AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 6(7): 7187–7198.
DOI: 10.3934/math.2021421
Received: 20 September 2020
Accepted: 25 April 2021
Published: 28 April 2021

Research article

Asymptotic behavior of the unique solution for a fractional Kirchhoff problem with singularity

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Abstract: In this paper, we consider the following fractional Kirchhoff problem with singularity

$$\begin{cases} \Big(1 + b\displaystyle\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|u(x)-u(y)|^2}{|x-y|^{3+2s}}\mathrm{d}x\mathrm{d}y\Big)(-\Delta)^s u + V(x)u = f(x)u^{-\gamma}, & x \in \mathbb{R}^3, \\ u > 0, & x \in \mathbb{R}^3, \end{cases}$$

where $(-\Delta)^s$ is the fractional Laplacian with $0 < s < 1$, $b \geq 0$ is a constant and $0 < \gamma < 1$. Under certain assumptions on V and f, we show the existence and uniqueness of positive solution u_b by using variational method. We also give a convergence property of u_b as $b \to 0$, where b is regarded as a positive parameter.

Keywords: fractional Kirchhoff problem; singularity; uniqueness; variational method; asymptotic behavior
Mathematics Subject Classification: 35A15, 35R11

1. Introduction

Nonlinear equations involving fractional powers of the Laplacian have attracted increasing attentions in recent years. The fractional Laplacian is the infinitesimal generator of Lévy stable diffusion process and arises in anomalous diffusion in plasma, population dynamics, geophysical fluid dynamics, flames propagation, chemical reactions in liquids and American options in finance, see [2] for instance. In this paper, we consider the following fractional Kirchhoff problem

$$\begin{cases} \Big(1+b\displaystyle\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|u(x)-u(y)|^2}{|x-y|^{3+2s}}\mathrm{d}x\mathrm{d}y\Big)(-\Delta)^s u+V(x)u=f(x)u^{-\gamma}, & x\in\mathbb{R}^3,\\ u>0, & x\in\mathbb{R}^3, \end{cases} \tag{P_b}$$

where $b\geq 0$ is a constant and $0<\gamma<1$. The fractional Laplacian operator $(-\Delta)^s$ in $\mathbb{R}^3$ is defined by

$$(-\Delta)^s u(x)=C(s)P.V.\int_{\mathbb{R}^3}\frac{u(x)-u(y)}{|x-y|^{3+2s}}\mathrm{d}y,\ \ u\in\mathbb{S}(\mathbb{R}^3),$$

where $P.V.$ stands for the Cauchy principal value, $C(s)$ is a normalized constant, $\mathbb{S}(\mathbb{R}^3)$ is the Schwartz space of rapidly decaying function. Throughout the paper, we suppose V and f satisfy:

(V_1) $V\in C(\mathbb{R}^3)$ satisfies $\inf_{x\in\mathbb{R}^3}V(x)>V_0>0$, where V_0 is a constant.

(V_2) meas$\{x\in R^3\ :\ -\infty<V(x)\leq h\}<+\infty$ for all $h\in\mathbb{R}$.

(f_1) $f\in L^{\frac{2}{1+\gamma}}(\mathbb{R}^3)$ is a nonnegative function.

The motivation for studying problem (P_b) comes from Kirchhoff equation of the form

$$-(a+b\int_{\Omega}|\nabla u|^2\mathrm{d}x)\Delta u=f(x,u),\quad x\in\Omega, \tag{1.1}$$

where $\Omega\subset\mathbb{R}^n$ is a bounded domain, $a>0, b\geq 0$ and u satisfies some boundary conditions. The problem (1.1) is related to the stationary analogue of the equation

$$\rho\frac{\partial^2 u}{\partial^2 t}-\Big(\frac{P_0}{h}+\frac{F}{2L}\int_0^L\Big|\frac{\partial u}{\partial x}\Big|^2\mathrm{d}x\Big)\frac{\partial^2 u}{\partial^2 x}=f(x,u), \tag{1.2}$$

which was introduced by Kirchhoff [14] in 1883. This equation is an extension of the classical d'Alembert's wave equation by considering the effects of the changes in the length of the string during the vibrations. For the meanings of parameters in (1.2), one can refer to [14]. After pioneering work of Lions [23], the Kirchhoff type equation began to receive the attention of many researchers.

