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Research article

Existence of φ-fixed point for generalized contractive mappings

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Abstract: In this paper, we first define the generalized (F, φ, $\alpha - \psi$)-contraction mappings. In the following, we consider the conditions in which these mappings have a φ-fixed point and also we present examples and applications of these mappings in partial metric space and integral equations.

Keywords: φ-fixed point; (F, φ)-contractive; generalized $(F, \varphi, \alpha - \psi)$-contractive mapping; partial metric spaces
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1. Introduction

Theory of fixed point is one of the old subjects in mathematics which is growing very rapidly. This theory has applications in various fields of science such as game theory, physics, equations, and so on. In 1886 Povancare was the first author that searched on fixed point. After that in 1922 Banach began his study on fixed point and got an important conclusion which is called Banach's contraction principle [1]. Some authors generalized these Banach mappings in different places, see [2–5] for details. In 2014, Jleli and his colleagues [6], using a family of functions called F, expressed the concept of φ-fixed point and (F, φ)-contraction mappings and proved theorems about this.

Supposing X is a nonempty set and $\varphi : X \longrightarrow [0, \infty]$ is a given function, we denote the center of the function φ by $Z_\varphi = \{x \in X; \varphi(x) = 0\}$.

Definition 1.1. [6] Let X is a nonempty set and $\varphi : X \longrightarrow [0, \infty]$ is a given function. An element $z \in X$ is called φ-fixed point of the mapping $T : X \longrightarrow X$ if and only if $Tz = z$ and $z \in Z_\varphi$.

Let $\mathcal{F}$ be the set of all functions $F : [0, \infty)^3 \longrightarrow [0, \infty)$ satisfying the following conditions:

(F_1) $\max\{a,b\} \le F(a,b,c),$ $for\ all\ a,b,c \in [0,\infty)$,

(F_2) $F(0,0,0) = 0$,

(F_3) F is continuous.

In 2017, Asadi replaced the condition of continuity with the following condition [7]:

$(F_3)'$ $\lim_{x\longrightarrow\infty} \sup F(a_n, b_n, 0) \le F(a, b, 0)$, when $a_n \longrightarrow a$ and $b_n \longrightarrow b$ as $n \longrightarrow \infty$.

The class of all functions satisfying properties (F_1), (F_2) and $(F_3)'$ is denoted by $\mathcal{F}_M$. Observe that $\mathcal{F} \subset \mathcal{F}_M$, [6,7].

[8] Also we denote by Ψ the class of all functions $\psi : [0,\infty) \longrightarrow [0,\infty)$ satisfying the following conditions:

(ψ_1) ψ is nondecreasing;

(ψ_2) $\sum_{n=1}^{+\infty} \psi^n(t) < \infty$, for all $t > 0$, where ψ^n is n-the iterate of ψ.

According to the properties of Ψ, we have the following results:

Lemma 1.2. *[9] If* $\sum_{n=1}^{+\infty} \psi^n(t) < \infty$*, then* $\lim_{n\longrightarrow\infty} \psi^n(t) = 0,$ *for all* $t \in (0,\infty)$.

Lemma 1.3. *[9] If* $\psi \in \Psi$*, then* $\psi(0) = 0$ *and* $\psi(t) < t$*, for all* $t > 0$.

The concept of (F,φ)-contractive maps was introduced by Jleli as follow:

Definition 1.4. [6] Let (X,d) be a metric space and $\varphi : X \to [0,\infty)$. A mapping $T : X \longrightarrow X$ is said to be an (F,φ)-contractive mapping if there exist $F \in \mathcal{F}_M$ and $k \in [0,\infty)$ such that

$$F(d(Tx,Ty), \varphi(Tx), \varphi(Ty)) \le k(F(d(x,y), \varphi(x), \varphi(y))). \tag{1.1}$$

for all $x,y \in X$.

Definition 1.5. Let (X,d) be a metric space, $T : X \to X$ be a given mapping and $\alpha : X \times X \longrightarrow [0,\infty)$ be a function. We say that T is α-admissible, if

$$x,y \in X, \alpha(x,y) \ge 1 \Longrightarrow \alpha(Tx,Ty) \ge 1.$$

In 2019, M. Imdad and his colleagues introduced the concept of $(F,\varphi,\alpha-\psi)$-contractive mappings as follow:

Definition 1.6. [9] Let (X,d) be a metric space, $F \in \mathcal{F}_M$ and $T : X \longrightarrow X$ be a given mapping. We say that T is a $(F,\varphi,\alpha-\psi)$-contractive mapping if there exist functions $\alpha : X \times X \longrightarrow [0,\infty), \varphi : X \longrightarrow [0,\infty)$ and $\psi \in \Psi$ satisfying the following inequality

$$\alpha(x,y)F(d(Tx,Ty),\varphi(Tx),\varphi(Ty)) \leq \Psi(F(d(x,y),\varphi(x),\varphi(y))). \tag{1.2}$$

for all $x, y \in X$.

2. φ-fixed points for generalized $(F, \varphi, \alpha - \psi)$-contractive mappings

In this section, we introduce the notion of generalized $(F, \varphi, \alpha - \psi)$-contractive mappings and prove some fixed point theorems related these mappings.

