(-d_2-\gamma) - S_0 e^{\sqrt{\sigma^2 T+\sigma^2 T^{2H}}\gamma}\Phi(-d_1-\gamma) \tag{4.17}$$

and

$$a_\gamma(P) = Ke^{-rT}\Phi(-d_2+\gamma) - S_0 e^{-\sqrt{\sigma^2 T+\sigma^2 T^{2H}}\gamma}\Phi(-d_1+\gamma), \tag{4.18}$$

where

$$d_1 = \frac{\ln\frac{S_0}{K} + rT + \frac{1}{2}\sigma^2 T + \frac{1}{2}\sigma^2 T^{2H}}{\sqrt{\sigma^2 T + \sigma^2 T^{2H}}},$$
$$d_2 = d_1 - \sqrt{\sigma^2 T + \sigma^2 T^{2H}},$$

and $\Phi(\cdot)$ denotes the standard normal cumulative distribution function.

Proof. Applying the arguments given in Theorem 1, we have the following expression for the bid price of the European put option:

$$\begin{aligned} b_\gamma(P) &= e^{-rT}\mathbb{E}^{\Psi_\gamma}\left[(K-S_T)^+\right] \\ &= e^{-rT}\int_0^{+\infty} x d\Psi_\gamma\left(1-F_{S_T}(K-x)\right) \\ &= -e^{-rT}\int_0^{K}(K-x)d\Psi_\gamma\left(1-F_{S_T}(x)\right) \\ &= \underbrace{-e^{-rT}\int_0^K Kd\Psi_\gamma\left(1-F_{S_T}(x)\right)}_{A_3} + \underbrace{e^{-rT}\int_0^K xd\Psi_\gamma\left(1-F_{S_T}(x)\right)}_{B_3}. \end{aligned} \tag{4.19}$$

Note from Eq (4.14) that the integrals A_3 and B_3 in Eq (28) can be, respectively, calculated by:

$$\begin{aligned} A_3 &= -e^{-rT}\int_0^K Kd\Phi\left(\frac{-\ln x + \ln S_0 + rT - \frac{1}{2}\sigma^2 T - \frac{1}{2}\sigma^2 T^{2H}}{\sqrt{\sigma^2 T+\sigma^2 T^{2H}}} + \gamma\right) \\ &= -Ke^{-rT}\Phi\left(\frac{-\ln x + \ln S_0 + rT - \frac{1}{2}\sigma^2 T - \frac{1}{2}\sigma^2 T^{2H}}{\sqrt{\sigma^2 T+\sigma^2 T^{2H}}} + \gamma\right)\Bigg|_0^K \\ &= Ke^{-rT}\Phi\left(-\frac{\ln\frac{S_0}{K} + rT - \frac{1}{2}\sigma^2 T - \frac{1}{2}\sigma^2 T^{2H}}{\sqrt{\sigma^2 T+\sigma^2 T^{2H}}} - \gamma\right) \end{aligned} \tag{4.20}$$

and

$$\begin{aligned} B_3 &= e^{-rT}\int_0^K xd\Psi_\gamma\left(1-F_{S_T}(x)\right) \\ &= e^{-rT}\frac{1}{\sqrt{2\pi}}\int_0^K \exp\left[-\frac{\left(\frac{-\ln x+\ln S_0+rT-\frac{1}{2}\sigma^2T-\frac{1}{2}\sigma^2T^{2H}}{\sqrt{\sigma^2T+\sigma^2T^{2H}}}+\gamma\right)^2}{2}\right]\frac{1}{-\sqrt{\sigma^2 T+\sigma^2 T^{2H}}}dx \\ &= -S_0 e^{\sqrt{\sigma^2T+\sigma^2T^{2H}}\gamma}\frac{1}{\sqrt{2\pi}}\int_{\frac{\ln\frac{S_0}{K}+rT+\frac{1}{2}\sigma^2T+\frac{1}{2}\sigma^2T^{2H}}{\sqrt{\sigma^2T+\sigma^2T^{2H}}}+\gamma}^{+\infty} e^{-\frac{y^2}{2}}dy. \end{aligned} \tag{4.21}$$

Substituting Eqs (4.20) and (4.21) to (4.19), we can obtain the bid price (4.17).

