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Research article

Anomalies of Lévy-based thermal transport from the Lévy-Fokker-Planck equation

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Abstract: Lévy-type behaviors are widely involved in anomalous thermal transport, yet generic investigations based on the mathematical descriptions of the confined Lévy flights are still lacking. In the frameworks of classical irreversible thermodynamics and Boltzmann-Gibbs statistical mechanics, the Lévy-Fokker-Planck equation is connected to near-equilibrium thermal transport. In this work, we show that thermal transport dominated by the confined Lévy flights will be paired with an anomaly, namely that the local effective thermal conductivity is nonlocal. It is demonstrated that the near-equilibrium assumption is not unconditionally valid, which relies on several thermodynamic restrictions expressed by the probability density function (PDF). It is illustrated that the Lévy-Fokker-Planck equation based on the Caputo operator will give rise to two signatures of anomalous thermal transport, the power-law size-dependence of the global effective thermal conductivity and nonlinear boundary asymptotics of the stationary temperature profile. These anomalies are interrelated with each other, and their quantitative relations can be considered as criteria for Lévy-based thermal transport.

Keywords: Lévy-Fokker-Planck equation; fractional-order derivative; anomalous thermal transport; irreversible thermodynamics; effective thermal conductivity
Mathematics Subject Classification: 34A08, 80A05, 80M60

1. Introduction

The Lévy process [1] is commonly defined by the characteristic function $\exp\left(-c_\alpha |k|^\alpha t\right)$, wherein α denotes the Lévy index, k is the variable in the Fourier space, and c_α is a constant. In recent years, Lévy-type behaviors have been widely used to interpret signatures of anomalous thermal transport in low-dimensional systems [2–7]. A typical example is the power-law size-dependence of the effective thermal conductivity κ_{eff} [4,8–11], namely,

$$\kappa_{eff} = \kappa_{eff}\left(L\right) \sim L^{\gamma} \tag{1.1}$$

with L denoting the system size. Based on the Monte Carlo technique for solving the phonon Boltzmann transport equation, Upadhyaya and Aksamija [5] have observed a Lévy-type (or heavy-tailed) distribution of the phonon mean free paths in Si-Ge alloy nanowires, which gives rise to a divergent exponent $\gamma = 1/3$. Denisov and co-authors [12] connected the size-dependence exponent to the Lévy index $\alpha \in (1,2)$ for one-dimensional dynamical channels, $\gamma = 2-\alpha$. This relation is supported by a recent investigation on the long-range interacting Fermi-Pasta-Ulam chains [7]. Furthermore, the results in Si-Ge alloy nanowires and one-dimensional dynamical channels also show that the Lévy processes will be paired with another signature of anomalous thermal transport, the superdiffusive growth of the mean-square energy displacement [5],

$$\left\langle \Delta x_e^2\left(t\right)\right\rangle \sim t^{\beta} \tag{1.2}$$

with $\beta \in (1,2)$. The coexistence of the Lévy-type regimes and superdiffusive thermal transport has also been acquired in semiconductor alloys [6] and two-dimensional nonlinear lattices [8].

There is another conceptual connection between the Lévy processes and anomalous thermal transport in low-dimensional systems, the spatial fractional-order operators [13–15]. For instance, the energy perturbation $\delta e(x,t)$ in the one-dimensional harmonic chains is commonly governed by a 3/4-fractional diffusion equation [14,15] as follows

$$\frac{\partial}{\partial t}\left[\delta e(x,t)\right] = -C_0\left(-\Delta\right)^{3/4}\left[\delta e(x,t)\right] \tag{1.3}$$

wherein C_0 is a positive constant and $\left(-\Delta\right)^{3/4}$ stands for the fractional Laplacian operator [16,17]. For infinite space like $\mathbb{R}$, $\left(-\Delta\right)^{3/4}$ is generally defined in terms of the Fourier transform, namely,

$$\mathcal{F}_k\left\{\left(-\Delta\right)^{3/4}\left[\delta e(x,t)\right]\right\} = \left|k\right|^{3/2}\mathcal{F}_k\left\{\delta e(x,t)\right\} \tag{1.4}$$

