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Research article

Three effective preconditioners for double saddle point problem

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Abstract: In this paper, we mainly propose three preconditioners for solving double saddle point problems, which arise from some practical problems. Firstly, the solvability of this kind of problem is investigated under suitable assumption. Next, we prove that all the eigenvalues of the three preconditioned matrices are 1. Furthermore, we analyze the eigenvector distribution and the upper bound of the minimum polynomial degree of the corresponding preconditioned matrix. Finally, numerical experiments are carried to show the effectiveness of the proposed preconditioners.

Keywords: double saddle point problems; preconditioning; spectral properties; Krylov subspace method
Mathematics Subject Classification: 65F10, 65F08, 65F50

1. Introduction

In this paper, we consider iterative methods for solving the following double saddle point problems:

$$\mathcal{B}u = \begin{pmatrix} A & B^T & C^T \\ B & 0 & 0 \\ C & 0 & -D \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} f \\ g \\ h \end{pmatrix} \equiv \bar{b}, \tag{1.1}$$

where $A \in \mathbb{R}^{n\times n}$ and $D \in \mathbb{R}^{p\times p}$ are symmetric positive definite (SPD) matrices, $B \in \mathbb{R}^{m\times n}$ is of full row rank, $C \in \mathbb{R}^{p\times n}$ is a rectangular matrix and $n \geq m + p$. Linear systems of the form (1.1) have wide application background in many fields of science and engineering, such as mixed element approximation of fluid flow problem [7, 13], finite element model of liquid crystal [9], the interior point method of quadratic programming problem [10, 15], mixed formula of second order elliptic equation [7, 9].

Obviously, we can decompose the coefficient matrix in (1.1) into block two-by-two forms and choose solution methods for standard saddle point problems [6] to solve the corresponding double saddle point problems. However, owing to its more complex structure, straightforward application of the solution method for standard block two-by-two saddle point problems usually leads to poor numerical performance. In recent years, solvers have devoted to finding a solution to the double saddle point problems (1.1). In 2018, Beik and Benzi [3] discussed the Krylov subspace method of block preconditioners in detail, besides, they also introduced the solvable conditions of coefficient matrix and proposed the block preconditioners for solving the double saddle point problems (1.1) [4]. In 2019, Benzi and Beik [5] proposed Uzawa-type and augmented Lagrangian method for solving double saddle point systems. Furthermore, Liang and Zhang [8] proposed alternating positive semi-definite splitting (APSS) preconditioners, and proved that the corresponding iterative scheme was unconditionally convergent; in order to improve its effectiveness, the relaxed APSS preconditioners was given.

It is worth noting that the linear systems (1.1) can be equivalently restated as

$$\mathcal{A}u = \begin{pmatrix} D & -C & 0 \\ C^T & A & B^T \\ 0 & -B & 0 \end{pmatrix} \begin{pmatrix} z \\ x \\ y \end{pmatrix} = \begin{pmatrix} -h \\ f \\ -g \end{pmatrix} \equiv b, \tag{1.2}$$

which assists that we can use some specific iteration solution methods. Based on the linear systems (1.2), we start our discussions in this paper. Some solvability conditions of double saddle point problems (1.2) are discussed in detail [4].

In this work, we establish three new preconditioners for solving the double saddle point problems (1.2). Theoretical analysis shows that the eigenvalues of the three preconditioned matrices are all 1. In addition, we also obtain the eigenvector distribution and the upper bound of the minimum polynomial degree of the three preconditioned matrices.

The arrangement of this work is as follows. In Section 2, we discuss the condition of invertibility of matrix $\mathcal{A}$. In Section 3, we propose three new block preconditioners for matrix $\mathcal{A}$ and derive that the eigenvalues of the corresponding preconditioned matrices are all 1. At the end of Section 3, we also analyze the distribution of eigenvectors and the upper bound of the minimum polynomial order of the corresponding preconditioned matrices. Brief discussions are given in Section 4 about practical implementation of three preconditioners. In Section 5, numerical experiments are performed to demonstrate the effectiveness of the proposed preconditioners.

Notations. For given arbitrary matrix A, we often shall write its transpose, null space and range space as A^T, $N(A)$ and $R(A)$, respectively. Moreover, if A is symmetric positive (semi) definite, we write $A \succ 0 (A \succeq 0)$. Finally, we write $(x; y; z)$ to denote the vector $(x^T, y^T, z^T)^T$.

2. Invertibility conditions

In this section, we mainly investigate the solvability of the double saddle point problems (1.2). We know that Beik and Benzi discussed the invertibility condition of coefficient matrix of the double saddle point problems (1.2) in the literature [4], which discussed four cases as follow: (1). $A \succ 0$, $D \succ 0$; (2). $A \succ 0$, $D \succeq 0$; (3). $A \succeq 0$, $D \succ 0$; (4). $A \succ 0$, $D = 0$. It is necessary to improve the invertibility condition of coefficient matrix in (1.2) on the basis of it. Therefore, we introduce two necessary proposition for solving double saddle point problem (1.2).

Proposition 2.1 Assume $A \succeq 0$ and $D \succeq 0$, both B and C have full row rank. Then $\mathcal{A}$ is nonsingular if $R(B^T) \bigcap R(C^T) = \{0\}$ and $N(C) \bigcap N(A) \bigcap N(B) = \{0\}$.

Proof. Assume that $\mathcal{A}u = 0$ for $u = (x; y; z)$, i.e.,

$$\begin{cases} Dx - Cy = 0, \\ C^T x + Ay + B^T z = 0, \\ -By = 0. \end{cases} \tag{2.1}$$

Multiplying the first equation in (2.1) by x^T from the left, we have

$$x^T Dx - x^T Cy = 0. \tag{2.2}$$

Then multiplying the second equation in (2.1) by y^T from the left, we can obtain

$$y^T C^T x + y^T Ay + y^T B^T z = 0. \tag{2.3}$$

Substituting the third equation in (2.1) into (2.3), we deduce that

$$y^T C^T x + y^T Ay = 0. \tag{2.4}$$

Combining (2.2) and (2.4), we have $x^T Dx = -y^T Ay$.