Definition 2.1. Let (X, d) be a metric space and $\varphi : X \longrightarrow [0, \infty)$. The given mapping $T : X \longrightarrow X$ is called generalized $(F, \varphi, \alpha - \psi)$-contractive mapping if there exist functions $\alpha : X \times X \longrightarrow [0, \infty)$, $\psi \in \Psi, F \in \mathcal{F}_M$ such that for all $x, y \in X$

$$\alpha(x,y)F(d(Tx,Ty),\varphi(Tx),\varphi(Ty)) \leq \psi(M_F^{\varphi}(x,y)), \tag{2.1}$$

where:

$$\begin{aligned} M_F^{\varphi}(x,y) = \max\{&F(d(x,y),\varphi(x),\varphi(y)), F(d(x,Tx),\\ &\varphi(x),\varphi(Tx)), F(d(y,Ty),\varphi(y),\varphi(Ty))\}. \end{aligned}$$

Remark 2.2. It is obvious that every $(F, \varphi, \alpha - \psi)$-contractive mapping is a generalized $(F, \varphi, \alpha - \psi)$-contractive mapping, but the converse is not true always (see example 2.8).

Theorem 2.3. *Let (X, d) be a complete metric space, $\varphi : X \to [0, \infty)$ and $T : X \longrightarrow X$ be a generalized $(F, \varphi, \alpha - \psi)$-contractive mapping satisfying the following conditions:*

(i) *T is α-admissible,*

(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$,*

(iii) *If $u, v \in F_T$, then $\alpha(u, v) \geq 1$,*

(iv) *T is continuous.*

Then T has a unique φ-fixed point.

Proof. Let $x_0 \in X$ be such that $\alpha(x_0, Tx_0) \geq 1$. Define a sequence $\{x_n\}$ in X such that for all $n \in \mathbb{N}$, $x_n = Tx_{n-1}$. First, we suppose $x_n \neq x_{n-1}$ for all $n \in \mathbb{N}$, otherwise T has trivially a fixed point. By using the fact that T is α-admissible, we write

$$\alpha(x_0, x_1) = \alpha(x_0, Tx_0) \geq 1 \Longrightarrow \alpha(Tx_0, Tx_1) = \alpha(x_1, x_2) \geq 1.$$

By induction, it follows that $\alpha(x_n, x_{n+1}) \geq 1$. Thus by applying (2.1), we have

$$\begin{aligned} F(d(x_n,x_{n+1}),\varphi(x_n),\varphi(x_{n+1})) &\leq \alpha(x_{n-1},x_n)F(d(x_n,x_{n+1}),\varphi(x_n),\varphi(x_{n+1})) \\ &\leq \psi(M_F^{\varphi}(x_{n-1},x_n)), \end{aligned} \tag{2.2}$$

where:

$$\begin{aligned}M_F^{\varphi}(x_{n-1},x_n) &= \max\{F(d(x_{n-1},x_n),\varphi(x_{n-1}),\varphi(x_n)),F(d(x_{n-1},Tx_{n-1}),\\ &\quad \varphi(x_{n-1}),\varphi(Tx_{n-1})),F(d(x_n,Tx_n),\varphi(x_n),\varphi(Tx_n))\}\\ &= \max\{F(d(x_{n-1},x_n),\varphi(x_{n-1}),\varphi(x_n)),F(d(x_n,x_{n+1}),\\ &\quad \varphi(x_n),\varphi(x_{n+1}))\}.\end{aligned}$$

If $M_F^{\varphi}(x_{n-1},x_n) = F(d(x_n,x_{n+1}),\varphi(x_n),\varphi(x_{n+1}))$, then by placing in (2.2) we have

$$F(d(x_n,x_{n+1}),\varphi(x_n),\varphi(x_{n+1})) \le \psi(F(d(x_n,x_{n+1}),\varphi(x_n),\varphi(x_{n+1}))).$$

Now since $F(d(x_n,x_{n+1}),\varphi(x_n),\varphi(x_{n+1})) \ge 0$, according to Lemma 1.3 and property F_1 of $\mathcal{F}$, we get contradiction. So

$$M_F^{\varphi}(x_{n-1},x_n) = F(d(x_{n-1},x_n),\varphi(x_{n-1}),\varphi(x_n)).$$

By placing in (2.2), we have

$$F(d(x_n,x_{n+1}),\varphi(x_n),\varphi(x_{n+1})) \le \psi(F(d(x_{n-1},x_n),\varphi(x_{n-1}),\varphi(x_n))).$$

Repeating the above procedure successively for $n \ge 1$, we obtain

$$F(d(x_n,x_{n+1}),\varphi(x_n),\varphi(x_{n+1})) \le \psi^n(F(d(x_0,x_1),\varphi(x_0),\varphi(x_1))). \tag{2.3}$$

By using the properties of $\mathcal{F}_M$, we have

$$\begin{aligned}d(x_n,x_{n+1}) &\le \max\{d(x_n,x_{n+1}),\varphi(x_n)\}\\ &\le \psi^n(F(d(x_0,x_1),\varphi(x_0),\varphi(x_1))).\end{aligned}$$

From above equations for all $k,n \in \mathbb{N}$,

$$\begin{aligned}d(x_n,x_{n+k}) &\le \sum_{p=n}^{n+k-1} d(x_p,x_{p+1})\\ &\le \sum_{p=n}^{n+k-1} \psi^p(F(d(x_0,x_1),\varphi(x_0),\varphi(x_1)))\\ &\le \sum_{p=n}^{\infty} \psi^p(F(d(x_0,x_1),\varphi(x_0),\varphi(x_1))).\end{aligned}$$