with $\mathcal{F}_k\{...\}$ the Fourier transform operator. At the microscopic level, Eq (1.3) can be obtained from the Boltzmann transport equation with a certain collision term [18–22]. In these studies, the Lévy-type behaviors are observed based on the specific physical regimes of the heat carriers, which differ from model to model, yet generic mathematical descriptions are not much involved with signatures of anomalous thermal transport. In mathematics, spatial fractional-order governing equations are widely applied to the Lévy processes [23–25], including the Lévy flights in a confined domain $[0,L]$. The main aim of this work is to address anomalous thermal transport which is dominated the confined Lévy flights, which has not been discussed by previous investigations.

The simplest mathematical description of the confined Lévy flights is the following symmetric Lévy-Fokker-Planck equation [23]

$$\begin{aligned}\frac{\partial P(x,t)}{\partial t} &= -\frac{K_\alpha}{2\cos\left(\frac{\pi\alpha}{2}\right)}\left[{}_0^{RL}D_x^\alpha + {}_L^{RL}D_x^\alpha\right]P(x,t) \\ &= -\frac{K_\alpha}{2\cos\left(\frac{\pi\alpha}{2}\right)\Gamma(2-\alpha)}\frac{\partial^2}{\partial x^2}\left[\int_0^L \frac{P(x',t)}{|x-x'|^{\alpha-1}}dx'\right],\end{aligned} \tag{1.5}$$

where $P(x,t)$ denotes the probability density function (PDF), K_α is the noise intensity with the dimension $|x|^\alpha t^{-1}$, ${}_L^{RL}D_x^\alpha$ and ${}_0^{RL}D_x^\alpha$ stand for the right-hand and left-hand Riemann-Liouville operators respectively. For engineering or experimental problems, the boundary points must be attained, which will give rise to infinite Lévy measure. In this work, we apply Eq (1.5) to one-dimensional thermal transport, wherein the PDF is defined in terms of the correlation function of the energy fluctuations [3], namely,

$$\begin{cases} P(x,t) = \left[\int_0^L C_u(x,t=0)dx\right]^{-1} C_u(x,t) \\ C_u(x,t) = \langle u(x,t)u(x=0,t=0)\rangle - \langle u(x,t)\rangle\langle u(x=0,t=0)\rangle \end{cases} \tag{1.6}$$

with $u(x,t)$ the density of the thermal energy. Eq (1.5) corresponds to nonlocal thermal transport, namely that the temporal evolution of the energy fluctuations at $x = x_0$ depends on the global distribution of the energy fluctuations in $[0,L]$. For arbitrary $\varepsilon \in (0, \min\{x_0, L-x_0\})$, the distributions in $[x_0-\varepsilon, x_0)$ and $(x_0, x_0+\varepsilon]$ have the same contribution to the temporal evolution at $x = x_0$, which indicates that the nonlocality is symmetric. Based on the entropic functionals, a connection between the evolution of the PDF and thermal transport is established. Anomalous features of thermal transport thereafter arise from the entropic connection, including the nonlocality of the local effective thermal conductivity, power-law size-dependence of the global effective thermal conductivity, and nonlinear boundary asymptotics of the stationary temperature profile. Thermal transport and confined Lévy flights.

2. Thermal transport and confined Lévy flights

The Lévy-Fokker-Planck equation describes the evolution of the PDF, while thermal transport focuses on thermodynamic quantities, i.e., the heat flux $J_q(x,t)$ and local temperature $T(x,t)$. In order to link the Lévy-Fokker-Planck equation to thermal transport, we consider the following entropy density in the framework of Boltzmann-Gibbs statistical mechanics,

$$s(x,t) = -k_B P(x,t) \ln P(x,t) \tag{2.1}$$

where k_B is the Boltzmann constant. The temporal derivative of $s(x,t)$ should be restricted by the entropy balance equation as follows

$$\begin{aligned} \frac{\partial s(x,t)}{\partial t} &= -k_B \frac{\partial P(x,t)}{\partial t}\left[\ln P(x,t)+1\right] \\ &= -\frac{\partial J_S(x,t)}{\partial x} + \sigma(x,t). \end{aligned} \tag{2.2}$$