Recently, many scholars pay attentions to the fractional Kirchhoff problem which was first studied by Fiscella and Valdinoci [9], where they proposed the following stationary Kirchhoff variational model in bounded regular domains of $\mathbb{R}^n$ ($n>2s$)

$$\begin{cases} M\Big(\displaystyle\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\mathrm{d}x\mathrm{d}y\Big)(-\Delta)^s u=\lambda f(x,u)+|u|^{2_s^*-2}u, & x\in\Omega,\\ u=0, & x\in\mathbb{R}^n\setminus\Omega, \end{cases} \tag{1.3}$$

with $2_s^*=\frac{2n}{n-2s}$ and proved the existence of nonnegative solutions when f and the Kirchhoff function M satisfy some suitable conditions. However, Fiscella and Valdinoci [9] only investigated the non-degenerate case, i.e. there exists $m_0>0$ such that Kirchhoff function M satisfies $M(t)\geq m_0=M(0)$ for all $t\in\mathbb{R}^+$, see also [5, 11, 34]. Autuori et al. [3] further considered the existence and the asymptotic behavior of non-negative solutions to problem (1.3) under different assumption that the Kirchhoff function M can be zero at zero, that is, the problem is degenerate case. Since then, several papers have been devoted to stationary fractional Kirchhoff problems in the degenerate case. We refer, e.g., to [8, 12] for degenerate problems in bounded regular domains of $\mathbb{R}^n$, to [1, 7, 13, 30] in all $\mathbb{R}^n$, and to [6, 10, 20] for quasilinear Kirchhoff problems involving the fractional p-Laplacian or Kirchhoff-Schrödinger-Poisson system and so on. In particular, Fiscella [11, 12] provided the existence of two solutions for a fractional Kirchhoff problem involving weak singularity (i.e. $0 < \gamma < 1$) and a critical nonlinearity on a bounded domain. We [35, 36] obtained the existence, uniqueness and asymptotical behavior of solutions to a Choquard equation and Schrödinger-Poisson system with singularity in $\mathbb{R}^3$, respectively.

In the local setting ($s = 1$), problem (P_b) is related to the following singular Kirchhoff type problem which was first considered by Liu and Sun [25]

$$\begin{cases} -(a + b\int_\Omega |\nabla u|^2 \mathrm{d}x)\Delta u = \lambda g(x)\dfrac{u^p}{|x|^\delta} + h(x)u^{-\gamma}, & x \in \Omega, \\ u > 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.4}$$

where Ω is a smooth bounded domain in $\mathbb{R}^3$, $0 \leq \delta < 1$ and $0 < \gamma < 1$. When $\lambda > 0$ is small, Liu and Sun [25] obtained two positive solutions for problem (1.4) with $3 < p < 5 - 2\delta$ and $g, h \in C(\overline{\Omega})$ are nontrivial nonnegative functions. Later, by the variational method and perturbation method, Lei et al. [15] obtained two positive solutions for problem (1.4) with $\delta = 0$, $p = 5$ i.e. singular Kirchhoff type equation with critical exponent. Liao et al. [21] investigated the existence and multiplicity of positive solutions for problem (1.4) with $\delta = 0$, $p = 3$. Liu et al. [26] studied the existence and multiplicity of positive solutions for the Kirchhoff type problem with singular and critical nonlinearities in dimension four. Liao et al. [22] obtained a uniqueness result of a class of singular Kirchhoff type problem. When $p = 3, \lambda = 1$ and $g \geq 0$ or g changes sign in Ω, Li et al. [18] showed the existence and multiplicity of positive solutions problem (1.4). By the perturbation method, variational method and some analysis techniques, Liu et al. [24], Tang et al. [32], Lei and Liao [16] established a multiplicity theorem for singular Kirchhoff type problem with critical Sobolev exponent, Hardy-Sobolev critical exponent and asymptotically linear nonlinearities, respectively. Mu and Lu [27], Li et al. [17] and Zhang [38] studied the existence, uniqueness and multiple results to singular Schrödinger-Kirchhoff-Poisson system. Li et al. [19], Sun and Tan [31], Zhang [39], Wang et al. [33] and we [37] established a necessary and sufficient condition on the existence of positive solutions for Kirchhoff problem, Kirchhoff-Schrödinger-Poisson system and fractional Kirchhoff problem with strong singularity (i.e. $\gamma > 1$) on a bounded domain Ω or in $\mathbb{R}^3$, respectively.

Motivated by the above results and Barilla et al. [4], we are concerned with the existence and convergence property of positive solutions for problem (P_b) in this paper. Before stating our main results, we first collect some basic results of fractional Sobolev spaces. In view of the presence of potential function $V(x)$, we will work in the space

$$E = \{u \in \mathcal{D}^{s,2}(\mathbb{R}^3) \,:\, \|u\|_E < +\infty\},$$

equipped with inner product and the norm

$$(u, v)_E = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{3+2s}} \mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}^3} V(x)u(x)v(x)\mathrm{d}x,\ \|u\|_E = (u, u)_E^{1/2}.$$

Here $\mathcal{D}^{s,2}(\mathbb{R}^3)$ is the homogeneous fractional Sobolev space as the completion of $C_0^\infty(\mathbb{R}^3)$ under the norm

$$\|u\|_{\mathcal{D}^{s,2}(\mathbb{R}^3)} = \left(\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u(x) - u(y)|^2}{|x - y|^{3+2s}} \mathrm{d}x\mathrm{d}y\right)^{1/2} \doteq [u]_s.$$

Moreover, by virtue of Proposition 3.4 and Proposition 3.6 in [29], we also have

$$\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 \mathrm{d}x = \frac{C(s)}{2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u(x)-u(y)|^2}{|x-y|^{3+2s}} \,\mathrm{d}x\,\mathrm{d}y.$$

Without loss of generality, we assume that $C(s) = 2$.