Now letting $n \to \infty$, we will conclude that $\{x_n\}$ is a Cauchy sequence. On the other hand (X,d) is a complete metric space, so there exists $u \in X$ such that

$$\lim_{n\to\infty} d(x_n,u) = 0, \tag{2.4}$$

so $\lim_{n\to\infty} x_n = u$. On the other hand, T is continuous, therefore

$$\lim_{n\to\infty} x_{n+1} = Tu \Longrightarrow Tu = u \Longrightarrow u \in F_T.$$

By hypothesis of theorem we have $\alpha(u,u) \ge 1$. Therefore we get

$$F(d(Tu,Tu),\varphi(Tu),\varphi(Tu)) \le \alpha(u,u)F(d(Tu,Tu),\varphi(Tu),\varphi(Tu)) \le \psi(M_F^{\varphi}(u,u)),$$

where

$$M_F^{\varphi}(u,u) = \max\{F(d(u,u),\varphi(u),\varphi(u)), F(d(u,Tu),\varphi(u),\varphi(Tu)),$$
$$F(d(u,Tu),\varphi(u),\varphi(Tu))\}$$
$$= F(0,\varphi(u),\varphi(u)),$$

hence we have $F(0,\varphi(u),\varphi(u)) \leq \psi(F(0,\varphi(u),\varphi(u)))$.

If $F(0,\varphi(u),\varphi(u)) > 0$, so by property of ψ, we get contradiction. So $F(0,\varphi(u),\varphi(u)) = 0$. According the property F_1 of $\mathcal{F}$ and since $\varphi(u) \geq 0$, we conclude that $\varphi(u) = 0$.

Now, we want to prove that φ-fixed point is unique. Let u, v be two distinctive φ-fixed points. So

$$F(d(u,v),\varphi(u),\varphi(v)) \leq \alpha(u,v)F(d(u,v),\varphi(u),\varphi(v)) \leq \psi(M_F^{\varphi}(u,v)).$$

On the other hand u, v are φ-fixed points, hence

$$M_F^{\varphi}(u,v) = \max\{F(d(u,v),\varphi(u),\varphi(v)), F(d(u,Tu),\varphi(u),\varphi(Tu)),$$
$$F(d(v,Tv),\varphi(v),\varphi(Tv))\}$$
$$= \max\{F(d(u,v),0,0), F(0,0,0), F(0,0,0)\}$$
$$= F(d(u,v),0,0).$$

Therefore:

$$F(d(u,v),\varphi(u),\varphi(v)) = F(d(u,v),0,0) \leq \psi(F(d(u,v),0,0)) < F(d(u,v),0,0).$$

That is contradiction, hence φ-fixed point is unique. □

Corollary 2.4. *[9] Let (X,d) be a complete metric space and $T : X \longrightarrow X$ be a continuous $(F,\varphi,\alpha-\psi)$-contractive mapping satisfying the following conditions:*

(i) *T is α-admissible,*

(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$,*

(iii) *if $u, v \in F_T$, then $\alpha(u,v) \geq 1$,*

Then T has a unique φ-fixed point.

Corollary 2.5. *[8] Let (X,d) be a complete metric space, $F \in \mathcal{F}$, $\psi \in \Psi$ and $T : X \longrightarrow X$ be a continuous mapping such that for all $x, y \in X$ we have:*

$$F(d(Tx,Ty),\varphi(Tx),\varphi(Ty)) \leq \psi(F(d(x,y),\varphi(x),\varphi(y))). \tag{2.5}$$

Then T has a unique φ-fixed point.

In Eq (2.1), if $\alpha(x, y) = 1$ for all $x, y \in X$, then we get the following definition, that is special case of definition 2.1.

Definition 2.6. Let (X, d) be a metric space and $T : X \longrightarrow X$ be a given mapping. We say that T is a generalized (F, φ, ψ)-contractive mapping if there exist functions $F \in \mathcal{F}_M$, $\psi \in \Psi$ satisfying the following inequality

$$F(d(Tx, Ty), \varphi(Tx), \varphi(Ty)) \le \psi(M_F^{\varphi}(x, y)).$$

for all $x, y \in X$.

Corollary 2.7. *Let (X, d) be a complete metric space and $T : X \longrightarrow X$ be a continuous generalized (F, φ, ψ)-contractive mapping and $F \in \mathcal{F}$. Then T has a unique φ-fixed point.*

The following example shows the usefulness of Definition 2.1. In fact in this example we show that the mapping T does not satisfy the Banach contraction principle but satisfies the inequality (2.1).

Example 2.8. Let $X = [0, \frac{3}{2}]$ and define $d : X \times X \to \mathbb{R}$ by $d(x, y) = |x - y|$, for all $x, y \in X$. It is obvious that (X, d) is a complete metric space. We define the mapping $T : X \to X$ by:

$$Tx = \begin{cases} \dfrac{x}{2(x+1)}, & 0 \le x \le 1 \\ 2x - \dfrac{7}{4}, & 1 < x \le \frac{3}{2}. \end{cases}$$

At first we show that T does not satisfy the Banach contraction principle. In order to see this, consider $x, y > 1$, then, by applying the following equation

$$d(Tx, Ty) \le kd(x, y),$$

we get $k \ge 2$, which gives a contradiction to the fact that $0 < k < 1$.