wherein $J_S(x,t)$ denotes the entropy flux and $\sigma(x,t)$ is the density of the entropy production rate. Besides the entropy balance equation, there is another restriction termed as continuity equation,

$$\frac{\partial P(x,t)}{\partial t} = -\frac{\partial J(x,t)}{\partial x} \tag{2.3}$$

where $J(x,t)$ is the probability current. Substituting Eq (2.3) into Eq (2.2) yields

$$\begin{aligned} \frac{\partial s(x,t)}{\partial t} &= k_B \frac{\partial J(x,t)}{\partial x}\left[\ln P(x,t)+1\right] \\ &= k_B \frac{\partial}{\partial x}\left\{J(x,t)\left[\ln P(x,t)+1\right]\right\} - k_B J(x,t)\frac{\partial}{\partial x}\left[\ln P(x,t)+1\right] \\ &= -\frac{\partial J_S(x,t)}{\partial x} + \sigma(x,t), \end{aligned} \tag{2.4}$$

and we thereafter arrive at

$$J_S(x,t) = -k_B J(x,t)\left[\ln P(x,t)+1\right] \tag{2.5}$$

$$\begin{aligned} \sigma(x,t) &= -k_B J(x,t)\frac{\partial}{\partial x}\left[\ln P(x,t)+1\right] \\ &= -k_B \frac{J(x,t)}{P(x,t)}\frac{\partial P(x,t)}{\partial x}. \end{aligned} \tag{2.6}$$

Then, $J(x,t)$ and $P(x,t)$ can be connected to thermal transport via the relationship between $\left\{J_S(x,t),\sigma(x,t)\right\}$ and $\left\{J_q(x,t),T(x,t)\right\}$.

For thermal transport not far from local equilibrium, Boltzmann-Gibbs statistical mechanics typically coincides with classical irreversible thermodynamics [26], which gives the following expressions for the above entropic functionals,

$$s(x,t)=\int^{T(x,t)} c\frac{dT}{T}+s_{eq} \tag{2.7}$$

$$J_S(x,t)=\frac{1}{T(x,t)}J_q(x,t) \tag{2.8}$$

$$\sigma(x,t)=J_q(x,t)\frac{\partial}{\partial x}\left[\frac{1}{T(x,t)}\right] \tag{2.9}$$

where s_{eq} is the entropy density independent of thermal transport, and c is the specific heat capacity per volume. Upon combining Eqs (2.8) and (2.9) with Eqs (2.5) and (2.6) respectively, one can derive the relations between $\{J(x,t),P(x,t)\}$ and $\{J_q(x,t),T(x,t)\}$, namely,

$$J_q(x,t)=-k_B T(x,t)J(x,t)\left[\ln P(x,t)+1\right] \tag{2.10}$$

$$cJ(x,t)\frac{\partial T(x,t)}{\partial x}=J_q(x,t)\frac{\partial P(x,t)}{\partial x} \tag{2.11}$$

The two relations do not rely on specific constitutive models between $J(x,t)$ and $P(x,t)$, which remains valid for various generalized Fokker-Planck equations besides the Lévy-Fokker-Planck equation.

For the Lévy-Fokker-Planck equation, the constitutive model between $J(x,t)$ and $P(x,t)$ is given by [27]

$$J(x,t)=\frac{K_\alpha}{2\cos\left(\frac{\pi\alpha}{2}\right)\Gamma(2-\alpha)}\frac{\partial}{\partial x}\left[\int_0^L \frac{P(x',t)}{|x-x'|^{\alpha-1}}dx'\right] \tag{2.12}$$

and substituting it into Eqs (2.10) and (2.11) leads to

$$J_q(x,t)=\frac{-k_B K_\alpha T(x,t)\left[\ln P(x,t)+1\right]}{2\cos\left(\frac{\pi\alpha}{2}\right)\Gamma(2-\alpha)}\frac{\partial}{\partial x}\left[\int_0^L \frac{P(x',t)}{|x-x'|^{\alpha-1}}dx'\right] \tag{2.13}$$

$$\frac{K_\alpha c\frac{\partial T(x,t)}{\partial x}}{2\cos\left(\frac{\pi\alpha}{2}\right)\Gamma(2-\alpha)}\frac{\partial}{\partial x}\left[\int_0^L \frac{P(x',t)}{|x-x'|^{\alpha-1}}dx'\right]=J_q(x,t)\frac{\partial P(x,t)}{\partial x} \tag{2.14}$$

