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Research article

On definition of solution of initial value problem for fractional differential equation of variable order

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Abstract: We propose a new definition of continuous approximate solution to initial value problem for differential equations involving variable order Caputo fractional derivative based on the classical definition of solution of integer order (or constant fractional order) differential equation. Some examples are presented to illustrate these theoretical results.

Keywords: variable order Caputo fractional derivative; variable order fractional integral; fractional differential equations; initial value problem; approximate solution
Mathematics Subject Classification: 26A33

1. Introduction

In this paper, we research a continuous approximate solution of the following variable order initial value problem

$$\begin{cases} {}^{C}D_{0+}^{p(t,x(t))}x(t) = f(t,x(t)), 0 < t \leq T, \\ x(0) = u_0, \end{cases} \tag{1.1}$$

where $0 < p(t,x(t)) < 1$, $u_0 \in \mathbb{R}$, $p(t,x(t))$ and $f(t,x(t))$ are given real-valued functions, ${}^{C}D_{0+}^{p(t,x(t))}$ denotes variable order Caputo fractional derivative defined by

$${}^{C}D_{0+}^{p(t,x(t))}x(t) = I_{0+}^{1-p(t,x(t))}x'(t), \tag{1.2}$$

and $I_{0+}^{1-p(t,x(t))}$ is variable order Riemann-Liouville fractional integral defined by

$$I_{0+}^{1-p(t,x(t))}x(t) = \int_0^t \frac{(t-s)^{-p(t,x(t))}}{\Gamma(1-p(t,x(t)))}x(s)ds, t > 0. \tag{1.3}$$

For details, please refer to [1, 2].

Fractional calculus has been acknowledged as an extremely powerful tool in describing the natural behavior and complex phenomena of practical problems due to its applications in [3–7]. However, the constant fractional order calculus is not the ultimate tool to model the phenomena in nature. Therefore, variable order fractional calculus is proposed. Moreover, variable order fractional differential equations provide better descriptions for nonlocal phenomena with varying dynamics than constant order differential equations and are extensively researched in [1, 2, 7–31]. Among these, there are many works dealing with numerical methods for some class of variable fractional order differential equations, for instance, [1, 2, 8–10, 12–19, 21–26, 30, 31]. In particular, variable order fractional boundary value problems are considered by numerical method base on reproducing kernel theory in [30, 31].

There are several definitions of variable order fractional integrals and derivatives in [1, 2]. We notice that when the order $p(t)$ is a constant function p, variable order Riemann-Liouville fractional derivative and integral are exactly constant order fractional derivative and integral. It is well known that the Riemann-Liouville fractional integral has the law of exponents, i.e. $I_{0+}^{\alpha}I_{0+}^{\beta}(\cdot) = I_{0+}^{\beta}I_{0+}^{\alpha}(\cdot) = I_{0+}^{\alpha+\beta}(\cdot), \alpha > 0, \beta > 0$. Based on the law of exponents, we can obtain some properties which are associated with fractional derivative and integral. For this reason, fractional order differential equations are transformed into equivalent integral equations. Thus some results of nonlinear functional analysis (for instance, some fixed point theorems) have been applied to considering the existence of solution of fractional order differential equations (see, e.g. [3, 20, 32–34] and the references therein). However, the law of exponents doesn't hold for variable order fractional integral. For example, in [21–24],

$$I_{0+}^{g(t)}I_{0+}^{h(t)}(\cdot) \neq I_{0+}^{h(t)}I_{0+}^{g(t)}(\cdot),$$

$$I_{0+}^{g(t)}I_{0+}^{h(t)}(\cdot) \neq I_{0+}^{h(t)+g(t)}(\cdot),$$

where $h(t)$ and $g(t)$ are both general nonnegative functions.

Then we will consider the properties which are interrelated with variable order fractional integral and variable order fractional derivative by given some examples.

Example 1.1. Let $p(t) = \frac{t}{4} + \frac{1}{4}$, $q(t) = \frac{3}{4} - \frac{t}{4}$, $f(t) = t, 0 \leq t \leq 2$. Now, we calculate $I_{0+}^{p(t)}I_{0+}^{q(t)}f(t)|_{t=1}$, $I_{0+}^{q(t)}I_{0+}^{p(t)}f(t)|_{t=1}$ and $I_{0+}^{p(t)+q(t)}f(t)|_{t=1}$, where $I_{0+}^{p(t)}$ and $I_{0+}^{q(t)}$ is defined in (1.3).

For $1 \leq t \leq 2$, we have

$$I_{0+}^{p(t)}I_{0+}^{q(t)}f(t) = \int_0^t \frac{(t-s)^{\frac{t}{4}+\frac{1}{4}-1}}{\Gamma(\frac{t}{4}+\frac{1}{4})}\int_0^s \frac{(s-\tau)^{\frac{3}{4}-\frac{s}{4}-1}\tau}{\Gamma(\frac{3}{4}-\frac{s}{4})}d\tau ds = \int_0^t \frac{(t-s)^{\frac{t}{4}-\frac{3}{4}}s^{\frac{7}{4}-\frac{s}{4}}}{\Gamma(\frac{t}{4}+\frac{1}{4})\Gamma(\frac{11}{4}-\frac{s}{4})}ds,$$

so

$$I_{0+}^{p(t)}I_{0+}^{q(t)}f(t)|_{t=1} = \int_0^1 \frac{(1-s)^{-\frac{1}{2}}s^{\frac{7}{4}-\frac{s}{4}}}{\Gamma(\frac{1}{2})\Gamma(\frac{11}{4}-\frac{s}{4})}ds \approx 0.4757,$$

yet

$$I_{0+}^{q(t)}I_{0+}^{p(t)}f(t) = \int_0^t \frac{(t-s)^{\frac{3}{4}-\frac{t}{4}-1}}{\Gamma(\frac{3}{4}-\frac{t}{4})}\int_0^s \frac{(s-\tau)^{\frac{s}{4}+\frac{1}{4}-1}\tau}{\Gamma(\frac{s}{4}+\frac{1}{4})}d\tau ds = \int_0^t \frac{(t-s)^{-\frac{1}{4}-\frac{t}{4}}s^{\frac{s}{4}+\frac{5}{4}}}{\Gamma(\frac{3}{4}-\frac{t}{4})\Gamma(\frac{9}{4}+\frac{s}{4})}ds,$$

$$I_{0+}^{q(t)} I_{0+}^{p(t)} f(t)|_{t=1} = \int_0^1 \frac{(1-s)^{-\frac{1}{2}} s^{\frac{s}{4}+\frac{5}{4}}}{\Gamma(\frac{1}{2})\Gamma(\frac{9}{4}+\frac{s}{4})} ds \approx 0.5283,$$

and

$$I_{0+}^{p(t)+q(t)} f(t)|_{t=1} = I_{0+}^{1} f(t)|_{t=1} = \int_0^1 s ds = 0.5.$$

Therefore,

$$I_{0+}^{p(t)} I_{0+}^{q(t)} f(t)|_{t=1} \neq I_{0+}^{p(t)+q(t)} f(t)|_{t=1},$$

$$I_{0+}^{q(t)} I_{0+}^{p(t)} f(t)|_{t=1} \neq I_{0+}^{p(t)+q(t)} f(t)|_{t=1},$$

$$I_{0+}^{p(t)} I_{0+}^{q(t)} f(t)|_{t=1} \neq I_{0+}^{q(t)} I_{0+}^{p(t)} f(t)|_{t=1}.$$

Example 1.1 illustrates that the law of exponents of the variable order Riemann-Liouville fractional integral doesn't hold when the order is non-constant continuous function.

Example 1.2. Let $p(t) = \begin{cases} \frac{1}{2} & 0 \le t \le 1, \\ \frac{1}{3}, & 1 < t \le 6, \end{cases}$, $q(t) = \begin{cases} \frac{1}{2}, & 0 \le t \le 1, \\ \frac{2}{3}, & 1 < t \le 6, \end{cases}$ and $f(t) = t, 0 \le t \le 6$. We'll consider $I_{0+}^{p(t)} I_{0+}^{q(t)} f(t)|_{t=4}$, $I_{0+}^{q(t)} I_{0+}^{p(t)} f(t)|_{t=4}$ and $I_{0+}^{p(t)+q(t)} f(t)|_{t=4}$, where $I_{0+}^{p(t)}$ and $I_{0+}^{q(t)}$ are defined in (1.3).

For $1 \le t \le 4$, we have

$$\begin{aligned} I_{0+}^{p(t)} I_{0+}^{q(t)} f(t) &= \int_0^1 \frac{(t-s)^{p(t)-1}}{\Gamma(p(t))} \int_0^s \frac{(s-\tau)^{\frac{1}{2}-1}\tau}{\Gamma(\frac{1}{2})} d\tau ds \\ &+ \int_1^t \frac{(t-s)^{p(t)-1}}{\Gamma(p(t))} \Big(\int_0^1 \frac{(s-\tau)^{\frac{1}{2}-1}\tau}{\Gamma(\frac{1}{2})} d\tau + \int_1^s \frac{(s-\tau)^{\frac{2}{3}-1}\tau}{\Gamma(\frac{2}{3})} d\tau \Big) ds \\ &= \int_0^1 \frac{(t-s)^{p(t)-1} s^{\frac{3}{2}}}{\Gamma(p(t))\Gamma(\frac{5}{2})} ds + \int_1^t \frac{(t-s)^{p(t)-1}}{\Gamma(p(t))} \frac{\frac{4s^{\frac{3}{2}}}{3} - \frac{2}{3}(s-1)^{\frac{1}{2}}(2s+1)}{\pi^{\frac{1}{2}}} ds \\ &+ \int_1^t \frac{(t-s)^{p(t)-1}}{\Gamma(p(t))} \frac{3(s-1)^{\frac{2}{3}}(3s+2))}{10\Gamma(\frac{2}{3})} ds, \end{aligned}$$

thus,

$$\begin{aligned} I_{0+}^{p(t)} I_{0+}^{q(t)} f(t)|_{t=4} &= \int_0^1 \frac{(4-s)^{-\frac{2}{3}} s^{\frac{3}{2}}}{\Gamma(\frac{1}{3})\Gamma(\frac{5}{2})} ds + \int_1^4 \frac{(4-s)^{-\frac{2}{3}}}{\Gamma(\frac{1}{3})} \frac{\frac{4s^{\frac{3}{2}}}{3} - \frac{2}{3}(s-1)^{\frac{1}{2}}(2s+1)}{\pi^{\frac{1}{2}}} ds \\ &+ \int_1^4 \frac{(4-s)^{-\frac{2}{3}}}{\Gamma(\frac{1}{3})} \frac{3(s-1)^{\frac{2}{3}}(3s+2))}{10\Gamma(\frac{2}{3})} ds \approx 7.8626. \end{aligned}$$