For the ask price of the European put option, we have

$$\begin{aligned} a_\gamma(P) &= -e^{-rT}\mathbb{E}^{\Psi_\gamma}\left[-(K-S_T)^+\right] \\ &= -e^{-rT}\int_{-\infty}^{0} x d\Psi_\gamma\left(F_{S_T}(K+x)\right) \\ &= -e^{-rT}\int_{0}^{\infty} x d\Psi_\gamma\left(F_{S_T}(K-x)\right) \\ &= e^{-rT}\int_{0}^{K} (K-x) d\Psi_\gamma\left(F_{S_T}(x)\right) \\ &= \underbrace{e^{-rT}\int_{0}^{K} K d\Psi_\gamma\left(F_{S_T}(x)\right)}_{A_4} - \underbrace{e^{-rT}\int_{0}^{K} x d\Psi_\gamma\left(F_{S_T}(x)\right)}_{B_4}. \end{aligned} \tag{4.22}$$

In a similar way, the integrals A_4 and B_4 in Eq (4.22) can be calculated by:

$$\begin{aligned} A_4 &= e^{-rT}\int_0^K K d\Phi\left(\frac{\ln x - \ln S_0 - rT + \frac{1}{2}\sigma^2 T + \frac{1}{2}\sigma^2 T^{2H}}{\sqrt{\sigma^2 T + \sigma^2 T^{2H}}} + \gamma\right) \\ &= Ke^{-rT}\Phi\left(\frac{\ln x - \ln S_0 - rT + \frac{1}{2}\sigma^2 T + \frac{1}{2}\sigma^2 T^{2H}}{\sqrt{\sigma^2 T + \sigma^2 T^{2H}}} + \gamma\right)\Bigg|_0^K \\ &= Ke^{-rT}\Phi\left(-\frac{\ln\frac{S_0}{K} + rT - \frac{1}{2}\sigma^2 T - \frac{1}{2}\sigma^2 T^{2H}}{\sqrt{\sigma^2 T + \sigma^2 T^{2H}}} + \gamma\right) \end{aligned} \tag{4.23}$$

and

$$\begin{aligned} B_4 &= e^{-rT}\int_0^K x d\Psi_\gamma\left(F_{S_T}(x)\right) \\ &= e^{-rT}\frac{1}{\sqrt{2\pi}}\int_0^K \exp\left[-\frac{\left(\frac{\ln x - \ln S_0 - rT + \frac{1}{2}\sigma^2 T + \frac{1}{2}\sigma^2 T^{2H}}{\sqrt{\sigma^2 T + \sigma^2 T^{2H}}} + \gamma\right)^2}{2}\right]\frac{1}{\sqrt{\sigma^2 T + \sigma^2 T^{2H}}}dx \\ &= S_0 e^{-\sqrt{\sigma^2 T + \sigma^2 T^{2H}}\gamma}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-\frac{\ln\frac{S_0}{K} + rT + \frac{1}{2}\sigma^2 T + \frac{1}{2}\sigma^2 T^{2H}}{\sqrt{\sigma^2 T + \sigma^2 T^{2H}}} + \gamma} e^{-\frac{y^2}{2}}dy. \end{aligned} \tag{4.24}$$

Finally, by combining Eqs (4.22)-(4.24), we can get the ask price (4.18). □

Remark: Note that the resulting bid and ask prices depend on the parameter γ. In the special case, where $\gamma = 0$, bid price equals ask price and we are again in the classical one-price framework. In other words, a value of γ equal to zero corresponds to a bid-ask spread of zero.

5. Numerical analysis

In this section, we will present the numerical results for the conic option pricing. The model input parameters values for this numerical analysis are assumed to be $S_0 = 100$, $T = 1.5$, $r = 0.05$, $\sigma = 0.2$.

Table 1. Bid and ask prices of European call option with respect to different values of strike price K, Hurst index H and market liquidity level γ. *Notes*: The values of model input parameters are assumed to be $S_0 = 100$, $T = 1.5$, $r = 0.05$, $\sigma = 0.2$. The bid-ask spreads (spread for short) are also presented for different groups.