Equation (2.13) exhibits a nonlocal behavior of the heat flux, namely that the heat flux at $x_0 \in [0, L]$ depends on not only the PDF and local temperature at x_0 but also all states in $[0, L]$. In other words, any points in $[0, L]$ will contribute to the heat flux at x_0. Such nonlocality will vanish in the limit $\alpha \to 2$, which leads to a degeneration into the standard diffusion equation. In this degenerate case, Eq (2.13) becomes

$$J_q(x,t) = K_{\alpha=2} k_B T(x,t) \frac{\partial P(x,t)}{\partial x} \left[\ln P(x,t) + 1 \right] \tag{2.15}$$

which illustrates that the diffusive heat flux is proportional to the PDF gradient. Note that the gradient of the entropy density is written as

$$\begin{aligned} \frac{\partial s(x,t)}{\partial x} &= -k_B \frac{\partial P(x,t)}{\partial x} \left[\ln P(x,t) + 1 \right] \\ &= \frac{c}{T(x,t)} \frac{\partial T(x,t)}{\partial x}. \end{aligned} \tag{2.16}$$

Combining Eqs (2.15) and (2.16) yields

$$\begin{aligned} J_q(x,t) &= -K_{\alpha=2} T(x,t) \frac{\partial s(x,t)}{\partial x} \\ &= -K_{\alpha=2} c \frac{\partial T(x,t)}{\partial x}. \end{aligned} \tag{2.17}$$

and we now obtain a constitutive relation between $J_q(x,t)$ and $\frac{\partial T(x,t)}{\partial x}$. Furthermore, the diffusive limit $\alpha \to 2$ implies normal thermal transport ($\beta = 1$), wherein $J_q(x,t)$ and $\frac{\partial T(x,t)}{\partial x}$ generally obey conventional Fourier's law, namely,

$$J_q(x,t) = -\kappa \frac{\partial T(x,t)}{\partial x} \tag{2.18}$$

Here, κ is the so-called thermal conductivity, which is an intrinsic material property and independent of geometric parameters such as the system size. It is found that Eqs (2.17) and (2.18) will possess a same formulation as if $\kappa \equiv K_{\alpha=2} c$. This degeneration to Fourier's law is physical reasonable and in agreement with existing understandings of anomalous thermal transport [2–4]. In the degeneration case, the Lévy process becomes the Gauss process. Meanwhile, Fourier's law corresponding to Eq (2.15) is paired with a parabolic governing equation of the local temperature [3], whose solution for initial thermal perturbation is Gaussian as well. Thus, Eq (2.15) also corresponds to the Gauss case. Nevertheless, $\kappa \equiv K_{\alpha=2} c$ is not unconditionally tenable. As material properties, κ and c generally vary as the local temperature changes, whereas K_α is assumed to be a constant. Therefore, $\kappa \equiv K_{\alpha=2} c$ is valid only if κ and c have a same temperature-dependence. This assumption commonly holds at the low temperature, yet is usually invalid in the high-temperature situations [28–30], wherein κ decays with the increasing temperature and c vanishingly varies.

For $\alpha \in (1,2)$, Eq (2.14) can still be reformed as a Fourier-like constitutive relation

$$\begin{cases} J_q(x,t) = -\kappa_{eff}^{loc} \dfrac{\partial T(x,t)}{\partial x} \\ \kappa_{eff}^{loc} = -\dfrac{K_\alpha c \left[\dfrac{\partial P(x,t)}{\partial x}\right]^{-1} \dfrac{\partial}{\partial x}\left[\displaystyle\int_0^L \dfrac{P(x',t)}{|x-x'|^{\alpha-1}} dx'\right]}{2\cos\left(\dfrac{\pi\alpha}{2}\right)\Gamma(2-\alpha)} \end{cases} \tag{2.19}$$