The energy functional corresponding to problem (P_b) given by

$$I_b(u) = \frac{1}{2}\|u\|_E^2 + \frac{b}{4}\Big(\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 \mathrm{d}x\Big)^2 - \frac{1}{1-\gamma}\int_{\mathbb{R}^3} f(x)|u|^{1-\gamma}\mathrm{d}x, \tag{1.5}$$

and a function $u \in E$ is called a solution of problem (P_λ) if $u > 0$ in $\mathbb{R}^3$ and for every $v \in E$,

$$(u, v)_E + b\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 \mathrm{d}x \int_{\mathbb{R}^3} (-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} v \,\mathrm{d}x - \int_{\mathbb{R}^3} f(x) u^{-\gamma} v \mathrm{d}x = 0. \tag{1.6}$$

To the best of our knowledge, there are no results on the existence of positive solutions for fractional Kirchhoff problem with weak singularity on unbounded domains. Here we need to overcome the lack of compactness as well as the non-differentiability of the functional I_b on E and indirect availability of critical point theory due to the presence of singular term. By variational method, we obtain the following existence and uniqueness of positive solution and the asymptotic behavior of solutions with respect to the parameter b.

Theorem 1. *Let $b \geq 0$ and $0 < \gamma < 1$. Assume (V_1), (V_2) and (f_1) hold. Then problem (P_b) admits a unique positive solution $u_b \in E$.*

Theorem 2. *Let $0 < \gamma < 1$. Assume (V_1), (V_2) and (f_1) hold. For every vanishing sequence $\{b_n\}$, let u_{b_n} be the unique positive solution to problem (P_b) provided by Theorem 1. Then, u_{b_n} converge to w_0 in E, where w_0 is the unique positive solution to problem*

$$\begin{cases} (-\Delta)^s u + V(x)u = f(x)u^{-\gamma}, & x \in \mathbb{R}^3, \\ u > 0, & x \in \mathbb{R}^3. \end{cases} \tag{P_0}$$

2. Preliminary results

Throughout the paper, we use the following notations.

- $L^p(\mathbb{R}^3)$ is a Lebesgue space whose norm is denoted by $\|u\|_p = (\int_{\mathbb{R}^3} |u|^p \mathrm{d}x)^{\frac{1}{p}}$.
- For any $\alpha \in (0, 1)$, $2^*_\alpha = \frac{6}{3-2\alpha}$ is the fractional critical exponent in dimension 3.
- $\to$ denotes the strong convergence and $\rightharpoonup$ denotes the weak convergence.
- $u^+ = \max\{u, 0\}$ and $u^- = \max\{-u, 0\}$ for any function u.
- C and C_i $(i = 1, 2, \ldots)$ denotes various positive constants, which may vary from line to line.

In this section, we mainly establish some preliminaries. Using conditions (V_1) and (V_2), we can obtain the following continuous or compact embedding theorem (see [20], Lemma 2.2).

Lemma 1. *Let $0 < s < 1$ and suppose that (V_1) and (V_2) hold. If $p \in [2, 2^*_s]$, then the embedding $E \hookrightarrow L^p(\mathbb{R}^3)$ is continuous and so there exists a constant $C_p > 0$ such that $\|u\|_p \leq C_p\|u\|_E$ for all $u \in E$. If $p \in [2, 2^*_s)$, then the embedding $E \hookrightarrow L^p(\mathbb{R}^3)$ is compact.*

We also give the following lemma which plays an important role in the proofs of our main results.

Lemma 2. *Let $b \geq 0$ and $0 < \gamma < 1$. Assume (V_1), (V_2) and (f_1) hold. The functional I_b defined in* (1.5) *attains its minimum in E, that is, there exists $u_b \in E$ such that $I_b(u_b) = m_b = \min_E I_b < 0$.*