Strike price K			70	80	90	100	110	120	130
H=0.76	$\gamma = 0.01$	bid	36.4336	29.1351	22.8681	17.6670	13.4719	10.1658	7.6083
		ask	37.1025	29.7508	23.4177	18.1437	13.8752	10.4999	7.8802
		spread	0.6689	0.6157	0.5495	0.4767	0.4034	0.3341	0.2719
	$\gamma = 0.05$	bid	35.1174	27.9279	21.7950	16.7401	12.6912	9.5221	7.0868
		ask	38.4620	31.0064	24.5428	19.1241	14.7084	11.1932	8.4468
		spread	3.3446	3.0785	2.7478	2.3840	2.0173	1.6711	1.3600
	$\gamma = 0.1$	bid	33.5125	26.4640	20.5019	15.6307	11.7631	8.7621	6.4753
		ask	40.2021	32.6214	25.9986	20.4005	15.8005	12.1079	9.1995
		spread	6.6895	6.1574	5.4967	4.7699	4.0374	3.3458	2.7242
H=0.86	$\gamma = 0.01$	bid	36.5626	29.3358	23.1315	17.9742	13.8006	10.4956	7.9231
		ask	37.2443	29.9631	23.6921	18.4618	14.2149	10.8406	8.2057
		spread	0.6817	0.6273	0.5606	0.4876	0.4143	0.3450	0.2826
	$\gamma = 0.05$	bid	35.2219	28.1063	22.0372	17.0264	12.9989	9.8308	7.3809
		ask	38.6303	31.2429	24.8403	19.4648	15.0708	11.5566	8.7947
		spread	3.4084	3.1366	2.8030	2.4384	2.0719	1.7258	1.4138
	$\gamma = 0.1$	bid	33.5883	26.6163	20.7193	15.8923	12.0460	9.0460	6.7451
		ask	40.4054	32.8900	26.3265	20.7713	16.1929	12.5013	9.5770
		spread	6.8171	6.2737	5.6073	4.8790	4.1469	3.4553	2.8319
H=0.96	$\gamma = 0.01$	bid	36.7038	29.5514	23.4119	18.2997	14.1486	10.8453	8.2583
		ask	37.3989	30.1910	23.9842	18.7989	14.5746	11.2020	8.5524
		spread	0.6951	0.6396	0.5723	0.4992	0.4259	0.3567	0.2941
	$\gamma = 0.05$	bid	35.3372	28.2984	22.2951	17.3296	13.3245	10.1581	7.6941
		ask	38.8129	31.4965	25.1568	19.8261	15.4547	11.9422	9.1652
		spread	3.4757	3.1981	2.8617	2.4964	2.1302	1.7840	1.4712
	$\gamma = 0.1$	bid	33.6733	26.7808	20.9507	16.1694	12.3453	9.3470	7.0323
		ask	40.6250	33.1778	26.6756	21.1645	16.6087	12.9189	9.9791
		spread	6.9517	6.3969	5.7248	4.9951	4.2635	3.5719	2.9468

Table 2. Bid and ask prices of European put option with respect to different values of strike price K, Hurst index H and market liquidity level γ. *Notes*: The values of model input parameters are assumed to be $S_0 = 100$, $T = 1.5$, $r = 0.05$, $\sigma = 0.2$. The bid-ask spreads (spread for short) are also presented for different groups.