Because the prefactor κ_{eff}^{loc} is determined by the all states in $[0,L]$, it cannot be formulated as a functional of the local temperature like $\kappa_{eff}^{loc} = \kappa_{eff}^{loc}[T(x,t)]$. It implies that κ_{eff}^{loc} is not a well-defined intrinsic property, and hence, Fourier's law no longer holds. From the viewpoint of physics, κ_{eff}^{loc} can be understood as the local effective thermal conductivity. There are several aspects which need careful discussion. First, the derivation of κ_{eff}^{loc} relies on the framework of classical irreversible thermodynamics, which requires the non-negative entropy production rate in Eq (2.9). This requirement is equivalent to $\kappa_{eff}^{loc} \ge 0$, which leads to

$$\frac{\partial P(x,t)}{\partial x} \frac{\partial}{\partial x}\left[\int_0^L \frac{P(x',t)}{|x-x'|^{\alpha-1}} dx'\right] \ge 0 \tag{2.20}$$

Furthermore, as a thermodynamically irreversible process, non-vanishing thermal transport ($J_q(x,t) \ne 0$) must be paired with a strictly positive value of the entropy production rate. Conversely, if the total entropy production rate of a system is zero, this system must be in thermal equilibrium, which indicates that $J_q(x,t) \equiv 0$ and $\dfrac{\partial}{\partial t} \equiv 0$. In the framework of classical irreversible thermodynamics, the thermodynamic restriction stated above corresponds to the following corollary

$$\begin{cases} \sup\limits_{0\le x\le L} |J_q(x,t)| > 0 \Rightarrow \displaystyle\int_0^L J_q(x,t) \frac{\partial}{\partial x}\left[\frac{1}{T(x,t)}\right] dx > 0 \\ \displaystyle\int_0^L J_q(x,t) \frac{\partial}{\partial x}\left[\frac{1}{T(x,t)}\right] dx = 0 \Rightarrow J_q(x,t) \equiv 0 \end{cases} \tag{2.21}$$

As a physically meaningful quantity, the supremum $\sup\limits_{0\le x\le L} |J_q(x,t)|$ should be attained. Singular κ_{eff}^{loc}

can arise from $J_q(x,t) \neq 0$ and $\frac{\partial T(x,t)}{\partial x} \equiv 0$, which will invalidate corollary (2.21). For the PDF, the above corollary becomes

$$\begin{cases} \sup_{0\le x\le L}\left|\frac{\partial}{\partial x}\left[\int_0^L \frac{K_\alpha P(x',t)}{|x-x'|^{\alpha-1}}dx'\right]\right| > 0 \Rightarrow \int_0^L \frac{\frac{\partial P(x,t)}{\partial x}\frac{\partial}{\partial x}\left[\int_0^L \frac{K_\alpha P(x',t)}{|x-x'|^{\alpha-1}}dx'\right]}{2\cos\left(\frac{\pi\alpha}{2}\right)\Gamma(2-\alpha)P(x,t)}dx < 0 \\ \int_0^L \frac{\frac{\partial P(x,t)}{\partial x}\frac{\partial}{\partial x}\left[\int_0^L \frac{K_\alpha P(x',t)}{|x-x'|^{\alpha-1}}dx'\right]}{2\cos\left(\frac{\pi\alpha}{2}\right)\Gamma(2-\alpha)P(x,t)}dx = 0 \Rightarrow \frac{\partial}{\partial x}\left[\int_0^L \frac{K_\alpha P(x',t)}{|x-x'|^{\alpha-1}}dx'\right] \equiv 0 \end{cases} \tag{2.22}$$

It is demonstrated that not all solutions of Eq (1.5) can coexist with classical irreversible thermodynamics in the near-equilibrium region. The coexistence of the Lévy-Fokker-Planck equation and classical irreversible thermodynamics relies on restrictions (2.20) and (2.22), which correspond to $0 \le \kappa_{eff}^{loc} < +\infty$.