Proof. For $u \in E$, by Hölder's inequality, Lemma 1 and $0 < \gamma < 1$, we have

$$\begin{aligned} I_b(u) &\geq \frac{1}{2}\|u\|_E^2 - \frac{1}{1-\gamma}\int_{\mathbb{R}^3} f(x)|u|^{1-\gamma}\mathrm{d}x \\ &\geq \frac{1}{2}\|u\|_E^2 - \frac{1}{1-\gamma}\|f\|_{\frac{2}{1+\gamma}}\Big[\int_{\mathbb{R}^3}|u|^2\mathrm{d}x\Big]^{\frac{1-\gamma}{2}} \\ &\geq \frac{1}{2}\|u\|_E^2 - \frac{1}{1-\gamma}\|f\|_{\frac{2}{1+\gamma}} C_2^{1-\gamma}\|u\|_E^{1-\gamma}, \end{aligned} \tag{2.1}$$

which implies that I_b is coercive and bounded from below on E for any $b \geq 0$. Therefore $m_b = \inf_E I_b$ is well defined. For $\eta > 0$ and given $u \in E \setminus \{0\}$, one has

$$I_b(\eta u) = \frac{\eta^2}{2}\|u\|_E^2 + \frac{b\eta^4}{4}\Big(\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u|^2\mathrm{d}x\Big)^2 - \frac{\eta^{1-\gamma}}{1-\gamma}\int_{\mathbb{R}^3} f(x)|u|^{1-\gamma}\mathrm{d}x,$$

so $I_b(\eta u) < 0$ for $\eta > 0$ small enough, then $m_b = \inf_E I_b < 0$ and there exists a minimizing sequence $\{u_n\} \subset E$ such that $\lim_{n\to\infty} I_b(u_n) = m_b < 0$. Since $I_b(|u_n|) \leq I_b(u_n)$, we could assume that $u_n \geq 0$. The coerciveness of I_b on E shows that $\{u_n\}$ is bounded in E. Going if necessary to a subsequence, we can assume that

$$\begin{array}{ll} u_n \rightharpoonup u_b, & \text{in } E, \\ u_n \to u_b, & \text{in } L^p(\mathbb{R}^3),\ p \in [2, 2_s^*), \\ u_n \to u_b, & \text{a.e. in } \mathbb{R}^3. \end{array} \tag{2.2}$$

Since $0 < \gamma < 1$ and $f \in L^{\frac{2}{1+\gamma}}(\mathbb{R}^3)$ is a nonnegative function, by Hölder's inequality, we have

$$\begin{aligned} \left|\int_{\mathbb{R}^3} f(x)|u_n|^{1-\gamma}\mathrm{d}x - \int_{\mathbb{R}^3} f(x)|u_b|^{1-\gamma}\mathrm{d}x\right| &\leq \int_{\mathbb{R}^3} f(x)\left||u_n|^{1-\gamma} - |u_b|^{1-\gamma}\right|\mathrm{d}x \\ &\leq \int_{\mathbb{R}^3} f(x)\,|u_n - u_b|^{1-\gamma}\,\mathrm{d}x \\ &\leq \|f\|_{\frac{2}{1+\gamma}}\Big[\int_{\mathbb{R}^3}|u_n - u_b|^2\,\mathrm{d}x\Big]^{\frac{1-\gamma}{2}}, \end{aligned}$$

which yields, by (2.2),

$$\lim_{n\to\infty}\int_{\mathbb{R}^3} f(x)|u_n|^{1-\gamma}\mathrm{d}x = \int_{\mathbb{R}^3} f(x)|u_b|^{1-\gamma}\mathrm{d}x. \tag{2.3}$$

Then by the weakly lower semi-continuity of the norm and (2.3), we have

$$\begin{aligned} I_b(u_b) &= \frac{1}{2}\|u_b\|_E^2 + \frac{b}{4}\Big(\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u_b|^2\mathrm{d}x\Big)^2 - \frac{1}{1-\gamma}\int_{\mathbb{R}^3} f(x)|u_b|^{1-\gamma}\mathrm{d}x \\ &\leq \liminf_{n\to\infty}\left[\frac{1}{2}\|u_n\|_E^2 + \frac{b}{4}\Big(\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u_n|^2\mathrm{d}x\Big)^2 - \frac{1}{1-\gamma}\int_{\mathbb{R}^3} f(x)|u_n|^{1-\gamma}\mathrm{d}x\right] \\ &= \liminf_{n\to\infty} I_b(u_n) = m_b. \end{aligned}$$

On the other hand, $I_b(u_b) \geq m_b$, so $I_b(u_b) = m_b < 0$. Therefore, u_b is a global minimum for I_b in E and this ends the proof of Lemma 2. □

3. Proofs of main results

Proof of Theorem 1.1. The proof will be complete in three steps.
Step 1. *For any* $0 \le \psi \in E$,

$$(u_b, \psi)_E + b\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u_b|^2 \mathrm{d}x \int_{\mathbb{R}^3} (-\Delta)^{\frac{s}{2}} u_b (-\Delta)^{\frac{s}{2}} \psi \, \mathrm{d}x - \int_{\mathbb{R}^3} f(x) u_b^{-\gamma} \psi \mathrm{d}x \ge 0.$$