$y^T Ay = 0$ and $x^T Dx = 0$ imply that $y \in N(A)$ and $x \in N(D)$ because of the symmetric positive semidefiniteness of A and D. Furthermore, the first equation in (2.1) becomes $-Cy = 0$, i.e., $y \in N(C)$. According to the third equation in (2.1), we can find $y \in N(B)$. Since $N(A) \bigcap N(B) \bigcap N(C) = \{0\}$, it is a fact that $y = 0$.

From the second equation in (2.1), we can get $C^T x + B^T z = 0$. Note that $C^T x + B^T z = 0$ together with the assumption $R(B^T) \bigcap R(C^T) = \{0\}$ implies that $C^T x = 0$ and $B^T z = 0$. Since B^T and C^T have full column rank, we have $x = 0$ and $z = 0$. Hence, $u = (x; y; z) = 0$, which implies that $\mathcal{A}$ is nonsingular. The proof is completed. □

Proposition 2.2 Let $D \succ 0$ and B has full row rank. Consider the linear system (1.2) with $A = 0$. Then $N(C) \bigcap N(B) = \{0\}$ is a necessary and sufficient condition for the coefficient matrix $\mathcal{A}$ to be invertible.

Proof. Assume that $\mathcal{A}u = 0$ for $u = (x; y; z)$, i.e.,

$$\begin{cases} Dx - Cy = 0, \\ C^T x + B^T z = 0, \\ -By = 0. \end{cases} \tag{2.5}$$

Multiplying the first equation in (2.5) by x^T from the left, we have

$$x^T Dx = x^T Cy. \tag{2.6}$$

Then multiplying by y^T from the left of the second equation in (2.5), we can obtain

$$y^T C^T x = -y^T B^T z = 0. \tag{2.7}$$

Substituting (2.6) into (2.7), we deduce that

$$x^T Dx = 0. \tag{2.8}$$

In view of the positive definiteness of D, the preceding equality implies that $x = 0$. Consequently, (2.5) reduces to

$$\begin{cases} -Cy = 0, \\ B^T z = 0, \\ -By = 0. \end{cases} \tag{2.9}$$

Because B has full row rank, it follows that $z = 0$ from the second equation in (2.9). From the first equation and the third equation in (2.9), we have $y \in N(C)$ and $y \in N(B)$. Since $N(C) \bigcap N(B) = \{0\}$, we can obtain $y \in N(C) \bigcap N(B) = \{0\}$. Hence, $u = (x; y; z)$ is the zero vector, which shows the invertibility of $\mathcal{A}$.

Finally, let us consider that $N(C) \bigcap N(B) = \{0\}$ is a necessary condition for the invertibility of $\mathcal{A}$. If there exists a nonzero vector $y \in N(C) \bigcap N(B)$, then for $u = (0; y; 0)$, we have $\mathcal{A}u = 0$, this leads to a contradiction. □

3. Three effective block preconditioners and spectral analysis of corresponding preconditioned matrix

In this section we describe three effective block preconditioners which can be used in Krylov subspace method for solving double saddle point peoblems (1.2). Inspired by the reference [12], we could establish three preconditioners. Firstly, by multiplying (1.2) from the left with the following invertible matrix

$$\mathcal{M} = \begin{pmatrix} I & 0 & 0 \\ -C^T D^{-1} & I & 0 \\ -B\widehat{A}^{-1} C^T D^{-1} & B\widehat{A}^{-1} & I \end{pmatrix},$$

the linear system (1.2) becomes to

$$\widetilde{\mathcal{A}} = \begin{pmatrix} D & -C & 0 \\ 0 & \widehat{A} & B^T \\ 0 & 0 & S_{\widehat{A}} \end{pmatrix} \begin{pmatrix} z \\ x \\ y \end{pmatrix} = \begin{pmatrix} -h \\ C^T D^{-1} h + f \\ -B\widehat{A}^{-1} C^T D^{-1} h + B\widehat{A}^{-1} f - g \end{pmatrix}, \tag{3.1}$$

where $\widehat{A} = A + C^T D^{-1} C$ and $S_{\widehat{A}} = B\widehat{A}^{-1} B^T$.

We propose the following three preconditioners for linear equation (3.1):

$$\widetilde{\mathcal{P}}_1 = \begin{pmatrix} D & 0 & 0 \\ 0 & \widehat{A} & 0 \\ 0 & 0 & S_{\widehat{A}} \end{pmatrix}, \quad \widetilde{\mathcal{P}}_2 = \begin{pmatrix} D & -C & 0 \\ 0 & \widehat{A} & 0 \\ 0 & 0 & S_{\widehat{A}} \end{pmatrix}, \quad \widetilde{\mathcal{P}}_3 = \begin{pmatrix} D & 0 & 0 \\ 0 & \widehat{A} & B^T \\ 0 & 0 & S_{\widehat{A}} \end{pmatrix}.$$

Furthermore, we propose three preconditioners for linear systems (1.2) by the expressions $\widetilde{\mathcal{P}}_1$, $\widetilde{\mathcal{P}}_2$ and $\widetilde{\mathcal{P}}_3$:

$$\mathcal{P}_1 = \begin{pmatrix} D & 0 & 0 \\ C^T & \widehat{A} & 0 \\ 0 & -B & S_{\widehat{A}} \end{pmatrix}, \quad \mathcal{P}_2 = \begin{pmatrix} D & -C & 0 \\ C^T & A & 0 \\ 0 & -B & S_{\widehat{A}} \end{pmatrix}, \quad \mathcal{P}_3 = \begin{pmatrix} D & 0 & 0 \\ C^T & \widehat{A} & B^T \\ 0 & -B & 0 \end{pmatrix}. \tag{3.2}$$

Next, we analyze the spectral properties of the preconditioned matrix using the preconditioner $\mathcal{P}_1, \mathcal{P}_2$ and $\mathcal{P}_3$. Firstly, the spectral distribution of $\mathcal{P}_1^{-1}\mathcal{A}$ is described as follow:

Theorem 3.1 Assume that $A \in \mathbb{R}^{n\times n}$ and $D \in \mathbb{R}^{p\times p}$ are symmetric positive definite matrix and $B \in \mathbb{R}^{m\times n}$ is of full row rank, $C \in \mathbb{R}^{p\times n}$ is a rectangular matrix and $n \geq m + p$. Then the eigenvalues of preconditioned matrix $\mathcal{P}_1^{-1}\mathcal{A}$ are all 1.