According to the result in [23], the equilibrium solution of the Lévy-Fokker-Planck equation is given by

$$P_{eq}(x) = \frac{\Gamma(\alpha)L^{1-\alpha}\left[x(L-x)\right]^{(\alpha-2)/2}}{\Gamma^2(\alpha/2)} \tag{2.23}$$

Non-uniform $P_{eq}(x)$ will give rise to a non-uniform temperature distribution, namely,

$$\begin{aligned} \left.\frac{ds(x)}{dx}\right|_{J_q\equiv 0} &= \frac{c}{\left.T(x)\right|_{J_q(x,t)\equiv 0}}\frac{d\left[\left.T(x)\right|_{J_q(x,t)\equiv 0}\right]}{dx} \\ &= -k_B\frac{dP_{eq}(x)}{dx}\left[\ln P_{eq}(x)+1\right] \\ &\Rightarrow \frac{d\left[\left.T(x)\right|_{J_q(x,t)\equiv 0, x\neq L/2}\right]}{dx} \neq 0. \end{aligned} \tag{2.24}$$

From a physical perspective, it is non-trivial that the non-vanishing temperature gradient coexists with the thermal equilibrium state, which means absolute thermal insulation, $\kappa_{eff}^{loc} \equiv 0$. Furthermore, $P_{eq}(x)$ is singular at the boundary, which will induces infinite boundary temperatures. These non-trivial behaviors have not been observed in existing studies on anomalous thermal transport [2–4].

3. Modification based on Caputo operator

If the temperature distribution is uniform in the absence of thermal transport, the equilibrium PDF should be written as

$$P(x)\big|_{J(x,t)\equiv 0} = L^{-1} \tag{3.1}$$

For the Lévy-Fokker-Planck equation, this equilibrium solution can be acquired via replacing the Riemann-Liouville operator by the Caputo operator [31], and the constitutive relation between $J(x,t)$ and $P(x,t)$ thereafter becomes

$$J(x,t) = \frac{K_\alpha}{2\cos\left(\frac{\pi\alpha}{2}\right)\Gamma(2-\alpha)}\left[\int_0^L \frac{1}{|x-x'|^{\alpha-1}}\frac{\partial P(x',t)}{\partial x'}dx'\right] \tag{3.2}$$

The corresponding local effective thermal conductivity reads

$$\kappa_{eff}^{loc} = -\frac{K_\alpha c\left[\frac{\partial P(x,t)}{\partial x}\right]^{-1}\left[\int_0^L \frac{1}{|x-x'|^{\alpha-1}}\frac{\partial P(x',t)}{\partial x'}dx'\right]}{2\cos\left(\frac{\pi\alpha}{2}\right)\Gamma(2-\alpha)} \tag{3.3}$$

which is still nonlocal. The thermodynamic restrictions for $\{J_q(x,t), T(x,t)\}$ remain unchanged, and the restrictions on the PDF take the following forms

$$\begin{cases} \frac{\partial P(x,t)}{\partial x}\left[\int_0^L \frac{1}{|x-x'|^{\alpha-1}}\frac{\partial P(x',t)}{\partial x'}dx'\right] \geq 0 \\ \sup\limits_{0\leq x\leq L}\left|\int_0^L \frac{K_\alpha}{|x-x'|^{\alpha-1}}\frac{\partial P(x',t)}{\partial x'}dx'\right| > 0 \Rightarrow \int_0^L\int_0^L\left[\frac{K_\alpha \frac{\partial P(x,t)}{\partial x}\frac{\partial P(x',t)}{\partial x'}}{|x-x'|^{\alpha-1}}\right]dx'dx < 0 \\ \int_0^L\int_0^L\left[\frac{K_\alpha \frac{\partial P(x,t)}{\partial x}\frac{\partial P(x',t)}{\partial x'}}{|x-x'|^{\alpha-1}}\right]dx'dx = 0 \Rightarrow \left[\int_0^L \frac{K_\alpha}{|x-x'|^{\alpha-1}}\frac{\partial P(x',t)}{\partial x'}dx'\right] \equiv 0 \end{cases} \tag{3.4}$$

which are equivalent to $0 \leq \kappa_{eff}^{loc} < +\infty$ likewise.