According to Lemma 2, $u_b \ge 0$ and $u_b \not\equiv 0$. For $0 \le \psi \in E$ and $\eta \ge 0$ satisfying $u_b + \eta\psi \in E$, since $I_b(u_b) = m_b = \min_E I_b$, one has

$$\begin{aligned} 0 &\le I_b(u_b + \eta\psi) - I_b(u_b) \\ &= \frac{1}{2}[\|u_b + \eta\psi\|_E^2 - \|u_b\|_E^2] + \frac{b}{4}\Big[\Big(\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}}(u_b + \eta\psi)|^2 \mathrm{d}x\Big)^2 - \Big(\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u_b|^2 \mathrm{d}x\Big)^2\Big] \\ &\quad - \frac{1}{1-\gamma}\int_{\mathbb{R}^3} f(x)[(u_b + \eta\psi)^{1-\gamma} - u_b^{1-\gamma}]\mathrm{d}x. \end{aligned}$$

Thus,

$$\begin{aligned} &\frac{1}{1-\gamma}\int_{\mathbb{R}^3} f(x)[(u_b + \eta\psi)^{1-\gamma} - u_b^{1-\gamma}]\mathrm{d}x \\ \le\ &\frac{1}{2}[\|u_b + \eta\psi\|_E^2 - \|u_b\|_E^2] + \frac{b}{4}\Big[\Big(\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}}(u_b + \eta\psi)|^2 \mathrm{d}x\Big)^2 - \Big(\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u_b|^2 \mathrm{d}x\Big)^2\Big]. \end{aligned} \tag{3.1}$$

Due to $\gamma \in (0,1)$ and $f(x)$ is nonnegative, we further obtain

$$f(x)[(u_b + \eta\psi)^{1-\gamma} - u_b^{1-\gamma}] \ge 0, \ \ \forall x \in \mathbb{R}^3.$$

Then

$$\liminf_{\eta\to 0^+} \int_{\mathbb{R}^3} \frac{f(x)[(u_b + \eta\psi)^{1-\gamma} - u_b^{1-\gamma}]}{\eta}\mathrm{d}x$$

exists. Dividing (3.1) by $\eta > 0$ and passing to the liminf as $\eta \to 0^+$, then we can get from Fatou's Lemma that

$$\begin{aligned} \int_{\mathbb{R}^3} f(x) u_b^{-\gamma}\psi \mathrm{d}x &\le \liminf_{\eta\to 0^+} \frac{1}{1-\gamma}\int_{\mathbb{R}^3} \frac{f(x)[(u_b + \eta\psi)^{1-\gamma} - u_b^{1-\gamma}]}{\eta}\mathrm{d}x \\ &\le (u_b, \psi)_E + b\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u_b|^2 \mathrm{d}x \int_{\mathbb{R}^3} (-\Delta)^{\frac{s}{2}} u_b (-\Delta)^{\frac{s}{2}} \psi \, \mathrm{d}x. \end{aligned} \tag{3.2}$$

Step 2. $u_b > 0$ *in* $\mathbb{R}^3$ *and* u_b *is a solution of problem* (P_b).

For given $\delta > 0$, define $g : [-\delta, \delta] \to \mathbb{R}$ by $g(\eta) = I_b(u_b + \eta u_b)$, then g attains its minimum at $\eta = 0$ by Lemma 2 which implies that

$$g'(0) = \|u_b\|_E + b\Big(\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u_b|^2 \mathrm{d}x\Big)^2 - \int_{\mathbb{R}^3} f(x) u_b^{1-\gamma} \mathrm{d}x = 0. \tag{3.3}$$

For any $v \in E$ and $\varepsilon > 0$, set $v_\varepsilon = u_b + \varepsilon v$, then

$$\begin{aligned} &(u_b(x) - u_b(y))(v_\varepsilon^+(x) - v_\varepsilon^+(y))(u_b(x) - u_b(y))(v_\varepsilon(x) + v_\varepsilon^-(x) - v_\varepsilon(y) - v_\varepsilon^-(y)) \\ =\ &|u_b(x) - u_b(y)|^2 + \varepsilon(u_b(x) - u_b(y))(v(x) - v(y)) + (u_b(x) - u_b(y))(v_\varepsilon^-(x) - v_\varepsilon^-(y)). \end{aligned} \tag{3.4}$$

According to the proof of Theorem 3.2 in [12],

$$\liminf_{\varepsilon\to 0^+}\frac{1}{\varepsilon}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{(u_b(x)-u_b(y))[v_\varepsilon^-(x)-v_\varepsilon^-(y)]}{|x-y|^{3+2s}}\mathrm{d}x\mathrm{d}y\le 0. \tag{3.5}$$