Strike price K			70	80	90	100	110	120	130
H=0.76	$\gamma = 0.01$	bid	1.6777	3.6034	6.5477	10.5512	15.5602	21.4623	28.1200
		ask	1.7412	3.7201	6.7306	10.8069	15.8892	21.8605	28.5804
		spread	0.0634	0.1167	0.1828	0.2556	0.3290	0.3982	0.4605
	$\gamma = 0.05$	bid	1.5563	3.3781	6.1920	10.0507	14.9125	20.6747	27.2058
		ask	1.8736	3.9616	7.1061	11.3287	16.5572	22.6655	29.5077
		spread	0.3173	0.5835	0.9141	1.2780	1.6447	1.9908	2.3019
	$\gamma = 0.1$	bid	1.4145	3.1113	5.7659	9.4453	14.1227	19.7075	26.0765
		ask	2.0501	4.2790	7.5943	12.0005	17.4104	23.6868	30.6774
		spread	0.6356	1.1677	1.8284	2.5552	3.2877	3.9793	4.6009
H=0.86	$\gamma = 0.01$	bid	1.8110	3.8072	6.8137	10.8608	15.8914	21.7945	28.4370
		ask	1.8786	3.9292	7.0024	11.1225	16.2263	22.1987	28.9036
		spread	0.0676	0.1220	0.1887	0.2616	0.3350	0.4042	0.4666
	$\gamma = 0.05$	bid	1.6816	3.5716	6.4464	10.3484	15.2318	20.9950	27.5106
		ask	2.0196	4.1814	7.3898	11.6564	16.9063	23.0157	29.8432
		spread	0.3380	0.6099	0.9434	1.3080	1.6745	2.0207	2.3326
	$\gamma = 0.1$	bid	1.5302	3.2922	6.0062	9.7284	14.4274	20.0133	26.3663
		ask	2.2072	4.5127	7.8931	12.3436	17.7747	24.0522	31.0287
		spread	0.6771	1.2205	1.8869	2.6152	3.3473	4.0389	4.6623
H=0.96	$\gamma = 0.01$	bid	1.9566	4.0262	7.0968	11.1890	16.2420	22.1469	28.7747
		ask	2.0286	4.1538	7.2916	11.4569	16.5832	22.5574	29.2478
		spread	0.0720	0.1275	0.1948	0.2679	0.3412	0.4105	0.4731
	$\gamma = 0.05$	bid	1.8185	3.7796	6.7174	10.6640	15.5701	21.3350	27.8355
		ask	2.1788	4.4174	7.6916	12.0036	17.2759	23.3869	30.2003
		spread	0.3603	0.6378	0.9742	1.3395	1.7058	2.0519	2.3648
	$\gamma = 0.1$	bid	1.6568	3.4870	6.2623	10.0286	14.7503	20.3379	26.6755
		ask	2.3784	4.7634	8.2108	12.7068	18.1601	24.4393	31.4021
		spread	0.7216	1.2764	1.9485	2.6782	3.4098	4.1014	4.7265

Since that the main objective of this paper is to discuss the pricing problem of European option in a two-price economy, we further investigate the bid and ask prices of European call and put options with respect to Hurst index H and market liquidity level γ. By utilizing the analytical formulas (4.6), (4.7), (4.17) and (4.18), Tables 1 and 2 reports the bid and ask prices of European call and put options with respect to different values of strike price K, Hurst index H and market liquidity level γ.

From Tables 1 and 2, we can be clearly observed that the bid and ask prices of European call and put option are decreasing and increasing with respect to strike price K, respectively, which are consistent with the payment functions of the European options. We also can see that the bid-ask spread is increasing with market liquidity level γ. As mentioned above, the market liquidity goes hand in hand with bid and ask spreads and highly liquid asset have a small spreads. In other words, the higher γ, the wider the bid-ask spread and hence less the liquidity. We can also find that the bid-ask spread is decreasing with respect to strike price K in theory , which is consistent with the empirical results of Leippold and Scharer [15].

In addition, it is worth noting that the bid-ask spread is increasing with respect to Hurst index H, which means that the stronger the long memory of underlying asset is, the higher bid-ask spread is. In spite of this, our numerical results presented here can at least illustrate that the valuation of bid and ask prices for European options considering the long-range dependence of underlying asset price may offer as a good competitor of the classical Black-Scholes [2] model, especially for some emerging markets.

6. Conclusions

In a two-price economy, we study the valuation of the bid and ask prices for the European options in this paper. Considering the long range dependence of the underlying asset returns in real markets, we assume the dynamic of the underlying asset price follows a mixed fractional Brownian motion with Hurst index $H > 3/4$. In fact, the Hurst exponent $H > 3/4$ ensures that the financial market does not allow arbitrage opportunity. Within the framework of conic finance, we then derive the closed-form solutions of the bid and ask prices for European call and put options by using WANG-transform as a distortion function. Moreover, numerical experiment is performed to illustrate the effects of the Hurst index and market liquidity level on bid and ask prices.

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Conflict of interest

The authors declare no conflict of interest in this paper.

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