Now we want to prove that mapping T satisfies the hypotheses of theorem 2.3. We define the mappings $F : [0, \infty)^3 \to [0, \infty)$ by $F(a, b, c) = a + b + c, \psi : [0, \infty) \to [0, \infty)$ by $\psi(t) = \dfrac{3}{4}t$ and $\varphi : X \to [0, \infty)$ by:

$$\varphi(x) = \begin{cases} 4, & x = \dfrac{1}{4} \\ x, & otherwise. \end{cases}$$

Also define:

$$\alpha(x, y) = \begin{cases} 0, & x \; or \; y \in (1, \frac{3}{2}] \\ 1, & otherwise. \end{cases}$$

To show that T is generalized $(F, \varphi, \alpha - \psi)$-contractive map, we have to check the inequality (2.1). We distinguish three cases:

(1) If $x = y = 1$ then

$$\begin{aligned} \alpha(x, y)F(d(Tx, Ty), \varphi(Tx), \varphi(Ty)) &= |Tx - Ty| + \varphi(Tx) + \varphi(Ty) \\ &= \left| \frac{1}{4} - \frac{1}{4} \right| + 1 + 1 = 2. \end{aligned} \tag{2.6}$$

On the other hand,

$$\begin{aligned}M_F^{\varphi}(x,y) &= \max\{F(d(x,y),\varphi(x),\varphi(y)), F(d(x,Tx),\varphi(x),\varphi(Tx)),\\ &\quad F(d(y,Ty),\varphi(y),\varphi(Ty))\}\\ &= \max\left\{2, \frac{11}{4}, \frac{11}{4}\right\} = \frac{11}{4}.\end{aligned}\tag{2.7}$$

By replacing (2.6) and (2.7) in Eq (2.1) we have $2 \leq \frac{3}{4} \times \frac{11}{4}$.

(2) If $x, y \in [0,1)$ and $x, y \neq \dfrac{1}{4}$, then we have

$$\begin{aligned}\alpha(x,y)F(d(Tx,Ty),\varphi(Tx),\varphi(Ty)) &= |Tx - Ty| + \varphi(Tx) + \varphi(Ty)\\ &= \left|\frac{x}{2(x+1)} - \frac{y}{2(y+1)}\right| + \frac{x}{2(x+1)} + \frac{y}{2(y+1)}\\ &= \frac{y}{y+1}.\end{aligned}\tag{2.8}$$

and

$$\begin{aligned}M_F^{\varphi}(x,y) &= \max\{|x-y| + \varphi(x) + \varphi(y), |x - Tx| + \varphi(x) + \varphi(Tx),\\ &\quad |y - Ty| + \varphi(y) + \varphi(Ty)\}\\ &= \max\{2y, \mid x - \frac{x}{2(x+1)} \mid +x + \frac{x}{2(x+1)}, \mid y - \frac{y}{2(y+1)} \mid +y + \frac{y}{2(y+1)}\}\\ &= \max\{2y, 2x, 2y\}.\end{aligned}\tag{2.9}$$

By replacing (2.8) and (2.9) in inequality (2.1), and since $\frac{1}{y+1} < \frac{3}{2}$, we have $\dfrac{y}{y+1} \leq \dfrac{3}{4}\max\{2x, 2y\}$.

(3) If $x, y \in [0,1)$ and $x = \dfrac{1}{4}$, and $y \neq \dfrac{1}{4}$ then we have:

$$\alpha(x,y)F(d(Tx,Ty),\varphi(Tx),\varphi(Ty)) = \left|\frac{1}{10} - \frac{y}{2(y+1)}\right| + \frac{1}{10} + \frac{y}{2(y+1)}.\tag{2.10}$$

Now, in (2.10), we have two cases:

(a) If assume that $\dfrac{1}{10} < \dfrac{y}{2(y+1)}$, then $\alpha(x,y)F(d(Tx,Ty),\varphi(Tx),\varphi(Ty)) = \dfrac{y}{y+1}$,

(b) If assume that $\dfrac{y}{2(y+1)} < \dfrac{1}{10}$, then $\alpha(x,y)F(d(Tx,Ty),\varphi(Tx),\varphi(Ty)) = \dfrac{1}{5}$.

On the other hand, we can see that

$$M_F^{\varphi}(x,y) = \max\left\{2y + \frac{3}{4}, \frac{5}{4}, 2y\right\},\tag{2.11}$$

and in (2.11), we have two cases:

(i) If $\dfrac{1}{4} < y < 1$, then $M_F^{\varphi}(x,y) = 2y + \dfrac{3}{4}$. So by applying (2.1), we get a true result.

(ii) If $0 \le y < \dfrac{1}{4}$ then $M_F^{\varphi}(x,y) = \dfrac{5}{4}$ and by replacing in (2.1), we get a true result.

In the same way, we can show that for other cases, inequality (2.1) holds and T is generalized $(F, \varphi, \alpha - \psi)$-contractive map. It is easy to see that T satisfies all conditions of theorem 2.3 and $x = 0$ is a φ-fixed point of T .

We can see that corollary 2.4 is not applicable in this example. To see this, consider $x = y = 1$, then by applying inequality (1.2), we get $2 \le \frac{3}{2}$, that is not true.

Also, T does not satisfy the inequality (1.1), indeed for $x = y = 1$, we get $k \ge 1$, which gives a contradiction to the fact that $0 < k < 1$.

Theorem 2.9. *Let (X, d) be a complete metric space, $\varphi : X \to [0, \infty)$, $F \in \mathcal{F}_{\mathcal{M}}$ and $T, S : X \longrightarrow X$ be two mappings such that*

$$\alpha(x,y)F(d(Sx,Ty),\varphi(Sx),\varphi(Ty)) \le \psi(M_F^{\varphi}(x,y)), \tag{2.12}$$

where:

$$\begin{aligned} M_F^{\varphi}(x,y) = \max\{&F(d(x,y),\varphi(x),\varphi(y)), F(d(x,Sx),\varphi(x),\varphi(Sx)),\\ &F(d(y,Ty),\varphi(y),\varphi(Ty))\}. \end{aligned}$$

Also assume that conditions below hold:

(i) *if $\alpha(x,y) \ge 1$ then $\alpha(Sx,Ty) \ge 1$ or $\alpha(Tx,Sy) \ge 1$, for all $x, y \in X$,*

(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Sx_0) \ge 1$,*

(iii) *S, T are continuous.*

Then S, T have common φ-fixed point.