Proof. According to the above analysis, we have the following equation:

$$\widetilde{\mathcal{P}}_1^{-1}\widetilde{\mathcal{A}} = \widetilde{\mathcal{P}}_1^{-1}\mathcal{M}\mathcal{A},$$

then the preconditioners $\mathcal{P}_1$ of linear system (1.2) can be reformulated as

$$\mathcal{P}_1 = (\widetilde{\mathcal{P}}_1^{-1}\mathcal{M})^{-1} = \mathcal{M}^{-1}\widetilde{\mathcal{P}}_1.$$

Therefore,

$$\begin{aligned}
\mathcal{P}_1^{-1} = \widetilde{\mathcal{P}}_1^{-1}\mathcal{M} &= \begin{pmatrix} D^{-1} & 0 & 0 \\ 0 & \widehat{A}^{-1} & 0 \\ 0 & 0 & S_{\widehat{A}}^{-1} \end{pmatrix} \begin{pmatrix} I & 0 & 0 \\ -C^T D^{-1} & I & 0 \\ -B\widehat{A}^{-1}C^T D^{-1} & B\widehat{A}^{-1} & I \end{pmatrix} \\
&= \begin{pmatrix} D^{-1} & 0 & 0 \\ -\widehat{A}^{-1}C^T D^{-1} & \widehat{A}^{-1} & 0 \\ -S_{\widehat{A}}^{-1}B\widehat{A}^{-1}C^T D^{-1} & S_{\widehat{A}}^{-1}B\widehat{A}^{-1} & S_{\widehat{A}}^{-1} \end{pmatrix}.
\end{aligned}$$

Then the preconditioned matrix:

$$\begin{aligned}
\mathcal{P}_1^{-1}\mathcal{A} &= \begin{pmatrix} D^{-1} & 0 & 0 \\ -\widehat{A}^{-1}C^T D^{-1} & \widehat{A}^{-1} & 0 \\ -S_{\widehat{A}}^{-1}B\widehat{A}^{-1}C^T D^{-1} & S_{\widehat{A}}^{-1}B\widehat{A}^{-1} & S_{\widehat{A}}^{-1} \end{pmatrix} \begin{pmatrix} D & -C & 0 \\ C^T & A & B^T \\ 0 & -B & 0 \end{pmatrix} \\
&= \begin{pmatrix} I & -D^{-1}C & 0 \\ 0 & I & \widehat{A}^{-1}B^T \\ 0 & 0 & I \end{pmatrix}.
\end{aligned} \tag{3.3}$$

Therefore, the eigenvalues of preconditioned matrix $\mathcal{P}_1^{-1}\mathcal{A}$ are all 1. □

It is well known that the convergence of Krylov subspace methods is not only related to the eigenvalue distribution of the preconditioned matrix, but also related to the number of corresponding linearly independent eigenvectors. The eigenvector distribution of the preconditioned matrix $\mathcal{P}_1^{-1}\mathcal{A}$ is presented in the following theorem.

Theorem 3.2 Let the preconditioner $\mathcal{P}_1$ be defined as in (3.2), then the preconditioned matrix $\mathcal{P}_1^{-1}\mathcal{A}$ has $p + n - r(C)$ linearly independent eigenvectors. There are

(1) p eigenvectors $(x_l; 0; 0)(l = 1, 2, ..., p)$ that correspond to the eigenvalue 1, where $x_l(l = 1, 2, ..., p)$ are arbitrary linearly independent vectors;

(2) $n - r(C)$ eigenvectors $(x_l^1; y_l^1; 0)(l = 1, 2, ..., n)$ that correspond to the eigenvalue 1, where $y_l(l = 1, 2, ..., n)$ are arbitrary linearly independent vectors.

Proof. Let λ be an eigenvalue of the preconditioned matrix $\mathcal{P}_1^{-1}\mathcal{A}$ and $(x; y; z)$ be the corresponding eigenvector. From (3.3), we have

$$\begin{pmatrix} I & -D^{-1}C & 0 \\ 0 & I & \widehat{A}^{-1}B^T \\ 0 & 0 & I \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \\ z \end{pmatrix}. \tag{3.4}$$

Based on (3.4), it follows that

$$\begin{cases} (1-\lambda)x + D^{-1}Cy = 0, \\ (1-\lambda)y + \widehat{A}^{-1}B^T z = 0, \\ (1-\lambda)z = 0. \end{cases} \tag{3.5}$$

Because $\lambda = 1$, then Eq (3.5) become

$$\begin{cases} D^{-1}Cy = 0 \\ \widehat{A}^{-1}B^T z = 0 \end{cases} \tag{3.6}$$

From the second equation in (3.6), we get that $B^T z = 0$, i.e. $z = 0$.

When $y = 0$, Eq (3.5) are always true. Hence, there are p linearly independent eigenvectors $(x_l; 0; 0)(l = 1, 2, ..., p)$ corresponding to the eigenvalue 1, where $x_l(l = 1, 2, ..., p)$ are arbitrary linearly independent vectors.

When $y \neq 0$, there will be $n - r(C)$ linearly independent eigenvectors $(0; y_l^1; 0)(l = 1, 2, ..., n - r(C))$ corresponding to the eigenvalue 1, where x_l^1 are arbitrary vectors and y_l^1 satisfies $y_l^1 \in N(C)$.