By the same way, we get

$$
\begin{aligned}
I_{0+}^{q(t)}I_{0+}^{p(t)}f(t) &= \int_0^1 \frac{(t-s)^{q(t)-1}}{\Gamma(q(t))}\int_0^s \frac{(s-\tau)^{\frac{1}{2}-1}\tau}{\Gamma(\frac{1}{2})}d\tau ds \\
&\quad + \int_1^t \frac{(t-s)^{q(t)-1}}{\Gamma(q(t))}\Big(\int_0^1 \frac{(s-\tau)^{\frac{1}{2}-1}\tau}{\Gamma(\frac{1}{2})}d\tau + \int_1^s \frac{(s-\tau)^{\frac{1}{3}-1}\tau}{\Gamma(\frac{1}{3})}d\tau\Big)ds \\
&= \int_0^1 \frac{(t-s)^{q(t)-1}s^{\frac{3}{2}}}{\Gamma(q(t))\Gamma(\frac{5}{2})}ds + \int_1^t \frac{(t-s)^{q(t)-1}}{\Gamma(q(t))}\frac{\frac{4s^{\frac{3}{2}}}{3}-\frac{2}{3}(s-1)^{\frac{1}{2}}(2s+1)}{\pi^{\frac{1}{2}}}ds \\
&\quad + \int_1^t \frac{(t-s)^{q(t)-1}}{\Gamma(q(t))}\frac{3(s-1)^{\frac{1}{3}}(3s+1)}{4\Gamma(\frac{1}{3})}ds,
\end{aligned}
$$

$$
\begin{aligned}
I_{0+}^{q(t)}I_{0+}^{p(t)}f(t)|_{t=4} &= \int_0^1 \frac{(4-s)^{-\frac{1}{3}}s^{\frac{3}{2}}}{\Gamma(\frac{2}{3})\Gamma(\frac{5}{2})}ds + \int_1^4 \frac{(4-s)^{-\frac{1}{3}}}{\Gamma(\frac{2}{3})}\frac{\frac{4s^{\frac{3}{2}}}{3}-\frac{2}{3}(s-1)^{\frac{1}{2}}(2s+1)}{\pi^{\frac{1}{2}}}ds \\
&\quad + \int_1^4 \frac{(4-s)^{-\frac{1}{3}}}{\Gamma(\frac{2}{3})}\frac{3(s-1)^{\frac{1}{3}}(3s+1)}{4\Gamma(\frac{1}{3})}ds \approx 8.1585,
\end{aligned}
$$

and

$$
I_{0+}^{p(t)+q(t)}f(t)|_{t=4} = \int_0^4 \frac{(4-s)^{p(4)+q(4)-1}s}{\Gamma(p(4)+q(4))}ds = \int_0^4 sds = 8.
$$

As a result, we deduce

$$
I_{0+}^{p(t)}I_{0+}^{q(t)}f(t)|_{t=4} \neq I_{0+}^{p(t)+q(t)}f(t)|_{t=4},
$$

$$
I_{0+}^{q(t)}I_{0+}^{p(t)}f(t)|_{t=4} \neq I_{0+}^{p(t)+q(t)}f(t)|_{t=4},
$$

$$
I_{0+}^{p(t)}I_{0+}^{q(t)}f(t)|_{t=4} \neq I_{0+}^{q(t)}I_{0+}^{p(t)}f(t)|_{t=4}.
$$

Example 1.2 shows that the law of exponents of the variable order Riemann-Liouville fractional integral doesn't hold when the order is piecewise constant function defined in the same partition.

Example 1.3. Let $p(t) = \frac{t}{4} + \frac{1}{4}$, $f(t) = t, 0 \le t \le 3$. Now, we consider $I_{0+}^{p(t)}{}^{C}D_{0+}^{p(t)}f(t)|_{t=2}$ and ${}^{C}D_{0+}^{p(t)}I_{0+}^{p(t)}f(t)|_{t=2}$.

By (1.2) and (1.3), we have

$$
I_{0+}^{p(t)}{}^{C}D_{0+}^{p(t)}f(t) = \int_0^t \frac{(t-s)^{\frac{t}{4}+\frac{1}{4}-1}}{\Gamma(\frac{t}{4}+\frac{1}{4})}\int_0^s \frac{(s-\tau)^{-\frac{s}{4}-\frac{1}{4}}}{\Gamma(\frac{3}{4}-\frac{s}{4})}d\tau ds = \int_0^t \frac{(t-s)^{\frac{t}{4}-\frac{3}{4}}s^{\frac{3}{4}-\frac{s}{4}}}{\Gamma(\frac{t}{4}+\frac{1}{4})\Gamma(\frac{7}{4}-\frac{s}{4})}ds,
$$

$$
I_{0+}^{p(t)}{}^{C}D_{0+}^{p(t)}f(t)|_{t=2} = \int_0^2 \frac{(2-s)^{-\frac{1}{4}}s^{\frac{3}{4}-\frac{s}{4}}}{\Gamma(\frac{3}{4})\Gamma(\frac{7}{4}-\frac{s}{4})}ds \approx 1.91596 \neq f(t)|_{t=2} - f(0) = 2,
$$

which implies that $I_{0+}^{p(t)C}D_{0+}^{p(t)}$ is different with the result of constant order fractional derivative and integral, that is,

$$I_{0+}^{\alpha}\,{}^{C}D_{0+}^{\alpha}g(t) = g(t) - g(0), 0 < t \le b, \tag{1.4}$$

where $0 < \alpha < 1$, $g \in C^1[0, b], 0 < b < +\infty$.

On the other hand, we have

$$\begin{aligned}
{}^{C}D_{0+}^{p(t)}I_{0+}^{p(t)}f(t)|_{t=2} &= I_{0+}^{1-p(t)}\frac{d}{dt}I_{0+}^{p(t)}f(t)\Big|_{t=2} = \int_0^t \frac{(t-s)^{-\frac{t}{4}-\frac{1}{4}}}{\Gamma(\frac{3}{4}-\frac{t}{4})}\frac{d}{ds}\Big(\frac{s^{\frac{5}{4}+\frac{s}{4}}}{\Gamma(\frac{9}{4}+\frac{s}{4})}\Big)ds\Big|_{t=2}\\
&= \int_0^t \frac{(t-s)^{-\frac{t}{4}-\frac{1}{4}}}{\Gamma(\frac{3}{4}-\frac{t}{4})}\Big[\frac{s^{\frac{s}{4}+\frac{5}{4}}(\frac{\frac{s}{4}+\frac{5}{4}}{s}+\frac{\log(s)}{4})}{\Gamma(\frac{s}{4}+\frac{9}{4})} - \frac{s^{\frac{s}{4}+\frac{5}{4}}\Gamma'(\frac{9}{4}+\frac{s}{4})}{4\Gamma^2(\frac{9}{4}+\frac{s}{4})}\Big]ds\Big|_{t=2}\\
&\approx 1.91365 \ne f(t)|_{t=2} = 2,
\end{aligned}$$

which illustrates that ${}^{C}D_{0+}^{p(t)}I_{0+}^{p(t)}$ is different with the result of constant order fractional derivative and integral, that is,

$${}^{C}D_{0+}^{\alpha}I_{0+}^{\alpha}h(t) = h(t), 0 < t \le b, \tag{1.5}$$

where $0 < \alpha < 1$, $h \in C^1[0, b], 0 < b < +\infty$.

Example 1.3 verifies that the properties (1.4) and (1.5) of constant order fractional calculus don't hold for variable order fractional calculus when the order is a continuous function.

Example 1.4. Let $p(t) = \begin{cases} \frac{1}{2}, & 0 \le t \le 1,\\ \frac{1}{3}, & 1 < t \le 6,\end{cases}$, $f(t) = t, 0 \le t \le 6$. Now, we consider $I_{0+}^{p(t)C}D_{0+}^{p(t)}f(t)|_{t=4}$ and ${}^{C}D_{0+}^{p(t)}I_{0+}^{p(t)}f(t)|_{t=4}$.

By (1.2) and (1.3), for $2 \le t \le 6$, we have

$$\begin{aligned}
I_{0+}^{p(t)C}D_{0+}^{p(t)}f(t) &= \int_0^t \frac{(t-s)^{p(t)-1}}{\Gamma(p(t))}\int_0^s \frac{(s-\tau)^{-p(s)}}{\Gamma(1-p(s))}d\tau ds\\
&= \int_0^1 \frac{(t-s)^{p(t)-1}}{\Gamma(p(t))}\int_0^s \frac{(s-\tau)^{-\frac{1}{2}}}{\Gamma(\frac{1}{2})}d\tau ds\\
&\quad + \int_1^t \frac{(t-s)^{p(t)-1}}{\Gamma(p(t))}\Big[\int_0^1 \frac{(s-\tau)^{-\frac{1}{2}}}{\Gamma(\frac{1}{2})}d\tau + \int_1^s \frac{(s-\tau)^{-\frac{1}{3}}}{\Gamma(\frac{2}{3})}d\tau\Big]ds\\
&= \int_0^t \frac{(t-s)^{p(t)-1}s^{\frac{1}{2}}}{\Gamma(p(t))\Gamma(\frac{3}{2})}ds - \int_1^t \frac{(t-s)^{p(t)-1}(s-1)^{\frac{1}{2}}}{\Gamma(p(t))\Gamma(\frac{3}{2})}ds + \int_1^t \frac{(t-s)^{p(t)-1}(s-1)^{\frac{2}{3}}}{\Gamma(p(t))\Gamma(\frac{5}{3})}ds,
\end{aligned}$$

so

$$\begin{aligned}
I_{0+}^{p(t)C}D_{0+}^{p(t)}f(t)|_{t=4} &= \int_0^4 \frac{(4-s)^{-\frac{2}{3}}s^{\frac{1}{2}}}{\Gamma(\frac{1}{3})\Gamma(\frac{3}{2})}ds - \int_1^4 \frac{(4-s)^{-\frac{2}{3}}(s-1)^{\frac{1}{2}}}{\Gamma(\frac{1}{3})\Gamma(\frac{3}{2})}ds\\
&\quad + \int_1^4 \frac{(4-s)^{-\frac{2}{3}}(s-1)^{\frac{2}{3}}}{\Gamma(\frac{1}{3})\Gamma(\frac{5}{3})}ds\\
&\approx 3.7194 \ne f(t)|_{t=4} - f(0) = 4.
\end{aligned}$$