We now consider stationary thermal transport in the presence of a small temperature difference, namely,

$$\left|\delta T\right| \ll \min\left\{T\left(x=L\right),T\left(x=0\right)\right\},\ \ \delta T=T\left(x=L\right)-T\left(x=0\right) \tag{3.5}$$

which yields

$$\left|\delta P\right| \ll \min\left\{P\left(x=L\right),P\left(x=0\right)\right\},\ \ \delta P=P\left(x=L\right)-P\left(x=0\right) \tag{3.6}$$

In this case, the solution of the modified Lévy-Fokker-Planck equation is written as

$$P\left(x\right)=P\left(x=0\right)+\frac{\left\{\int_0^x\left[y\left(L-y\right)\right]^{\frac{\alpha}{2}-1}dy\right\}\delta P}{\int_0^L\left[y\left(L-y\right)\right]^{\frac{\alpha}{2}-1}dy} \tag{3.7}$$

Preconditions (3.5) and (3.6) enable us to employ the following expansion

$$\begin{aligned}\delta s&=s\left(x=L\right)-s\left(x=0\right)\\&=\frac{c_0}{T_0}\left[\delta T+o\left(\delta T\right)\right]\\&=-k_B\left[\delta P\left(\ln P_0+1\right)+o\left(\delta P\right)\right],\end{aligned} \tag{3.8}$$

where c_0 is the specific heat capacity at T_0, T_0 and P_0 are the averaged temperature and probability density respectively. With the remainder term neglected, we arrive at

$$\delta T\approx-\frac{k_BT_0}{c_0}\delta P\left(\ln P_0+1\right) \tag{3.9}$$

Similarly, the entropy flux can be expanded as

$$\begin{aligned}J_S&=J_q\left[\frac{1}{T_0}+\frac{o\left(\delta T\right)}{T_0^2}\right]\\&=-\frac{\pi k_BK_\alpha\left[\ln P_0+1+o\left(\delta P\right)\right]\delta P}{\cos\left(\frac{\pi\alpha}{2}\right)\sin\left(\frac{\pi\alpha}{2}\right)\Gamma\left(\alpha\right)L^{\alpha-1}\left\{\int_0^1\left[z\left(1-z\right)\right]^{\frac{\alpha}{2}-1}dz\right\}}.\end{aligned} \tag{3.10}$$

From Eq (3.10), one can derive the following expression of the heat flux

$$J_q\approx-\frac{\pi k_BT_0K_\alpha\left(\ln P_0+1\right)\delta P}{\cos\left(\frac{\pi\alpha}{2}\right)\sin\left(\frac{\pi\alpha}{2}\right)\Gamma\left(\alpha\right)L^{\alpha-1}\left\{\int_0^1\left[z\left(1-z\right)\right]^{\frac{\alpha}{2}-1}dz\right\}} \tag{3.11}$$

Stationary thermal transport is typically characterized by the global effective thermal conductivity as follows [2–4],

$$\kappa_{eff}^{glo} = -\frac{J_q L}{\delta T} \tag{3.12}$$

which can be obtained through combining Eqs (3.9) and (3.11), namely,

$$\kappa_{eff}^{glo} = -\frac{\pi K_\alpha c_0 L^{2-\alpha}}{\cos\left(\frac{\pi\alpha}{2}\right)\sin\left(\frac{\pi\alpha}{2}\right)\Gamma(\alpha)\int_0^1 \left[y(1-y)\right]^{\frac{\alpha}{2}-1} dy} \tag{3.13}$$

The power-law size-dependence of the effective thermal conductivity presently occurs, while the size-dependence exponent is $\gamma = 2-\alpha$. This relation between γ and α formally coincides with Ref. [12], but it is derived from the confined Lévy flight rather than the Lévy walk model. In existing numerical and experimental investigations [2–4], the range of the size-dependence exponent is observed as $\gamma \le 1$. This range will not allow the case of $0 < \alpha < 1$, that is why the Lévy exponent is restricted as $1 < \alpha < 2$.

