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Research article

A class of dissipative differential operators of order three

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Abstract: In this paper we find a class of boundary conditions which determine dissipative differential operators of order three and prove that these operators have no real eigenvalues. The completeness of the system of eigenfunctions and associated functions is also established.

Keywords: third order boundary value problems; dissipative operators; eigenvalues; limit circle case
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1. Introduction

In this paper we study a class of third order dissipative differential operators. Dissipative operators are of general interest in mathematics, for example in the study of the Cauchy problems in partial differential equations and in infinite dimensional dynamical systems. Even order dissipative operators and the boundary conditions generating them have been investigated by many authors, see [1–9] and their references. Odd order problems arise in physics and other areas of applied mathematics and have also been studied, e.g., in [10–16].

Non-self-adjointness of spectral problems can be caused by one or more of the following factors: the non-linear dependence of the problems on the spectral parameter, the non-symmetry of the differential expressions used, and the non-self-adjointness of the boundary conditions(BCs) involved. Many scholars focus on the non-self-adjoint differential operators caused by non-self-adjoint BCs. Bairamov, Uğurlu, Tuna and Zhang et al. considered the even order dissipative operators and their spectral properties in [5–9], respectively. However, these results all restricted in some special boundary conditions. In 2012, Wang and Wu [2] found all boundary conditions which generate dissipative operators of order two and proved the completeness of eigenfunctions and associated functions for these operators. In [3] the authors studied a class of non-self-adjoint fourth order differential operators in Weyl's limit circle case with general separated BCs, and they proved the completeness of eigenfunctions and associated functions. Here we find a class of such general conditions for the third order case, which may help to classify the dissipative boundary conditions of third order differential operators.

As is mentioned above, there are many results for dissipative Sturm-Liouville operators and fourth order differential operators, however, there are few studies on the odd order dissipative operators. Thus, in this paper, we study a class of non-self-adjoint third order differential operators generated by the symmetric differential expression in Weyl's limit circle case together with non-self-adjoint BCs.

This paper is organized as follows. In Section 2 we introduce third order dissipative operators and develop their properties. Section 3 discusses some general properties of dissipative operators in Hilbert space and some particular properties of the third order operators studied here. The completeness of eigenfunctions and associated functions is given in Section 4. Brief concluding remarks on the obtained results in this present paper and the comparison with other works are reported in Section 5.

2. Third order boundary value problems

Consider the third order differential expression

$$l(u) = iu^{(3)} + q(x)u, \quad x \in I = (a, b), \tag{2.1}$$

where $-\infty \leq a < b \leq +\infty$, $q(x)$ is a real-valued function on I and $q(x) \in L^1_{loc}(I)$. Suppose that the endpoints a and b are singular, i.e., $a = -\infty$ or for any $c \in (a, b)$ $q(x)$ is not absolutely integrable in $(a, c]$ (the same statement holds for endpoint b), and Weyl's limit-circle case holds for the differential expression $l(u)$, i.e., the deficiency indices at both endpoints are $(3, 3)$.

Let

$$\Omega = \{u \in L^2(I) : u, u', u'' \in AC_{loc}(I),\ l(u) \in L^2(I)\}.$$

For all $u, v \in \Omega$, we set

$$[u, v]_x = iu\overline{v''} - iu'\overline{v'} + iu''\overline{v} = R_{\overline{v}}(x)QC_u(x), \quad x \in I,$$

where the bar over a function denotes its complex conjugate, and

$$Q = \begin{pmatrix} 0 & 0 & i \\ 0 & -i & 0 \\ i & 0 & 0 \end{pmatrix},\ R_v(x) = (v(x), v'(x), v''(x)),\ C_{\overline{v}}(x) = R^*_v(x),$$

and $R^*_v(x)$ is the complex conjugate transpose of $R_v(x)$.