On the other hand, for $2 \le t \le 6$, we get

$$
\begin{aligned}
{}^{C}D_{0+}^{p(t)}I_{0+}^{p(t)}f(t) =& I_{0+}^{1-p(t)}\frac{d}{dt}I_{0+}^{p(t)}f(t)\\
=& \int_0^t \frac{(t-s)^{-p(t)}}{\Gamma(1-p(t))}\frac{d}{ds}\int_0^s \frac{(s-\tau)^{p(s)-1}\tau}{\Gamma(p(s))}d\tau ds\\
=& \int_0^1 \frac{(t-s)^{-p(t)}}{\Gamma(1-p(t))}\frac{d}{ds}\int_0^s \frac{(s-\tau)^{p(s)-1}\tau}{\Gamma(p(s))}d\tau ds\\
&+\int_1^t \frac{(t-s)^{-p(t)}}{\Gamma(1-p(t))}\frac{d}{ds}\int_0^s \frac{(s-\tau)^{p(s)-1}\tau}{\Gamma(p(s))}d\tau ds\\
=& \int_0^1 \frac{(t-s)^{-p(t)}s^{\frac{1}{2}}}{\Gamma(1-p(t))\Gamma(\frac{3}{2})}ds + \int_1^t \frac{(t-s)^{-p(t)}}{\Gamma(1-p(t))}\frac{d}{ds}\int_0^1 \frac{(s-\tau)^{-\frac{1}{2}}\tau}{\Gamma(\frac{1}{2})}d\tau ds\\
&+\int_1^t \frac{(t-s)^{-p(t)}}{\Gamma(1-p(t))}\frac{d}{ds}\int_1^s \frac{(s-\tau)^{-\frac{2}{3}}\tau}{\Gamma(\frac{1}{3})}d\tau ds\\
=& \int_0^1 \frac{(t-s)^{-p(t)}s^{\frac{1}{2}}}{\Gamma(1-p(t))\Gamma(\frac{3}{2})}ds + \int_1^t \frac{(t-s)^{-p(t)}}{\Gamma(1-p(t))}\frac{1-2s+2(s-1)^{\frac{1}{2}}s^{\frac{1}{2}}}{\pi^{\frac{1}{2}}(s-1)^{\frac{1}{2}}}ds\\
&+\int_1^t \frac{(t-s)^{-p(t)}}{\Gamma(1-p(t))}\frac{(3s-2)(s-1)^{-\frac{2}{3}}}{\Gamma(\frac{1}{3})}ds,
\end{aligned}
$$

so

$$
\begin{aligned}
{}^{C}D_{0+}^{p(t)}I_{0+}^{p(t)}f(t)|_{t=4} =& \int_0^1 \frac{(4-s)^{-\frac{1}{3}}s^{\frac{1}{2}}}{\Gamma(\frac{2}{3})\Gamma(\frac{3}{2})}ds + \int_1^4 \frac{(4-s)^{-\frac{1}{3}}}{\Gamma(\frac{2}{3})}\frac{1-2s+2(s-1)^{\frac{1}{2}}s^{\frac{1}{2}}}{\pi^{\frac{1}{2}}(s-1)^{\frac{1}{2}}}ds\\
&+\int_1^4 \frac{(4-s)^{-\frac{1}{3}}}{\Gamma(\frac{2}{3})}\frac{(3s-2)(s-1)^{-\frac{2}{3}}}{\Gamma(\frac{1}{3})}ds\\
\approx& 4.0331 \ne f(t)|_{t=4} = 4.
\end{aligned}
$$

Example 1.4 verifies the properties (1.4) and (1.5) of constant order fractional calculus is impossible for $I_{0+}^{p(t)}{}^{C}D_{0+}^{p(t)}f(t)$ and ${}^{C}D_{0+}^{p(t)}I_{0+}^{p(t)}f(t)$ when the order is a piecewise function.

Hence, we can claim that variable order fractional integral defined by (1.3) has no law of exponents. Moreover, for general functions $p(t)$ and $f(t)$, the representations of ${}^{C}D_{0+}^{p(t)}I_{0+}^{p(t)}f(t)$ and $I_{0+}^{p(t)}{}^{C}D_{0+}^{p(t)}f(t)$ are not clear. These obstacles make it difficult for us to transform variable order fractional differential equation into equivalent integral equation. As a result, it is almost impossible that some nonlinear functional analysis classical methods such as fixed point theorems are applied to prove the existence of solution of the corresponding integral equation. To the best of our knowledge, there are few works ([8,23,24,27]) to deal with the existence of solutions to variable order fractional differential equations.

In [18], authors considered the following numerical solutions for variable order fractional functional boundary value problem

$$
\begin{cases} D^{\alpha(x)}u(x) + a(x)u'(x) + b(x)u(x) + c(x)u(\tau(x)) = f(x), \quad 0 \le x \le 1,\\ u(0) = \lambda_0, \quad u(1) = \lambda_1, \end{cases} \tag{1.6}
$$

where $D^{\alpha(x)}$ is the variable order Caputo fractional derivative defined by

$$
D^{\alpha(x)}u(x) = \frac{1}{\Gamma(2-\alpha(x))}\int_0^x (x-s)^{1-\alpha(x)}u''(s)ds, 1 \le \alpha(x) < 2;
$$

$a(x), b(x), c(x) \in C^2[0,1]$; $\alpha(x), \tau(x), f(x) \in C[0,1]$; and $\lambda_0, \lambda_1 \in \mathbb{R}$. When $\alpha(x) = \frac{5+\sin(x)}{4}$, $a(x) = cos(x)$, $b(x) = 4$, $c(x) = 5$, $\tau(x) = x^2$, $f(x) = \frac{2x^{2-\alpha(x)}}{\Gamma(3-\alpha(x))} + 5x^4 + 4x^2 + 2x\cos(x)$ and $\lambda_0 = 0, \lambda_1 = 1$, the exact solution of problem (1.6) was given by

$$u(x) = x^2.$$

In (1.6), if we take $f(x) = 6x^6+5x^4+4x^2+2x\cos(x)$, it is almost impossible to obtain its exact solution. In fact, we don't even know whether the solution to problem (1.6) exists.

In [8], authors discussed the existence of solution for a generalized fractional differential equation with non-autonomous variable order operators

$$\begin{cases} {}_cD_t^{q(t,x(t))}x(t) = f(t, x(t)), \\ x(c) = x_0, \end{cases} \tag{1.7}$$

where $x \in \mathbb{R}^n$ is the state vector, $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is a vector field, $c \in \mathbb{R}$ is the lower terminal, x_0 is the initial value, $0 < q(t, x(t)) \leqslant 1$, and ${}_cD_t^{q(t,x(t))}$ is the variable order Riemann-Liouville fractional differential operator defined as follows

$${}_cD_t^{q(t,x(t))}x(t) = \frac{d}{dt}\int_c^t \frac{(t-s)^{-q(s,x(s))}}{\Gamma(1-q(s,x(s)))}x(s)ds.$$

In [8] , authors claimed that, by [35], the initial value problem (1.7) is equivalent to the integral equation

$$\begin{cases} x(t) = \int_c^t \frac{(t-s)^{q(s,x(s))-1}}{\Gamma(q(s,x(s)))}f(s, x(s))ds + \Psi(f, -q, a, c, t) \qquad t > c, \\ x(c) = x_0, \end{cases} \tag{1.8}$$

where $\Psi : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^n$ is the intialization function [35] which is needed in the general form given in (1.8) and the scalar a is used for characterizing the initial period such that for $a < t < c$, the initial information is given, and for $t < a$ we consider $f(t) = 0$. Given a, one can develop a closed analytical formula for Ψ [35].

However, in our opinion, there is no theoretical basis for this assertion. Because there are not contents related to variable order fractional integral and derivative in [35].

In [27], authors considered the existence results of solution to the initial value problem (1.1), in which $f(t, x(t)) : [0,T] \times \mathbb{R} \to \mathbb{R}$ and $p : [0,T] \times \mathbb{R} \to (0,1)$ are both continuous functions, and $0 < p_1 \leq p(t, x(t)) \leq p_2 < 1$, $u_0 \in \mathbb{R}$. By means of Arzela-Ascoli theorem, authors obtained that the following sequence

$$\begin{cases} x_n(t) = x_{n-1}(t) + \int_0^{t-\frac{T}{n}} \frac{(t-s)^{-p(t,x(t))}}{\Gamma(1-p(t,x(t)))}x_{n-1}(s)ds - f[t, \int_0^{t-\frac{T}{n}} x_{n-1}(s)ds + u_0], t \in (\frac{T}{n}, T], \\ x_n(t) = 0, t \in [0, \frac{T}{n}] \end{cases} \tag{1.9}$$

existed a subsequence still denoted by the sequence $\{x_n\}$ which uniformly converged to a continuous function x^*. Set $x(t) = \int_0^t x^*(s)ds+u_0$, then authors obtained the problem (1.1) existed one solution $x(t)$. But, we find that it has fatal errors in these analysis procedure, that is the result of uniformly bounded of sequence $\{x_n\}$. This is easy to be overlooked. In our opinion the sequence $\{x_n\}$ is non-uniformly bounded. As a result, the existence result of solution to the initial value problem (1.1) is not obtained by Arzela-Ascoli theorem. On the other hand, it is almost impossible to transform the initial value problem (1.1) into equivalent integral equation.

Base on these facts, how to deal with the existence of solution of variable order fractional differential equations is a principal problem to be solved. In this paper, according to the classical definition of solution of integer order(or constant fractional order) differential equation, we propose a new definition of continuous approximate solution to the problem (1.1) under two kinds of variable order $p(t, x(t))$ and $p(t)$.