Proof. Define two sequences $\{x_{2n}\}$ and $\{x_{2n+1}\}$ such as :

$$x_{2n+1} = Sx_{2n}, x_{2n} = Tx_{2n-1}.$$

According the assumption

$$\alpha(x_0,x_1) = \alpha(x_0,Sx_0) \ge 1 \Longrightarrow \alpha(Sx_0,TSx_0) = \alpha(x_1,x_2) \ge 1.$$

By induction, we get

$$\alpha(x_n,x_{n+1}) \ge 1.$$

By the assumption, we have

$$\begin{aligned} &F(d(x_{2n+1},x_{2n+2}),\varphi(x_{2n+1}),\varphi(x_{2n+2}))\\ &= F(d(Sx_{2n},Tx_{2n+1}),\varphi(Sx_{2n}),\varphi(Tx_{2n+1})\\ &\le \alpha(x_{2n},x_{2n+1})F(d(Sx_{2n},Tx_{2n+1}),\varphi(Sx_{2n}),\varphi(Tx_{2n+1})\\ &\le \psi(M_F^{\varphi}(x_{2n},x_{2n+1})), \end{aligned}$$

where

$$\begin{aligned}M_F^{\varphi}(x_{2n},x_{2n+1}) = \max\{&F(d(x_{2n},x_{2n+1}),\varphi(x_{2n}),\varphi(x_{2n+1})),\\&F(d(x_{2n},Sx_{2n}),\varphi(x_{2n}),\varphi(Sx_{2n})),\\&F(d(x_{2n+1},Tx_{2n+1}),\varphi(x_{2n+1}),\varphi(Tx_{2n+1}))\}\\=\max\{&F(d(x_{2n},x_{2n+1}),\varphi(x_{2n}),\varphi(x_{2n+1})),\\&F(d(x_{2n+1},x_{2n+2}),\varphi(x_{2n+1}),\varphi(x_{2n+2}))\}.\end{aligned}$$

Now if

$$M_F^{\varphi}(x_{2n},x_{2n+1}) = F(d(x_{2n+1},x_{2n+2}),\varphi(x_{2n+1}),\varphi(x_{2n+2})),$$

that is contradiction, so

$$M_F^{\varphi}(x_{2n},x_{2n+1}) = F(d(x_{2n},x_{2n+1}),\varphi(x_{2n}),\varphi(x_{2n+1})).$$

Hence

$$F(d(x_{2n+1},x_{2n+2}),\varphi(x_{2n+1}),\varphi(x_{2n+2})) \leq \psi(F(d(x_{2n},x_{2n+1}),\varphi(x_{2n}),\varphi(x_{2n+1}))).$$

Now fixed $\epsilon > 0$, so there exists $n(\epsilon) \in \mathbb{N}$ such that

$$\sum_{n\geq n(\varepsilon)} \psi^n(F(d(x_0,x_1),\varphi(x_0),\varphi(x_1))) < \varepsilon.$$

By repeating this process for each $n \in \mathbb{N}$, we get

$$\begin{aligned}\max\{d(x_n,x_{n+1}),\varphi(x_n)\} &\leq F(d(x_n,x_{n+1}),\varphi(x_n),\varphi(x_{n+1}))\\&\leq \psi^n(F(d(x_0,x_1),\varphi(x_0),\varphi(x_1))).\end{aligned}\tag{2.13}$$

Assume that $n, m \in \mathbb{N}$ and $m > n > n(\epsilon)$. By using (2.13), we get

$$\begin{aligned}d(x_n,x_m) &\leq \sum_{k=n}^{m-1} d(x_k,x_{k+1})\\&\leq \sum_{k=n}^{m-1} \psi^k(F(d(x_0,x_1),\varphi(x_0),\varphi(x_1)))\\&\leq \sum_{k=n}^{\infty} \psi^k(F(d(x_0,x_1),\varphi(x_0),\varphi(x_1))) < \varepsilon.\end{aligned}$$

Therefore $\{x_n\}$ is a Cauchy sequence in complete metric space (X,d). So there exists $u \in X$ such that

$$\lim_{n\to\infty} d(x_n,u) = 0.$$

So $\lim_{n\to\infty} x_n = u$. On the other hand, T is continuous, therefore

$$\lim_{n\to\infty} d(T_{x_{2n}},u) = 0 \Longrightarrow \lim_{n\to\infty} d(T_{x_{2n-1}},u) = 0.$$

On the other hand $\lim_{n\to\infty} d(T_{x_{2n-1}},Tu) = 0$. So $Tu = u$. By the same, we have $Su = u$. Such as proof of theorem 2.3 we can see that u is a common φ-fixed point for S, T.

□

The following example is concerned with theorem 2.9.