Let $\psi_j(x, \lambda)$, $j = 1, 2, 3$ represent a set of linearly independent solutions of the equation $l(u) = \lambda u$, where λ is a complex parameter. Then $\psi_j(x, 0)$, $j = 1, 2, 3$ represent the linearly independent solutions of the equation $l(u) = 0$. From Naimark's Patching Lemma, we can choose the solutions above mentioned satisfying any initial conditions, for future conveniences, here we set $z_j(x) = \psi_j(x, 0)$, $j = 1, 2, 3$ satisfying the condition

$$([z_j, z_k]_a) = J, \quad j, k = 1, 2, 3, \tag{2.2}$$

where

$$J = \begin{pmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & -i \end{pmatrix}.$$

From [17], the solutions $z_j(x),\ j=1,2,3$ as described above exist and are linearly independent. Since Weyl's limit-circle case holds for the differential expression $l(u)$ on I, the solutions $z_j(x),\ j=1,2,3$ must belong to $L^2(I)$. Furthermore, because $z_j(x),\ j=1,2,3$ are solutions of equation $l(u)=0$, thus according to the Green's formula, it is easy to get $[z_j,z_k]_x = const$ for any $x\in I$, hence for any $x\in I$, $([z_j,z_k]_x)=J$.

Let $l(u)=\lambda u$ and we consider the boundary value problem consisting of the differential equation

$$iu^{(3)}+q(x)u=\lambda u,\quad x\in I, \tag{2.3}$$

and the boundary conditions:

$$l_1(u)=[u,z_1]_a+\gamma_1[u,z_2]_a+\gamma_2[u,z_3]_a=0, \tag{2.4}$$

$$l_2(u)=\overline{\gamma_2}[u,z_2]_a+[u,z_3]_a+r\overline{\gamma_4}e^{-2i\theta}[u,z_2]_b+re^{-2i\theta}[u,z_3]_b=0, \tag{2.5}$$

$$l_3(u)=[u,z_1]_b+\gamma_3[u,z_2]_b+\gamma_4[u,z_3]_b=0, \tag{2.6}$$

where λ is a complex parameter, r is a real number with $|r|\geq 1$, $\theta\in(-\pi,\pi]$, $\gamma_j,\ j=1,2,3,4$ are complex numbers with $2\Re\gamma_1\geq|\gamma_2|^2$ and $2\Re\gamma_3\leq|\gamma_4|^2$, here $\Re$ denotes the real part of a value.

In $L^2(I)$, let us define the operator L as $Lu=l(u)$ on $D(L)$, where the domain $D(L)$ of L is given by

$$D(L)=\{u\in\Omega:\ \ l_j(u)=0,\ j=1,2,3\}.$$

Let $\Psi(x)$ be the Wronskian matrix of the solutions $z_j(x),\ j=1,2,3$ in I, then ones have

$$\Psi(x)=(C_{z_1}(x),C_{z_2}(x),C_{z_3}(x)).$$

Now let us introduce several lemmas.

Lemma 1.

$$Q=(\Psi^*(x))^{-1}J\,\Psi^{-1}(x),\quad x\in I.$$

Proof. From

$$[z_j,z_k]_x=R_{\bar{z}_k}(x)QC_{z_j}(x),\quad j,k=1,2,3,$$

we have

$$J=J^T=([z_j,z_k]_x)^T=\Psi^*(x)Q\Psi(x),\quad j,k=1,2,3.$$

Then the conclusion can be obtained by left multiplying $(\Psi^*(x))^{-1}$ and right multiplying $\Psi^{-1}(x)$ on the two ends of the above equality. □

Lemma 2. *For arbitrary $u\in D(L)$*

$$([u,z_1]_x,\ [u,z_2]_x,\ [u,z_3]_x)^T=J\,\Psi^{-1}(x)C_u(x),\quad x\in I.$$

Proof. From

$$[u,z_j]_x=R_{\bar{z}_j}(x)QC_u(x),\quad j=1,2,3,$$

one has

$$\begin{aligned}([u,z_1]_x,\ [u,z_2]_x,\ [u,z_3]_x)^T&=\Psi^*(x)QC_u(x)\\&=\Psi^*(x)(\Psi^*(x))^{-1}J\,\Psi^{-1}(x)C_u(x)\\&=J\,\Psi^{-1}(x)C_u(x).\end{aligned}$$