The paper is organized as follows. In section 2, a new definition of approximate solution to the problem (1.1) for variable order $p(t, x(t))$ is proposed and we provide an example to demonstrate the definition. Section 3 is devoted to introduce another definition of approximate solution for $p(t)$, then an example is given to illustrate the theoretical result.

2. Definition of approximate solution for $p(t, x(t))$

Throughout this section, we assume that

(A_1): $p : [0, T] \times \mathbb{R} \to (0, 1)$ is a continuous function;

(A_2): $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ is a continuous differential function.

We begin with definitions and characters of solution of integer order and constant fractional order.

Remark 2.1. *According to the definition of solution $x(t)$ of differential equation, it should be defined in the interval, in which the equation is satisfied. For instance, a function $x(t)$ is called a solution of the following initial value problem*

$$\begin{cases} x'(t) = t^2 sin x + t^3 x^3, 0 < t \leq T, \\ x(0) = 0 \end{cases}$$

if x is defined in the interval $[0, T]$, satisfying the equation and the initial value condition $x(0) = 0$.

Remark 2.2. *Fractional operators are typical nonlocal operators, which can well describe the memory and global correlation of physical processes. Hence, for initial value problem of fractional order differential equations, its solution in given interval is affected by the state of solution in the preceding intervals.*

By Remark 2.1, a function $x : [0, T] \to \mathbb{R}$ is called a solution of the initial value problem (1.1) if $x(t)$ satisfies the equation (1.1) and $x(0) = u_0$. However, based on analysis above, we are faced with extreme difficulties in considering the existence of solution of initial value problem (1.1) in sense of this definition. Thus, a new continuous approximate solution of problem (1.1) is proposed.

We start off by analyzing the problem (1.1) based on the facts above.

Let

$$p_1 = p(0, u_0). \tag{2.1}$$

We consider the initial value problem defined in the interval $[0, T]$ as following

$$\begin{cases} {}^C D_{0+}^{p_1} x(t) = f(t, x), \quad 0 < t \leq T, \\ x(0) = u_0. \end{cases} \tag{2.2}$$

Let $x_1 \in C^1(0,T] \cap C[0,T]$ be a solution of the initial value problem (2.2)(by standard way, we know that the initial value problem (2.2) exists continuous solution in $[0,T]$ under some assumptions on f). Since $x_1(t)$ is right continuous at point 0, then for arbitrary small $\varepsilon > 0$, there exists $\delta_{01} > 0$ such that

$$|x_1(t) - x_1(0)| < \varepsilon, \text{ for } 0 < t \le \delta_{01}. \tag{2.3}$$

And because $p(t, x_1(t))$ is right continuous at point $(0, x_1(0)) = (0, u_0)$, then together with (2.3) and (2.1), for the above ε, there exists $\delta_0 > 0$ such that

$$|p(t, x_1(t)) - p(0, x_1(0))| = |p(t, x_1(t)) - p(0, u_0)| = |p(t, x_1(t)) - p_1| < \varepsilon, \text{ for } 0 < t \le \delta_0. \tag{2.4}$$

If $\delta_0 < T$, we take $\delta_0 \doteq T_1$ and continue next procedure. Otherwise, we take $T_1 = T$ and end this procedure.

We assume that $\delta_0 < T$, and then let

$$p_2 = p(T_1, x_1(T_1)). \tag{2.5}$$

In order to consider the existence of solution to (1.1) in the interval $[T_1, T_2]$, we let

$$x'(t) = \int_0^t x_1(s)ds, \quad t \in (0, T_1]. \tag{2.6}$$

By (2.6) and integration by parts, we denote

$$\int_0^{T_1} \frac{(t-s)^{-p_2} x'(s)}{\Gamma(1-p_2)} ds = \int_0^{T_1} \frac{(t-s)^{1-p_2} x_1(s)}{\Gamma(2-p_2)} ds - \frac{(t-T_1)^{1-p_2}}{\Gamma(2-p_2)} \int_0^{T_1} x_1(s)ds \doteq \varphi_{x_1}(t). \tag{2.7}$$

Hence, we may consider the initial value problem defined in the interval $[T_1, T]$ as following

$$\begin{cases} {}^C D_{T_1+}^{p_2} x(t) = f(t,x) - \varphi_{x_1}(t), \quad T_1 < t \le T, \\ x(T_1) = x_1(T_1), \end{cases} \tag{2.8}$$

where $x_1(t)$ is the solution of the initial value problem (2.2) and φ_{x_1} is the function defined by (2.7).

Let $x_2 \in C^1(T_1,T] \cap C[T_1,T]$ be a solution of the problem (2.8) (by standard way, we know that the problem (2.8) exists continuous solution in $[T_1,T]$ under some assumptions on f). Since $x_2(t)$ is right continuous at point T_1, then for the above ε, there exists $\delta_{11} > 0$ such that

$$|x_2(t) - x_2(T_1)| < \varepsilon, \text{ for } T_1 < t \le T_1 + \delta_{11}. \tag{2.9}$$

And because $p(t, x_2(t))$ is right continuous at point $(T_1, x_2(T_1)) = (T_1, x_1(T_1))$, then together with (2.9), for the above ε, there exists $\delta_1 > 0$ such that

$$|p(t, x_2(t)) - p(T_1, x_2(T_1))| = |p(t, x_2(t)) - p(T_1, x_1(T_1))| < \varepsilon, \text{ for } T_1 < t \le T_1 + \delta_1. \tag{2.10}$$

If $T_1 + \delta_1 < T$, we take $T_1 + \delta_1 \doteq T_2$ and continue next procedure. Otherwise, we take $T_2 = T$ and end this procedure. Obviously, according to (2.10) and (2.5), we obtain

$$|p(t, x_2(t)) - p_2| < \varepsilon, \text{ for } T_1 < t \le T_2. \tag{2.11}$$

We assume that $T_1 + \delta_1 < T$, and then let

$$p_3 = p(T_2, x_2(T_2)). \tag{2.12}$$

We let

$$x'(t) = \begin{cases} \int_0^t x_1(s)ds, & t \in (0, T_1], \\ \int_{T_1}^t x_2(s)ds, & t \in (T_1, T_2]. \end{cases} \tag{2.13}$$

Thus, by (2.13) and integration by parts, we have

$$\int_{T_{i-1}}^{T_i} \frac{(t-s)^{-p_3} x'(s)}{\Gamma(1-p_3)} ds = \int_{T_{i-1}}^{T_i} \frac{(t-s)^{1-p_3} x_i(s)}{\Gamma(2-p_3)} ds - \frac{(t-T_i)^{1-p_3}}{\Gamma(2-p_3)} \int_{T_{i-1}}^{T_i} x_i(s)ds \doteq \phi_{x_i}(t), \tag{2.14}$$

$i = 1, 2, T_0 = 0$.

We may consider the initial value problem in the interval $[T_2, T]$ as following

$$\begin{cases} {}^C D_{T_2+}^{p_3} x(t) = f(t, x) - \phi_{x_1}(t) - \phi_{x_2}(t), & T_2 < t \le T, \\ x(T_2) = x_2(T_2), \end{cases} \tag{2.15}$$

where $x_i(t)$ is the solution of the initial value problems (2.2) and (2.8) respectively, and $\phi_{x_i}(i = 1, 2)$ is the function defined by (2.14).

Let $x_3 \in C^1(T_2, T] \cap C[T_2, T]$ be a solution of the problem (2.15)(by standard way, we know that the problem (2.15) exists continuous solution in $[T_2, T]$ under some assumptions on f). Since $x_3(t)$ is right continuous at point T_2, then for the above ε, there exists $\delta_{21} > 0$ such that

$$|x_3(t) - x_3(T_2)| < \varepsilon, \text{ for } T_2 < t \le T_2 + \delta_{21}. \tag{2.16}$$

And because $p(t, x_3(t))$ is right continuous at point $(T_2, x_3(T_2)) = (T_2, x_2(T_2))$, then together with (2.16), for the above ε, there exists $\delta_2 > 0$ such that

$$|p(t, x_3(t)) - p(T_2, x_3(T_2))| = |p(t, x_3(t)) - p(T_2, x_2(T_2))| < \varepsilon, \text{ for } T_2 < t \le T_2 + \delta_2. \tag{2.17}$$

If $T_2 + \delta_2 < T$, we take $T_2 + \delta_2 \doteq T_3$ and continue next procedure. Otherwise, we take $T_3 = T$ and end this procedure. Obviously, according to (2.17) and (2.12), it holds

$$|p(t, x_3(t)) - p_3| < \varepsilon, \text{ for } T_2 < t \le T_3. \tag{2.18}$$

We assume $T_2 + \delta_2 < T$, and then let

$$p_4 = p(T_3, x_3(T_3)). \tag{2.19}$$

We let

$$x'(t) = \begin{cases} \int_0^t x_1(s)ds, & t \in (0, T_1], \\ \int_{T_1}^t x_2(s)ds, & t \in (T_1, T_2], \\ \int_{T_2}^t x_3(s)ds, & t \in (T_2, T_3]. \end{cases} \tag{2.20}$$

By (2.20) and integration by parts, we get

$$\omega_{x_i}(t) \doteq \int_{T_{i-1}}^{T_i} \frac{(t-s)^{-p_4}x'(s)}{\Gamma(1-p_4)}ds = \int_{T_{i-1}}^{T_i} \frac{(t-s)^{1-p_4}x_i(s)}{\Gamma(2-p_4)}ds - \frac{(t-T_i)^{1-p_4}}{\Gamma(2-p_4)} \int_{T_{i-1}}^{T_i} x_i(s)ds, \tag{2.21}$$

$i = 1, 2, 3, T_0 = 0$.

We may consider the initial value problem in the interval $[T_3, T]$ as following

$$\begin{cases} {}^C D_{T_3+}^{p_4} x(t) = f(t, x) - \omega_{x_1}(t) - \omega_{x_2}(t) - \omega_{x_3}(t), & T_3 < t \le T, \\ x(T_3) = x_3(T_3), \end{cases} \tag{2.22}$$

where $x_i(t)$ are solutions of the initial value problems (2.2), (2.8) and (2.15) respectively, and $\omega_{x_i}(i = 1, 2, 3)$ is the function defined by (2.21).