Set $\Omega_\varepsilon=\{x\in\mathbb{R}^3 : v_\varepsilon\le 0\}$, then using (3.3)–(3.4) and applying inequality (3.2) with $\psi=v_\varepsilon^+$ lead to

$$\begin{aligned}
0 &\le \frac{1}{\varepsilon}\Big\{\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{(u_b(x)-u_b(y))(v_\varepsilon^+(x)-v_\varepsilon^+(y))}{|x-y|^{3+2s}}\mathrm{d}x\mathrm{d}y+\int_{\mathbb{R}^3}V(x)u_bv_\varepsilon^+\mathrm{d}x\\
&\quad+b\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u_b|^2\mathrm{d}x\int_{\mathbb{R}^3}(-\Delta)^{\frac{s}{2}}u_b(-\Delta)^{\frac{s}{2}}v_\varepsilon^+\,\mathrm{d}x-\int_{\mathbb{R}^3}f(x)u_b^{-\gamma}v_\varepsilon^+\,\mathrm{d}x\Big\}\\
&= \frac{1}{\varepsilon}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|u_b(x)-u_b(y)|^2}{|x-y|^{3+2s}}\,\mathrm{d}x\mathrm{d}y+\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{(u_b(x)-u_b(y))(v(x)-v(y))}{|x-y|^{3+2s}}\mathrm{d}x\mathrm{d}y\\
&\quad+\frac{1}{\varepsilon}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{(u_b(x)-u_b(y))(v_\varepsilon^-(x)-v_\varepsilon^-(y))}{|x-y|^{3+2s}}\mathrm{d}x\mathrm{d}y+\frac{1}{\varepsilon}\Big(\int_{\mathbb{R}^3}-\int_{\Omega_\varepsilon}\Big)\Big\{V(x)u_b(u_b+\varepsilon v)\\
&\quad+b\Big(\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u_b|^2\mathrm{d}x\Big)(-\Delta)^{\frac{s}{2}}u_b(-\Delta)^{\frac{s}{2}}(u_b+\varepsilon v)-f(x)u_b^{-\gamma}(u_b+\varepsilon v)\Big\}\,\mathrm{d}x\\
&= \frac{1}{\varepsilon}\Big\{\|u_b\|_E^2+b\Big(\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u_b|^2\mathrm{d}x\Big)^2-\int_{\mathbb{R}^3}f(x)u_b^{1-\gamma}\mathrm{d}x\Big\}\\
&\quad+\Big\{(u_b,v)_E+b\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u_b|^2\mathrm{d}x\int_{\mathbb{R}^3}(-\Delta)^{\frac{s}{2}}u_b(-\Delta)^{\frac{s}{2}}v\,\mathrm{d}x-\int_{\mathbb{R}^3}f(x)u_b^{-\gamma}v\mathrm{d}x\Big\}\\
&\quad-\frac{1}{\varepsilon}\int_{\Omega_\varepsilon}\Big\{V(x)u_b(u_b+\varepsilon v)+b\Big(\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u_b|^2\mathrm{d}x\Big)(-\Delta)^{\frac{s}{2}}u_b(-\Delta)^{\frac{s}{2}}(u_b+\varepsilon v)\\
&\quad-f(x)u_b^{-\gamma}(u_b+\varepsilon v)\Big\}\,\mathrm{d}x+\frac{1}{\varepsilon}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{(u_b(x)-u_b(y))(v_\varepsilon^-(x)-v_\varepsilon^-(y))}{|x-y|^{3+2s}}\mathrm{d}x\mathrm{d}y\\
&\le \Big\{(u_b,v)_E+b\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u_b|^2\mathrm{d}x\int_{\mathbb{R}^3}(-\Delta)^{\frac{s}{2}}u_b(-\Delta)^{\frac{s}{2}}v\,\mathrm{d}x-\int_{\mathbb{R}^3}f(x)u_b^{-\gamma}v\mathrm{d}x\Big\}\\
&\quad-\frac{1}{\varepsilon}\int_{\Omega_\varepsilon}\Big[V(x)u_b^2+b\Big(\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u_b|^2\mathrm{d}x\Big)|(-\Delta)^{\frac{s}{2}}u_b|^2\Big]\mathrm{d}x\\
&\quad-\int_{\Omega_\varepsilon}\Big[V(x)u_bv+b\Big(\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u_b|^2\mathrm{d}x\Big)(-\Delta)^{\frac{s}{2}}u_b(-\Delta)^{\frac{s}{2}}v\Big]\mathrm{d}x\\
&\quad+\frac{1}{\varepsilon}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{(u_b(x)-u_b(y))(v_\varepsilon^-(x)-v_\varepsilon^-(y))}{|x-y|^{3+2s}}\mathrm{d}x\mathrm{d}y\\
&\le \Big\{(u_b,v)_E+b\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u_b|^2\mathrm{d}x\int_{\mathbb{R}^3}(-\Delta)^{\frac{s}{2}}u_b(-\Delta)^{\frac{s}{2}}v\,\mathrm{d}x-\int_{\mathbb{R}^3}f(x)u_b^{-\gamma}v\mathrm{d}x\Big\}\\
&\quad-\int_{\Omega_\varepsilon}\Big[V(x)u_bv+b\Big(\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u_b|^2\mathrm{d}x\Big)(-\Delta)^{\frac{s}{2}}u_b(-\Delta)^{\frac{s}{2}}v\Big]\mathrm{d}x\\
&\quad+\frac{1}{\varepsilon}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{(u_b(x)-u_b(y))(v_\varepsilon^-(x)-v_\varepsilon^-(y))}{|x-y|^{3+2s}}\mathrm{d}x\mathrm{d}y.
\end{aligned}$$