This complete the proof. □

Corollary 1. *For arbitrary $y_1, y_2, y_3 \in D(L)$, let $Y(x) = (C_{y_1}(x), C_{y_2}(x), C_{y_3}(x))$ be the Wronskian matrix of y_1, y_2, y_3, then*

$$J\Psi^{-1}(x)Y(x) = \begin{pmatrix} [y_1, z_1]_x & [y_2, z_1]_x & [y_3, z_1]_x \\ [y_1, z_2]_x & [y_2, z_2]_x & [y_3, z_2]_x \\ [y_1, z_3]_x & [y_2, z_3]_x & [y_3, z_3]_x \end{pmatrix}, \quad x \in I.$$

Lemma 3. *For arbitrary $u, v \in D(L)$, we have*

$$[u, v]_x = i([u, z_1]_x\overline{[v, z_2]_x} + [u, z_2]_x\overline{[v, z_1]_x} - [u, z_3]_x\overline{[v, z_3]_x}), \quad x \in I. \tag{2.7}$$

Proof. From Lemma 1 and Lemma 2, it is easy to calculate that

$$\begin{aligned}
[u, v]_x &= R_{\overline{v}}(x)QC_u(x) \\
&= R_{\overline{v}}(x)(\Psi^*(x))^{-1}J\Psi^{-1}(x)C_u(x) \\
&= (J\Psi^{-1}(x)C_v(x))^* J(J\Psi^{-1}(x)C_u(x)) \\
&= (\overline{[v, z_1]_x}, \overline{[v, z_2]_x}, \overline{[v, z_3]_x})J\,([u, z_1]_x,\ [u, z_2]_x,\ [u, z_3]_x)^T \\
&= i([u, z_1]_x\overline{[v, z_2]_x} + [u, z_2]_x\overline{[v, z_1]_x} - [u, z_3]_x\overline{[v, z_3]_x}).
\end{aligned}$$

This completes the proof. □

3. Dissipative operators

We start with the definition of dissipative operators.

Definition 1. *A linear operator L, acting in the Hilbert space $L^2(I)$ and having domain $D(L)$, is said to be dissipative if $\Im(Lf, f) \geq 0,\ \forall f \in D(L)$, where $\Im$ denotes the imaginary part of a value.*

Theorem 1. *The operator L is dissipative in $L^2(I)$.*

Proof. For $u \in D(L)$, we have

$$2i\Im(Lu, u) = (Lu, u) - (u, Lu) = [u, u](b) - [u, u](a), \tag{3.1}$$

then, applying (2.7), it follows that

$$\begin{aligned}
2i\Im(Lu, u) =& i([u, z_1]_b\overline{[u, z_2]_b} + [u, z_2]_b\overline{[u, z_1]_b} - [u, z_3]_b\overline{[u, z_3]_b}) \\
&-i([u, z_1]_a\overline{[u, z_2]_a} + [u, z_2]_a\overline{[u, z_1]_a} - [u, z_3]_a\overline{[u, z_3]_a}).
\end{aligned} \tag{3.2}$$

From (2.4)–(2.6), it has

$$[u, z_1]_a = (\gamma_2\overline{\gamma_2} - \gamma_1)[u, z_2]_a + r\gamma_2\overline{\gamma_4}e^{-2i\theta}[u, z_2]_b + r\gamma_2 e^{-2i\theta}[u, z_3]_b, \tag{3.3}$$

$$[u, z_3]_a = -\overline{\gamma_2}[u, z_2]_a - r\overline{\gamma_4}e^{-2i\theta}[u, z_2]_b - re^{-2i\theta}[u, z_3]_b, \tag{3.4}$$