Since $[0, T]$ is a finite interval, we continue this procedure and could finish it by finite steps. That is, there exists $\delta_{n^*-2} > 0$, $\delta_{n^*-1} > 0$ $(n^* \in N)$ such that $T_{n^*-2} + \delta_{n^*-2} \doteq T_{n^*-1} < T$, $T_{n^*-1} + \delta_{n^*-1} \ge T \doteq T_{n^*}$. Then we have intervals $[0, T_1], [T_1, T_2], [T_2, T_3], \cdots, [T_{n^*-2}, T_{n^*-1}], [T_{n^*-1}, T]$, and solutions $x_i \in C^1(T_{i-1}, T] \cap C[T_{i-1}, T]$ of the following initial value problem defined in the interval $[T_{i-1}, T]$

$$\begin{cases} {}^C D_{T_{i-1}+}^{p_i} x(t) = f(t, x) - \psi_{x_1}(t) - \psi_{x_2}(t) - \cdots - \psi_{x_{i-1}}(t), & T_{i-1} < t \le T, \\ x(T_{i-1}) = x_{i-1}(T_{i-1}), \end{cases} \tag{2.23}$$

where $p_i = p(T_{i-1}, x_{i-1}(T_{i-1}))$ satisfying

$$|p(t, x_i(t)) - p_i| < \varepsilon, \text{ for } T_{i-1} < t \le T_i, \tag{2.24}$$

and

$$\psi_{x_j}(t) \doteq \int_{T_{j-1}}^{T_j} \frac{(t-s)^{-p_i}x'(s)}{\Gamma(1-p_i)}ds = \int_{T_{j-1}}^{T_j} \frac{(t-s)^{1-p_i}x_j(s)}{\Gamma(2-p_i)}ds - \frac{(t-T_j)^{1-p_i}}{\Gamma(2-p_i)} \int_{T_{j-1}}^{T_j} x_j(s)ds, \tag{2.25}$$

$j = 1, 2, \ldots, i-1$, $i = 5, 6, \cdots, n^*$, $T_0 = 0, T_{n^*} = T$. For details, please refer to the analysis of problems (2.2), (2.8), (2.15) and (2.22).

From the arguments above, we obtain a function $x^* \in C[0, T]$ defined by

$$x^*(t) = \begin{cases} x_1(t), & 0 \le t \le T_1, \\ x_2(t), & T_1 \le t \le T_2, \\ \vdots \\ x_{n^*}(t), & T_{n^*-1} \le t \le T. \end{cases} \tag{2.26}$$

Now, we propose the new definition of approximate solution to the initial value problem (1.1), which is crucial in our work.

Definition 2.1. *If there exists natural number $n^* \in N$ and intervals $[0, T_1], (T_1, T_2], \cdots, (T_{n^*-1}, T]$, and initial value problems (2.2), (2.8), (2.15), (2.22), (2.23) exist solutions $x_1 \in C^1(0, T] \cap C[0, T]$, $x_2 \in C^1(T_1, T] \cap C[T_1, T]$, $x_3 \in C^1(T_2, T] \cap C[T_2, T]$, $x_4 \in C^1(T_3, T] \cap C[T_3, T]$, $x_i \in C^1(T_{i-1}, T] \cap C[T_{i-1}, T]$ $(i = 5, \cdots, n^*)$ respectively, then the function $x \in C[0, T]$ defined by*

$$x(t) = \begin{cases} x_1(t), 0 \le t \le T_1, \\ x_2(t), T_1 \le t \le T_2, \\ x_3(t), T_2 \le t \le T_3, \\ \vdots \\ x_{n^*}(t), T_{n^*-1} \le t \le T \end{cases} \tag{2.27}$$

is called an approximate solution of the initial value problem (1.1).

Remark 2.3. *If $x_1(t), x_2(t), \cdots, x_{n^*}(t)$ are both unique, then we say $x(t)$ defined by (2.27) is unique approximate solution of the initial value problem* (1.1).

Remark 2.4. *Based on Remark 2.1, Definition 2.1 seems suitable.*

Remark 2.5. *From Definition 2.1, we notice that approximate solution $x(t)$ of the initial value problem (1.1) in interval $[T_2, T_3]$ is $x_3(t)$ which is a solution of the initial value problem (2.15). Obviously, the state of $x_3(t)$ is affected by the state of $x_1(t)$ and $x_2(t)$. That is, the state of $x(t)$ in interval $[T_2, T_3]$ is affected by the state of $x(t)$ in interval $[0, T_2]$. Hence, Definition 2.1 is suitable and reasonable according to Remark 2.2.*

Remark 2.6. *In our previous analysis, we chose functions (2.6), (2.13), etc, so that we obtain the initial value problems (2.8), (2.15), etc. Such a choice must meet the following three reasons at the same time. The first reason is operability, for instance, choosing function (2.13) enable us to calculate functions $\phi_{x_1}(t), \phi_{x_2}(t)$, and obtain the initial value problem (2.15) defined in $[T_2, T]$. The second reason is for fitting Remark 2.1 and Remark 2.5. If we take $x'(t) = x_1(T_1)$ for $0 \le t \le T_1$ (here $x_1(t)$ is the solution of the initial value problem (2.2)), we may easily calculate function $\varphi_{x_1}(t) = \frac{x_1(T_1)(t^{1-p_1}-(t-T_1)^{1-p_1})}{\Gamma(2-p_1)}$, and then have the initial value problem (2.8) with such $\varphi_{x_1}(t)$. However, we see that the state of solution of problem (2.8) is only affected by $x_1(T_1)$, but not affected by the state of $x_1(t), 0 \le t \le T_1$. The third reason is the rationality of the obtained equation. For instance, according to the definition the Caputo fractional derivative ${}^C D^{p_2}_{T_1+} x(t) = \frac{1}{\Gamma(1-p_2)} \int_{T_1}^{t} (t-s)^{-p_2} x'(s)\, ds$, the solution of initial value problem (2.8) exists only if the term $f(t, \cdot) - \varphi_{x_1}(t)$ is absolutely continuous in the interval $[T_1, T]$, otherwise one can not obtain the existence of solution of the initial value problems (2.8). If we take $x(t) = x_1(t)$ for $0 \le t \le T_1$ (here $x_1(t)$ is the solution of initial value problem (2.2)), we may easily get function $\varphi_{x_1}(t) = \int_0^{T_1} \frac{(t-s)^{-p_2} x_1'(s) ds}{\Gamma(1-p_2)}$, and then obtain initial value problem (2.8) with such $\varphi_{x_1}(t)$. However, we can't obtain the existence of solution of initial value problem (2.8) with this $\varphi_{x_1}(t)$.*

Example 2.1. According to Definition 2.1, we consider the approximate solution of the following initial value problem

$$\begin{cases} {}^C D_{0+}^{\frac{1}{2}+\frac{t}{1000(1+t^2)}+\frac{x(t)}{3(1+x^2(t))}} x(t) = \frac{1}{2000\Gamma(\frac{3}{2})} t^{\frac{1}{2}}, \quad 0 < t \le 1, \\ x(0) = 0. \end{cases} \tag{2.28}$$

By the definition of the variable order Caputo fractional derivative, there is no way to obtain explicit expression of its solution, even hardly conventional methods to study the existence of its solution. Next, according to Definition 2.1, we try to seek its approximate solution.

Here $p(t, x(t)) = \frac{1}{2} + \frac{t}{1000(1+t^2)} + \frac{x(t)}{3(1+x^2(t))}$, then we take $p_1 = p(0, 0) = \frac{1}{2}$.

First, we consider initial value problem as following

$$\begin{cases} {}^C D_{0+}^{\frac{1}{2}} x(t) = \frac{1}{2000\Gamma(\frac{3}{2})} t^{\frac{1}{2}}, \quad 0 < t \le 1, \\ x(0) = 0. \end{cases} \tag{2.29}$$

By simple calculation, the solution of the problem (2.29) is given by

$$x_1(t) = \frac{1}{2000\Gamma(\frac{3}{2})\Gamma(\frac{1}{2})} \int_0^t (t-s)^{-\frac{1}{2}} s^{\frac{1}{2}} ds = \frac{1}{2000} t, \quad 0 \le t \le 1. \tag{2.30}$$

Since $x_1(t)$ is right continuous at point 0, then for $\varepsilon = 0.002$, we take $\delta_{01} = \frac{1}{2}$ such that

$$|x_1(t) - x_1(0)| = \frac{1}{2000}t \le \frac{1}{4000} < 0.002, \text{ for } 0 < t \le \delta_{01} = \frac{1}{2}. \tag{2.31}$$

Notice that $p(t, x_1(t))$ is right continuous at point $(0, x_1(0)) = (0, 0)$, then together with (2.31), for the above $\varepsilon = 0.002$, we take $\delta_0 = \frac{1}{2}$, such that

$$|p(t, x_1(t)) - p(0,0)| = |p(t, x_1(t)) - p_1| \le \frac{t}{1000} + \frac{x_1(t)}{3} = \frac{1}{2000} + \frac{1}{12000} < 0.002 = \varepsilon, \text{ for } 0 < t \le \frac{1}{2}.$$

We take point $T_1 = \delta_0 = \frac{1}{2}$, and let

$$p_2 = p(T_1, x_1(T_1)) = p\Big(\frac{1}{2}, x_1\Big(\frac{1}{2}\Big)\Big) = \frac{1}{2} + \frac{1}{2500} + \frac{4000}{48000003}. \tag{2.32}$$

We denote

$$\varphi_{x_1}(t) = \frac{\int_0^{\frac{1}{2}} (t-s)^{1-p_2} x_1(s)ds}{\Gamma(2-p_2)} - \frac{(t-\frac{1}{2})^{1-p_2} \int_0^{\frac{1}{2}} x_1(s)ds}{\Gamma(2-p_2)}, \quad \frac{1}{2} < t \le 1. \tag{2.33}$$

Next, we consider the following initial value problem

$$\begin{cases} {}^C D^{p_2}_{\frac{1}{2}+} x(t) = \frac{1}{2000\Gamma(\frac{3}{2})} t^{\frac{1}{2}} - \varphi_{x_1}(t), & \frac{1}{2} < t \le 1, \\ x(\frac{1}{2}) = x_1(\frac{1}{2}) = \frac{1}{4000}. \end{cases} \tag{2.34}$$