Passing to the liminf as $\varepsilon\to 0^+$ to the above inequality and using (3.5) and the fact that $|\Omega_\varepsilon|\to 0$ as $\varepsilon\to 0^+$, we have

$$(u_b,v)_E+b\int_{\mathbb{R}^3}|(-\Delta)^{\frac{s}{2}}u_b|^2\mathrm{d}x\int_{\mathbb{R}^3}(-\Delta)^{\frac{s}{2}}u_b(-\Delta)^{\frac{s}{2}}v\,\mathrm{d}x-\int_{\mathbb{R}^3}f(x)u_b^{-\gamma}v\mathrm{d}x\ge 0,\ \ \forall v\in E.$$

This inequality also holds for $-v$, hence we obtain

$$(u_b, v)_E + b\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u_b|^2 \mathrm{d}x \int_{\mathbb{R}^3} (-\Delta)^{\frac{s}{2}} u_b (-\Delta)^{\frac{s}{2}} v\, \mathrm{d}x - \int_{\mathbb{R}^3} f(x) u_b^{-\gamma} v \mathrm{d}x = 0,\ \forall v \in E. \tag{3.6}$$

From an argument similar to [28, Theorem 6.3], we know that $u_b \in C^{\alpha}_{loc}(\mathbb{R}^3)$ for some $\alpha \in (0, s)$. On the other hand, the (3.6) implies that

$$\Big[1 + b\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u_b|^2 \mathrm{d}x\Big](-\Delta)^s u_b + V(x) u_b \geq 0. \tag{3.7}$$

Assume that there exists $x_0 \in \mathbb{R}^3$ such that $u_b(x_0) = 0$, then from (3.7), we have $(-\Delta)^s u_b(x_0) \geq 0$. On the other hand, since $u_b \geq 0$ and $u_b \not\equiv 0$, we can get from Lemma 3.2 in [29] that

$$\begin{aligned} (-\Delta)^s u_b(x_0) &= -\int_{\mathbb{R}^3} \frac{u_b(x_0 + y) + u_b(x_0 - y) - 2u_b(x_0)}{|y|^{3+2s}} \mathrm{d}y \\ &= -\int_{\mathbb{R}^3} \frac{u_b(x_0 + y) + u_b(x_0 - y)}{|y|^{3+2s}} \mathrm{d}y < 0, \end{aligned}$$

a contradiction. Therefore, $u_b > 0$ in $\mathbb{R}^3$ and $u_b \in E$ is a solution of problem (P_b).

Step 3. *u_b is a unique solution of problem (P_b).*

Suppose $u_* \in E$ is also a solution of problem (P_b), then we have

$$(u_*, v)_E + b\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u_*|^2 \mathrm{d}x \int_{\mathbb{R}^3} (-\Delta)^{\frac{s}{2}} u_* (-\Delta)^{\frac{s}{2}} v\, \mathrm{d}x - \int_{\mathbb{R}^3} f(x) u_*^{-\gamma} v \mathrm{d}x = 0,\ \forall v \in E. \tag{3.8}$$

Taking $v = u_b - u_*$ in both equations (3.6)-(3.8) and subtracting term by term, we obtain

$$\begin{aligned} 0 &\geq \int_{\mathbb{R}^3} f(x)(u_b^{-\gamma} - u_*^{-\gamma})(u_b - u_*) \mathrm{d}x \\ &= \|u_b - u_*\|_E^2 + b\Big[\Big(\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u_b|^2 \mathrm{d}x\Big)^2 - \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u_b|^2 \mathrm{d}x \int_{\mathbb{R}^3} (-\Delta)^{\frac{s}{2}} u_b (-\Delta)^{\frac{s}{2}} u_*\, \mathrm{d}x \\ &\quad - \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u_*|^2 \mathrm{d}x \int_{\mathbb{R}^3} (-\Delta)^{\frac{s}{2}} u_* (-\Delta)^{\frac{s}{2}} u_b\, \mathrm{d}x + \Big(\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u_*|^2 \mathrm{d}x\Big)^2\Big] \\ &\geq \|u_b - u_*\|_E^2 + b\Big([u_b]_s^4 - [u_b]_s^3 [u_*]_s - [u_*]_s^3 [u_b]_s + [u_*]_s^4\Big) \\ &= \|u_b - u_*\|_E^2 + b([u_b]_s - [u_*]_s)^2([u_b]_s^2 + [u_b]_s [u_*]_s + [u_*]_s^2) \\ &\geq \|u_b - u_*\|_E^2 \geq 0, \end{aligned}$$