Example 2.10. Let $X = [0, 0.9]$ and define $d : X \times X \to \mathbb{R}$ by $d(x, y) = |x - y|$. It is obvious that (X, d) is a complete metric space. We define the mapping $T, S : X \to X$ by $Tx = \sqrt{1 - x^2} - \frac{1}{5}$ and $Sy = \frac{3}{5}\log(11 - \frac{5}{3}y)$. Now consider the mappings $F : [0, \infty)^3 \to [0, \infty)$ by $F(a, b, c) = a + b + c$, $\varphi : X \to [0, \infty)$ by

$$\varphi(x) = \begin{cases} -\frac{5}{3}x + 1, & 0 \le x \le 0.6 \\ \\ 2x, & 0.6 < x \le 0.9, \end{cases}$$

and $\psi : [0, \infty) \to [0, \infty)$ by $\psi(t) = \frac{9}{10}t$. Also we have

$$\alpha(x, y) = \begin{cases} 0, & (x, y) \in 0.6 \times [0, 0.4] \cup (0.5, 0.6) \ or \ 0 \le x < 0.6 \\ 1, & otherwise. \end{cases}$$

We want to show S, T satisfy all conditions of theorem 2.9. First we show that S, T satisfy the inequality (2.12). To see this, consider the following cases:

(1) If $x = 0.6$ and $0.4 \le y \le 0.5$, then $Tx = 0.6$ and $0.604 \le Sy \le 0.609$. So we have

$$\begin{aligned} d(Tx, Sy) + \varphi(Tx) + \varphi(Sy) &= \frac{3}{5}\log(11 - \frac{5}{3}y) - 0.6 + 0 + 2(\frac{3}{5}\log(11 - \frac{5}{3}y)) \\ &< 1.227. \end{aligned}$$

On the other hand

$$d(x, y) + \varphi(x) + \varphi(y) = 0.6 - y + 0 - \frac{5}{3}y + 1 = 1.6 - \frac{8}{3}y > 0.266, \tag{2.14}$$

and

$$\begin{aligned} d(y, Sy) + \varphi(y) + \varphi(Sy) &= \frac{3}{5}\log(11 - \frac{5}{3}y) - y - \frac{5}{3}y + 1 + 2(\frac{3}{5}\log(11 - \frac{5}{3}y)) \\ &> 1.478, \end{aligned} \tag{2.15}$$

From (2.14) and (2.15), we get $M_F^{\varphi}(x, y) > 1.478$. Thus we have

$$\begin{aligned} \alpha(x, y)F(d(Tx, Sy), \varphi(Tx), \varphi(Sy)) &< 1.227 \\ &< \frac{9}{10} \times 1.478 \\ &\le \psi(M_F^{\varphi}(x, y)). \end{aligned}$$

(2) If $x, y \in (0.6, 0.9)$, we see $Tx < Sy$, hence we have

$$d(Tx, Sy) + \varphi(Tx) + \varphi(Sy) = \frac{3}{5}\log(11 - \frac{5}{3}y) - \sqrt{1 - x^2} + \frac{1}{5}. \tag{2.16}$$

Since $0.225 < Tx < 0.6$ and $0.586 < Sy < 0.6$, in (2.16) we get

$$d(Tx, Sy) + \varphi(Tx) + \varphi(Sy) < 0.984.$$

On the other hand

$$d(x, Tx) + \varphi(x) + \varphi(Tx) = x - \sqrt{1-x^2} + \frac{1}{5} + 2x - \frac{5}{3}(\sqrt{1-x^2} - \frac{1}{5}) + 1 > 1.2.$$

So, $M_F^{\varphi}(x, y) > 1.2$. Thus we have

$$\begin{aligned} \alpha(x, y)F(d(Tx, Sy), \varphi(Tx), \varphi(Sy)) &< 0.984 \\ &< \frac{9}{10} \times 1.2 \\ &\leq \frac{9}{10}\psi(M_F^{\varphi}(x, y)). \end{aligned}$$

(3) If $x = 0.6$ and $y = 0.9$, then $Tx = 0.6$ and $Sy = 0.586$. So we have:

$$d(Tx, Sy) + \varphi(Tx) + \varphi(Sy) = 0.6 - 0.586 + 0 + 0.023 = 1.209$$

On the other hand

$$d(y, Sy) + \varphi(y) + \varphi(Sy) = 2.137,$$

hence $M_F^{\varphi}(x, y) = 2.137$. So we have

$$\begin{aligned} \alpha(x, y)F(d(Tx, Sy), \varphi(Tx), \varphi(Sy)) &= 1.209 \\ &< 2.137 \\ &\leq \psi(M_F^{\varphi}(x, y)). \end{aligned}$$

In the same way, we can show that for other cases, inequality (2.12) holds. It is easy to see that T and S satisfy all conditions of theorem 2.9 and have a common φ-fixed point $x = 0.6$.

3. Applications

3.1. Results in partial metric space

In this section we have some applications of contractive mappings in partial metric space. Here, we recall some definitions and some properties of partial metric spaces.

Definition 3.1. [10] Supposing X is a nonempty set, the function $\rho : X \times X \to [0, \infty)$ is the partial metric on X, if for all $x, y, z \in X$ we have:

1. $\rho(x, x) = \rho(y, y) = \rho(x, y)$ if and only if $x = y$
2. $\rho(x, x) \leq \rho(x, y)$
3. $\rho(x, y) = \rho(y, x)$
4. $\rho(x, y) \leq \rho(x, z) + \rho(z, y) - \rho(z, z)$

and (X,ρ) is a partial metric space.

Corollary 3.2. *[10] If ρ is a partial metric space and X is a nonempty set, then the function $d_\rho : X\times X \to X$ by*

$$d_\rho(x,y) := 2\rho(x,y) - \rho(x,x) - \rho(y,y)$$

is a metric on X.