$$[u, z_1]_b = -\gamma_3[u, z_2]_b - \gamma_4[u, z_3]_b, \tag{3.5}$$

substituting (3.3)–(3.5) into (3.2) one obtains

$$2i\Im(Lu, u) = (Lu, u) - (u, Lu) = i(\overline{[u, z_2]_a}, \overline{[u, z_2]_b}, \overline{[u, z_3]_b}) \tag{3.6}$$

$$\begin{pmatrix} -\gamma_2\overline{\gamma_2}+\gamma_1+\overline{\gamma_1} & 0 & 0 \\ 0 & r^2\gamma_4\overline{\gamma_4}-\gamma_3-\overline{\gamma_3} & \gamma_4(r^2-1) \\ 0 & \overline{\gamma_4}(r^2-1) & (r^2-1) \end{pmatrix}\begin{pmatrix} [u,z_2]_a \\ [u,z_2]_b \\ [u,z_3]_b \end{pmatrix},$$

and hence

$$2\Im(Lu,u) = (\overline{[u,z_2]_a}, \overline{[u,z_2]_b}, \overline{[u,z_3]_b})\begin{pmatrix} s & 0 & 0 \\ 0 & c & f \\ 0 & \overline{f} & d \end{pmatrix}\begin{pmatrix} [u,z_2]_a \\ [u,z_2]_b \\ [u,z_3]_b \end{pmatrix}, \tag{3.7}$$

where

$$s = 2\Re\gamma_1 - |\gamma_2|^2, \quad f = \gamma_4(r^2-1), \quad c = r^2|\gamma_4|^2 - 2\Re\gamma_3, \quad d = r^2 - 1.$$

Note that the 3 by 3 matrix in (3.7) is Hermitian, its eigenvalues are

$$s, \quad \frac{c+d \pm \sqrt{(c-d)^2+4|f|^2}}{2},$$

and they are all non-negative if and only if

$$s \geq 0, \ \ c+d \geq 0, \ \ cd \geq |f|^2.$$

Since $|r| \geq 1$, $2\Re\gamma_1 \geq |\gamma_2|^2$ and $2\Re\gamma_3 \leq |\gamma_4|^2$, we have

$$\Im(Lu,u) \geq 0, \ \forall u \in D(L).$$

Hence L is a dissipative operator in $L^2(I)$. □

Theorem 2. *If $|r| > 1$, $2\Re\gamma_1 > |\gamma_2|^2$ and $2\Re\gamma_3 < |\gamma_4|^2$, then the operator L has no real eigenvalue.*

Proof. Suppose λ_0 is a real eigenvalue of L. Let $\phi_0(x) = \phi(x,\lambda_0) \neq 0$ be a corresponding eigenfunction. Since

$$\Im(L\phi_0,\phi_0) = \Im(\lambda_0\|\phi_0\|^2) = 0,$$

from (3.7), it follows that

$$\Im(L\phi_0,\phi_0) = \frac{1}{2}(\overline{[\phi_0,z_2]_a}, \overline{[\phi_0,z_2]_b}, \overline{[\phi_0,z_3]_b})\begin{pmatrix} s & 0 & 0 \\ 0 & c & f \\ 0 & \overline{f} & d \end{pmatrix}\begin{pmatrix} [\phi_0,z_2]_a \\ [\phi_0,z_2]_b \\ [\phi_0,z_3]_b \end{pmatrix} = 0,$$

since $|r| > 1$, $2\Re\gamma_1 > |\gamma_2|^2$ and $2\Re\gamma_3 < |\gamma_4|^2$, the matrix

$$\begin{pmatrix} s & 0 & 0 \\ 0 & c & f \\ 0 & \overline{f} & d \end{pmatrix}$$

is positive definite. Hence $[\phi_0,z_2]_a = 0$, $[\phi_0,z_2]_b = 0$ and $[\phi_0,z_3]_b = 0$, and by the boundary conditions (2.4)–(2.6), we obtain that $[\phi_0,z_1]_b = 0$. Let $\phi_0(x) = \phi(x,\lambda_0)$, $\tau_0(x) = \tau(x,\lambda_0)$ *and* $\eta_0(x) = \eta(x,\lambda_0)$ be the linearly independent solutions of $l(y) = \lambda_0 y$. Then from Corollary 1 one has