Finally, we just need to verify that the $p + n - r(C)$ eigenvectors are linearly independent. Let $k = (k_1, k_2, ..., k_p)^T$, $k^1 = (k_1^1, k_2^1, ..., k_{n-r(C)}^1)^T$ be two vectors. Then we need to show that

$$\begin{pmatrix} x_1 & \cdots & x_p \\ 0 & \cdots & 0 \\ 0 & \cdots & 0 \end{pmatrix}\begin{pmatrix} k_1 \\ \vdots \\ k_p \end{pmatrix} + \begin{pmatrix} x_1^1 & \cdots & x_{n-r(C)}^1 \\ y_1^1 & \cdots & y_{n-r(C)}^1 \\ 0 & \cdots & 0 \end{pmatrix}\begin{pmatrix} k_1^1 \\ \vdots \\ k_{n-r(C)}^1 \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \tag{3.7}$$

holds true if and only if the vectors k, k^1 are all zero vectors, where the first matrix consists of the eigenvectors corresponding to the eigenvalue 1 for the case (1), the second matrix consists of those for the case (2). Because $y_1^1 k_1^1 + y_2^1 k_2^1 + ... + y_{n-r(C)}^1 k_{n-r(C)}^1 = 0$ and $y_l^1(l = 1, 2, ..., n - r(C))$ are linearly independent, we know that $k_l^1 = 0(l = 1, ..., n - r(C))$. Thus, Eq. (3.7) reduces to

$$\begin{pmatrix} x_1 & \cdots & x_p \\ 0 & \cdots & 0 \\ 0 & \cdots & 0 \end{pmatrix}\begin{pmatrix} k_1 \\ \vdots \\ k_p \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

Since $x_l(l = 1, 2, ..., p)$ are linearly independent, we have $k_l = 0(l = 1, ...p)$. Therefore, the $p + n - r(C)$ eigenvectors are linearly independent. □

The function of preconditioning is to make the eigenvalue distribution more clustered and to reduce the iteration number necessary for solving the corresponding peoblems within certain tolerance. Among all the iteration methods, Krylov subspace methods with favorable properties, such as the degree of the minimal polynomial is equal to the dimension of the corresponding Krylov subspace [1, 2, 14], were found to be extremely useful when combined with an appropriate preconditioner in solving the underlying system. In the following, we study an upper bound of the degree of the minimal polynomial of the preconditioned matrix $\mathcal{P}_1^{-1}\mathcal{A}$.

Theorem 3.3 Let the preconditioner $\mathcal{P}_1$ be defined as in (3.2). Then the degree of the minimal polynomial of the preconditioned matrix $\mathcal{P}_1^{-1}\mathcal{A}$ is at most 3.

Proof. From (3.3), we can get the characteristic polynomial of the matrix $\mathcal{P}_1^{-1}\mathcal{A}$ is

$$\Phi_{\mathcal{P}_1^{-1}\mathcal{A}}(\lambda) = det(\mathcal{P}_1^{-1}\mathcal{A} - \lambda I) = (-1)^{p+n+m}(\lambda - 1)^{p+n+m}.$$

Moreover, it is easy to check

$$\mathcal{P}_1^{-1}\mathcal{A} - I = \begin{pmatrix} 0 & -D^{-1}C & 0 \\ 0 & 0 & \widehat{A}^{-1}B^T \\ 0 & 0 & 0 \end{pmatrix},$$

$$(\mathcal{P}_1^{-1}\mathcal{A} - I)^2 = \begin{pmatrix} 0 & 0 & -D^{-1}C\widehat{A}^{-1}B^T \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and $(\mathcal{P}_1^{-1}\mathcal{A} - I)^3 = \mathbf{0}$. Therefore, the degree of the minimal polynomial of the preconditioned matrix $\mathcal{P}_1^{-1}\mathcal{A}$ is at most 3. □

Theorem 3.4 Under the assumptions of Theorem 3.1, the eigenvalues and degree of the minimal polynomial of preconditioned matrix $\mathcal{P}_2^{-1}\mathcal{A}$ are 1 and is at most 2, respectively. And the preconditioned matrix $\mathcal{P}_2^{-1}\mathcal{A}$ has $p + n$ linearly independent eigenvectors. There are

(1) p eigenvectors $(x_l; 0; 0)(l = 1, 2, ..., p)$ that correspond to the eigenvalue 1, where $x_l(l = 1, 2, ..., p)$ are arbitrary linearly independent vectors;

(2) n eigenvectors $(0; y_l^1; 0)(l = 1, 2, ..., n)$ that correspond to the eigenvalue 1, where $y_l(l = 1, 2, ..., n)$ are arbitrary linearly independent vectors.

Proof. Its proof is similar to Theorem 3.1, which can be obtained according to the above proof. □

Theorem 3.5 Under the assumptions of Theorem 3.1, the eigenvalues and degree of the minimal polynomial of preconditioned matrix $\mathcal{P}_3^{-1}\mathcal{A}$ are 1 and is at most 2, respectively. Then the preconditioned matrix $\mathcal{P}_3^{-1}\mathcal{A}$ has $p + m + n - r(C)$ linearly independent eigenvectors. There are

(1) p eigenvectors $(x_l; 0; 0)(l = 1, 2, ..., p)$ that correspond to the eigenvalue 1, where $x_l(l = 1, 2, ..., p)$ are arbitrary linearly independent vectors;

(2) m eigenvectors$(0; 0; z_l^1)(l = 1, 2, ..., m)$ that correspond to the eigenvalue 1, where $z_l(l = 1, 2, ..., m)$ are arbitrary linearly independent vectors;

(3) $n - r(C)$ eigenvectors $(x_l^2; y_l^2; z_l^2)(l = 1, 2, ..., n - r(C))$ that correspond to the eigenvalue 1, where $y_l^2 \in N(C)$, and $x_l^2, z_l^2(l = 1, 2, ..., m)$ are arbitrary vectors.