From the facts of constant order fractional calculus, the solution of the initial value problem (2.34) is

$$x_2(t) = \frac{1}{4000} + \frac{1}{2000\Gamma(\frac{3}{2})\Gamma(p_2)} \int_{\frac{1}{2}}^t (t-s)^{p_2-1} s^{\frac{1}{2}} ds - \frac{1}{\Gamma(p_2)} \int_{\frac{1}{2}}^t (t-s)^{p_2-1} \varphi_{x_1}(s)ds. \tag{2.35}$$

We notice that

$$\begin{aligned} |\varphi_{x_1}(t)| =& \frac{1}{2000\Gamma(2-p_2)} \Big| \int_0^{\frac{1}{2}} (t-s)^{1-p_2} sds - \Big(t - \frac{1}{2}\Big)^{1-p_2} \int_0^{\frac{1}{2}} sds \Big| \\ \le& \frac{1}{2000\Gamma(2-p_2)} \Big[\int_0^{\frac{1}{2}} sds + \int_0^{\frac{1}{2}} sds \Big] \\ <& \frac{1}{2000\Gamma(2-p_2)}. \end{aligned}$$

Since $x_2(t)$ is right continuous at point $T_1 = \frac{1}{2}$, then for $\varepsilon = 0.002$, we take $\delta_{11} = \frac{1}{2}$ such that for $\frac{1}{2} < t \le 1$, we have

$$\begin{aligned} \Big| x_2(t) - x_2\Big(\frac{1}{2}\Big) \Big| \le& \frac{1}{2000\Gamma(\frac{3}{2})\Gamma(p_2)} \int_{\frac{1}{2}}^t (t-s)^{p_2-1} s^{\frac{1}{2}} ds + \frac{1}{\Gamma(p_2)} \int_{\frac{1}{2}}^t (t-s)^{p_2-1} |\varphi_{x_1}(s)| ds \\ \le& \frac{1}{2000\Gamma(\frac{3}{2})\Gamma(p_2)} \int_{\frac{1}{2}}^t (t-s)^{p_2-1} ds + \frac{1}{2000\Gamma(2-p_2)\Gamma(p_2)} \int_{\frac{1}{2}}^t (t-s)^{p_2-1} ds \\ \le& \frac{1}{2000\Gamma(\frac{3}{2})\Gamma(1+p_2)} \Big(t - \frac{1}{2}\Big)^{p_2} + \frac{1}{2000\Gamma(2-p_2)\Gamma(1+p_2)} \Big(t - \frac{1}{2}\Big)^{p_2} \\ \le& \frac{1}{2000\Gamma(\frac{3}{2})\Gamma(1+p_2)} + \frac{1}{2000\Gamma(2-p_2)\Gamma(1+p_2)} \\ \approx& 0.00096 \\ <& 0.002. \end{aligned}$$

Because $p(t, x_2(t))$ is right continuous at point $(T_1, x_2(T_1)) = (\frac{1}{2}, x_2(\frac{1}{2}))$, then together with the estimation above, for $\varepsilon = 0.002$, we take $\delta_1 = \frac{1}{2}$ such that for $\frac{1}{2} < t \leq 1$,

$$\begin{aligned}|p(t, x_2(t)) - p(T_1, x_2(T_1))| =&\left|\frac{t}{1000(1+t^2)} - \frac{\frac{1}{2}}{1000(1+(\frac{1}{2})^2)} + \frac{x_2(t)}{3(1+x_2^2(t))} - \frac{x_2(\frac{1}{2})}{3(1+x_2^2(\frac{1}{2}))}\right| \\ \leq&\frac{1}{1000}\left|t-\frac{1}{2}\right| + \frac{1}{3}\left|x_2(t) - x_2\left(\frac{1}{2}\right)\right| \\ \leq&\frac{1}{2000} + \frac{2}{3000} \\ <&0.002.\end{aligned}$$

From the arguments above, the function $x^* \in C[0, 1]$ defined by

$$x^*(t) = \begin{cases} \frac{1}{2000}t, & 0 \leq t \leq \frac{1}{2}, \\ \frac{1}{4000} + \frac{\int_{\frac{1}{2}}^{t}(t-s)^{p_2-1}s^{\frac{1}{2}}ds}{2000\Gamma(\frac{3}{2})\Gamma(p_2)} - \frac{\int_{\frac{1}{2}}^{t}(t-s)^{p_2-1}\varphi_{x_1}(s)ds}{\Gamma(p_2)}, & \frac{1}{2} \leq t \leq 1 \end{cases} \tag{2.36}$$

is the continuous approximate solution of problem (2.28) according to Definition 2.1(see Figure 1).

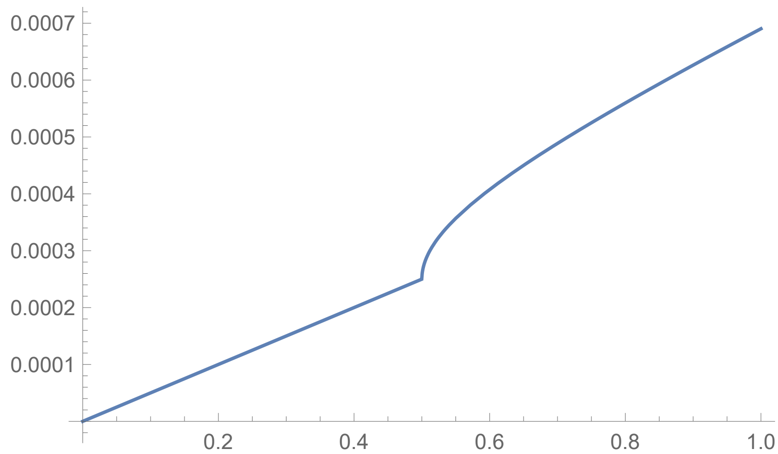

Figure 1. $x^*(t)$ is given by (2.36).

3. Definition of approximate solution for $p(t)$

In this section, we study the initial value problem (1.1) for $p(t, x(t)) \equiv p(t)$. Since $p(t)$ is a function of one variable, we may propose another definition of approximate solution of the initial value problem (1.1) such that we can simplify the analysis process in the section 2.

The following result is crucial for us to propose another definition of approximate solution of the initial value problem (1.1) for the particular order $p(t)$.

Lemma 3.1. *Assume that* $p : [0, T] \to (0, 1)$ *is a continuous function, then for arbitrary small* $\varepsilon > 0$*, there exists natural number* n^* *and intervals* $[0, T_1], (T_1, T_2], \cdots, (T_{n^*-1}, T]$ *and a piecewise function* $\alpha : [0, T] \to (0, 1)$ *defined by*

$$\alpha(t) = \begin{cases} p_1, & t \in [0, T_1], \\ p_2, & t \in (T_1, T_2], \\ \vdots & \\ p_{n^*}, & t \in (T_{n^*-1}, T], \end{cases} \tag{3.1}$$

where $p_k = p(T_{k-1})(k = 1, 2, \cdots, n^*, T_0 = 0)$*, such that for arbitrary small* $\varepsilon > 0$*,*

$$|p(t) - \alpha(t)| < \varepsilon, \quad 0 \le t \le T. \tag{3.2}$$

Proof. Since $p(t)$ is right continuous at point 0, then for arbitrary small $\varepsilon > 0$, there exists $\delta_0 > 0$ such that

$$|p(t) - p(0)| < \varepsilon, \quad \text{for } 0 \le t \le \delta_0. \tag{3.3}$$

We take point $\delta_0 \doteq T_1$ (if $T_1 < T$, we consider continuity of $p(t)$ at point T_1, otherwise, we end this procedure). Since $p(t)$ is right continuous at point T_1, then for the above ε, there exists $\delta_1 > 0$ such that

$$|p(t) - p(T_1)| < \varepsilon, \quad \text{for } T_1 < t \le T_1 + \delta_1. \tag{3.4}$$

We take point $T_1 + \delta_1 \doteq T_2$ (if $T_2 < T$, we consider continuity of $p(t)$ at point T_2, otherwise, we end this procedure). Since $p(t)$ is right continuous at point T_2, then for the above ε, there exists $\delta_2 > 0$ such that

$$|p(t) - p(T_2)| < \varepsilon, \quad \text{for } T_2 < t \le T_2 + \delta_2. \tag{3.5}$$

Since $[0, T]$ is a finite interval, we continue this analysis procedure and could finish it by finite steps. That is, there exists $\delta_{n^*-2} > 0$ and $\delta_{n^*-1} > 0$ ($n^* \in N$) such that $T_{n^*-2} + \delta_{n^*-2} \doteq T_{n^*-1} < T$ and $T_{n^*-1} + \delta_{n^*-1} \ge T$. Then we have intervals $[0, T_1], [T_1, T_2], \cdots, [T_{n^*-2}, T_{n^*-1}], [T_{n^*-1}, T]$, such that for the above ε,

$$|p(t) - p(T_{i-1})| < \varepsilon, \text{ for } T_{i-1} < t \le T_i, \tag{3.6}$$

$i = 1, 2, \ldots, n^*$, $T_0 = 0$ and $T_{n^*} = T$.

We denote

$$p(0) \doteq p_1, p(T_1) \doteq p_2, p(T_2) \doteq p_3, p(T_3) \doteq p_4, \cdots, p(T_{n^*-1}) \doteq p_{n^*}. \tag{3.7}$$

Thus, we define a piecewise function $\alpha : [0, T] \to (0, 1)$ as following

$$\alpha(t) = \begin{cases} p_1, & t \in [0, T_1], \\ p_2, & t \in (T_1, T_2], \\ \vdots & \\ p_{n^*}, & t \in (T_{n^*-1}, T]. \end{cases} \tag{3.8}$$

From (3.3)–(3.8), we obtain that for the arbitrary small $\varepsilon > 0$,

$$|p(t) - \alpha(t)| < \varepsilon, \quad 0 \le t \le T.$$

The proof is completed. □

Similar to the analysis in the section 2, we consider approximate solution of the initial value problem (1.1) in the following sense: if $p(t)$ and $\alpha(t)$ satisfy (3.2), then solution $x(t)$ of the following initial value problem

$$\begin{cases} {}^{C}D_{0+}^{\alpha(t)}x(t) = f(t,x), \quad 0 < t \le T, \\ x(0) = u_0 \end{cases} \tag{3.9}$$

is called the approximate solution of the initial value problem (1.1).