$$\begin{pmatrix} [\phi_0,z_1]_b & [\tau_0,z_1]_b & [\eta_0,z_1]_b \\ [\phi_0,z_2]_b & [\tau_0,z_2]_b & [\eta_0,z_2]_b \\ [\phi_0,z_3]_b & [\tau_0,z_3]_b & [\eta_0,z_3]_b \end{pmatrix} = Q\Psi^{-1}(b)(C_{\phi_0}(b), C_{\tau_0}(b), C_{\eta_0}(b)).$$

It is evident that the determinant of the left hand side is equal to zero, the value of the Wronskian of the solutions $\phi(x,\lambda_0)$, $\tau(x,\lambda_0)$ and $\eta(x,\lambda_0)$ is not equal to zero, therefore the determinant on the right hand side is not equal to zero. This is a contradiction, hence the operator L has no real eigenvalue. □

4. Completeness theorems

In this section we start with a result of Gasymov and Guseinov [18]. It also can be found in many literatures, for instance in [19] and [20].

Lemma 4. *For all $x \in [a,b]$, the functions $\phi_{jk} = [\psi_k(\cdot,\lambda), z_j](x)$, $j,k = 1,2,3$, are entire functions of λ with growth order ≤ 1 and minimal type: for any $j,k = 1,2,3$ and $\varepsilon \ge 0$, there exists a positive constant $C_{j,k,\varepsilon}$ such that*

$$|\phi_{jk}| \le C_{j,k,\varepsilon} e^{\varepsilon|\lambda|}, \quad \lambda \in \mathbb{C}.$$

Let

$$A = \begin{pmatrix} 1 & \gamma_1 & \gamma_2 \\ 0 & \overline{\gamma_2} & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & r\overline{\gamma_4}e^{-2i\theta} & re^{-2i\theta} \\ 1 & \gamma_3 & \gamma_4 \end{pmatrix}$$

denote the boundary condition matrices of boundary conditions (2.4)–(2.6), and set $\Phi = (\phi_{jk})_{3\times 3}$. Then, a complex number is an eigenvalue of the operator L if and only if it is a zero of the entire function

$$\Delta(\lambda) = \begin{vmatrix} l_1(\psi_1(\cdot,\lambda)) & l_1(\psi_2(\cdot,\lambda)) & l_1(\psi_3(\cdot,\lambda)) \\ l_2(\psi_1(\cdot,\lambda)) & l_2(\psi_2(\cdot,\lambda)) & l_2(\psi_3(\cdot,\lambda)) \\ l_3(\psi_1(\cdot,\lambda)) & l_3(\psi_2(\cdot,\lambda)) & l_3(\psi_3(\cdot,\lambda)) \end{vmatrix} = \mathbf{det}(A\Phi(a,\lambda) + B\Phi(b,\lambda)). \tag{4.1}$$

Remark 1. *Note that $a = -\infty$ or $b = \infty$ have not been ruled out. Since the limit circle case holds, the functions ϕ_{jk} and $\Phi(a,\lambda)$, $\Phi(b,\lambda)$ are well defined at $a = -\infty$ and $b = \infty$, i.e., $\phi_{jk}(\pm\infty) = [\psi_k(\cdot,\lambda), z_j](\pm\infty) = \lim_{x\to\pm\infty} = [\psi_k(\cdot,\lambda), z_j](x)$ exist and are finite.*

Corollary 2. *The entire function $\Delta(\lambda)$ is also of growth order ≤ 1 and minimal type: for any $\varepsilon \ge 0$, there exists a positive constant C_ε such that*

$$|\Delta(\lambda)| \le C_\varepsilon e^{\varepsilon|\lambda|}, \quad \lambda \in \mathbb{C}, \tag{4.2}$$

and hence

$$\limsup_{|\lambda|\to\infty} \frac{ln|\Delta(\lambda)|}{|\lambda|} \le 0. \tag{4.3}$$

From Theorem 2 it follows that zero is not an eigenvalue of L, hence the operator L^{-1} exists. Let's give an analytical representation of L^{-1}.