Proof. Its proof is similar to Theorem 3.1, which is omitted here. □

4. Implementation of the three preconditioners

In practical implementation of the $\mathcal{P}_1$ within Krylov subspace acceleration, we need to solve the residual equation

$$\begin{pmatrix} D & 0 & 0 \\ C^T & \widehat{A} & 0 \\ 0 & -B & S_{\widehat{A}} \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix},$$

where $\widehat{A} = A + C^T D^{-1} C$ and $S_{\widehat{A}} = B\widehat{A}^{-1}B^T$. we obtain the following algorithm implementation for the $\mathcal{P}_1$:

Algorithm 4.1: Solution of $\mathcal{P}_1 z = r$, where $r = (r_1; r_2; r_3)$ and $z = (z_1; z_2; z_3)$ are given residual vector and the current vector, respectively; and $r_1,\ z_1 \in \mathbb{R}^p,\ r_2,\ z_2 \in \mathbb{R}^n,\ r_3,\ z_3 \in \mathbb{R}^m$, from the following procedures:
(1) solve z_1 from $Dz_1 = r_1$;
(2) solve z_2 from $(A + C^T D^{-1} C)z_2 = r_2 - C^T z_1$;
(3) compute $B(A + C^T D^{-1} C)^{-1} B^T z_3 = r_3 + Bz_2$.

Similarly, the implementation of the $\mathcal{P}_2$ and $\mathcal{P}_3$ with a Krylov subspace method can be described as follow.

Algorithm 4.2: Solution of $\mathcal{P}_2 z = r$, where $r = (r_1; r_2; r_3)$ and $z = (z_1; z_2; z_3)$ are given residual vector and the current vector, respectively; and $r_1,\ z_1 \in \mathbb{R}^p,\ r_2,\ z_2 \in \mathbb{R}^n,\ r_3,\ z_3 \in \mathbb{R}^m$, from the following procedures:
(1) solve z_2 from $(A + C^T D^{-1} C)z_2 = r_2 - C^T D^{-1} r_1$;
(2) solve z_1 from $Dz_1 = r_1 + Cz_2$;
(3) compute $B(A + C^T D^{-1} C)^{-1} B^T z_3 = r_3 + Bz_2$.

Algorithm 4.3: We solve $\mathcal{P}_3 z = r$ for the preconditioner $\mathcal{P}_3$ by the following steps:
(1) solve z_1 from $Dz_1 = r_1$;
(2) solve z_3 from $B(A + C^T D^{-1} C)^{-1} B^T z_3 = r_3 + B\widehat{A}^{-1}(r_2 - C^T D^{-1} z_1)$;
(3) compute $(A + C^T D^{-1} C)z_2 = r_2 - C^T z_1 - B^T z_3$.

5. Numerical experiments

In this section, we give two numerical examples to prove the effectiveness of the three preconditioners $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3$ proposed in Section 3, their structures are generalizations of the examples in [11]. In order to better show the advantages of the proposed preconditioners, we adopt the GMRES method incorporated with no preconditioner, APSS and RAPSS preconditioners proposed by [8] and our proposed three preconditioners to solve the linear system (1.2). All experiments are performed in MATLAB 2016a on an Intel Core (8G RAM) Windows 10 system. We define CPU times (denoted as 'CPU'), iteration steps (denoted by 'IT') and iteration residual (denoted by 'RES') to show the effect of preconditioners applied to GMRES method. In all tests, all runs are started from the zero vectors and stopped once the relative residual satisfies

$$RES = \frac{\|b - \mathcal{A}z_k\|}{\|b\|} < 10^{-8}$$

or if it exceeds the prescribed iteration steps $k_{max} = 2000$, the iteration is stopped. It should be noted that we use the preconditioned GMRES method to solve the linear system (1.2). In Example 1, we use sparse Cholesky decomposition to solve the linear subsystem when we start to solve the residual equation. Similarly, in example 2, we use sparse LU decomposition to solve the linear subsystem.

Example 1. Consider the double saddle point peoblems (1.2), in which

$$A = diag(2W^T W + D_1, D_2, D_3) \in \mathbb{R}^{n\times n}$$

is a block diagonal matrix,

$$B = [E, -I_{2\tilde{p}}, I_{2\tilde{p}}] \in \mathbb{R}^{m\times n},\ C = (C_1 \otimes I_p) \in \mathbb{R}^{l\times n}\ and\ D = I \in \mathbb{R}^{l\times l},$$

where $\tilde{p} = p^2$, $\hat{p} = p(p+1)$; $W = (w_{ij}) \in \mathbb{R}^{\hat{p}\times\hat{p}}$ with $w_{ij} = e^{-2((i/3)^2+(j/3)^2)}$; $D_1 = I_{\hat{p}}$ is an identity matrix; $D_i = diag(d_j^{(i)}) \in \mathbb{R}^{2\tilde{p}\times 2\tilde{p}},\ i = 2, 3$, are diagonal matrices, with

$$d_j^2 = \begin{cases} 1, & for\ 1 \le j \le \tilde{p}, \\ 10^{-5}(j-\tilde{p})^2, & for\ \tilde{p}+1 \le j \le 2\tilde{p}, \end{cases}$$

$$d_j^3 = 10^{-5}(j+\tilde{p})^2,\ \ for\ 1 \le j \le 2\tilde{p};$$

and

$$E = \begin{pmatrix} \hat{E} \otimes I_p \\ I_p \otimes \hat{E} \end{pmatrix},$$

$$\hat{E} = \begin{pmatrix} 2 & -1 & & & \\ & 2 & -1 & & \\ & & \ddots & \ddots & \\ & & & 2 & -1 \end{pmatrix} \in \mathbb{R}^{p\times(p+1)},\ C_1 = \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \end{pmatrix} \in \mathbb{R}^{(p+1)\times(5p+1)}.$$

Example 2. Consider the double saddle point peoblems (1.2), in which

$$A = \begin{pmatrix} F\otimes T + T\otimes F & 0 \\ 0 & F\otimes T + T\otimes F \end{pmatrix} \in \mathbb{R}^{2p^2\times 2p^2},$$

$$B = C = \begin{pmatrix} I\otimes E & E\otimes I \end{pmatrix} \in \mathbb{R}^{p^2\times 2p^2}, \qquad D = I\otimes I \in \mathbb{R}^{p^2\times p^2},$$

where

$$T = \frac{1}{h^2} tridiag(-1, 2, -1), \qquad F = \frac{1}{h} tridiag(0, 1, -1),$$
$$E = tridiag(1, p+1, \cdots, p^2 - p + 1),$$

$\otimes$ means the Kronecker product symbol and $h = \frac{1}{p+1}$. For this peoblems, the total dimension is $4p^2$.