We start off by analyzing the problem (3.9), and then propose a new definition of continuous approximate solutions to the initial value problem (1.1) with $p(t, x(t)) = p(t)$.

For the initial value problem (3.9) in the interval $[0, T_1]$, by (3.1), we have the initial value problem

$$\begin{cases} {}^{C}D_{0+}^{p_1}x(t) = f(t,x), \quad 0 < t \le T_1, \\ x(0) = u_0. \end{cases} \tag{3.10}$$

By similar analysis in section 2, we may consider the initial value problem defined in the interval $[T_1, T_2]$ as following

$$\begin{cases} {}^{C}D_{T_1+}^{p_2}x(t) = f(t,x) - \varphi_{x_1}(t), \quad T_1 < t \le T_2, \\ x(T_1) = x_1(T_1), \end{cases} \tag{3.11}$$

where $x_1(t)$ is the solution of the initial value problem (3.10) and

$$\int_0^{T_1} \frac{(t-s)^{-p_2}x'(s)}{\Gamma(1-p_2)}ds = \int_0^{T_1} \frac{(t-s)^{1-p_2}x_1(s)}{\Gamma(2-p_2)}ds - \frac{(t-T_1)^{1-p_2}}{\Gamma(2-p_2)} \int_0^{T_1} x_1(s)ds \doteq \varphi_{x_1}(t),$$

in which $x'(s) = \int_0^s x_1(\tau)d\tau$.

Using the same method, we may consider the initial value problem defined in the interval $[T_2, T_3]$ as following

$$\begin{cases} {}^{C}D_{T_2+}^{p_3}x(t) = f(t,x) - \phi_{x_1}(t) - \phi_{x_2}(t), T_2 < t \le T_3, \\ x(T_2) = x_2(T_2), \end{cases} \tag{3.12}$$

where $x_1(t)$ is the solution of the initial value problem (3.10), $x_2(t)$ is the solution of the initial value problem (3.11) and $\phi_{x_i}(t)$ is the function defined by

$$\int_{T_{i-1}}^{T_i} \frac{(t-s)^{-p_3}x'(s)}{\Gamma(1-p_3)}ds = \int_{T_{i-1}}^{T_i} \frac{(t-s)^{1-p_3}x_i(s)}{\Gamma(2-p_3)}ds - \frac{(t-T_i)^{1-p_3}}{\Gamma(2-p_3)} \int_{T_{i-1}}^{T_i} x_i(s)ds \doteq \phi_{x_i}(t),$$

in which $x'(s) = \int_{T_{i-1}}^s x_i(\tau)d\tau$, $i = 1, 2, T_0 = 0$.

Similarly, we may consider the initial value problem defined in the interval $[T_{i-1}, T_i]$ as following

$$\begin{cases} {}^{C}D_{T_{i-1}+}^{p_i}x(t) = f(t,x) - \psi_{x_1}(t) - \psi_{x_2}(t) - \cdots - \psi_{x_{i-1}}(t), T_{i-1} < t \le T_i, \\ x(T_{i-1}) = x_{i-1}(T_{i-1}), \end{cases} \tag{3.13}$$

where $x_1(t)$ is the solution of the initial value problem (3.10), $x_2(t)$ is the solution of the initial value problem (3.11), $x_3(t)$ is the solution of the initial value problem (3.12), $x_i(t)$ is the solution of the initial value problem (3.13) and ψ_{x_j} is the function defined by

$$\int_{T_{j-1}}^{T_j} \frac{(t-s)^{-p_i}x'(s)}{\Gamma(1-p_i)}ds = \int_{T_{j-1}}^{T_j} \frac{(t-s)^{1-p_i}x_j(s)}{\Gamma(2-p_i)}ds - \frac{(t-T_j)^{1-p_i}}{\Gamma(2-p_i)} \int_{Tj-1}^{T_j} x_j(s)ds \doteq \psi_{x_j}(t),$$

in which $x'(s) = \int_{T_{j-1}}^{s} x_j(\tau)d\tau$, $j = 1, 2, \cdots, i-1$, $i = 4, \cdots, n^*$, $T_0 = 0, T_{n^*} = T$.

Based on the arguments above, we propose the definition of solution to the initial value problem (3.9), which is crucial in our work.

Definition 3.1. *If the initial value problems (3.10), (3.11), (3.12) and (3.13) exist solutions* $x_1 : [0, T_1] \to \mathbb{R}$, $x_2 : [T_1, T_2] \to \mathbb{R}$, $x_3 : [T_2, T_3] \to \mathbb{R}$, $x_i : [T_{i-1}, T_i] \to \mathbb{R}$*(*$i = 4, \cdots, n^*$, $T_{n^*} = T$*) respectively, then we call function* $x : [0, T] \to \mathbb{R}$ *defined by*

$$x(t) = \begin{cases} x_1(t), & 0 \le t \le T_1, \\ x_2(t), & T_1 \le t \le T_2, \\ x_3(t), & T_2 \le t \le T_3, \\ \vdots \\ x_{n^*}(t), & T_{n^*-1} \le t \le T \end{cases} \tag{3.14}$$

is a solution of the initial value problem (3.9).

Remark 3.1. *If* $x_1(t), x_2(t), \cdots, x_{n^*}(t)$ *are unique, then we say* $x(t)$ *defined by (3.14) is unique solution of the initial value problem (3.9).*

The following is the definition of the approximate solution of the initial value problem (1.1) with $p(t, x(t)) \equiv p(t)$.

Definition 3.2. *We call the solution of initial value problem (3.9) defined by Definition 3.1 is an (unique) approximate solution of initial value problem (1.1) with* $p(t, x(t)) = p(t)$ *if* $\alpha(t)$ *is defined by Lemma 3.1.*

Remark 3.2. *For the initial value problem (1.1) with* $p(t, x(t)) \equiv p(t)$*, its approximate solution defined by Definition 3.2 is consistent with by Definition 2.1.*

Example 3.1. We consider the following initial value problem for linear equation

$$\begin{cases} {}^{C}D_{0+}^{\frac{1}{3}+\frac{t}{1000(1+t^2)}} x(t) = t, & 0 < t \le 1, \\ x(0) = 0. \end{cases} \tag{3.15}$$

By the definition of the variable order Caputo fractional derivative, there is no way to obtain explicit expression of its solution, even hardly conventional methods to study the existence of its solution. Next, we try to seek its approximate solution in the sense of Definition 3.2.

Here $p(t) = \frac{1}{3} + \frac{t}{1000(1+t^2)}$. Obviously, $p(t)$ is continuous on $[0, 1]$ and $0 < p(t) < 1$.

By the right continuity of function $p(t)$ at point 0, for given arbitrary small $\varepsilon = 0.00035$, taking $\delta_0 = \frac{1}{3}$, when $0 \le t \le \delta_0 = \frac{1}{3}$, we have

$$|p(t) - p(0)| = \left|\frac{t}{1000(1+t^2)}\right| \le \frac{t}{1000} \le \frac{\delta_0}{1000} < \frac{1}{3000} < \varepsilon.$$

Then, we get $T_1 = \delta_0 = \frac{1}{3}$. By the right continuity of function $p(t)$ at point T_1, for the above ε, taking $\delta_1 = \frac{1}{3}$, when $T_1 < t \leq T_1 + \delta_1 (\frac{1}{3} < t \leq \frac{2}{3})$, by differential mean value theorem, we have

$$\begin{aligned}|p(t) - p(T_1)| =& \left| \frac{t}{1000(1+t^2)} - \frac{T_1}{1000(1+T_1^2)} \right| \\ \leq& \left| \frac{1-\xi^2}{1000(1+\xi^2)^2} \right| |t - T_1| \\ \leq& \frac{1+\xi^2}{1000(1+\xi^2)^2} |t - T_1| \\ \leq& \frac{1}{1000} |t - T_1| \\ \leq& \frac{\delta_1}{1000} \\ <& \frac{1}{3000} < \varepsilon,\end{aligned}$$

where $T_1 < \xi < t < \frac{2}{3}$.

We let $T_2 = T_1 + \delta_1 = \frac{2}{3}$. By the right continuity of function $p(t)$ at point T_2, for $\varepsilon = 0.00035$, taking $\delta_2 = \frac{1}{3}$, when $T_2 < t \leq T_2 + \delta_2 (\frac{2}{3} < t \leq 1)$, we have

$$|p(t) - p(T_2)| = \left| \frac{t}{1000(1+t^2)} - \frac{T_2}{1000(1+T_2^2)} \right| \leq \frac{\delta_2}{1000} < \frac{1}{3000} < \varepsilon.$$

We see that $T_2 + \delta_2 = 1$, hence, we obtain three intervals $[0, \frac{1}{3}]$, $(\frac{1}{3}, \frac{2}{3}]$, $(\frac{2}{3}, 1]$ and piecewise constant function $\alpha(t)$ defined by

$$\alpha(t) = \begin{cases} p_1 = p(0) = \frac{1}{3}, & t \in [0, \frac{1}{3}], \\ \\ p_2 = p(\frac{1}{3}) = \frac{10009}{30000}, & t \in (\frac{1}{3}, \frac{2}{3}], \\ \\ p_3 = p(\frac{2}{3}) = \frac{6509}{19500}, & t \in (\frac{2}{3}, 1], \end{cases}$$

which satisfies

$$|\alpha(t) - p(t)| < 0.00035 = \varepsilon. \tag{3.16}$$

Thus, according to Definition 3.2, we first consider the following initial value problem

$$\begin{cases} {}^C D_{0+}^{p_1} x(t) = t, & 0 < t \leq \frac{1}{3}, \\ x(0) = 0. \end{cases} \tag{3.17}$$

By the facts of constant order fractional calculus, the unique solution of the initial value problem (3.17) is

$$x_1(t) = \frac{1}{\Gamma(\frac{7}{3})} t^{\frac{4}{3}}, \quad 0 \leq t \leq \frac{1}{3}. \tag{3.18}$$