Consider the non-homogeneous boundary value problem composed of the equation $l(u) = f(x)$ and the boundary conditions (2.4)–(2.6), where $x \in I = (a,b)$, $f(x) \in L^2(I)$.

Let $u(x)$ be the solution of the above non-homogeneous boundary value problem, then

$$u(x) = C_1 z_1(x) + C_2 z_2(x) + C_3 z_3(x) + u^*(x),$$

where C_j, $j = 1,2,3$ are arbitrary constants and $u^*(x)$ is a special solution of $l(u) = f(x)$.

It can be obtained by the method of constant variation,

$$u^*(x) = C_1(x) z_1(x) + C_2(x) z_2(x) + C_3(x) z_3(x),$$

where $C_j,\ j = 1, 2, 3$ satisfies

$$\begin{cases} C_1'(x)z_1(x) + C_2'(x)z_2(x) + C_3'(x)z_3(x) = 0, \\ C_1'(x)z_1'(x) + C_2'(x)z_2'(x) + C_3'(x)z_3'(x) = 0, \\ i(C_1'(x)z_1''(x) + C_2'(x)z_2''(x) + C_3'(x)z_3''(x)) = f(x). \end{cases}$$

By proper calculation, we have

$$u^*(x) = \int_a^b K(x,\xi)f(\xi)d\xi,$$

where

$$K(x,\xi) = \begin{cases} \frac{1}{i|\Psi(x)|}\begin{vmatrix} z_1(\xi) & z_2(\xi) & z_3(\xi) \\ z_1'(\xi) & z_2'(\xi) & z_3'(\xi) \\ z_1(x) & z_2(x) & z_3(x) \end{vmatrix}, & a < \xi \le x < b, \\ 0, & a < x \le \xi < b, \end{cases} \tag{4.4}$$

then the solution can be written as

$$u(x) = C_1z_1(x) + C_2z_2(x) + C_3z_3(x) + \int_a^b K(x,\xi)f(\xi)d\xi,$$

substituting $u(x)$ into the boundary conditions one obtains

$$C_j(x) = \frac{1}{\Delta(0)}\int_a^b F_j(\xi)f(\xi)d\xi, \quad j = 1, 2, 3,$$

where

$$F_1(\xi) = -\begin{vmatrix} l_1(K) & l_1(z_2) & l_1(z_3) \\ l_2(K) & l_2(z_2) & l_2(z_3) \\ l_3(K) & l_3(z_2) & l_3(z_3) \end{vmatrix}, \tag{4.5}$$

$$F_2(\xi) = -\begin{vmatrix} l_1(z_1) & l_1(K) & l_1(z_3) \\ l_2(z_1) & l_2(K) & l_2(z_3) \\ l_3(z_1) & l_3(K) & l_3(z_3) \end{vmatrix}, \tag{4.6}$$

$$F_3(\xi) = -\begin{vmatrix} l_1(z_1) & l_1(z_2) & l_1(K) \\ l_2(z_1) & l_2(z_2) & l_2(K) \\ l_3(z_1) & l_3(z_2) & l_3(K) \end{vmatrix}, \tag{4.7}$$

thus

$$u(x) = \int_a^b \frac{1}{\Delta(0)}[F_1(\xi)z_1(x) + F_2(\xi)z_2(x) + F_3(\xi)z_3(x) + K(x,\xi)\Delta(0)]f(\xi)d\xi.$$