Numerical experiments are formed by setting different dimensions. It is noticed that we take $b = \mathcal{A} \cdot \mathbf{1}$. To show the effectiveness of the new preconditioners, we compare proposed preconditioners with APSS and RAPSS preconditioners by applying them to GMRES method.

In Tables 1 and 2, we list the numerical results of different preconditioned GMRES methods for Examples 1 and 2. Here, I denotes the GMRES method without preconditioning. Besides, we omit the iteration results if iteration steps exceed 2000. From these numerical results in Table 1, we can see that the no-preconditioning GMRES method converges very slowly, the five preconditioned methods are all effective. By comparing their iteration steps, residuals and CPU times, it follows that the three preconditioners $\mathcal{P}_1$, $\mathcal{P}_2$ and $\mathcal{P}_3$ are more effective than the other two preconditioners, and their iteration steps are obviously reduced. Especially, the iteration steps of $\mathcal{P}_1$, $\mathcal{P}_2$ and $\mathcal{P}_3$ preconditioned GMRES methods remain constant even if the dimension of the double saddle point system increases,which shows that three proposed preconditioners are advantage for solving the double saddle point problems (1.2). It can be seen from Table 2 that with increasing of dimension, the iteration time of GMRES method under the three proposed preconditioners increases greatly. The reason may be that in addition to the solutions of sublinear systems with matrix D, it also involves the solutions of those with matrices $A + C^T D^{-1}C$ and $B(A + C^T D^{-1}C)B^T$ as sloving the residual equations under three preconditioners. We can find that both $A + C^T D^{-1}C$ and $B(A + C^T D^{-1}C)B^T$ are full matrices, and it is expensive to solve sunlinear systems with using matrix decomposition. However, we can still find that the CPU times of three preconditioners are less than those of APSS and RAPSS preconditioners. Theoretical analysis shows that the degree of the minimum polynomial of the three preconditioned matrices is at most 3, 2 and 2, respectively. However, the iteration steps of numerical results corresponding to the three preconditioners are 4, 3 and 3, respectively. In fact, this is also according with convention. We can refer to [14].

Table 1. The numerical results for preconditioned GMRES methods with I, $\mathcal{P}_{APSS}$, $\mathcal{P}_{RAPSS}$, $\mathcal{P}_1$, $\mathcal{P}_2$ and $\mathcal{P}_3$ for Example 1.

size		I	$\mathcal{P}_{APSS}$	$\mathcal{P}_{RAPSS}$	$\mathcal{P}_1$	$\mathcal{P}_2$	$\mathcal{P}_3$
528	α	-	0.1	0.1	-	-	-
	IT	1223	39	23	4	3	3
	CPU	0.0905	0.0289	0.0192	0.0112	0.0100	0.0093
	RES	9.3e-09	6.6e-09	2.6e-09	3.9e-10	6.8e-11	1.0e-12
1176	α	-	0.1	0.1	-	-	-
	IT	1170	47	33	4	3	3
	CPU	0.0795	0.1738	0.1273	0.0281	0.0266	0.0342
	RES	9.5e-09	9.8e-09	5.3e-09	9.7e-11	3.0e-11	2.6e-13
2080	α	-	0.1	0.1	-	-	-
	IT	1164	50	39	4	3	3
	CPU	0.1012	0.6601	0.5238	0.1104	0.0906	0.0995
	RES	9.3e-09	8.8e-09	5.1e-09	4.3e-12	1.5e-11	7.8e-12
4656	α	-	0.1	0.1	-	-	-
	IT	1286	51	45	4	3	3
	CPU	1.2453	3.5699	3.1501	0.8276	0.7222	0.7637
	RES	9.6e-09	9.4e-09	9.2e-09	6.8e-13	9.6e-13	8.2e-14
8256	α	-	0.1	0.1	-	-	-
	IT	1553	52	49	4	3	3
	CPU	7.6373	11.0295	11.1070	4.0296	3.9548	4.2159
	RES	9.9e-09	7.3e-09	9.5e-09	8.2e-14	3.5e-13	7.2e-14

Table 2. The numerical results for preconditioned GMRES methods with $\mathcal{I}$, $\mathcal{P}_{APSS}$, $\mathcal{P}_{RAPSS}$, $\mathcal{P}_1$, $\mathcal{P}_2$ and $\mathcal{P}_3$ for Example 2.

size		I	$\mathcal{P}_{APSS}$	$\mathcal{P}_{RAPSS}$	$\mathcal{P}_1$	$\mathcal{P}_2$	$\mathcal{P}_3$
256	α	-	0.01	0.01	-	-	-
	IT	1192	88	87	4	3	3
	CPU	0.0827	0.0409	0.0302	0.0142	0.0155	0.0144
	RES	9.0e-09	2.8e-09	4.0e-09	9.1e-15	9.4e-15	5.5e-14
1024	α	-	0.1	0.1	-	-	-
	IT	1638	202	202	4	3	3
	CPU	0.9797	0.4381	0.3235	0.0945	0.0884	0.0890
	RES	9.6e-09	9.4e-09	6.8e-09	1.6e-13	4.6e-14	1.0e-12
2500	α	-	0.1	0.1	-	-	-
	IT	-	285	285	4	3	3
	CPU	-	2.4462	2.6105	1.0909	1.0916	1.0764
	RES	-	9.5e-09	9.8e-09	7.3e-13	1.4e-13	1.0e-11
4096	α	-	0.1	0.1	-	-	-
	IT	-	348	348	4	3	3
	CPU	-	6.9239	8.1949	4.9200	4.7040	4.7255
	RES	-	9.3e-09	9.3e-09	5.0e-12	2.2e-13	4.4e-11
7056	α	-	1	1	-	-	-
	IT	-	697	697	4	3	3
	CPU	-	63.6368	65.3049	20.9885	20.9199	21.1170
	RES	-	9.9e-09	9.0e-09	1.3e-11	4.1e-13	8.5e-11