Definition 3.3. [10] Let (X,ρ) be a partial metric space.

1. A sequence $\{x_n\}$ in X is called Cauchy sequence if there exists $r \in \mathbb{R}^+$ such that $\lim_{n,m\to\infty} \rho(x_n, x_m) = r$.
2. A sequence $\{x_n\}$ in X is called to converge in (X,ρ) if there exists $x \in X$ such that $\rho(x,x) = \lim_{n\to\infty} \rho(x, x_n) = \lim_{n\to\infty} \rho(x_n, x)$.
3. (X,ρ) is called to be complete if every Cauchy sequence in X converges in (X,ρ).

Lemma 3.4. *[10] Let (X,ρ) is a partial metric space. Then (X,ρ) is complete if and only if (X,d_ρ) be complete.*

More properties and results concerned with completeness of partial metric space can be found in [11].

We denote by $\mathbb{G}$ the family of all functions $g : [0,\infty) \longrightarrow [0,\infty)$ satisfying the following conditions:

(g_1) $\max\{a,b\} \le g(a+b+c), \forall a,b,c \in [0,\infty)$;

(g_2) $g(0) = 0$;

(g_2) $\lim_{n\longrightarrow\infty} \sup g(a_n) \le g(a)$, when $a_n \to a$ as $n \to \infty$.

Example 3.5. Consider $g : [0,\infty) \longrightarrow [0,\infty)$ by $g(a) = a$ for all $a \in [0,\infty)$. It is obvious that $g \in \mathbb{G}$.

Now, we are trying to obtain a partial metric version of theorem 2.3, as follow:

Theorem 3.6. *Let (X,ρ) be a complete partial metric space, $g \in \mathbb{G}$ and $T : X \to X$ be a α-admissible mapping such that for $\psi \in \Psi$ and for all $x,y \in X$:*

$$\alpha(x,y)g(\rho(Tx,Ty)) \le \psi(\max\{g(\rho(x,y)), g(\rho(x,Tx)), g(\rho(y,Ty))\}). \tag{3.1}$$

Also we assume that:

(i) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \ge 1$,*

(ii) *if $u,v \in F_T$, then $\alpha(u,v) \ge 1$,*

(iii) *T is continuous.*

Then T has a unique fixed point x^* and $\rho(x^*, x^*) = 0$.

Proof. According the Lemma 3.4, (X,d_ρ) is a complete metric space. Also by using the corollary 3.2, we have

$$\rho(x,y)=\frac{d_\rho(x,y)}{2}+\frac{\rho(y,y)}{2}+\frac{\rho(x,x)}{2}$$

Now set: $\varphi(x)=\dfrac{\rho(x,x)}{2}$, $d(x,y)=\dfrac{d_\rho(x,y)}{2}$. So

$$\rho(x,y)=d(x,y)+\varphi(x)+\varphi(y).$$

Now by applying (3.1), we have

$$\begin{aligned}&\alpha(x,y)g(d(Tx,Ty)+\varphi(Tx)+\varphi(Ty))\\&\le\psi\Big(\max\{g(d(x,y)+\varphi(x)+\varphi(y)),g(d(x,Tx)+\varphi(x)+\varphi(Tx)),\\&g(d(y,Ty)+\varphi(y)+\varphi(Ty))\}\Big).\end{aligned}\tag{3.2}$$

Now define the mapping $F:[0,\infty)^3\to[0,\infty)$ by $F(a,b,c)=g(a+b+c)$. If $a_n\to a, b_n\to b$, then

$$\lim_{n\to\infty}\sup F(a_n,b_n,0)=\lim_{n\to\infty}\sup g(a_n+b_n+0)\le g(a+b)=F(a,b,0).$$

Hence $F\in F_M$ and by (3.2) we get

$$\alpha(x,y)F(d(Tx,Ty),\varphi(Tx),\varphi(Ty))\le\psi(M_F^\varphi(x,y)).$$

So according to the theorem 2.3, T has a φ-fixed point x^*. Hence $Tx^*=x^*$ and therefore $\rho(x^*,x^*)=0$. □

Corollary 3.7. *Let (X,ρ) be a complete partial metric space, $g\in\mathbb{G}$ and $T:X\to X$ be a α-admissible mapping such that for $\psi\in\Psi$ and for all $x,y\in X$:*

$$\alpha(x,y)g(\rho(Tx,Ty))\le\psi(g(\rho(x,y))).\tag{3.3}$$

also we assume that the conditions below hold:

(i) *there exists $x_0\in X$ such that $\alpha(x_0,Sx_0)\ge 1$,*

(ii) *if $u,v\in F_T$, then $\alpha(u,v)\ge 1$,*

(iii) *T is continuous,*

then T has a unique fixed point x^* and $\rho(x^*,x^*)=0$.

Remark 3.8. It is obvious that if mapping T satisfies the (3.3), it also satisfies the (3.1), but the converse is not true always.

In the theorem 3.6, if $g(a)=a$, we get the following result:

Theorem 3.9. *Let (X,ρ) be a complete partial metric space, $g\in\mathbb{G}$ and $T:X\to X$ be a α-admissible mapping such that for $\psi\in\Psi$ and for all $x,y\in X$:*

$$\alpha(x,y)\rho(Tx,Ty) \leq \psi(\max\{\rho(x,y),\rho(x,Tx),\rho(y,Ty)\}), \tag{3.4}$$

also we assume that:

(i) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$,*

(ii) *if $u, v \in F_T$, then $\alpha(u, v) \geq 1$,*

(iii) *T is continuous,*

then T has a unique fixed point x^* and $\rho(x^*, x^*) = 0$.