Let

$$G(x,\xi) = -\frac{1}{\Delta(0)}\begin{vmatrix} z_1(x) & z_2(x) & z_3(x) & K(x,\xi) \\ l_1(z_1) & l_1(z_2) & l_1(z_3) & l_1(K) \\ l_2(z_1) & l_2(z_2) & l_2(z_3) & l_2(K) \\ l_3(z_1) & l_3(z_2) & l_3(z_3) & l_3(K) \end{vmatrix}, \tag{4.8}$$

then one obtains

$$u(x) = \int_a^b G(x,\xi)f(\xi)d\xi.$$

Now define the operator T as

$$Tu = \int_a^b G(x,\xi)u(\xi)d\xi, \quad \forall u \in L^2(I), \tag{4.9}$$

then T is an integral operator and $T = L^{-1}$, this implies that the root vectors of the operators T and L coincide, since $z_j(x) \in L^2(I)$, $j = 1,2,3$, then the following inequality holds

$$\int_a^b \int_a^b |G(x,\xi)|^2 dx d\xi < +\infty, \tag{4.10}$$

this implies that the integral operator T is a Hilbert-Schmidt operator [21].

The next theorem is known as Krein's Theorem.

Theorem 3. *Let S be a compact dissipative operator in $L^2(I)$ with nuclear imaginary part $\Im S$. The system of all root vectors of S is complete in $L^2(I)$ so long as at least one of the following two conditions is fulfilled:*

$$\lim_{m\to\infty} \frac{n_+(m,\Re S)}{m} = 0, \quad \lim_{m\to\infty} \frac{n_-(m,\Re S)}{m} = 0, \tag{4.11}$$

where $n_+(m,\Re S)$ and $n_-(m,\Re S)$ denote the number of characteristic values of the real component $\Re S$ of S in the intervals $[0,m]$ and $[-m,0]$, respectively.

Proof. See [22]. □

Theorem 4. *If an entire function $h(\mu)$ is of order ≤ 1 and minimal type, then*

$$\lim_{\rho\to\infty} \frac{n_+(\rho,h)}{\rho} = 0, \quad \lim_{\rho\to\infty} \frac{n_-(\rho,h)}{\rho} = 0, \tag{4.12}$$

where $n_+(\rho,h)$ and $n_-(\rho,h)$ denote the number of the zeros of the function $h(\mu)$ in the intervals $[0,\rho]$ and $[-\rho,0]$, respectively.

Proof. See [23]. □

The operator T can be written as $T = T_1 + iT_2$, where $T_1 = \Re T$ and $T_2 = \Im T$, T and T_1 are Hilbert-Schmidt operators, T_1 is a self-adjoint operator in $L^2(I)$, and T_2 is a nuclear operator (since it is a finite dimensional operator) [22]. It is easy to verify that T_1 is the inverse of the real part L_1 of the operator L.

Since the operator L is dissipative, it follows that the operator $-T$ is dissipative. Consider the operator $-T = -T_1 - iT_2$, the eigenvalues of the operator $-T_1$ and L_1 coincide. Since the characteristic function of L_1 is an entire function, therefore using Theorem 4 and Krein's Theorem we arrive at the following results.

Theorem 5. *The system of all root vectors of the operator $-T$ (also of T) is complete in $L^2(I)$.*

Theorem 6. *The system of all eigenvectors and associated vectors of the dissipative operator L is complete in $L^2(I)$.*

5. Concluding remarks

This paper considered a class of third order dissipative operator generated by symmetric third order differential expression and a class of non-self-adjoint boundary conditions. By using the well known Krein's Theorem and theoretical analysis the completeness of eigenfunctions system and associated functions is proved.

The similar results already exist for second order S-L operators and fourth order differential operators, see e.g., [2] and [3]. For third order case, the corresponding discussions about dissipative operators can be found in most recent works in [15, 19], where the maximal dissipative extension and the complete theorems of eigenvectors system are given. The boundary conditions at the present work are much general and the methods are different from those in [15, 19]. These boundary conditions may help us to classify all the analytical representations of dissipative boundary conditions of third order differential operators.

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Conflict of interest

The authors declare no conflict of interest in this paper.

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