In Figures 1 and 2, we plot the eigenvalue distribution of the original coefficient matrix and the $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$ preconditioned matrices for Example 1. In Figure 1, we test problems size of 1176. In Figure 2, we test problems size of 4656. From these two figures, we observe that the $\mathcal{P}_1$, $\mathcal{P}_2$ and $\mathcal{P}_3$ preconditioners improve the eigenvalue distribution of the original coefficient matrix greatly. Most importantly, we find that the eigenvalues distribution of these three preconditioned matrices all clustered at a point, which is the same as the theoretical analysis in Section 3. Moreover, the three preconditioned matrices have tight spectrums, which lead to stable numerical performances. In Figures 3 and 4, we plot the eigenvalue distribution of the original coefficient matrix and the $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$ preconditioned matrices with different dimension sizes for Example 2. From these Figures, it is easy to observe that the $\mathcal{P}_1$, $\mathcal{P}_2$ and $\mathcal{P}_3$ preconditioners improve the eigenvalue distribution of the original coefficient matrix greatly.

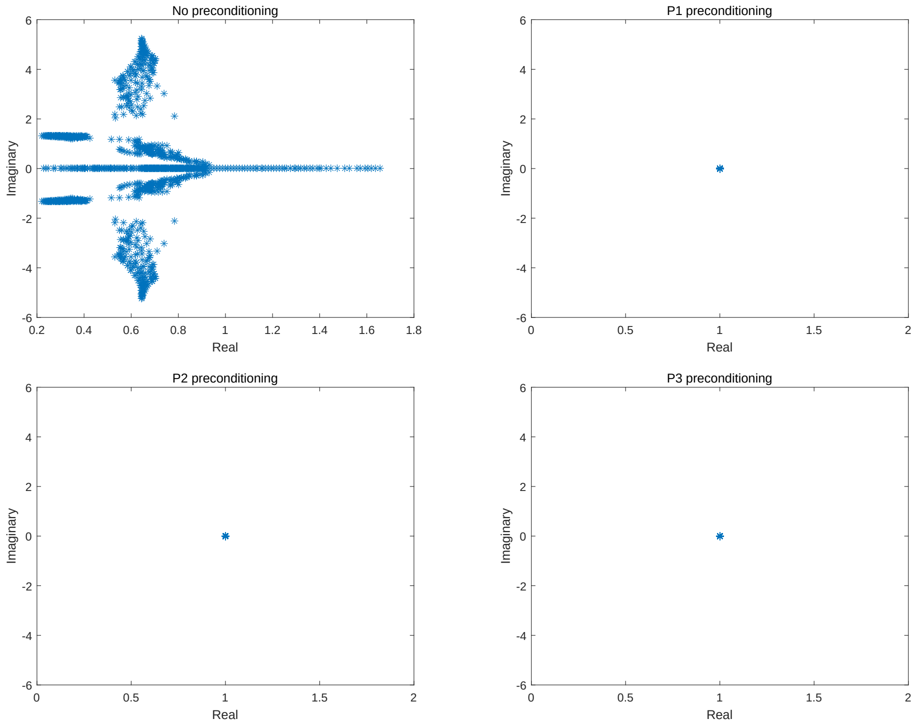

Figure 1. Eigenvalue distributions of the original coefficient and $\mathcal{P}_1$, $\mathcal{P}_2$ and $\mathcal{P}_3$ preconditioned matrices with dimension size 1176 for Example 1.

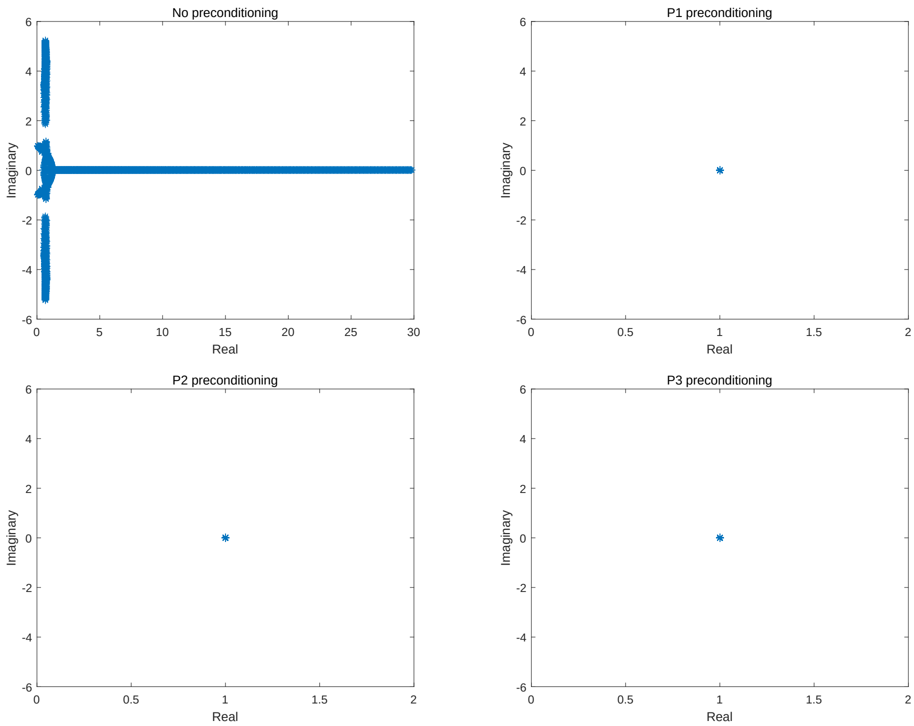

Figure 2. Eigenvalue distributions of the original coefficient and $\mathcal{P}_1$, $\mathcal{P}_2$ and $\mathcal{P}_3$ preconditioned matrices with dimension size 4656 for Example 1.