Remark 3.10. If $\max\{\rho(x,y),\rho(x,Tx),\rho(y,Ty)\}) = \rho(x,y)$, then condition (3.4) reduces to condition (3.5) given below.

Corollary 3.11. *Let (X,ρ) be a complete partial metric space, $g \in \mathbb{G}$ and $T : X \to X$ be a α-admissible mapping such that for $\psi \in \Psi$ and for all $x, y \in X$:*

$$\alpha(x,y)\rho(Tx,Ty) \leq \psi(\rho(x,y)). \tag{3.5}$$

also we assume that:

(i) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$,*

(ii) *if $u, v \in F_T$, then $\alpha(u, v) \geq 1$,*

(iii) *T is continuous,*

then T has a unique fixed point x^* and $\rho(x^*, x^*) = 0$.

Recently, partial metric spaces have been generalized to partial tvs-cone metric spaces (see [12]), now an open question arises: Can we prove the above results in partial tvs-cone metric space or in generalized metric space? (see [13])

3.2. Results on integral equations

Now, we discuss the existence of solution for the following nonlinear integral equation:

$$u(t) = v(t) + \int_a^t f(t,z,u(z))dz \tag{3.6}$$

such that $a \in \mathbb{R}, u \in C[a,b]$, $v : [a,b] \to \mathbb{R}$ and $f : [a,b] \times [a,b] \times \mathbb{R} \to \mathbb{R}$. Let $X = C([a,b],\mathbb{R})$ and $d : X \times X \to [0,\infty)$ is defined by $d(u,w) = \|u-w\|_\infty$. It is obvious that (X,d) is a complete metric space.

Theorem 3.12. *Assume that $T : X \to X$ is a mapping such that for all $u \in X$:*

$$Tu(t) = v(t) + \int_a^t f(t,z,u(z))dz.$$

Let $\xi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a given function and for Eq (3.6)

(i) *$f : [a,b] \times [a,b] \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous.*

(ii) *there exists $\psi \in \Psi$ such that, for $u, w \in X$:*

$$|f(t,z,u(z)) - f(t,z,w(z))| \leq \frac{1}{b-a}\psi(\max\{| u(z) - w(z) |, | u(z) - Tu(z) |, | w(z) - Tw(z) |\}),$$

(iii) *if $\xi(u(t), w(t)) \geq 0$ then $\xi(Tu(t), Tw(t)) \geq 0$, for all $u, w \in X$ and $t \in [a,b]$*

(iv) *there exists $u_0 \in X$, such that $\xi(u_0(t), Tu_0(t)) \geq 0$ for all $t \in [a,b]$,*

(v) *if $u, w \in F_T$ then $\xi(u(t), w(t)) \geq 0$ for all $t \in [a,b]$.*

Then the nonlinear integral (3.6) *has a unique solution.*

Proof. Consider the mapping $F : [0,\infty)^3 \to [0,\infty)$ by $F(a,b,c) = a+b+c$, for all $a,b,c \in [0,\infty)$. Also we define $\varphi : X \to [0,\infty)$ by $\varphi(u) = 0$, for all $u \in X$. Let $u, v \in X$ such that $\xi(u(t), w(t)) \geq 0$ for all $t \in [a,b]$. We assume that $\alpha : X \times X \to [0,\infty)$:

$$\alpha(u,w) = \begin{cases} 1, & \xi(u(t), w(t)) \geq 0, t \in [a,b] \\ 0, & otherwise. \end{cases}$$

Now we want to show that T is a generalized $(F, \varphi, \alpha - \psi)$-contractive mapping. Assume that $u, w \in X$ and $t \in [a,b]$. Hence

$$\begin{aligned} |Tu(t) - Tw(t)| &= |\int_a^t f(t,z,u(z))dz - \int_a^t f(t,z,w(z))dz| \\ &\leq \int_a^t |f(t,z,u(z)) - f(t,z,w(z))|dz \\ &\leq \frac{1}{b-a}\psi(\max\{| u(z) - w(z) |, | u(z) - Tu(z) |, | w(z) - Tw(z) |\}) \int_a^t dz \\ &\leq \psi(\max\{d(u,w), d(u,Tu), d(w,Tw)\}). \end{aligned}$$

Therefore for all $u, w \in X$:

$$d(Tu, Tw) \leq \psi(\max\{d(u,w), d(u,Tu), d(w,Tw)\}).$$

So

$$\begin{aligned} \alpha(u,w)F(d(Tu,Tw), \varphi(Tu), \varphi(Tw)) \leq \psi(\max\{&F(d(u,w), \varphi(u), \varphi(w)), F(d(u,Tu), \varphi(u), \varphi(Tu)), \\ &F(d(w,Tw), \varphi(w), \varphi(Tw))\}) \\ = \psi(&M_F^{\varphi}(u,w)). \end{aligned}$$

So, according to the theorem 2.3, T has a φ-fixed point, therefore the integral equation has a unique solution. We can see easily that the theorem 3.12 does not hold in the case of inequality (1.2). □

4. Conclusions

In this paper, we investigate the existence of φ-fixed point for a generalized $(F, \varphi, \alpha - \psi)$-contractive mappings. Also we provide the conditions which ensure that two maps have a common φ-fixed point. On the other hand, we present some applications of contractive mappings in partial metric space and existence of solution for the nonlinear integral equation.

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Conflict of interest

The author declares no conflict of interest in this paper.

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