In the following, the local effective thermal conductivity will be discussed. Since J_q is already known, we only need to consider the expression of $\frac{dT(x)}{dx}$, which can be acquired from the following expansion

$$\begin{aligned} \frac{ds(x)}{dx} &= c_0\left[\frac{1}{T_0} + \frac{o(\delta T)}{T_0^2}\right]\frac{dT(x)}{dx} \\ &= -k_B\left[\ln P_0 + 1 + o(\delta P)\right]\frac{dP(x)}{dx} \\ \Rightarrow \frac{dT(x)}{dx} &\approx -\frac{k_B T_0(\ln P_0 + 1)\left[x(L-x)\right]^{\frac{\alpha}{2}-1}\delta P}{c_0\left\{\int_0^L \left[y(L-y)\right]^{\frac{\alpha}{2}-1} dy\right\}}. \end{aligned} \tag{3.14}$$

The local effective thermal conductivity is subsequently presented as follows

$$\begin{aligned} \kappa_{eff}^{loc} &= -\left[\frac{dT(x)}{dx}\right]^{-1} J_q \\ &= \frac{\pi K_\alpha c_0 \left[x(L-x)\right]^{1-\frac{\alpha}{2}}}{\cos\left(\frac{\pi\alpha}{2}\right)\sin\left(\frac{\pi\alpha}{2}\right)\Gamma(\alpha)}. \end{aligned} \tag{3.15}$$

which depends on not only the system size but also the location. Eq (3.14) also exhibits another signature of anomalous thermal transport, the nonlinear boundary asymptotics of the stationary temperature profile [4], namely,

$$\begin{cases} \lim_{x\to 0^+} \left| T(x) - T(x=0) \right| \sim x^{\frac{\alpha}{2}} \\ \lim_{x\to L^-} \left| T(x=L) - T(x) \right| \sim (L-x)^{\frac{\alpha}{2}} \end{cases} \tag{3.16}$$

In the diffusive limit $\alpha \to 2$, κ_{eff}^{loc} will be independent of the system size and location, and meanwhile, the asymptotic exponent $\chi = \alpha/2$ becomes linear. All of these degenerate behaviors agree with Fourier's law, which is physically reasonable. It should be underlined that the expanding approach stated above is inapplicable to the standard Lévy-Fokker-Planck equation based on the Riemann-Liouville operator. That is because the assumption of sufficiently small temperature difference ($\delta T \ll T_0$) is invalid for the Riemann-Liouville operator.

4. Concluding remarks

The symmetric Lévy-Fokker-Planck equation is applied to investigating anomalous thermal transport in a one-dimensional confined domain. Based on the frameworks of classical irreversible thermodynamics and Boltzmann-Gibbs statistical mechanics, we establish a connection between the evolution of the probability density function and thermal transport dominated by the confined Lévy flights. The expression of the local effective thermal conductivity is derived as a nonlocal formula, which depends on all states in the domain. The thermal transport process therefore becomes anomalous. It is demonstrated that the diffusive limit $\alpha \to 2$ will lead to the degeneration into conventional Fourier's law of heat conduction as if the thermal conductivity and specific heat capacity possess the same temperature-dependence. The thermodynamic connection between the Lévy-Fokker-Planck equation and anomalous thermal transport relies on the near-equilibrium assumption, which needs certain physical restrictions on the evolution of the probability density function. It is found that the Riemann-Liouville operator will be paired with thermodynamically non-trivial behaviors, namely that the equilibrium state corresponds to the non-uniform temperature distribution and infinite boundary temperature. In order to avoid the non-uniform equilibrium state, the Lévy-Fokker-Planck equation is modified in terms of the Caputo operator. It is shown that the modified Lévy-Fokker-Planck equation will give rise to two signatures of anomalous thermal transport, the power-law size-dependence of the global effective thermal conductivity and nonlinear boundary asymptotics of the stationary temperature profile. The results illustrate that the anomalies of Lévy-based thermal transport are not independent of each other, and should fulfill certain quantitative relations. For instance, the size-dependence exponent of the global effective thermal conductivity and asymptotic exponent of the stationary temperature profile are constrained by $\gamma = 2 - 2\chi$. The quantitative relations can be used to test whether a specific thermal transport process is dominated by the confined Lévy flights.

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Conflict of interest

The authors declare no conflict of interest.

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