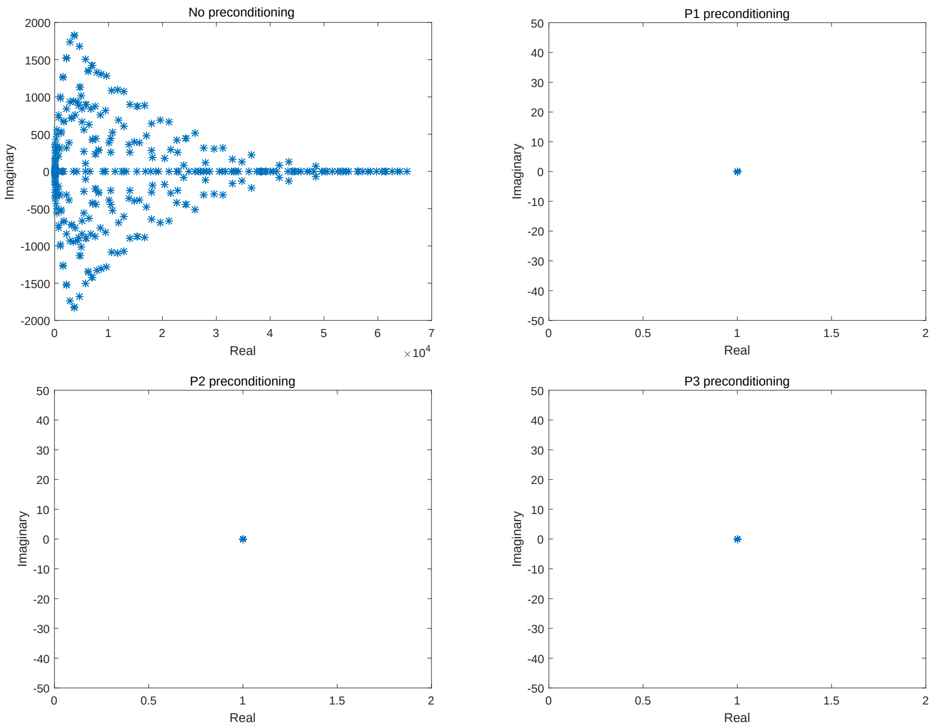

Figure 3. Eigenvalue distributions of the original coefficient and $\mathcal{P}_1$, $\mathcal{P}_2$ and $\mathcal{P}_3$ preconditioned matrices with dimension size 1024 for Example 2.

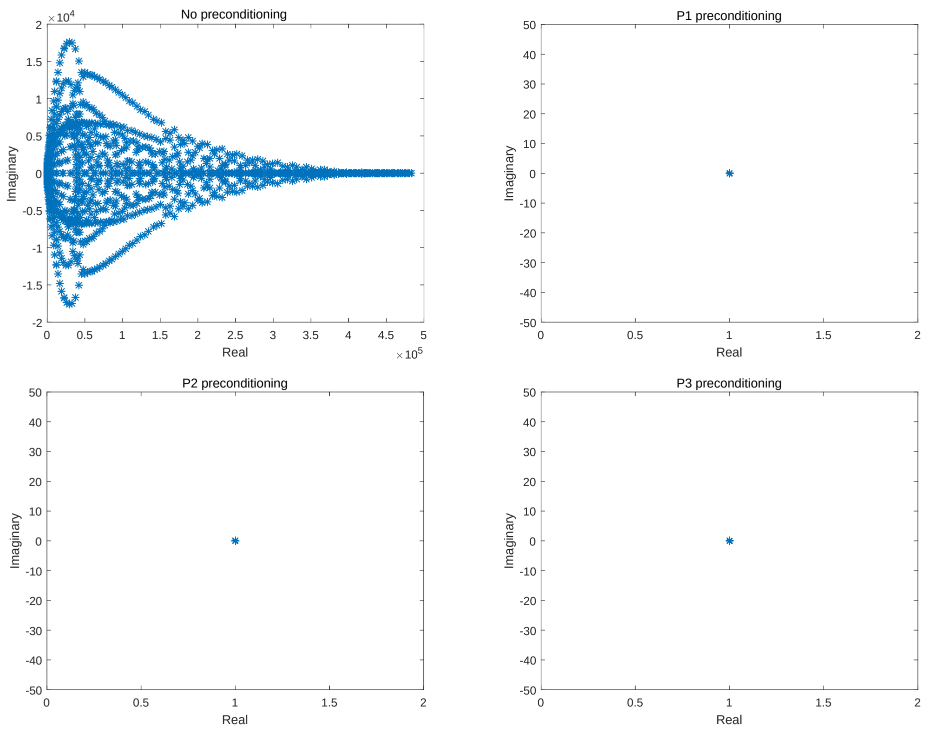

Figure 4. Eigenvalue distributions of the original coefficient and $\mathcal{P}_1$, $\mathcal{P}_2$ and $\mathcal{P}_3$ preconditioned matrices with dimension size 4096 for Example 2.

6. Conclusions

In this paper, three effective preconditioners are proposed for double saddle point problem (1.2). We give the solvability of this linear system and discuss the spectral properties of the preconditioned matrices under three preconditioners $\mathcal{P}_1$, $\mathcal{P}_2$ and $\mathcal{P}_3$. Besides, compared with the existing APSS and RAPSS preconditioners, three preconditioners established in this paper have better computing efficiency, which may be because the spectral sets of the three preconditioners have tight spectrums. In the end, some numerical experiments also verify the validity of the theoretical results.

Note that we use the Krylov subspace method under exact algorithm to solve the double saddle point problem in this paper. In fact, the inexact solvers are more efficient that exact solvers especially for large problem. Thus, we will consider three preconditioned GMRES method with inexact solvers, which are very crucial and interesting in further study.

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Conflict of interest

The authors declare there is no conflict of interest.

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