where we use Hölder's inequality. So $\|u_b - u_*\|_E^2 = 0$, then $u_b = u_*$ and u_b is the unique solution of problem (P_b). This ends the proof of Theorem 1. □

Proof of Theorem 1.2. In the proofs of Lemma 2 and Theorem 1, $b = 0$ is allowed. Hence, under the assumptions of Theorem 2, there exists a unique positive solution $w_0 \in E$ to problem (P_0), that is for any $v \in E$, it holds

$$(w_0, v)_E = \int_{\mathbb{R}^3} f(x) w_0^{-\gamma} v \mathrm{d}x. \tag{3.9}$$

For every vanishing sequence $\{b_n\}$, since $\{u_{b_n}\}$ is a positive solution sequence to problem (P_b) provided by Theorem 1, then for every $v \in E$ and $n \in \mathbb{N}$, one has

$$(u_{b_n}, v)_E + b_n \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u_{b_n}|^2 \mathrm{d}x \int_{\mathbb{R}^3} (-\Delta)^{\frac{s}{2}} u_{b_n} (-\Delta)^{\frac{s}{2}} v \, \mathrm{d}x = \int_{\mathbb{R}^3} f(x) u_{b_n}^{-\gamma} v \mathrm{d}x. \tag{3.10}$$

By Lemma 2 and the proof of Theorem 1, we have $I_{b_n}(u_{b_n}) = m_{b_n} < 0$ and then $\{u_{b_n}\}$ is bounded in E since I_{b_n} is coercive according to (2.1). So, there exists a subsequence of $\{u_{b_n}\}$ (still denoted by $\{u_{b_n}\}$) and a nonnegative function $u_0 \in E$ such that

$$\begin{aligned} &u_{b_n} \rightharpoonup u_0, && \text{in } E, \\ &u_{b_n} \to u_0, && \text{in } L^p(\mathbb{R}^3), \ p \in [2, 2_s^*), \\ &u_{b_n} \to u_0, && \text{a.e. in } \mathbb{R}^3. \end{aligned}$$

Choosing $v = u_{b_n}$ in (3.10) and passing to the liminf as $n \to \infty$, one can get from (2.3) and the weakly lower semicontinuity of the norm that

$$\|u_0\|_E^2 \le \int_{\mathbb{R}^3} f(x) u_0^{1-\gamma} \mathrm{d}x. \tag{3.11}$$

On the other hand, passing to the liminf as $n \to \infty$ in (3.10) and using Fatou's Lemma, for any $0 \le v \in E$, we have

$$(u_0, v)_E \ge \int_{\mathbb{R}^3} f(x) u_0^{-\gamma} v \mathrm{d}x. \tag{3.12}$$

Similarly to Step 1 in the proof of Theorem 1, we have $u_0 > 0$ in $\mathbb{R}^3$. Choosing $v = u_0$ in (3.12) leads to

$$\|u_0\|_E^2 \ge \int_{\mathbb{R}^3} f(x) u_0^{1-\gamma} \mathrm{d}x.$$

This combined with (3.11) leads to

$$\|u_0\|_E^2 = \int_{\mathbb{R}^3} f(x) u_0^{1-\gamma} \mathrm{d}x \text{ and } u_{b_n} \to u_0 \text{ in } E. \tag{3.13}$$

Using (3.13) and similar to Step 2 in the proof of Theorem 1, we can further obtain that $u_0 \in E$ is also a solution of problem (P_0). By the uniqueness of solution to problem (P_0), $u_0 = w_0$. Hence $u_{b_n} \to w_0$ in E where w_0 is the unique positive solution to problem (P_0). This completed the proof of Theorem 2. □

4. Conclusions

In this work, we investigate a weak singular Kirchhoff-type fractional Laplacian problem. By using variational method, the existence, uniqueness and asymptotic behavior of positive solution are established. The results presented in this paper supplement some recent ones.

Acknowledgments

We appreciate the editor and referees for their invaluable comments. This work was supported by the National Natural Science Foundation of China (No. 11871152, 11671085), Natural Science Foundation of Fujian Province (No. 2019J01089) and Program for New Century Excellent Talents in Fujian Province University (2018).

Conflict of interest

The authors declare no conflict of interest.

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