Let

$$\varphi_{x_1}(t) = \int_0^{\frac{1}{3}} \frac{(t-s)^{1-p_2} x_1(s)}{\Gamma(2-p_2)} ds - \frac{(t-\frac{1}{3})^{1-p_2}}{\Gamma(2-p_2)} \int_0^{\frac{1}{3}} x_1(s) ds. \tag{3.19}$$

Next, we consider the following initial value problem

$$\begin{cases} {}^{C}D^{p_2}_{\frac{1}{3}+}x(t) = t - \varphi_{x_1}(t), & \frac{1}{3} < t \leq \frac{2}{3}, \\ x(\frac{1}{3}) = x_1(\frac{1}{3}), \end{cases} \tag{3.20}$$

where x_1 and ψ_{x_1} are given by (3.18) and (3.19) respectively. By simple calculation, we get the solution of the initial value problem (3.20) as following

$$x_2(t) = \int_{\frac{1}{3}}^{t} \frac{(t-s)^{p_2-1}}{\Gamma(p_2)}(s - \varphi_{x_1}(s))ds + x_1\Big(\frac{1}{3}\Big), \quad \frac{1}{3} \leq t \leq \frac{2}{3}. \tag{3.21}$$

Let

$$\begin{cases} \psi_{x_1}(t) = \int_0^{\frac{1}{3}} \frac{(t-s)^{1-p_3}x_1(s)}{\Gamma(2-p_3)}ds - \frac{(t-\frac{1}{3})^{1-p_3}}{\Gamma(2-p_3)} \int_0^{\frac{1}{3}} x_1(s)ds, \\ \\ \psi_{x_2}(t) = \int_{\frac{1}{3}}^{\frac{2}{3}} \frac{(t-s)^{1-p_3}x_2(s)}{\Gamma(2-p_3)}ds - \frac{(t-\frac{2}{3})^{1-p_3}}{\Gamma(2-p_3)} \int_{\frac{1}{3}}^{\frac{2}{3}} x_2(s)ds. \end{cases} \tag{3.22}$$

Finally, we consider the following initial value problem

$$\begin{cases} {}^{C}D^{p_3}_{\frac{2}{3}+}x(t) = t - \psi_{x_1}(t) - \psi_{x_2}(t), & \frac{2}{3} < t \leq 1, \\ x(\frac{2}{3}) = x_2(\frac{2}{3}), \end{cases} \tag{3.23}$$

where x_1 is given by (3.18), x_2 is presented by (3.21) and $\psi_{x_i}(i = 1, 2)$ is given by (3.22). Thus, the solution of the initial value problem (3.23) is obtained by

$$x_3(t) = \int_{\frac{2}{3}}^{t} \frac{(t-s)^{p_3-1}}{\Gamma(p_3)}(s - \psi_{x_1}(s) - \psi_{x_2}(s))ds + x_2\Big(\frac{2}{3}\Big), \quad \frac{2}{3} \leq t \leq 1. \tag{3.24}$$

According to Definition 3.2, the approximate solution of initial value problem (3.15) is given by

$$x(t) = \begin{cases} x_1(t) = \frac{1}{\Gamma(\frac{7}{3})}t^{\frac{4}{3}}, \quad 0 \leq t \leq \frac{1}{3}, \\ \\ x_2(t) = \int_{\frac{1}{3}}^{t} \frac{(t-s)^{p_2-1}}{\Gamma(p_2)}(s - \varphi_{x_1}(s))ds + x_1(\frac{1}{3}), \quad \frac{1}{3} \leq t \leq \frac{2}{3} \\ \\ x_3(t) = \int_{\frac{2}{3}}^{t} \frac{(t-s)^{p_3-1}}{\Gamma(p_3)}(s - \psi_{x_1}(s) - \psi_{x_2}(s))ds + x_2(\frac{2}{3}), \quad \frac{2}{3} \leq t \leq 1. \end{cases} \tag{3.25}$$

Obviously, $x(t)$ defined by (3.25) is continuous.

Example 3.2. We consider the continuous approximate solution of initial value problem (3.15) in the sense of Definition 2.1.

According Definition 2.1, let

$$p_0 = p(0) = \frac{1}{3}. \tag{3.26}$$

We first consider the following initial value problem

$$\begin{cases} {}^{C}D^{p_1}_{0+}x(t) = t, & 0 < t \leq 1, \\ x(0) = 0. \end{cases} \tag{3.27}$$

By the arguments above, we know that the solution of initial value problem (3.27) is

$$x_1(t) = \frac{1}{\Gamma(\frac{7}{3})} t^{\frac{4}{3}}, \quad 0 \le t \le 1. \tag{3.28}$$

Since $p(t)$ is right continuous at point 0, for $\varepsilon = 0.00035$, there exists $\delta_0 = \frac{1}{3}$ such that

$$|p(t) - p(0)| < \varepsilon, \quad 0 < t \le \delta_0 = \frac{1}{3}.$$

We take $T_1 = \delta_0 = \frac{1}{3}$. Let

$$p_2 = p(T_1) = p\Big(\frac{1}{3}\Big). \tag{3.29}$$

By Definition 2.1, we next seek the solution of the initial value problem

$$\begin{cases} {}^C D^{p_2}_{\frac{1}{3}+} x(t) = t - \varphi_{x_1}(t), & \frac{1}{3} < t \le 1, \\ x(\frac{1}{3}) = x_1(\frac{1}{3}), \end{cases} \tag{3.30}$$

where $x_1(t)$ is the function given by (3.28), and $\varphi_{x_1}(t)$ is the function defined by

$$\varphi_{x_1}(t) = \int_0^{\frac{1}{3}} \frac{(t-s)^{1-p_2} x_1(s)}{\Gamma(2-p_2)} ds - \frac{(t-\frac{1}{3})^{1-p_2}}{\Gamma(2-p_2)} \int_0^{\frac{1}{3}} x_1(s) ds. \tag{3.31}$$

Thus, the solution of the initial value problem (3.30) is

$$x_2(t) = \int_{\frac{1}{3}}^t \frac{(t-s)^{p_2-1}}{\Gamma(p_2)} (s - \varphi_{x_1}(s)) ds + x_1\Big(\frac{1}{3}\Big), \quad \frac{1}{3} \le t \le 1. \tag{3.32}$$

Since $p(t)$ is right continuous at point T_1, for $\varepsilon = 0.00035$, there exists $\delta_1 = \frac{1}{3}$ such that

$$|p(t) - p(T_1)| = \left|p(t) - p\Big(\frac{1}{3}\Big)\right| < \varepsilon, \quad \frac{1}{3} < t \le \frac{1}{3} + \delta_1 = \frac{2}{3}. \tag{3.33}$$

We take $T_2 = T_1 + \delta_1 = \frac{2}{3}$. Let

$$p_3 = p(T_2) = p\Big(\frac{2}{3}\Big). \tag{3.34}$$

According to Definition 2.1, then we seek the solution of the initial value problem

$$\begin{cases} {}^C D^{p_3}_{\frac{2}{3}+} x(t) = t - \psi_{x_1}(t) - \psi_{x_2}(t), & \frac{2}{3} < t \le 1, \\ x(\frac{2}{3}) = x_2(\frac{2}{3}), \end{cases} \tag{3.35}$$

where $x_1(t)$ is given by (3.28), $x_2(t)$ is presented by (3.32) and $\psi_{x_i}(t)(i = 1, 2)$ is the function as followings

$$\begin{cases} \psi_{x_1}(t) = \int_0^{\frac{1}{3}} \frac{(t-s)^{1-p_3} x_1(s)}{\Gamma(2-p_3)} ds - \frac{(t-\frac{1}{3})^{1-p_3}}{\Gamma(2-p_3)} \int_0^{\frac{1}{3}} x_1(s) ds, \\ \psi_{x_2}(t) = \int_{\frac{1}{3}}^{\frac{2}{3}} \frac{(t-s)^{1-p_3} x_2(s)}{\Gamma(2-p_3)} ds - \frac{(t-\frac{2}{3})^{1-p_3}}{\Gamma(2-p_3)} \int_{\frac{1}{3}}^{\frac{2}{3}} x_2(s) ds. \end{cases} \tag{3.36}$$

Thus, the solution of the initial value problem (3.35) is

$$x_3(t) = \int_{\frac{2}{3}}^{t} \frac{(t-s)^{p_3-1}}{\Gamma(p_3)}(s - \psi_{x_1}(s) - \psi_{x_2}(s))ds + x_2\Big(\frac{2}{3}\Big), \quad \frac{2}{3} \le t \le 1, \tag{3.37}$$

where $\psi_{x_i}(i = 1, 2)$ is given by (3.36).

Hence, the approximate solution of initial value problem (3.15) in the sense of Definition 2.1 is given by

$$x(t) = \begin{cases} x_1(t) = \frac{1}{\Gamma(\frac{7}{3})} t^{\frac{4}{3}}, \quad 0 \le t \le \frac{1}{3}, \\ x_2(t) = \int_{\frac{1}{3}}^{t} \frac{(t-s)^{p_2-1}}{\Gamma(p_2)}(s - \varphi_{x_1}(s))ds + x_1(\frac{1}{3}), \quad \frac{1}{3} \le t \le \frac{2}{3}, \\ x_3(t) = \int_{\frac{2}{3}}^{t} \frac{(t-s)^{p_3-1}}{\Gamma(p_3)}(s - \psi_{x_1}(s) - \psi_{x_2}(s))ds + x_2(\frac{2}{3}), \quad \frac{2}{3} \le t \le 1. \end{cases} \tag{3.38}$$

Obviously, $x(t)$ defined by (3.38) is continuous.

From the analysis above, for the same ε and the same interval $[0, \frac{1}{3}]$, $[\frac{1}{3}, \frac{2}{3}]$, $[\frac{2}{3}, 1]$, continuous approximate solution of initial value problem (3.15) obtained by Definition 3.2 is consistent with which is obtained by Definition 2.1.

4. Conclusions

This paper propose a new definition of continuous approximate solution to initial value problem for differential equations involving variable order Caputo fractional derivative on finite interval. It provide a new method to consider the existence of solution to more general variable order fractional differential equations. This method is different from common numerical methods and it also can be apply to variable order fractional boundary value problem.The new results generalize some existing results in the literature.

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Conflict of interest

The authors declare no conflict of interest.

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